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Quasiregular mappings and cohomology

by

MARIO BONK and JUHA HEINONEN

University of Michigan
Ann Arbor, MI, U.S.A.

University of Michigan
Ann Arbor, MI, U.S.A.

1. Introduction

The main result of this paper is the following statement.

THEOREM 1.1. *Let N be a closed, connected and oriented Riemannian n-manifold, $n\geqslant 2$. If there exists a nonconstant K-quasiregular mapping $f\colon \mathbf{R}^n \to N$, then*

$$\dim H^*(N) \leqslant C(n,K), \tag{1.2}$$

where $\dim H^*(N)$ is the dimension of the de Rham cohomology ring $H^*(N)$ of N, and $C(n,K)$ is a constant only depending on n and K.

As will be discussed shortly, Theorem 1.1 provides first examples of compact manifolds with small fundamental group that do not receive nonconstant quasiregular mappings from Euclidean space.

Recall that a continuous mapping $f\colon X\to Y$ between connected and oriented Riemannian n-manifolds, $n\geqslant 2$, is *K-quasiregular*, $K\geqslant 1$, if the first distributional derivatives of f in local charts are locally n-integrable and if the (formal) differential $Df(x)$: $T_xX\to T_{f(x)}Y$ satisfies

$$|Df(x)|^n \leqslant K \det Df(x) \tag{1.3}$$

for almost every $x\in X$. In (1.3), and throughout this paper, $|Df(x)|$ denotes the operator norm of the linear map $Df(x)$, and $\det Df(x)$ its determinant. We say that a mapping is *quasiregular* if it is K-quasiregular for some $K\geqslant 1$. The synonymous term a *mapping of bounded distortion* is also used in the literature.

Nonconstant quasiregular mappings are discrete (the preimage of each point is a discrete set) and open according to a deep theorem of Reshetnyak [Re1]. Thus, quasiregular mappings are generalized branched coverings with geometric control given by (1.3).

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Note that the mild smoothness assumption, the membership in the Sobolev class $W^{1,n}_{\mathrm{loc}}$, is essential for an interesting theory; every smooth quasiregular mapping in dimension $n \geqslant 3$ is locally injective. We refer to [Re2], [Ri4], [MR] for the basic theory of quasiregular mappings.

Following a terminology suggested by Gromov [Gr4], we call an arbitrary (oriented and connected) Riemannian n-manifold N *K-quasiregularly elliptic* if there exists a nonconstant K-quasiregular mapping $f: \mathbf{R}^n \to N$. Also, N is *quasiregularly elliptic*, or simply *elliptic*, if it is K-quasiregularly elliptic for some $K \geqslant 1$.

Few examples of closed quasiregularly elliptic manifolds are known. In fact, the only examples that are known to the authors are the quotients of products of spheres, tori and complex projective spaces. We note that Gromov in [Gr4, pp. 63 ff.] defines a notion of ellipticity based on Lipschitz mappings with "nonzero degree at infinity". In particular, he raises the interesting question whether the ellipticity of a manifold in his sense is equivalent to quasiregular ellipticity as defined above.

Before describing the idea behind the proof of Theorem 1.1, we review what was known before about the quasiregular ellipticity problem, and point out some corollaries.

First of all, if $n=2$ and $K=1$ in (1.3), we recover precisely the holomorphic functions of one variable; by the uniformization theorem, in dimension $n=2$, we may assume without loss of generality that X and Y have a conformal structure. Up to diffeomorphism, there are only two compact Riemann surfaces that receive nonconstant quasiregular maps from $\mathbf{R}^2 = \mathbf{C}$, namely the sphere S^2 and the torus T^2. This follows from the quasiregular Liouville theorem; since the universal covering space of every compact Riemann surface of genus at least two is conformally equivalent to the unit disk in $\mathbf{R}^2$, no such surface is elliptic.

Indeed, if we restrict ourselves to Riemannian manifolds which are 1-quasiregularly elliptic, then a complete classification is possible in all dimensions. The following proposition may well be known to the experts, but we have not found it explicitly stated in the literature.

PROPOSITION 1.4. *Let N be a closed, connected and oriented Riemannian n-manifold, $n \geqslant 3$. If there exists a nonconstant 1-quasiregular mapping $f: \mathbf{R}^n \to N$, then N is conformally equivalent to a quotient of the standard sphere S^n or a flat torus T^n.*

Proof. It is well-known that, for $n \geqslant 3$, there exists a constant $K(n) > 1$ such that, for $1 \leqslant K \leqslant K(n)$, every K-quasiregular map between Riemannian n-manifolds is a local homeomorphism [Re2, Theorem II.10.5, p. 232]. Moreover, a locally homeomorphic quasiregular map from $\mathbf{R}^n$ into a simply-connected oriented Riemannian manifold is a homeomorphism onto its image, and omits at most one point [Gr4, p. 336].

We apply the latter remark to the Riemannian universal cover $\widetilde{N}$ of N, and to a lift $\tilde{f}\colon \mathbf{R}^n \to \widetilde{N}$ of f. Then $\tilde{f}$ will be a 1-quasiregular (i.e. conformal) homeomorphism onto its image, which is either $\widetilde{N}$ minus a point p_0 or all of $\widetilde{N}$. In the first case, the inverse map $\tilde{f}^{-1}$ has a removable singularity at p_0 and can be extended to a conformal map from $\widetilde{N}$ to the standard sphere S^n. Hence N is conformally equivalent to a quotient of S^n by a group of Möbius transformations that acts without fixed points. If $\tilde{f}(\mathbf{R}^n)=\widetilde{N}$, then $\tilde{f}$ gives a conjugation of the group action of the deck transformations on $\widetilde{N}$ to a Bieberbach group acting on $\mathbf{R}^n$ cocompactly and without fixed points. It follows that N is conformally equivalent to a quotient of a flat torus. The proposition follows. □

Most of the previous results on quasiregularly elliptic manifolds were based on the growth behavior of the fundamental group. We next summarize what is known in this respect in the following theorem due to Varopoulos (see [VSC, pp. 146–147]).

THEOREM 1.5. *Let N be a closed, connected, quasiregularly elliptic Riemannian n-manifold, $n\geqslant 2$. Then the fundamental group of N is virtually nilpotent and has polynomial growth of degree at most n.*

The proof of Theorem 1.5 is based on Gromov's theorem that characterizes the virtually nilpotent finitely generated groups in terms of their growth [Gr3], and on the analysis of isoperimetric inequalities and n-parabolicity. See also [CHS].

The following corollary to Theorem 1.5 seems to have gone unnoticed in the literature.

COROLLARY 1.6. *Let N be a closed, connected, quasiregularly elliptic Riemannian n-manifold, $n\geqslant 2$. Then $\dim H^1(N)\leqslant n$. In particular, if $n=2$ or $n=3$, then*

$$\dim H^*(N) \leqslant 2^n. \tag{1.7}$$

Proof. For the volume growth function $V(r)$ on the universal cover $\widetilde{N}$ of N, we have $V(r)=O(r^D)$ as $r\to\infty$ for some integer $D\leqslant n$. This follows from Theorem 1.5. We claim that $k=\dim H^1(N)\leqslant D\leqslant n$.

To see this, let M be a positive integer, and observe that there are $(2M+1)^k$ lattice points in $\mathbf{Z}^k\subseteq\mathbf{R}^k\cong H^1(N)$ with coordinates bounded in absolute value by M. Let $\gamma_1, \dots, \gamma_k\in\Gamma\cong\pi_1(N)$ be deck transformations corresponding to a basis of $H_1(N,\mathbf{R})\cong H^1(N)$ under the canonical map $\pi_1(N)\to H_1(N,\mathbf{R})$. By applying $\gamma_1^{a_1}\dots\gamma_k^{a_k}$ with $|a_i|\leqslant M$ to a point $x_0\in\widetilde{N}$, we see that there are at least $(2M+1)^k$ images of x_0 under Γ contained in a ball centered at x_0 with radius at most CM. But these images are uniformly separated, so that

$$(2M+1)^k \leqslant C\,\mathrm{Vol}(B(x_0, CM)) \leqslant CM^D,$$

where $C>0$ is independent of M. Because M was arbitrary, we obtain $k \leqslant D \leqslant n$ as required.

The second claim follows from the first by Poincaré duality. □

If we denote by $Q(n,K)$ the smallest integer that we can take in (1.2), then it is an interesting problem to decide whether in fact $Q(n,K) \leqslant Q(n)$ for each $K \geqslant 1$. One could even ask whether the torus T^n gives the extremal answer, i.e., whether

$$Q(n,K) = \dim H^*(T^n) = 2^n \tag{1.8}$$

for each $n \geqslant 2$ and $K \geqslant 1$. By Corollary 1.6, this is true in dimensions $n=2$ and $n=3$.

Theorem 1.1 bears a (modest) similarity to Gromov's celebrated result stating that the sum of the Betti numbers (over any coefficient field) of each closed manifold with nonnegative sectional curvature is bounded from above by a dimensional constant. It has been conjectured that this constant is 2^n as in the case of the torus. (See [Gr2].)

Quasiregularly elliptic manifolds in dimension $n=3$ have been studied by Jormakka [J]. His analysis is based on the growth behavior of the fundamental group and follows from Theorem 1.5. In particular, the only 3-dimensional elliptic manifold that can be expressed as a nontrivial connected sum is $\mathbf{RP}^3 \# \mathbf{RP}^3$ (which is a quotient of $S^1 \times S^2$). Jormakka also shows that if the geometrization conjecture for 3-manifolds is true (including the Poincaré conjecture, see [T]), then all compact quasiregularly elliptic 3-manifolds are quotients of either S^3, T^3 or $S^1 \times S^2$.

In dimension $n=4$, there already are simply-connected manifolds that can be expressed as nontrivial connected sums. In particular, Gromov and Rickman have asked whether $S^2 \times S^2 \# S^2 \times S^2$ (or, more generally, any simply-connected oriented and closed manifold) is quasiregularly elliptic [Ri3], [Gr1, p. 200], [Gr4, 2.41]. Theorem 1.1 does not answer this interesting question, but it does follow from Theorem 1.1 that if

$$S^l \times S^l \# \dots \# S^l \times S^l, \quad l \geqslant 2,$$

receives a nonconstant K-quasiregular mapping from $\mathbf{R}^{2l}$, then the number of summands has an upper bound depending only on l and K. We note that $S^l \times S^l$ is elliptic [Ri3, p. 183].

Theorem 1.1 can be used to decide the nonellipticity of some compact manifolds with small fundamental group. The following corollary answers an explicit question of Rickman [Ri3].

COROLLARY 1.9. *Let X and Y be arbitrary closed, connected and oriented Riemannian manifolds of dimension n and $n-1$, respectively. Assume that X has nontrivial cohomology in some dimension $l=1, \dots, n-1$. Then the manifold $N = X \# S^1 \times Y$ does not receive nonconstant quasiregular maps from $\mathbf{R}^n$.*

Proof. Assume that $f: \mathbf{R}^n \to X \# S^1 \times Y$ is K-quasiregular. Then f has a K-quasiregular lift $f_1: \mathbf{R}^n \to N_1$, where N_1 is a connected sum of $\mathbf{R} \times Y$ with infinitely many copies of X. This lift can be projected further to a K-quasiregular map $f_2: \mathbf{R}^n \to N_2$ to a compact manifold N_2 which is a connected sum of $S^1 \times Y$ with a finite, but arbitrarily large, number of copies of X. So we can make the dimension of the cohomology ring of N_2 exceed any prescribed bound. By Theorem 1.1, f_2 is constant as soon as $\dim H^*(N_2)$ is large enough, depending only on n and K. Thus f is constant and the corollary follows. $\square$

In the special case where $Y = T^{n-1}$ (so that $N = X \# T^n$), Corollary 1.9 was proved by Peltonen [P]. She used path family methods that are not strong enough to give the more general result of Corollary 1.9.

Naturally, Corollary 1.9 admits a more general formulation. One can take a closed (oriented and connected) n-manifold Z with the property that the fundamental group $\pi_1(Z)$ has subgroups of arbitrarily large (finite) index that act cocompactly on the universal cover $\widetilde{Z}$. Then $X \# Z$ is not quasiregularly elliptic if X is as in Corollary 1.9.

One can also consider noncompact manifolds in a similar vein. For example, if N is the connected sum of infinitely many copies of $S^l \times S^l$, with a (natural) isometric $\mathbf{Z}$-action, then N does not receive nonconstant quasiregular maps from $\mathbf{R}^{2l}$; i.e., N is nonelliptic. The reader can imagine more examples of this sort.

Recall that Rickman [Ri1] has proved that if $a_1, ..., a_p$ are distinct points in S^n, then, in our terminology, the manifold $S^n \setminus \{a_1, ..., a_p\}$ is K-quasiregularly elliptic only if $p \leqslant R(n, K) < \infty$. Although Theorem 1.1 can be viewed as a Picard-type theorem about the existence of quasiregular mappings, it does not imply this result of Rickman. It would be interesting to have a more general statement that would simultaneously embrace both Theorem 1.1 and the Rickman–Picard theorem. Note that Rickman's theorem is qualitatively sharp in dimension $n=3$ [Ri2].

We do not know what noncompact manifolds can replace $\mathbf{R}^n$ in Theorem 1.1. Holopainen and Rickman [HR] have established the following general Picard-type theorem: Assume that M is a complete Riemannian n-manifold satisfying both a Poincaré-type inequality and a doubling condition on its volume. If $f: M \to N$ is a nonconstant K-quasiregular mapping to an arbitrary (oriented and connected) manifold N, then the number of ends of N is bounded by a constant that depends only on n and K, and on the data associated with the Poincaré and doubling conditions. It is not hard to construct manifolds M satisfying the hypotheses of Holopainen and Rickman, and such that M covers compact manifolds of arbitrarily large cohomology. For instance, one can take M to be the universal cover of $N = S^l \times S^l \# T^{2l}$.

Finally, we state a result which shows that there is a lower growth bound for every nonconstant quasiregular map $f: \mathbf{R}^n \to N$, if N is compact and not a homology sphere. Recall that the *averaged counting function* $A(r)$ of f is defined as

$$A(r) = \frac{1}{\mathrm{Vol}(N)} \int_{B(0,r)} \det Df(x)\, dx, \tag{1.10}$$

where $\mathrm{Vol}(N)$ is the total volume of N, and $B(0,r)$ is the open n-ball in $\mathbf{R}^n$ centered at the origin with radius $r>0$. The following theorem answers another question of Rickman.

THEOREM 1.11. *Let $f: \mathbf{R}^n \to N$ be a nonconstant K-quasiregular mapping into a closed, connected and oriented Riemannian n-manifold N, $n \geqslant 2$. If the l-th cohomology group $H^l(N)$ of N is nontrivial for some $l=1, \dots, n-1$, then there exists a positive constant $\alpha = \alpha(n,K) > 0$ such that*

$$\liminf_{r\to\infty} \frac{A(r)}{r^\alpha} > 0, \tag{1.12}$$

where $A(r)$ is the averaged counting function defined in (1.10).

Theorem 1.11 is sharp in the sense that if N is a homology n-sphere covered by S^n, then one can exhibit quasiregular mappings $f: \mathbf{R}^n \to N$ with arbitrarily slow but prescribed growth for $A(r)$. Indeed, it is not hard to find such maps to S^n, and the general case is obtained by postcomposing with a covering projection $S^n \to N$. We also note that the exponent α in (1.12) can be as small as one pleases if we make K large enough: take a slowly growing quasiconformal mapping $g: \mathbf{R}^n \to \mathbf{R}^n$ and postcompose g with a fixed uniformly continuous quasiregular mapping $\mathbf{R}^n \to N$, if N is elliptic.

To illustrate Theorem 1.11 by an example, let us consider the case $N = S^l \times S^l$, $n = 2l$. Then N is elliptic. If $l=1$, then $N = T^2$ and every K-quasiregular mapping $f: \mathbf{R}^2 \to T^2$ factors, $f = \pi \circ g$, where $g: \mathbf{R}^2 \to \mathbf{R}^2$ is K-quasiregular and π is the covering projection. By the properties of π, it is easy to see that (1.12) holds in this case. On the other hand, if $l \geqslant 2$, then N is simply-connected and no covering space argument can be used.

We next discuss the idea behind the proofs of Theorem 1.1 and Theorem 1.11.

To give the main idea, consider first the case where $f: \mathbf{C} \to N$ is a holomorphic map to a compact surface of genus $g \geqslant 2$. A simple but important first step is to perform a *rescaling argument* to replace f by a holomorphic map $F: \mathbf{C} \to N$ with bounded derivative; if f is nonconstant, then so is F. This sort of rescaling in classical complex analysis goes back to Bloch. In recent times, it has been advocated by Gromov [Gr4, p. 344] and Zalcman [Z]. We review the rescaling argument in §2. Thus, without loss of generality, we may assume that $f: \mathbf{C} \to N$ has bounded derivative. Now, if $g \geqslant 2$, there are two nonzero linearly independent holomorphic 1-forms on N, say ξ_1 and ξ_2. Because N is compact and the derivative of f is bounded, the pullback forms $\eta_1 = f^*\xi_1$ and $\eta_2 = f^*\xi_2$ can be written as

$$\eta_i(z) = a_i(z)\,dz, \quad i = 1, 2,$$

with $a_i(z)$ bounded holomorphic; in particular, $a_i \equiv$ constant for $i = 1, 2$, which implies that f must be constant, for else η_1 and η_2 should be linearly independent.

We implement the above idea to quasiregular mappings as follows. Let $f: \mathbf{R}^n \to N$ be K-quasiregular. Upon rescaling, we may assume that f is uniformly continuous as a map between metric spaces. Fix an integer $l = 1, ..., n-1$. By nonlinear Hodge theory (reviewed in §3 below), each cohomology class $[\alpha]$ of closed l-forms contains a unique p-harmonic representative ξ_α; we choose the conformally invariant case $p = n/l$. If $\dim H^l(N) = k$, we pick p-harmonic l-forms $\xi_1, ..., \xi_k$ that have unit L^p-norm and are sufficiently separated in L^p. If $l = \frac{1}{2}n$, we could simply choose an orthogonal basis consisting of harmonic forms. The pullbacks $\eta_i = f^*\xi_i$ satisfy a fixed nonlinear elliptic system with measurable coefficients in $\mathbf{R}^n$. Such generalized Cauchy–Riemann systems in connection with quasiconformal geometry have been studied by Donaldson and Sullivan [DS], and Iwaniec and Martin [IM] (see also [Ma]). From the fact that f is uniformly continuous, it follows that the averaged counting function as defined in (1.10) satisfies

$$A(r) = O(r^n), \quad r \to \infty. \tag{1.13}$$

Using (1.13) and an equidistribution result of Mattila and Rickman [MR], we are able to show that the L^p-norms of the forms η_i and $\eta_i - \eta_j$ are controlled on suitably chosen balls (Lemma 5.2).

After another rescaling, we may assume that we have forms η_i on the unit ball $\mathbf{B}^n$ in $\mathbf{R}^n$ with uniformly bounded norm, uniformly separated in L^p. By the L^p-Poincaré lemma, there exist forms α_i on the unit ball with $d\alpha_i = \eta_i$. Moreover, a Caccioppoli-type argument shows that the forms $\alpha_1, ..., \alpha_k$ form an ε-separated set in L^p, where $\varepsilon = \varepsilon(n, K) > 0$. The final bound for the number k comes from this separation together with the L^p-compactness of a homotopy operator T that can be used to produce the forms α_i from the given forms η_i.

A detailed proof for Theorem 1.1 is presented in §§2–5 below.

The idea behind Theorem 1.11 is much simpler. If $H^l(N) \neq 0$, as above we choose a p-harmonic l-form $\xi \neq 0$ on N, $p = n/l$. Its pullback $\eta = f^*\xi$ satisfies an elliptic system in $\mathbf{R}^n$ as discussed above. In particular, η satisfies a certain type of reverse Hölder inequality from which a lower growth condition can be derived by rather standard methods. The proof is presented in §6.

The notation used throughout the paper is standard, or self-explanatory. The expression "$A \simeq B$ with constants of comparability depending only on $a, b, ...$" means that there exist constants $C_1, C_2 > 0$, depending only on $a, b, \ldots$, such that $C_1 A \leqslant B \leqslant C_2 A$.

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2. Rescaling principle

Suppose that $g: U \to \mathbf{R}^n$ is a K-quasiregular mapping defined on an open set $U \subseteq \mathbf{R}^n$. It is well-known that g is locally Hölder continuous with exponent $\alpha = 1/K$. Let N be a closed, connected and oriented Riemannian n-manifold N, $n \geqslant 2$. There exists $\varepsilon > 0$ so that every open geodesic ball $B(p, \varepsilon)$ in N can be mapped onto an open subset in $\mathbf{R}^n$ by a 2-bi-Lipschitz homeomorphism. By using these local bi-Lipschitz homeomorphisms we can reduce questions about the local regularity of K-quasiregular mappings into compact Riemannian manifolds to the corresponding question for mappings between subsets of Euclidean spaces. In particular, it follows that if $f: \mathbf{R}^n \to N$ is K-quasiregular, then it is locally Hölder continuous with exponent $\beta = \beta(K) > 0$.

The *rescaling principle* asserts that if $f: \mathbf{R}^n \to N$ is nonconstant, then *there exists a nonconstant K-quasiregular mapping $F: \mathbf{R}^n \to N$ that is uniformly Hölder continuous*; more precisely, we have that

$$d(F(x), F(y)) \leqslant C|x-y|^{\beta} \tag{2.1}$$

for all $x, y \in \mathbf{R}^n$. Here d is the Riemannian distance on N, $\beta = \beta(K)$ is as above, and $C \geqslant 1$ is a constant independent of x and y.

This rescaling principle for quasiregular mappings $f: \mathbf{R}^n \to S^n$ has been proved by Miniowitz [Mi]. The general case follows along similar lines, and is discussed in [Gr4, pp. 344–345].

For the reader's convenience, we sketch the main point in the proof. We define a function

$$Q_f(x) = \sup_{|x-y| \leqslant 1} \frac{d(f(x), f(y))}{|x-y|^{\beta}}, \quad x \in \mathbf{R}^n,$$

where β is as in (2.1). This function is locally bounded by the local β-Hölder continuity of f. If Q_f is not bounded on $\mathbf{R}^n$, one can choose a sequence (x_ν) of points with $x_\nu \to \infty$ and $Q_f(x_\nu) \to \infty$ as $\nu \to \infty$.

Next, let

$$\varphi_\nu(x) = Q_f(x)\left[\frac{\operatorname{dist}(x, \partial B(x_\nu, \nu))}{\nu}\right]^\beta, \quad x \in B(x_\nu, \nu).$$

We can find points $a_\nu \in B(x_\nu, \nu)$ such that

$$\varphi_\nu(a_\nu) > \tfrac{1}{2} \sup_{x \in B(x_\nu, \nu)} \varphi_\nu(x).$$

It is then easy to see that a subsequence of the K-quasiregular mappings

$$f_\nu(x) = f(a_\nu + \varrho_\nu x), \quad \varrho_\nu = Q_f(a_\nu)^{-1/\beta},$$

converges locally uniformly to a nonconstant K-quasiregular mapping $F: \mathbf{R}^n \to N$ with Q_F bounded.

COROLLARY 2.2. *If a closed, connected and oriented Riemannian n-manifold N receives a nonconstant K-quasiregular mapping from* $\mathbf{R}^n$, *then it receives one whose averaged counting function as defined in* (1.10) *satisfies*

$$A(r) = O(r^n), \quad r \to \infty. \tag{2.3}$$

Indeed, (2.3) *holds for each uniformly continuous quasiregular mapping* $f: \mathbf{R}^n \to N$.

Proof. By the rescaling principle, we may assume that $f: \mathbf{R}^n \to N$ is a uniformly continuous nonconstant K-quasiregular mapping. Let $\varepsilon > 0$ be such that each open geodesic ball of radius ε in N is 2-bi-Lipschitz to a round ball in $\mathbf{R}^n$. By the uniform continuity of f we can find $\delta > 0$ such that f maps every ball of radius 2δ in $\mathbf{R}^n$ into a ball of radius ε in N. Every ball $B(0, r)$, $r \geqslant 1$, can be covered by at most $C_0 r^n$ balls of radius δ, where $C_0 = C_0(n, \delta) > 0$. The assertion will follow if we can show that there exists a constant $C > 0$ such that

$$\int_{B(p,\delta)} \det Df(x)\, dx \leqslant C \tag{2.4}$$

for each $p \in \mathbf{R}^n$. By using our local 2-bi-Lipschitz homeomorphisms, in order to show (2.4), we may without loss of generality assume that f maps $B(p, 2\delta)$ into a Euclidean ball of radius 2ε and is K'-quasiregular with $K' = K(n, K)$. In this situation, (2.4) with $C = C(n, K', \varepsilon) = C(n, K, \varepsilon) > 0$ follows from a Caccioppoli-type estimate for quasiregular mappings (see [BI, p. 274, (3.6)] or [Ri4, p. 141], for example). This proves the corollary. □

3. Nonlinear Hodge theory and quasiregular mappings

In this section, we briefly discuss nonlinear Hodge theory as far as it is relevant for the proofs of our main results. A thorough presentation of the facts quoted below can be found in the paper [S].

Let N be a closed, connected and oriented Riemannian n-manifold, $n\geqslant 2$. We fix an integer $l=1,\dots,n-1$, and consider the lth cohomology group $H^l(N)$. Each element in $H^l(N)$ is represented by a smooth closed l-form α. Let us denote by $[\alpha]$ the corresponding equivalence class: $\alpha\sim\beta$ if and only if $\beta=\alpha+d\gamma$. Here β is a smooth l-form and γ is a smooth $(l-1)$-form. The Riemannian metric on N induces natural inner products, and in particular norms, on the fibers $\bigwedge^l T_a^*N$, $a\in N$, of the lth exterior power $\bigwedge^l T^*N$ of the cotangent bundle T^*N.

We need the following fact: *If* $1<p<\infty$, *then for each equivalence class* $[\alpha]\in H^l(N)$ *there exists a unique* p*-integrable* l*-form* ξ_α *that minimizes the* p*-energy within the equivalence class*; i.e.,

$$\|\xi_\alpha\|_p^p=\inf\int_N|\alpha+d\gamma|^p\,dV, \tag{3.1}$$

where the infimum is taken over all smooth $(l-1)$-forms γ on N. Here (and hereafter) dV denotes integration with respect to the Riemannian volume. We have that both

$$d\xi_\alpha=0 \tag{3.2}$$

and

$$d^*(|\xi_\alpha|^{p-2}\xi_\alpha)=0 \tag{3.3}$$

hold in the sense of distributions, where d^* is the formal adjoint of d. We call a p-integrable l-form on N p*-harmonic* if it satisfies (3.2) and (3.3). Note that (3.3) is the Euler equation for the minimization problem (3.1).

We also require the following result due to Ural'tseva [Ur], [Uh]: *Each* p*-harmonic form on* N *is Hölder continuous, a fortiori bounded.*

For the purposes of this paper, we define a norm on $H^l(N)$ by the formula

$$\|[\alpha]\|:=\|\xi_\alpha\|_p=\left(\int_N|\xi_\alpha|^p\,dV\right)^{1/p},\quad p=\frac{n}{l}. \tag{3.4}$$

It is easy to see that $\|\cdot\|$ defines a norm. Indeed, it is nothing but the quotient norm on the space

$$\{L^p\text{-integrable closed } l\text{-forms}\}/\{L^p\text{-integrable exact } l\text{-forms}\},$$

where, of course, the terminology should be understood in the sense of distributions. The (finite-dimensional) normed space $(H^l(N), \|\cdot\|)$ will be featured more prominently in the next section.

For the relationship between quasiregular mappings and nonlinear Hodge theory, see [DS], [IM], [ISS, §7.2]. We need the following facts. Assume that $f: \mathbf{R}^n \to N$ is a nonconstant K-quasiregular mapping, and that ξ is a p-harmonic l-form on N in the conformally invariant case $p = n/l$. Then the pullback form

$$\eta = f^*(\xi)$$

satisfies both

$$d\eta = f^*(d\xi) = 0 \tag{3.5}$$

and

$$d^*(\langle G\eta, \eta\rangle^{(p-2)/2} G\eta) = 0 \tag{3.6}$$

in the sense of distributions, where $G: \bigwedge^l T^*\mathbf{R}^n \to \bigwedge^l T^*\mathbf{R}^n$ is a measurable bundle map which on almost every fiber $\bigwedge^l T^*_x\mathbf{R}^n$ is induced by the linear map

$$(\det Df(x))^{2/n} (Df(x)^t Df(x))^{-1}$$

on $T_x\mathbf{R}^n$. If we denote by G_x the restriction of G to a fiber $\bigwedge^l T^*_x\mathbf{R}^n$, then we have that $\det G_x = 1$, and that, for each l-form ζ,

$$\langle G_x\zeta(x), \zeta(x)\rangle \simeq |\zeta(x)|^2 \tag{3.7}$$

for almost every $x \in \mathbf{R}^n$, with constants of comparability depending only on n and K. The *ellipticity* condition (3.7) will crucially be used in what follows; indeed, the fact that (3.7) holds is equivalent to the quasiregularity of f, as is easily seen.

4. Equidistribution

Let us assume that we are in the situation of Theorem 1.1. Thus, $f: \mathbf{R}^n \to N$ is a nonconstant K-quasiregular mapping, and we want to bound the dimension of the cohomology ring of N. By the rescaling principle described in §2 (see Corollary 2.2), we may assume that the averaged counting function $A(r)$ of f satisfies the growth condition (2.3). In this section, we shall show how to combine this growth condition with an equidistribution theorem of Mattila and Rickman [MR] to obtain more information about the behavior of f.

To that end, fix $l=1,\dots,n-1$, and consider the vector space $H^l(N)$ equipped with the norm $\|\cdot\|$ as in (3.4). Let $k=\dim H^l(N)$. As $(H^l(N),\|\cdot\|)$ is a finite-dimensional normed space, we can find linearly independent elements $[\alpha_1],\dots,[\alpha_k]\in H^l(N)$ so that

$$\|[\alpha_i]\|=1 \quad \text{and} \quad \|[\alpha_i]-[\alpha_j]\|\geqslant 1 \tag{4.1}$$

for all $i,j=1,\dots,k$, $i\neq j$. It follows that for the associated p-harmonic forms $\xi_i=\xi_{\alpha_i}$ we have

$$\|\xi_i\|_p=1. \tag{4.2}$$

Since $\xi_i-\xi_j$ is cohomologous (in the distributional sense) to $\alpha_i-\alpha_j$, we see that

$$\|[\alpha_i]-[\alpha_j]\|=\|[\alpha_i-\alpha_j]\|\leqslant\|\xi_i-\xi_j\|_p.$$

Hence

$$1\leqslant\|\xi_i-\xi_j\|_p\leqslant 2 \quad \text{for } i\neq j. \tag{4.3}$$

Next, consider measures μ_i and μ_{ij} on N given by

$$\begin{aligned} d\mu_i &= |\xi_i|^p\,dV,\\ d\mu_{ij} &= |\xi_i-\xi_j|^p\,dV, \end{aligned} \tag{4.4}$$

where $i,j=1,\dots,k$, $i\neq j$. Note that by (4.2) and (4.3), and since $p=n/l$, we have

$$1\leqslant\mu_*(N)\leqslant 2^n. \tag{4.5}$$

Here and below we shall write μ_* for any of the measures μ_i and μ_{ij} defined in (4.4).

Because the p-harmonic forms ξ_i are continuous by Ural'tseva's theorem cited in §3, we have the growth condition

$$\mu_*(B_r)\leqslant Cr^n \tag{4.6}$$

for each of the above measures and for each Riemannian ball B_r of radius r on N with a constant $C\geqslant 1$ independent of the ball. The averaged counting function $\nu_*=\nu_{\mu_*}$ with respect to a measure μ_* is defined by

$$\nu_*(r)=\frac{1}{\mu_*(N)}\int_N n(r,a)\,d\mu_*(a), \tag{4.7}$$

where $n(r,a)$ is the number of a-points of f with multiplicity regarded in the closed ball $\overline{B}(0,r)$ in $\mathbf{R}^n$.

The growth condition (4.6) guarantees that the hypotheses of the main theorem, Theorem 5.11, in [MR] are satisfied, and we can record the following result.

PROPOSITION 4.8. *For each measure* μ_* *there exists a set* $E_* \subseteq (0,\infty)$ *of finite logarithmic measure,*

$$\int_{E_*} \frac{dt}{t} < \infty, \tag{4.9}$$

so that

$$\lim_{\substack{r\to\infty \\ r\notin E_*}} \frac{\nu_*(r)}{A(r)} = 1. \tag{4.10}$$

Now set

$$E = \Big(\bigcup_{i=1}^{k} E_i\Big) \cup \Big(\bigcup_{\substack{i,j=1 \\ i\neq j}}^{k} E_{ij}\Big), \tag{4.11}$$

so that E is the union of the exceptional sets E_* as in Proposition 4.8. Obviously, by (4.9), we have that E has finite logarithmic measure as well; i.e.,

$$\int_E \frac{dt}{t} < \infty. \tag{4.12}$$

Moreover, we deduce from (4.10) that there exists a radius $r'>0$ such that

$$\frac{1}{2} \leqslant \frac{\nu_*(r)}{A(r)} \leqslant 2 \quad \text{whenever } r \geqslant r',\ r \notin E, \tag{4.13}$$

and whenever $\nu_* = \nu_{\mu_*}$ is the averaged counting function for any of the measures μ_i and μ_{ij} given in (4.4).

Next, we require the following real analysis lemma:

LEMMA 4.14. *Let* $h\colon (0,\infty)\to(0,\infty)$ *be an increasing function such that for some* $n>0$ *we have*

$$h(r) = O(r^n) \tag{4.15}$$

as $r\to\infty$. *Then there exists a constant* $L=L(n)>0$ *such that*

$$\limsup_{R\to\infty} \frac{1}{\log R} \int_{E_L\cap[1,R]} \frac{dt}{t} \leqslant \frac{1}{2}, \tag{4.16}$$

where $E_L = \{r\in[1,\infty) : h(2r) > Lh(r)\}$.

Proof. Results of this sort are typical in the value distribution theory of entire and meromorphic functions; rather than searching for a reference, we provide the elementary proof.

For given $L>1$ and $R\geqslant 1$, the set $E_L\cap[1,R]$ can be covered by intervals $I_r = \left[\frac{1}{2}r, 2r\right]$, $r\in E_L$, $1\leqslant r\leqslant R$. By standard covering arguments, one can select finitely many intervals $I_{r_1}, \dots, I_{r_M}$ of this type that cover $E_L\cap[1,R]$ and so that no point in $(0,\infty)$ lies in more than two of the sets I_{r_i}. We may also assume, obviously, that none of the intervals I_{r_i} contains another interval I_{r_j}. We then find that

$$L^{M/2}h(1) \leqslant h(2R) \leqslant CR^n,$$

and hence that

$$M \leqslant \frac{C'+2n\log R}{\log L},$$

where $C' \geqslant 1$ is independent of R and L. In particular,

$$\int_{E_L\cap[1,R]} \frac{dt}{t} \leqslant M\log 4 \leqslant \left(\frac{C'+2n\log R}{\log L}\right)\log 4,$$

so that (4.16) holds as soon as $L \geqslant 4^{4n}$. □

To finish this section, we state the following crucial corollary to Proposition 4.8 and Lemma 4.14. The hypotheses and the notation are as in the beginning of the section.

COROLLARY 4.17. *There exist arbitrarily large values r such that*

$$r, 2r \notin E \quad \text{and} \quad A(2r) \leqslant LA(r), \tag{4.18}$$

where $L = L(n)$.

Proof. We apply Lemma 4.14 to the function $h(r) = A(r)$. This function is increasing, and condition (4.15) holds, so that the lemma is applicable.

Let $L = L(n) \geqslant 1$ and E_L be as in the lemma, and let E be the set given in (4.11). If $\frac{1}{2}E = \{\frac{1}{2}r : r \in E\}$, then $E \cup \frac{1}{2}E$ has finite logarithmic measure by (4.12). So we deduce from (4.16) that

$$\liminf_{R\to\infty} \frac{1}{\log R}\int_{G\cap[1,R]} \frac{dt}{t} \geqslant \frac{1}{2},$$

where $G = (0,\infty)\setminus(E_L \cup E \cup \frac{1}{2}E)$. Since (4.18) holds for all $r \in G$, the lemma follows. □

5. Conclusion of the proof of Theorem 1.1

In this section, we finish the proof of Theorem 1.1. We keep the notation of the previous section. Fix $l = 1, ..., n-1$. Our task is to bound $k = \dim H^l(N)$. Since $\dim H^l(N) = \dim H^{n-l}(N)$ by Poincaré duality, we may without loss of generality assume that $l \leqslant \frac{1}{2}n$. Then $p = n/l \geqslant 2$. We consider the p-harmonic forms $\xi_1, ..., \xi_k$ as in §4 and their pullbacks

$$\eta_i = f^*(\xi_i), \quad i = 1, ..., k.$$

These forms satisfy the elliptic system described in (3.5) and (3.6). By Corollary 4.17 and by (4.13) we can find a number $r_0>0$ such that

$$\nu_*(r_0) \simeq A(r_0) \simeq A(2r_0) \simeq \nu_*(2r_0), \tag{5.1}$$

where the constants of comparability depend only on n. This leads to the following estimate for the L^p-norms of the forms η_i.

LEMMA 5.2. *The forms $\eta_1, ..., \eta_k$ satisfy*

$$\int_{B(0,r_0)} |\eta_i|^p\,dx \simeq \int_{B(0,2r_0)} |\eta_i|^p\,dx \simeq A(r_0), \quad i=1,...,k, \tag{5.3}$$

and

$$\int_{B(0,r_0)} |\eta_i-\eta_j|^p\,dx \simeq \int_{B(0,2r_0)} |\eta_i-\eta_j|^p\,dx \simeq A(r_0), \quad i,j=1,...,k,\ i\neq j, \tag{5.4}$$

where the constants of comparability depend only on n and K.

Proof. The quasiregularity of f implies that

$$|\eta_i(x)| = |f^*\xi_i(x)| \simeq |\xi_i(f(x))|\,(\det Df(x))^{l/n}$$

for almost every $x\in\mathbf{R}^n$. By the change of variables formula for quasiregular mappings [Ri4, pp. 20–21], and by (4.5), we therefore obtain for all $r>0$ that

$$\int_{B(0,r)} |f^*\xi_i|^p\,dx \simeq \int_N n(r,a)|\xi_i(a)|^p\,dV(a) \simeq \nu_i(r).$$

By the choice of r_0, we see that (5.3) holds. A similar computation shows (5.4), and the lemma follows. □

The next step is to scale everything to the unit ball $\mathbf{B}^n$ of $\mathbf{R}^n$. Indeed, the system described in (3.5) and (3.6) is conformally invariant, as are the integrals in (5.3) and (5.4). After dividing the forms η_i by $A(r_0)^{1/p}$, and after performing the scaling, we can assume without loss of generality that we have L^p-integrable l-forms $\eta_1, ..., \eta_k$ in the unit ball $\mathbf{B}^n$ of $\mathbf{R}^n$ such that each of the forms satisfy (3.5) and (3.6), and, in addition,

$$\int_{\frac{1}{2}\mathbf{B}^n} |\eta_i|^p\,dx \simeq \int_{\mathbf{B}^n} |\eta_i|^p\,dx \simeq 1, \quad i=1,...,k, \tag{5.5}$$

and

$$\int_{\frac{1}{2}\mathbf{B}^n} |\eta_i-\eta_j|^p\,dx \simeq \int_{\mathbf{B}^n} |\eta_i-\eta_j|^p\,dx \simeq 1, \quad i,j=1,...,k,\ i\neq j, \tag{5.6}$$

with constants of comparability only depending on n and K.

This reduction understood, we apply the L^p-Poincaré lemma and find p-integrable $(l-1)$-forms $\alpha_1, ..., \alpha_k$ in $\mathbf{B}^n$ such that

$$\eta_i = d\alpha_i, \quad i = 1, ..., k. \tag{5.7}$$

In a moment we shall choose the forms α_i in a particular way, but with an arbitrary choice we have the following lemma which will be derived from a Caccioppoli-type inequality.

LEMMA 5.8. *In the situation described above in* (5.5), (5.6) *and* (5.7), *we have that*

$$\int_{\mathbf{B}^n} |\alpha_i - \alpha_j|^p \, dx \geqslant c(n, K) > 0 \tag{5.9}$$

for $i, j = 1, ..., k$, $i \neq j$.

Proof. Let us write

$$A\zeta = \langle G\zeta, \zeta \rangle^{(p-2)/2} G\zeta$$

for an l-form ζ, where G is given in (3.6). A simple linear algebra computation (see [BI, p. 288]), using $p \geqslant 2$ and (3.7), gives that

$$\langle A\zeta_1 - A\zeta_2, \zeta_1 - \zeta_2 \rangle \geqslant c|\zeta_1 - \zeta_2|^p$$

for some constant $c = c(n, K) > 0$. We thus have

$$\begin{aligned} \int_{\frac{1}{2}\mathbf{B}^n} |\eta_i - \eta_j|^p \, dx &\leqslant C \int_{\frac{1}{2}\mathbf{B}^n} \langle A\eta_i - A\eta_j, \eta_i - \eta_j \rangle \, dx \\ &\leqslant C \int_{\mathbf{B}^n} \langle A\eta_i - A\eta_j, \varphi(\eta_i - \eta_j) \rangle \, dx, \end{aligned} \tag{5.10}$$

where $C \geqslant 1$ depends only on n and K, and where $\varphi \in C_0^\infty(\mathbf{B}^n)$ is a fixed cut-off function with $0 \leqslant \varphi \leqslant 1$, $|d\varphi| \leqslant 2$ and $\varphi \equiv 1$ on $\frac{1}{2}\mathbf{B}^n$. But now

$$\varphi(\eta_i - \eta_j) = \varphi(d\alpha_i - d\alpha_j) = d(\varphi(\alpha_i - \alpha_j)) - d\varphi \wedge (\alpha_i - \alpha_j),$$

and so it follows from (3.6) that the right-hand side of (5.10) equals

$$-C \int_{\mathbf{B}^n} \langle A\eta_i - A\eta_j, d\varphi \wedge (\alpha_i - \alpha_j) \rangle \, dx.$$

This in turn can be estimated from above by

$$
\begin{aligned}
C&\int_{\mathbf{B}^n}|A\eta_i-A\eta_j|\,|d\varphi|\,|\alpha_i-\alpha_j|\,dx\\
&\leqslant C\biggl(\int_{\mathbf{B}^n}|\eta_i|^{p-1}|d\varphi|\,|\alpha_i-\alpha_j|\,dx+\int_{\mathbf{B}^n}|\eta_j|^{p-1}|d\varphi|\,|\alpha_i-\alpha_j|\,dx\biggr)\\
&\leqslant C\biggl(\biggl(\int_{\mathbf{B}^n}|\eta_i|^p\,dx\biggr)^{(p-1)/p}+\biggl(\int_{\mathbf{B}^n}|\eta_j|^p\,dx\biggr)^{(p-1)/p}\biggr)\biggl(\int_{\mathbf{B}^n}|\alpha_i-\alpha_j|^p\,dx\biggr)^{1/p}\\
&\leqslant C\biggl(\int_{\mathbf{B}^n}|\alpha_i-\alpha_j|^p\,dx\biggr)^{1/p},
\end{aligned}
$$

where $C\geqslant 1$ depends only on n and K. Note that (3.7) and (5.5) were used here. The assertion (5.9) now follows from the above estimation and from (5.6), and the lemma is proved. □

We are now ready to give the punch line for the proof of Theorem 1.1. For each dimension n there exists a compact linear homotopy operator

$$T: L^p(\textstyle\bigwedge^l T^*\mathbf{B}^n)\to L^p(\textstyle\bigwedge^{l-1} T^*\mathbf{B}^n)$$

such that $dT(\eta)=\eta$ (in the sense of distributions) for each closed p-integrable l-form η on the unit ball $\mathbf{B}^n$. (See e.g. [IL, p. 39, Remark 4.1]. It is not clear to us how to derive [IL, Remark 4.1] from [IL, (4.14)], but in any case the compactness of T for $1<p<\infty$ immediately follows from [IL, Proposition 4.1] and from the compactness of the embedding of the Sobolev space $W^{1,p}$ in L^p.) It follows from (5.9) and (5.5) that the forms

$$\alpha_1=T\eta_1,\ \ldots,\ \alpha_k=T\eta_k$$

form a separated set in $L^p=L^p(\bigwedge^{l-1}T^*\mathbf{B}^n)$ and are contained in the image of a ball under T, where the separation and the size of the ball depend only on n and K. By the compactness of T, this implies that the number k of the forms α_i is bounded by a constant that only depends on n and K.

This completes the proof of Theorem 1.1.

6. Proof of Theorem 1.11

Let $f:\mathbf{R}^n\to N$ be as in the statement of the theorem. Assuming that $H^l(N)\neq 0$ for some $l=1,\ldots,n-1$, we can fix a p-harmonic l-form ξ on N, $\xi\neq 0$, for $p=n/l$. As explained in §3, the pullback

$$\eta=f^*(\xi)$$

satisfies the system described in (3.5) and (3.6) in $\mathbf{R}^n$. Since $\det Df(x)$ is nonsingular at almost every point $x\in\mathbf{R}^n$, we have that $\eta\neq 0$.

We claim the following well-known reverse Hölder inequality for solutions of such systems.

LEMMA 6.1. *There exist numbers $q=q(n,K)>p=n/l$ and $C=C(n,K)\geqslant 1$ such that*

$$\left(r^{-n}\int_{B(0,r/2)}|\eta|^q\,dx\right)^{1/q}\leqslant C\left(r^{-n}\int_{B(0,r)}|\eta|^p\,dx\right)^{1/p} \tag{6.2}$$

holds for all $r>0$.

Proof. The proof for (6.2) runs in two steps. First one uses a Caccioppoli-type inequality (as in [BI, p. 274] or [Ri4, p. 141], for example) together with the Sobolev–Poincaré inequality for differential forms (see [IL, Corollary 4.2], for example) to conclude that

$$\left(r^{-n}\int_{B_{r/2}}|\eta|^p\,dx\right)^{1/p}\leqslant C\left(r^{-n}\int_{B_r}|\eta|^{n/(l+1)}\,dx\right)^{(l+1)/n}, \tag{6.3}$$

where B_r is an arbitrary ball of radius r, and $B_{r/2}$ is the ball concentric to B_r but half its radius. The constant $C\geqslant 1$ depends only on n and K. The crucial point is that the exponent $n/(l+1)$ on the right is strictly less than $p=n/l$. Inequality (6.3) is an example of a weak reverse Hölder inequality; it is called "weak" because the ball on the right is twice as big as that on the left. A nontrivial but by now standard real variable argument (originating in [Ge]) can be used to show that (6.3) self-improves itself to (6.2). See [BI, pp. 281 ff.]. This proves the lemma. □

We now show how (6.2) implies the desired growth (1.12) for the counting function $A(r)$. Indeed, by a computation similar to the one in the proof of Lemma 5.2, the boundedness of ξ on N implies

$$\int_{B(0,r)}|\eta|^p\,dx\leqslant CA(r),$$

and hence

$$\left(r^{-n}\int_{B(0,r/2)}|\eta|^q\,dx\right)^{1/q}\leqslant Cr^{-n/p}A(r)^{1/p},$$

where $C\geqslant 1$ is independent of r. Since $q>p$ and $\eta\neq 0$, we have that

$$\liminf_{r\to\infty}\frac{A(r)}{r^\alpha}>0$$

for $\alpha=n(1-p/q)>0$. This completes the proof of Theorem 1.11.

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MARIO BONK
Department of Mathematics
University of Michigan
Ann Arbor, MI 48109
U.S.A.
mbonk@math.lsa.umich.edu

JUHA HEINONEN
Department of Mathematics
University of Michigan
Ann Arbor, MI 48109
U.S.A.

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