A NOTE ON DERIVATES AND DIFFERENTIAL COEFFICIENTS.

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§ 1.

The main theorem obtained in the present note is the following: — *Except at a countable set of points, the lower derivate on either side is not greater than the upper derivate on the other side;* i. e. using an accepted notation which explains itself[1]

$$f_-(x) \leq f^+(x),$$

and also

$$f_+(x) \leq f^-(x).$$

The primitive fonction $f(x)$ may be any function whatever of the single real variable x. If $f(x)$ is a continuous function, this theorem enables us to assert that, except at a countable set of points, $f(x)$ has at least one symmetric derivate, that is to say there is at least one sequence of positive, and one sequence of negative values of h, both with zero as limit, corresponding to each point x, such that the incrementary ration $(f(x+h)-f(x))/h$ has the same limit for the two sequences. I define accordingly *the mean symmetric derivate of a continuous function* $f(x)$ to be the trigonometric mean (§ 7) between the greatest and least symmetric derivate at each point; *the mean symmetric derivate of a continuous function then exists except at most at a countable set of points; it agrees with the differential coefficient, wherever this exists, and is finite except at a set of points of content zero.*

[1] W. H. YOUNG and the present author, »On Derivates and the Theorem of the Mean», 1908, Quart. Jour. of Pure and Applied Math., § 2, p. 4. SCHEEFFER, who first introduced the concept of a derivate, used $D_-f(x)$, etc. »Allgemeine Untersuchungen über Rectification der Curven», 1884, Acta Math. 5, p. 52.

From the main theorem a number of interesting corollaries are at once evident.

§ 2.

The theorem from the Theory of Sets of Points on which the following discussion is based is due to W. H. YOUNG,[1] and runs as follows: —

Given a set of intervals, overlapping in any manner, those points which are end-points of the intervals, without being internal to any of them are countable.

This theorem may be proved in various ways; the following proof is perfectly direct, and depends only on CANTOR's theorem[2] that any set of non-overlapping intervals is countable.

Let us divide the set of points in question into two sets, according as the point under consideration is a left-hand end-point of one of the given intervals, or not. It will be shown that each of these sets is countable, and therefore the whole set is countable.

Corresponding to each point P, which is a left-hand end-point of one of the given intervals, without being internal to any interval of the given set, we have an interval d_P of the given set with P as left-hand end-point. The point P will not then, by hypothesis, be internal to any of the intervals d_Q belonging to a different point Q of the same type, nor will Q be internal to d_P. Hence the two intervals d_P and d_Q cannot overlap; for, if they did, the left-hand end-point of one would have to be internal to the other. Thus the intervals d_P form a set of non-overlapping intervals, and are therefore countable. The points P therefore also form a countable set, which is evidently dense nowhere, and contains no point which is a limiting point of the set on the left.

Similarly those points which are right-hand end-points of intervals of the given set, without being internal to any of the given intervals, form a countable set, nowhere dense, and containing no point which is a limiting point of the set on the right.

Thus *the whole set of points in question is countable, dense nowhere and contains no sub-set which is dense in itself on both sides.*

§ 3.

Let $e_1, e_2, \ldots$ be a monotone descending sequence with zero as limit, $k_1, k_2, \ldots$ a monotone ascending sequence with k as limit, where k may be finite or $+\infty$, and let D_n denote the set of overlapping intervals consisting of all those intervals which have the following three properties: —

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[1] »On overlapping intervals«, 1902, § 5. Proc. L. M. S., (1) Vol. XXXV, p. 387.

[2] Math. Ann. Vol. XXII, p. 117, (1882).

1) *the length of the interval is less than* e_n;

2) *when* x *and* $x+h$ *are the end-points of the interval,*

$$(f(x+h)-f(x))/h > k_n;$$

3) *the same inequality holds when* x *is the left-hand end-point and* $x+h$ *any internal point of the interval.*

Now consider a point y at which

$$f_+(y) \geq k$$

then, by the definition of the lower right-hand derivate $f_+(y)$, we can find an interval with y as left-hand end-point, satisfying the conditions (1), (2) and (3). Thus each such point y is a left-hand end-point of the intervals of D_n. Hence, by the theorem stated at the end of § 2, there is at most a countable set of these points y, which are not internal to the intervals of D_n.

Since this is true for all values of n, it appears that all but a countable set of the points y belong to the set of points G internal to all the sets of intervals

$$D_1, D_2, \ldots, D_n, \ldots$$

On the other hand, if X is any point of the set G, (supposing this set to be other than a null-set),[1] and $(x, x+h)$ be an interval of the set D_n containing X as internal point, it follows from the property 3), that (x, X) is also an interval of the set D_n, and therefore, by the properties 1) and 2),

$$X - x < e_n, \qquad \{f(X) - f(x)\}/(X - x) > k_n.$$

Since this is true for all values of n, this proves that the upper limit of $\{f(X) - f(X-h)\}/h$ as h decreases towards zero as limit, is not less than k; that is

$$f^-(X) \geq k.$$

Thus the set of points at which $f^- \geq k$ includes the set G, and therefore includes all the points at which $f_+ \geq k$, except possibly a countable set of these latter points.

Hence we have the following result: —

There is at most a countable set of the points at which $f_+(x) \geq k$, *at which we do not have also* $f^-(x) \geq k$.

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[1] If $f(x)$ is continuous, and there are any points y at all, there are c such, where c is the potency of the continuum. Thus for suitable values of k, the set G certainly is not a null-set. W. H. Young »Term by term integration of oscillating series» 1909. Proc. L. M. S. (2) Vol. 8. p. 106.

Exchanging right and left we have the alternative result.

Similarly we have the following: —

There is at most a countable set of the points at which $f^+ \leq k$, *at which we do not also have* $f_- \leq k$.

Here again we may interchange right and left.

In the first of the above statements k may be finite or $+\infty$, in the second it may be finite or $-\infty$. When k has one of these infinite values the sign $\geq$, or $\leq$, must, of course be taken to be the sign of equality.

§ 4.

Now the points at which $f_+(x) > k$ at which we do not also have $f^- > k$, may be divided into two classes; 1) the points $f_+ > k$, $f^- < k$, which are among the points at which $f_+ \geq k$ but not $f^- \geq k$, and therefore are countable, by the first of the results of the preceding article, and 2) the points at which $f_+ > k$, $f^- = k$, which are among the points at which $f^- \leq k$, but not $f_+ \leq k$, and are therefore countable by the second of the above results, after we have exchanged left and right. Thus *in the statements of the preceding article the sign of equality may be omitted.* For instance, omitting the sign of equality in the first of the theorems, we have the theorem: —

There is at most a countable set of the points at which $f_+ > k$, *at which we do not also have* $f^- > k$.

§ 5.

We can now prove the main theorem: —

Theorem. *Except at a countable set of points, the lower derivate on either side is less than, or equal to, the upper derivate on the other side*; i. e.

$$f_-(x) \leq f^+(x), \textit{ and also } f_+(x) \leq f^-(x).$$

To prove this let G_{k, e_r} denote the set of points at which

$$f^+(x) \leq k e_r \quad \dots\dots\dots\dots\dots \quad (1)$$

where k is any integer, positive, negative or zero, and $e_1, e_2, \dots, e_r, \dots$ is a monotone descending sequence of constants with zero as limit. Then, by the second of the theorems of § 3

$$G_{k, e_r} = C_{k, e_r} + F_{k, e_r},$$

where C_{k, e_r} is a countable set, and, at every point of F_{k, e_r}

$$f_-(x) \leq k e_r \qquad (2)$$

Let C denote the set consisting of all the sets C_{k, e_r} for all values of the integers k and r. Then C is a countable set, since it consists of doubly infinite set of countable sets.

Now let P be any point not belonging to the set C. Then either at P

a) $f^+(x)$ has a finite value, or

b) $f^+(x)$ has the value $+\infty$, or

c) $f^+(x)$ has the value $-\infty$.

Take the first case a), and let

$$f^+(x) = p. \qquad (3)$$

Then there is a perfectly definite succession of values of k, say $k_1, k_2, \ldots, k_r, \ldots$ such that, for every value of r,

$$(k_r - 1) e_r < p \leq k_r e_r \qquad (4)$$

so that, the quantities $k_r e_r$ form, as r increases, a monotone descending sequence with p as limit.

By (3) and (4) the point P belongs to the set G_{k_r, e_r} for all values of r; and therefore, since P is not a point of C_{k_r, e_r}, it is a point of F_{k_r, e_r} for all values of r. Hence, by (2)

$$f_-(P) \leq k_r e_r \qquad (5)$$

for all values of r.

But, as we let r increase indefinitely, the right hand side of (5) has the unique limit p, by (4). Hence

$$f_-(P) \leq p \qquad (6)$$

Thus, by (3)

$$f_-(P) \leq f^+(P) \qquad (7)$$

In case b) the relation (7) is also, of course true.

In case c) we have already seen that, except at a countable set of points (7) holds as an equality (§ 3).

Thus (7) holds for all points P, excepting only a countable set.

Similarly, except at a countable set,

$$f^-(P) \geq f_+(P) \qquad (8)$$

except at a countable set of points. This proves the theorem.

§ 6.

The relations (7) and (8) are precisely equivalent to the statement that there is at least one number which lies both between $f_-(P)$ and $f^-(P)$, both inclusive, and also between $f_+(P)$ and $f^+(P)$, both inclusive. But when $f(x)$ is a continuous function, its derivates on the right, or on the left, being the limits of the continuous function $(f(x+h)-f(x))/h$, when $0 < h$, or when $h < 0$, fill up respectively a whole closed interval, including, of course, a point as a special case. Thus we have the following corollary: —

Cor. *If $f(x)$ is a continuous fonction, it has at least one symmetric derivate, except at a countable set of points.*

By a symmetric derivate, I mean a limit of $(f(x+h)-f(x))/h$, which is the same when h describes a certain sequence of positive quantities with zero as limit, or a certain sequence of negative quantities with zero as limit, these two sequences depending on the point x.

§ 7.

On the other hand it is clear that, if P is a point at which a function, continuous or not, has a symmetric derivate, then the lower derivate on each side is less than, or equal to, the upper derivate on the other side; for the lower derivate on either side is not greater than the symmetric derivate, and the upper derivate on either side is not less than the symmetric derivate.

If the function is continuous, the symmetric derivates at any point not belonging to the exceptional countable set fill up a closed interval of values; for the upper and lower bounds of the symmetric derivates will lie between the upper and lower derivates both inclusive, on either side, so that every value between them is a derivate on either side.

Let us define as *the mean symmetric derivate* the trigonometric mean, as I should call it, between the greatest and least symmetric derivates in the case of a continuous function; that is to say, writing $\tan a$ for the greatest symmetric derivate, and $\tan b$ for the least, the mean symmetric derivate is $\tan \frac{1}{2}(a+b)$. Then the mean symmetric derivate of a continuous function is one of its derivates both on the right and also on the left, and, by the corollary of the preceding article, a continuous function has a mean symmetric derivate, except at a countable set of points.

At a point at which a differential coefficient exists, whether $f(x)$ is or is not continuous, there is only one symmetric derivate, which is the differential coefficient itself, and may be considered to be the mean symmetric derivate. But at a point at which the mean symmetric derivate, as defined above for a continuous function, exists, there need not be a differential coefficient. If accordingly I use the notation $f'(x)$ for the mean symmetric derivate, there will be no confusion with the usual notation for the differential coefficient.

If the mean symmetric derivate of a continuous function is $+\infty$, one or other of the lower derivates $f_-(x)$, or $f_+(x)$, must be $+\infty$. For the upper derivate on each side must of course be $+\infty$, hence, if the lower derivates were both less than a finite quantity $\tan c$, where $0 < c < \frac{1}{2}\pi$, every value greater than this would be a symmetric derivate, and therefore the mean symmetric derivate would be not greater than the finite quantity $\tan \frac{1}{2}(\frac{1}{2}\pi + c)$. At such a point accordingly there is, at least on one side, an infinite differential coefficient. The same is true at a point where the mean symmetric derivate is $-\infty$.

Now, as we have seen in § 3 when $k = \pm\infty$, the points at which a forward or backward differential coefficient exists and is infinite, without there being a proper differential coefficient are countable. Also by LUSIN's Theorem,[1] the points at which a continuous function has an infinite differential coefficient form a set of content zero, while, by HAHN's example[2] we know that they may form a perfect set. Thus *the mean symmetric derivate of a continuous function exists except at a countable set of points, and is finite except possibly at a set of content zero, which may, however, be perfect.*

§ 8.

As a special case of the theorem of § 5 we get immediately the following: —

Theorem. *If x is a point, not belonging to the exceptional countable set, and such that at it a forward differential coefficient $f'_+(x)$ exists, this lies between the upper and lower derivates on the other side, i. e.*

$$f_-(x) \leq f'_+(x) \leq f^-(x);$$

similarly if at x a backward differential coefficient f'_- exists,

$$f_+(x) \leq f'_-(x) \leq f^+(x).$$

Cor. *Except at a countable set of points, a function $f(x)$ cannot have a forward and a backward differential coefficient which are unequal in value, whether finite or infinite with determinate sign.*

In the case when $f(x)$ is a continuous function, this corollary has, with

[1] N. LUSIN. »Sur un théorème fondamental du calcul intégral», 1911. Recueil de la Société mathématique de Moscou, Vol. XXVIII, 2.

[2] H. HAHN. Mon. f. Math. 16. (1905), p. 317.

certain possible restrictions, not clearly stated, been given by BEPPO LEVI.[1] It is an immediate consequence that if $f(x)$ has both a forward and a backward differential coefficient, of value finite or infinite with determinate sign, at every point of an interval or set S, excepting perhaps at a countable set of points, then $f(x)$ has a differential coefficient $f'(x)$, of value finite, or infinite with determinate sign, at every point of S, excepting only a countable set.

With respect to this last result I may perhaps call attention to the fact that in HILBERT's proof[2] of the existence of a set of constants (Eigenwerte) $\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots$ for a quadric in space of an infinite number of dimensions, corresponding to the principal axes in the case of a quadric in a finite number of dimensions, a proof *ad hoc* was given that, in the special case of the functions there utilised, which have in fact a forward and a backward differential coefficient at every point, the differential coefficient exists except at a countable set.

§ 9.

The theorem of § 5 permits us also to extend a known theorem[3] as follows: —

Theorem. *If $f(x)$ is a continuous function which is zero at a and at b, then the points at which both the upper derivates and one of the lower derivates are ≥ 0 in (a, b) have the potency c, and the same is true of the points at which both the lower derivates and one of the upper derivates are ≤ 0.*

If $f(x)$ is identically zero, the theorem is obviously true. If not $f(x)$ must assume either a positive or a negative value, and it is clearly only necessary to discuss the former case. Let then p be the upper bound of the values

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[1] BEPPO LEVI »Ricerche sulle funzioni derivate», 1906. Rend. dei Lincei, (5), Vol. XV, p. 437.

Since finishing the present paper in August 1912 the following additional references have come to my notice. A. ROSENTHAL, »Ueber die Singularitäten der reellen ebenen Kurven», Habilitationsschrift, München, July 4th, 1912. W. SIERPINSKI, »Sur l'ensemble des points angulaires d'une courbe $y = f(x)$», Bull. de l'Acad. des Sciences de Cracovie, (A), October 1912, pp. 850—854. The theorem of SIERPINSKI and that of ROSENTHAL, given on p. 28, loc cit. are worded similarly as follows: — »*The set of points at which a curve has an angular point or cusp is at most countable*». The interpretation attached by the two authors to the geometrical concepts involved is, however, different, and, in consequence the theorem of ROSENTHAL is more general. SIERPINSKI has in fact reproved BEPPO LEVI's Theorem, without leaving any doubt as to restrictions. ROSENTHAL works with a general Jordan curve. If we interpret the definition of angular point given by ROSENTHAL on p. 5, loc. cit. in the sense in which I understand from him it was intended, and apply it to the curve $y = f(x)$, where $f(x)$ is a *continuous* function, we obtain the corresponding case of the Theorem of § 5, supra.

[2] D. HILBERT, »Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen», vierte Mitteilung, 1906, Gött. Nachr. p. 168.

[3] H. LEBESGUE. Leçons sur l'Intégration, p. 72. See also § 5 of »On Derivates and the Theorem of the Mean», loc. cit.

of $f(x)$, and let P be the first point at which $f(x)$, being continuous, assumes the value p.

Let $k_1, k_2, \ldots k_n, \ldots$ be the values of $f(x)$ at the exceptional points at which at least one of the lower derivates is greater than the upper derivate on the other side. Also let k be any value in the completely open interval $(0, p)$, other than one of the exceptional values k_n. Then, since $f(x)$ is continuous, and $f(a)=0$, $f(P)=p$, there is a point K at which $f(x)$ has the value k, and K is not one of the exceptional points. If $f(x)=k$ at more than one point of the interval (a, P), such points form a closed set, and we take the point K to be the nearest such point to P. Then in the completely open interval (K, P), $f(x)$ is never equal to k, but at P it is greater than k, therefore, throughout the completely open interval (K, P), $f(x)$ is greater than k. Thus, if x is the coordinate of K, and $x+h$ that of a point in (K, P),

$$\frac{f(x+h)-f(x)}{h} > 0,$$

and therefore

$$f_+(x) \geq 0.$$

Since the point K is not one of the exceptional points, it follows that

$$f^-(x) \geq 0,$$

moreover, corresponding to each value of k, other than $k_1, k_2, \ldots, k_n, \ldots$ we get such a point.

Similarly, working on the right, instead of on the left of the point P, we find a point at which

$$f^-(x) \leq 0,$$

which is not one of the exceptional points, so that here we have also

$$f_+(x) \leq 0.$$

This proves the theorem.

§ 10.

Hence also, by the usual method of deducing the Theorem of the Mean from Rolle's Theorem, we have the following: —

Theorem. *If $f(x)$ is a continuous function, and*

$$m(a, b) = \frac{f(b)-f(a)}{b-a},$$

then the points of the interval (a, b) *at which both the upper derivates and one of the lower derivates are* $\geq m(a, b)$ *have the potency c, and so have the points at which both the lower derivates and one of the upper derivates are* $\leq m(a, b)$.

From this we have the following corollary: —

Cor. *If* $f(x)$ *is continuous in the interval* (a, b) *without lines of invariability, there is an everywhere dense set of points of potency c at which both the upper derivates and one lower derivate are positive, (not zero), or both the lower derivates and one upper derivate are negative (not zero).*

§ 11.

Hence we have the following theorem: —

Theorem. *If, except at a countable set, we know of a function* $f(x)$ *that*

a) *the two upper derivates,*

or b) *the two lower derivates,*

or c) *the two right-hand derivates,*

or d) *the two left-hand derivates,*

never have the same sign, then $f(x)$, *if continuous, is a constant.*

The cases c) and d), when there is no exceptional set, are given in LEBESGUE's Leçons sur l'intégration (p. 72), and he deduces (p. 74) the theorem that a continuous[1] function is determined, to an additive constant près, when we know the finite value of one of its extreme[2] derivates for every finite value of the variable. The theorem, as above stated, gives us the shortest proof of a theorem of SCHEEFFER's, beginning as LEBESGUE's proof (loc. cit. p. 78) does, but ending after the third line, instead of requiring an additional sixteen lines. It is as follows: —

Theorem. *A continuous function* $f(x)$ *is determined, to anadditive constant près, if, except at the points of a countable set, we know that one of its extreme derivates is finite, and we have its value.*

In fact, if possible let there be two such functions $f(x)$ and $g(x)$, and let us write

$$F(x) = f(x) - g(x).$$

Then $F(x)$ has, on the side on which we know one of the extreme derivates of $f(x)$ and $g(x)$ to be equal, its extreme derivates never of the same sign, except perhaps at the points of the exceptional set. Thus, by the preceding theorem, $F(x)$ is a constant, which proves the theorem.

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[1] This word has inadvertently apparently dropped out of the enunciation.

[2] That is an upper or lower left or right hand derivate.

Similarly from cases a) and b) of the theorem at the beginning of the present article, we have the following new theorem: —

Theorem. *If, except at a countable set at which we may be doubtful, we know of two continuous functions $f(x)$ and $g(x)$ that on one side the upper derivate of $f(x)$ is not greater than the lower derivate of $g(x)$, and, if equal, is not infinite, while on the other side one of the extreme derivates of $f(x)$ is, at each point, not less than the corresponding derivate of $g(x)$, and, if equal, is not infinite, then the two functions only differ by a constant.*

In symbols we have, taking the first known fact to refer to the left-side,

$$f^{-}(x) - g_{-}(x) \leq 0, \tag{1}$$

and, by the second known fact, at each point x either

$$f^{+}(x) - g^{+}(x) \geq 0, \tag{2}$$

or

$$f_{+}(x) - g_{+}(x) \geq 0. \tag{3}$$

Now, using familiar inequalities, and writing $F(x) = f(x) + h(x)$, where $h(x) = -g(x)$, we have

$$F^{-}(x) \leq f^{-}(x) + h^{-}(x) \leq f^{-}(x) - g_{-}(x),$$

so that, except at the doubtful points, by (1),

$$F^{-}(x) \leq 0, \tag{4}$$

while

$$F^{+}(x) \geq f^{+}(x) + h_{+}(x) \geq f^{+}(x) - g^{+}(x)$$

and also

$$F^{+}(x) \geq f_{+}(x) + h^{+}(x) \geq f_{+}(x) - g_{+}(x),$$

so that, by (2) and (3), we have, at each point not belonging to the doubtful set,

$$F^{+}(x) \geq 0. \tag{5}$$

By (4) and (5) $F(x)$ is a constant, using case a) of the theorem at the beginning of the present article. This porves the theorem.

This theorem seems to me worthy of notice; we had hitherto, as far as I am aware, no theorem except SCHEEFFER's Theorem, which enabled us to identify from a knowledge of their derivates two continuous functions of a perfectly general character, not necessarily of bounded variation, or even belonging to the more general classes of continuous functions which have recently been studied.

§ 12.

From the theorem of § 9 we have immediately certain theorems for the mean symmetrie derivate, which were known to be true for the differential coefficient.

Theorem. *If $f(x)$ is a continuous function which is zero at a and at b, then the points at which the mean symmetric derivate $f'(x) \geq 0$ in (a, b) have the potency c, and the same is true of the points at which $f'(x) \leq 0$.*

Theorem. *If $f(x)$ is a continuous function, and*

$$m(a, b) = \frac{f(b) - f(a)}{b - a},$$

then the points of the interval (a, b) at which the mean symmetric derivate $f'(x) \geq m(a, b)$ have the potency c, and so have the points at which $f'(x) \leq m(a, b)$.

Theorem. *If we know that, except possibly at a countable set the mean symmetric derivate $f'(x) = 0$, then $f(x)$, if continuous, is a constant.*

Theorem. *If two continuous functions have the same finite symmetric derivate except possibly at a countable set of points, then the two functions only differ by a constant.*

§ 13.

It has been already mentioned that, except at a countable set of points, the mean symmetric derivate is only infinite where the function possesses a differential coefficient which is infinite with determinate sign. Combining this fact with W. H. Young's extension of Lebesgue's theorem,[1] which states that, *if $f(x)$ denotes one of the derivates, extreme or intermediate, of a continuous function $F(x)$, and $f(x)$ is $+\infty$ at most at a countable set of points, then if, and only if, $f(x)$ is summable over the set of points S where it is positive, $F(x)$ is a function of bounded variation, and, in fact a lower semi-integral,[2] whose positive variation is* $\int_S f(x)\,dx$, we have the following theorem: —

If $F(x)$ is a continuous function, having at most at a countable set of points a differential coefficient which is $+\infty$, then if, and only if, the mean symmetric derivate $F'(x)$ is summable over the set of points S where it is positive, $F(x)$ is a

[1] «On Derivates and their Primitive Functions», 1912, Proc. L. M. S.

[2] A lower semi-integral is the sum of a Lebesgue integral and a monotone decreasing function. See W. H. Young, «On Semi-integrals and Oscillating Successions of Functions», 1910, Proc. L. M. S., (2), Vol. 9, p. 294.

function of bounded variation, and indeed a lower semi-integral, and its positive variation is $\int_S F'(x)\,dx$.

A similar theorem holds changing $+\infty$ into $-\infty$ and positive into negative. Combining these we have the following:

If $F(x)$ is a continuous function, having at most at a countable set of points a differential coefficient which is infinite with determinate sign, then if, and only if, the mean symmetric derivate $F'(x)$ is summable, $F(x)$ is the Lebesgue integral of any one of its derivates.

Hence: —

If $F(x)$ is a continuous function of bounded variation, but not a Lebesgue integral, it has a differential coefficient which is infinite with determinate sign at a more than countable set of points.

§ 14.

The exceptional set which has furnished the keynote of the present paper, has not been shown to have any particularities beyond that of being countable. That the set may be any countable set whatever is evident by the principle of the Condensation of Singularities. If we consider the class of functions for which the exceptional set is dense nowhere, we can give further results. Thus, for instance, it follows from the Theorem of the Mean for Derivates[1] that, *if (a, b) is an interval free of points of the exceptional set, there is a point x in the completely open interval (a, b) at which there is one and only one symmetric derivate, and its value is $m(a,b)=\{f(b)-f(a)\}/(b-a)$; the function $f(x)$ is here of course*

[1] The theorem states that «*if $f(x)$ is continuous throughout the closed interval (a, b), and the incrementary ratio $m(a, b)$ is finite, then there is a point x, $(a<x<b)$, at which one of the upper derivates is not greater than $m(a, b)$ while the other lower derivate is not less than $m(a, b)$, that is*

$$f^{+}(x)\leq m(a,b)\leq f_{-}(x),$$

or

$$f^{-}(x)\leq m(a,b)\leq f_{+}(x).»$$

See «On Derivates and the Theorem of the Mean», loc. cit., p. 10.

The proof of this theorem only depends on the continuity of $f(x)$ in so far that $f(x)$ and $f(x)-x\,.\,m(a,b)$ have to be such functions of x that they assume all values between their upper and lower bounds. Hence it follows that the result stated in the text for continuous functions holds for functions of a more general nature. In particular it follows that *if $f(x)$ is a finite differential coefficient throughout (a, b), and the exceptional countable set is absent, there is a point of the completely open interval (a, b) at which one and only one symmetric derivate of $f(x)$ exists and it is equal to $m(a, b)$.*

supposed to be continuous. Hence the usual properties of the differential coefficient of a continuous function, deduced from the Theorem of the Mean, may be carried over to the mean symmetric derivate in such an interval. In particular the mean symmetric derivate will be one of the limits of its values on the right and on the left.

§ 15.

I have only to add that, in the present paper, I have not considered functions other than general ones and continuous ones. But corresponding results hold to those given in W. H. Young's paper on «Term-by-term integration of Oscillating Series».[1] Thus, for instance we have the following theorem: —

Theorem. *If $f(x)$ is upper semi-continuous on the right, and lower semi-continuous on the left, throughout the interval (a, b), and is zero at the end-points, then the points at which both the upper derivates and one of the lower derivates are ≥ 0, have the potency c.*

[1] loc. cit., pp. 104—107.