ON INVARIANT AND SEMI-INVARIANT ABERRATIONS OF THE SYMMETRICAL OPTICAL SYSTEM.

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Introduction.

An attempt was made[1], by the present writer, to examine theoretically the diffraction patterns associated with the symmetrical optical system, as modified by the presence of the geometrical aberrations of the system, and the investigation was carried out as far as the first order aberrations were concerned, and for the region in the neighbourhood of the axis of the system. But, in order to consider the effects in the outer parts of the field it appeared necessary, as a preliminary measure, to examine the higher order geometrical aberrations themselves, and this[2], accordingly, has been undertaken in several papers. The five first order aberrations, commonly known as the 'five aberrations of von SEIDEL' — although these had all been discussed fully by HAMILTON, by AIRY, and by CODDINGTON long before the time of von SEIDEL — are *spherical aberration*, *coma*, *curvature of the field and astigmatism*, and *distortion*. And, in a detailed examination of these it becomes evident that one of them stands altogether apart from the others, and this in several respects: this aberration is *curvature of the field and astigmatism*. The condition for the absence of curvature of the field, the condition, that is to say, that a flat field should, in the absence of astigmatism, be reproduced as flat, is found to be independent of the positions of the object-image planes and of the positions of the pupil-planes, and also of the separa-

[1] *Aberration Diffraction Effects:* Phil. Trans. Royal Soc. (Lond.) A. 225.

[2] see, for example, *The Aberrations of a Symmetrical Optical System,* Trans. Camb. Phil. Soc. XXIII, No. IX, and also *The Symmetrical Optical System:* Camb. Tracts in Mathematics and Mathematical Physics, No. 25.

tions of the several refracting surfaces of the optical system. Moreover, the condition has a peculiarly simple form, especially when compared with the conditions for the freedom from the other geometrical aberrations of the optical system. The condition referred to is, of course, the vanishing of the well-known and so called »Petzval-sum»: that is, $\varpi \equiv \Sigma \varkappa/\mu\mu' = 0$, $\varkappa$ being the power of the surface separating media of optical indices μ and μ', and the summation extending throughout the optical system.

It will be noticed that the condition involved in the expression given above is in form very simple, especially when compared with the conditions for freedom from even the other first order aberrations; and the aberrations of higher orders lead, for the most part, to increasingly complicated expressions. The very simplicity of this condition suggests that it has a meaning more extended than that commonly assigned to it; just as the well-known 'sine-condition', and also 'Herschel's condition', have definite geometrical meanings not only, as they are commonly presented, with regard to the first order aberrations alone, but also with regard to certain aberrations of all orders: and, indeed, they are themselves but special cases of the recently discovered and very general 'optical cosine-law'.

Accordingly, in the present paper the clue afforded by the 'Petzval-condition' is followed up, and the extent and the meaning of this condition are investigated more fully: and, in particular, a complete generalisation of the 'Petzval-condition' is obtained, for the higher order aberrations. And this is found to raise another and a more general problem, namely, that of the separation, into three types, of the geometrical aberrations of the general symmetrical optical system, of all orders, according as these aberrations possess properties which we have named 'invariant', or 'semi-invariant', or else are completely unrestricted. The conditions attaching to the aberrations of the first two types, and, more especially, to those of the first type, are of a peculiarly simple nature, — and this for aberrations of *all* orders. And a corresponding simplicity of geometrical meaning is found. It is hoped, then, that the results obtained, themselves of theoretical interest and importance, may be of use in the design of optical systems.

The investigation falls naturally into three parts, namely:

Part I: in which is undertaken a qualitative investigation of the geometrical aberrations of the general symmetrical optical system. Here each several aberration is shewn to fall under one or other of three categories; the properties of each category are examined, and the total number of aberrations falling under each is found.

Part II: in which is undertaken a quantitative investigation of the various conditions obtained qualitatively in Part I. These various conditions are found, explicitly, for the general symmetrical optical system.

Part III: in which is undertaken an investigation of the geometrical meanings and implications associated with the conditions obtained qualitatively and quantitatively in Parts I and II.

The only papers known to me and bearing in any manner upon the subjects of this paper are the following, namely,

The Changes in Aberrations when Object and Stop are Moved: T. SMITH, Trans. Opt. Soc. (Lond.) (1921—22), No. 5.

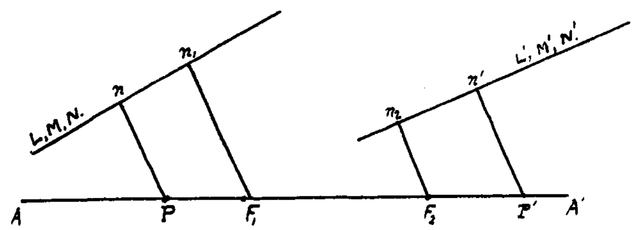
The Additon of Aberrations: T. SMITH, Trans. Opt. Soc. (Lond.) (1923—24), No. 4.

Part I.

1. The method underlying the present investigation is based upon a modification of the Characteristic Function introduced by HAMILTON[1], known more commonly as the *Eikonal* of *Bruns.* It is not without interest to notice that, although the name 'Eikonal' is due to Bruns, the function itself appeared at a much earlier date, in Hamilton's original series of Papers. The detailed development of these functions, and their application to the theory of the symmetrical optical system, have been given elsewhere by the present writer[2], so that an outline only, in briefest possible form, is necessary in this first paragraph.

[1] The surprising extent to which Sir WILLIAM HAMILTON had applied his very general theory to the actual consideration of particular optical systems, whether symmetrical or quite unsymmetrical, is only revealed by a careful perusal of his celebrated Papers on *The Theory of Systems of Rays.* These have recently been published in the Edition of his Collected Works, Volume 1, *Geometrical Optics,* by the Cambridge University Press, under the very able and joint Editorship of Professors Conway and Synge: here certain papers are published for the first time. And in them the general functions introduced by Hamilton are applied to the symmetrical optical system, a project which frequently he mentioned in his published works, but to which, in them, he never seems to have addressed himself. But even in the papers published, for example, in 1833—34 there is given an investigation of the aberration known afterwards as *coma,* and this for a system quite unsymmetrical; and the discovery of this aberration has commonly been attributed to KIRCHHOFF, at a much later date, who himself was working with functions akin to those introduced by Hamilton. For additional information concerning these matters, and other matters connected with them, reference may be made to a paper, by the present writer, *On the Optical Writings of Sir William Rowan Hamilton,* Mathematical Gazette, July 1932, Vol. XVI, No. 219, pp. 179—191.

[2] *The Symmetrical Optical System:* Camb. Tracts in Mathematics and Mathematical Physics, No. 25.



We consider a symmetrical optical system of which AA' is the axis, P and P' two conjugate points upon AA', and F_1 and F_2 the principal foci. Pn and $P'n'$ are the perpendiculars from P and P' upon the incident and emergent portions of a ray of light, the direction cosines of which, referred to convenient and parallel axes, the one set in the 'object-space' and the other set in the 'image-space', are respectively L, M, N, and L', M', N'. Then the Eikonal E, with base points P and P', is defined as being equal to the optical path from n to n', measured along the ray; that is,

$$E = \int_{n}^{n'} \mu \, d s,$$

where μ is the optical index of the medium in which the element of length ds is measured. A function of great theoretical and practical importance is the 'focal-eikonal', E_0, defined by means of the base points F_1 and F_2, the principal foci of the optical system; that is,

$$E_0 = \int_{n_1}^{n_2} \mu \, d s,$$

where n_1 and n_2 are the feet of the perpendiculars upon the ray from the points F_1 and F_2. Further, if we denote by e and e_0 respectively the values of these functions when the ray coincides with the axis AA' of the system, we have

$$e = \int_{P}^{P'} \mu \, d s, \quad \text{and} \quad e_0 = \int_{F_1}^{F_2} \mu \, d s,$$

in each case the path of integration being the axis of the optical system; and it is convenient to absorb the constants e and e_0 in the more general eikonal-functions. We write then,

$$J(E-e)=u-\Phi, \quad \text{and} \quad J(E_0-e_0)=U,$$

where J is the 'modified power' of the optical system, and is given by the relation

$$\mu\mu' J = K,$$

K being the power as commonly defined, and μ and μ' the optical indices of the end media. The form $u-\Phi$ is explained subsequently.

In general a ray of light, as presented above, has *four* degrees of freedom, but, owing to the axial symmetry of the optical system, *three* variables only are needed, and each of the preceding functions, namely, E, and E_0, $u-\Phi$ and U, may be regarded as depending upon three variables alone. The choice of these variables is of considerable importance. We may choose, for example, a, b, and c, given by the relations

$$a=M^2+N^2, \ b=MM'+NN', \text{ and } c=M'^2+N'^2.$$

Another choice, the explanation of which is indicated later, is the following, namely, α, β, and γ, where

$$\left.\begin{aligned} \alpha d^2 &= a-2sb+s^2c, \\ \beta d^2 &= a-(s+m)b+smc, \\ \gamma d^2 &= a-2mb+m^2c, \end{aligned}\right\} \text{ that is, } \left\{\begin{aligned} a &= m^2\alpha-2sm\beta+s^2\gamma, \\ b &= m\alpha-(s+m)\beta+s\gamma, \\ c &= \alpha-2\beta+\gamma; \end{aligned}\right.$$

here d is a certain convenient constant which may be taken to be $d\equiv s-m$, where m and s are respectively the paraxial, or Gaussian, (reduced) magnifications associated with the conjugate points P and P', and with the pupil-planes of the system. Thus d is equal to the (reduced and modified) distance between the exit-pupil and the paraxial image plane.

The conjugate and normal planes through P and P' will not, in general, be free from aberration; incident rays, that is to say, passing through a point upon one plane will not, in general, pass through any corresponding point upon the other plane. But we may shew that if it were indeed possible for these two normal planes to be free from geometrical aberration, that is, if there could be a one-to-one correspondence in points between them, then the eikonal-function $E-e$, and also the function $u-\Phi$, would depend only upon the variable γ; and conversely. This property gives the suggestion for the form of the variable

γ, and then the forms of the variables α and β follow from considerations of symmetry.

The eikonal for an actual optical system — one subject to geometrical aberrations — will contain terms involving also the variables α and β, so that, if we write $u = f(\gamma)$, the form of this function f being at present undetermined (and it may be determined subsequently so as to satisfy other conditions), then in the expression $u - \Phi$ we may regard the function Φ as containing all the terms involving α and β, and so as summing up in itself the departure of the system from 'ideal' imagery, for the particular pair of conjugate planes chosen. We may, therefore, appropriately name Φ the *aberration-function*, and we observe that it gives completely the aberrations of the optical system for the conjugate planes through P and P', at paraxial magnification m: and that it depends upon these aberrations alone. In other words, we have separated the Gaussian performance of the system from the departures from this performance.

Actually, the aberrations are given by the relations

$$Y' - m\,Y = \frac{\partial \Phi}{\partial M'} + m\frac{\partial \Phi}{\partial M}, \text{ and } Z' - m\,Z = \frac{\partial \Phi}{\partial N'} + m\frac{\partial \Phi}{\partial N},$$

where Y and Z are the co-ordinates of the point of intersection of the incident ray with the normal plane through P, and Y' and Z' are the co-ordinates of the point of intersection of the emergent ray with the normal plane through the conjugate point P'.

Now, we may write

$$\Phi(\alpha, \beta, \gamma) = \Phi_2(\alpha, \beta, \gamma) + \Phi_3(\alpha, \beta, \gamma) + \cdots + \Phi_n(\alpha, \beta, \gamma) + \cdots$$

where $\Phi_n(\alpha, \beta, \gamma)$ is a homogeneous function, of degree n, in the three variables α, β, and γ: Φ_0 and Φ_1 are omitted, since the aberrations depend essentially upon the terms of the second and higher orders in α, β, and γ. The coefficients appearing in the various functions $\Phi_n(\alpha, \beta, \gamma)$ give completely the aberrations of the optical system of the several orders, and we name them therefore 'aberration-coefficients'. For example, if we write

$$8\,\Phi_2 = \sigma_1\alpha^2 - 4\,\sigma_2\,\alpha\,\beta + 2\,\sigma_3\,\alpha\,\gamma + 4\,\sigma_4\,\beta^2 - 4\,\sigma_5\,\beta\,\gamma + \sigma_6\,\gamma^2,$$

the σ-coefficients give completely the *five* first order geometrical aberrations, for the term in σ_6 depends only upon the variable γ, and so is annihilated by each of the operators

$$\frac{\partial}{\partial M'} + m\frac{\partial}{\partial M}, \quad \text{and} \quad \frac{\partial}{\partial N'} + m\frac{\partial}{\partial N}.$$

In particular, the coefficients σ_3 and σ_4 together give the astigmatism and the curvature of the field, each of the first order. So we have outlined a method of investigating the qualitative nature of the geometrical aberrations, and we have now to consider their quantitative aspect. But, in passing, it will be noticed that we have separated these aberrations into various 'orders', depending successively upon the functions $\Phi_n(\alpha, \beta, \gamma)$; thus, $\Phi_2(\alpha, \beta, \gamma)$ gives the aberrations of the *first* order, $\Phi_3(\alpha, \beta, \gamma)$ those of the *second* order, and, more generally $\Phi_{n+1}(\alpha, \beta, \gamma)$ gives the aberrations of the *n'th* order. And this is the manner in which the aberrations of a symmetrical optical system are commonly presented.

It is clear that the focal-eikonal U, introduced in the preceding scheme, is a constant of the optical system; that is, U is independent both of the positions of the conjugate axial points P and P', and also of the positions of the pupil-planes of the system. In other words, U does not depend either upon m or upon s. We may regard U as a function of the three variables a, b, and c alone, and the coefficients of the various terms in the expansion of this function are the quantities which we calculate in the computation of the optical system.

Moreover, the functions $u - \Phi$ and U differ only by reason of their differing base points, and there is therefore a purely geometrical relation between them, namely, the following relation,

$$u - \Phi = U + (1 - L)/m + (1 - L')\, m.$$

If, then, we know the function U we can calculate immediately the aberration coefficients, and so the aberrations themselves, for any symmetrical optical system, for any conjugate planes and for any pupil-planes.

It is convenient to calculate the function U step by step, and we make here, for the first time, the assumption that the surfaces of the system are *spherical:* and we may shew that the focal-eikonal for a *single* spherical surface, separating media of optical indices μ and μ', is given, without any approximation, by the relation,

$$v\, U = \sqrt{(1 + v\chi)} + v(L + L' - 2) - 1,$$ [1]

where

$$v = \mu\mu'/(\mu' - \mu)^2, \quad \text{and} \quad \chi + 2(LL' + b - 1) = 0.$$

[1] *The Symmetrical Optical System,* Camb. Tracts in Mathematics and Mathematical Physics, No. 25, Ch. V.

The radius of curvature does not appear explicitly, since the eikonal is supposed to have been multiplied by the 'modified' power of the system. The expression is clearly a function of the variables a, b, and c, since $L^2 = 1 - a$, and $L'^2 = 1 - c$; and the general focal-eikonal, for any number of co-axial spherical surfaces, is similarly a function of the variables a, b, and c, — where now, however, these variables refer to the system as a whole.

2. The essential relation, upon which we concentrate, is the purely geometrical equation of paragraph 1, namely,

$$u - \Phi = U + (1 - L)/m + (1 - L')\,m. \tag{1}$$

Here the expression upon the left-hand side of the equation is a function of the variables α, β, and γ, while the expression upon the right-hand side is a function of the variables a, b, and c; and between these two sets of variables there exist linear relations given in paragraph 1: moreover neither s nor m appears in the function U. We expect, therefore, various invariant relations between the coefficients appearing upon the two sides of the equation, and these we proceed to investigate.

From the relations of the preceding paragraph we have,

$$\left.\begin{aligned} \frac{\partial}{\partial \alpha} &= m^2\frac{\partial}{\partial a} + m\frac{\partial}{\partial b} + \frac{\partial}{\partial c}, \\ -\frac{\partial}{\partial \beta} &= 2sm\frac{\partial}{\partial a} + (s+m)\frac{\partial}{\partial b} + 2\frac{\partial}{\partial c}, \\ \frac{\partial}{\partial \gamma} &= s^2\frac{\partial}{\partial a} + s\frac{\partial}{\partial b} + \frac{\partial}{\partial c}, \end{aligned}\right| \text{ and } \left\{\begin{aligned} d^2\frac{\partial}{\partial a} &= \frac{\partial}{\partial \alpha} + \frac{\partial}{\partial \beta} + \frac{\partial}{\partial \gamma}, \\ -d^2\frac{\partial}{\partial b} &= 2s\frac{\partial}{\partial \alpha} + (s+m)\frac{\partial}{\partial \beta} + 2m\frac{\partial}{\partial \gamma}, \\ d^2\frac{\partial}{\partial c} &= s^2\frac{\partial}{\partial \alpha} + sm\frac{\partial}{\partial \beta} + m^2\frac{\partial}{\partial \gamma}; \end{aligned}\right.$$

that is, we have relations between the operations of differentiation with respect to the several variables. We define new operators Π, P, Ω and O, by the following,

$$\Pi \equiv 4\frac{\partial^2}{\partial \gamma\,\partial \alpha} - \frac{\partial^2}{\partial \beta^2}, \qquad P \equiv 4\frac{\partial^2}{\partial c\,\partial a} - \frac{\partial^2}{\partial b^2},$$

$$\Omega \equiv \frac{\partial}{\partial \alpha} + \frac{\partial}{\partial \beta} + \frac{\partial}{\partial \gamma}, \qquad O \equiv \frac{\partial}{\partial a};$$

and from the above relations we have immediately,

$$\Pi \equiv d^2 P, \quad \text{and} \quad \Omega \equiv d^2 O.$$

The operators Π and Ω are commutative, as also are the operators P and O. It follows that

$$\Omega^p \Pi^q \equiv d^{2(p+q)} O^p P^q.$$

where p and q are any positive integers.

Let us now apply these operators, the one to the left-hand side and the other to the right-hand side of the relation (1), considering only the terms of degree $p + 2q$ in the variables α, β, and γ, or the variables a, b, and c; terms of lower degree will, of course, be annihilated, while terms of higher degree we omit for the present. On the left-hand side we shall obtain a *linear* expression involving the aberration coefficients appearing in that part of the aberration-function Φ which we have written Φ_{p+2q}, that is, a linear relation between the aberration-coefficients of order $p + 2q - 1$. Let this expression be written φ. On the right-hand side we have a function of the coefficients appearing in U, that is, a function of certain constants of the optical system — independent, that is to say, of the quantities s and m; and we have, in addition, a multiplying factor $d^{2(p+q)}$, or $(s-m)^{2(p+q)}$, so that, if we write the resulting function f, we have the relation

$$\varphi = d^{2(p+q)} f,$$

or

$$\varphi = (s-m)^{2(p+q)} f.$$

We have assumed that $q \neq 0$, for then the terms $(1 - L)/m + (1 - L')\,m$, appearing in (1) are annihilated; otherwise, if $q = 0$, we have always in φ terms involving m.

It follows that if, for any particular optical system, $f = 0$, then also will $\varphi = 0$, and this latter result will be true for *all* values of s and m: that is, we have a relation between the aberration coefficients of order $p + 2q - 1$, which is independent of s and of m. While, if $f \neq 0$, the function φ will depend upon s and upon m, but only through the factor $(s - m)$. Now, φ denotes an aberration of order $p + 2q - 1$, more strictly, a linear relation between the various aberration-coefficients of this order, as we defined them above. We have obtained, then, an aberration which we may name an 'invariant aberration', in the sense that if this aberration vanish for *any* particular single pair of positions of the conjugate planes and of the pupil-planes of the optical system then the aberration will vanish for *all* pairs of positions of these planes. On the other hand, if $\varphi \neq 0$, then the magnitude of this aberration will depend upon the positions of these planes through the factor $(s - m)$ alone; that is, it will depend only upon the *relative* positions of these planes.

As the simplest example in illustration of the preceding general theory, we may write $p = 0$ and $q = 1$, that is, we consider simply the operator Π alone; and we know that

$$\Pi \equiv d^2 P.$$

Applying the operator Π to the left-hand side of (1) we have, taking only the second order terms, which are written out at length in paragraph 1,

$$\Pi (u - \Phi) = - \Pi \Phi_2 = \sigma_3 - \sigma_4,$$

and the operator P, applied to the second order terms in U, will give some constant quantity, a constant of the optical system, which we may write ϖ; we have then

$$P\, U = \varpi.$$

Thus

$$(\sigma_3 - \sigma_4)/(s - m)^2 = \varpi;$$

the quantity ϖ is in fact the 'Petzval-sum', and, subsequently, we shall prove that

$$\varpi = \Sigma\, \varkappa / \mu\, \mu'$$

in the usual notation. If $\varpi = 0$, then, from the preceding relation, $\sigma_3 - \sigma_4 = 0$, for all values of s and m: that is, a flat field is, in the absence of astigmatism, reproduced as flat, — as far as the *first* order aberrations are concerned.

3. Again it is seen, from the relations of paragraph 2, that the operator $\frac{\partial}{\partial \alpha}$ involves only the quantity m, and not the quantity s, since

$$\frac{\partial}{\partial \alpha} \equiv m^2 \frac{\partial}{\partial a} + m \frac{\partial}{\partial b} + \frac{\partial}{\partial c}.$$

Also, we have, from these same relations,

$$\Omega^p\, \Pi^q \left(\frac{\partial}{\partial \alpha}\right)^r \equiv d^{2(p+q)}\, O^p\, P^q \left(m^2 \frac{\partial}{\partial a} + m \frac{\partial}{\partial b} + \frac{\partial}{\partial c}\right)^r.$$

If we apply the operator $\Omega^p\, \Pi^q \left(\frac{\partial}{\partial \alpha}\right)^r$ to the left-hand side of the equation in paragraph 1, retaining therein only the terms of degree $p + 2q + r$ in the variables α, β, and γ, we obtain a linear function of the aberration coefficients of order $p + 2q + r - 1$. Let this function be written ψ. Applying now, to the other side of the equation, the equivalent operator, retaining therein only the terms of degree $p + 2q + r$ in the variables a, b, and c, and assuming that $q \neq 0$, we have a function of the optical constants of the optical system, containing the quantity m but *not* the quantity s; let this function be written $f(m)$. Then, by reason of the above relation between the operators employed, we have

$$\psi = (s - m)^{2(p+q)} f(m).$$

If, therefore, $f(m) = 0$ the function ψ will vanish for *all* values of s; otherwise the value of ψ will depend upon s and upon m through the factor $(s - m)$, and also directly through the value of m, but *not* directly through the value of s. We have here, then, an aberration ψ which we may name a 'semi-invariant' aberration. Clearly, in the same manner, we may define 'semi-invariant' aberrations the vanishing of which depends only upon s, and not upon m. And, finally, we have entirely unrestricted aberrations, the vanishing of which depends *both* upon s *and* upon m.

We may sum up the results of the two preceding paragraphs as follows. There exist linear relations between the aberration coefficients, of every 'order', of each of the three following types, namely:

1. the *invariant type:* the vanishing of which is independent of the conjugate planes chosen, and also of the pupil-planes (and, as we find subsequently, in the case of the most important sub-class of the invariant type, independent also of the separations of the component surfaces of the optical system).
2. the *semi-invariant type:* the vanishing of which depends *either* upon the positions of the conjugate planes, *or* upon the positions of the pupil-planes, but *not* upon both of these, and
3. the *general*, or *unrestricted*, *type:* the vanishing of which depends both upon the positions of the conjugate planes, and also upon the positions of the pupil-planes of the optical system.

It will be seen that the preceding classification of the aberrations of the symmetrical optical system cuts altogether across the usual division of these aberrations into 'orders', — based, as this division is, upon the idea of the 'orders' of small quantities. But the new classification corresponds, in the first place, to certain physical properties of the optical system, and, in the second place, to a certain striking simplicity of calculation; for we shall find, in the sequel, that the aberrations of the various types have certain geometrical peculiarities, and also that the conditions attaching to the invariant type are of an exceedingly simple form.

4. We proceed now to enquire how the geometrical aberrations of any particular 'order' are distributed amongst the three general types to which we have been led, namely, the invariant type, the semi-invariant type, and the unrestricted type. And we consider, in the first place, the function $\Phi_{2n}(\alpha, \beta, \gamma)$ homogeneous and of degree $2n$ in the variables α, β, and γ; this function gives then completely the aberrations of 'order' $2n-1$.

Now, from the preceding paragraphs, the operators

$$\Omega^{2n-2q}\,\Pi^q,$$

where q takes successively the values $1, 2, 3, \ldots n$, when applied to the function $\Phi_{2n}(\alpha, \beta, \gamma)$ lead to invariant aberrations. We have therefore n invariant aberrations of order $2n-1$. If we wish to consider the aberrations of order $2n$ we must use the function $\Phi_{2n+1}(\alpha, \beta, \gamma)$, homogeneous and of degree $2n+1$ in the variables α, β, and γ. In this case the appropriate operators are the following, namely,

$$\Omega^{2n-2q+1}\,\Pi^q,$$

where q takes successively the values $1, 2, 3, \ldots n$. And again we have n invariant aberrations of order n.

We consider next the semi-invariant aberrations, which will follow from applications of the operators Ω and Π, together with $\frac{\partial}{\partial \alpha}$ for the s-invariants, or $\frac{\partial}{\partial \gamma}$ for the m-invariants. We take then the general operator

$$\Omega^p\,\Pi^q \left(\frac{\partial}{\partial \alpha}\right)^r \left(\frac{\partial}{\partial \gamma}\right)^t,$$

where $p + 2q + r + t = 2n$, and apply this to the function $\Phi_{2n}(\alpha, \beta, \gamma)$. This will lead immediately to some linear function of the aberration coefficients, of order $2n-1$; an m-invariant if $r = 0$, or an s-invariant if $t = 0$: while, of course, if $r = 0$ and $s = 0$ we obtain invariant aberrations of the first type, already investigated.

We consider, in the first place, the s-invariants, for which $t = 0$: then, since $p + 2q + r = 2n$, there are $2n - 2q + 1$ sets of values of p and r satisfying this condition, for every value of q; so that the total number of invariants obtained in this way is

$$\sum_{q=1}^{n} (2n - 2q + 1) = n^2,$$

since q takes successively the values $1, 2, 3, \ldots n$. But of these there will be *one*, for each value of q, for which $r = 0$, and which therefore is an invariant of the first type; the total number of s-invariants is therefore $n^2 - n$. There is an equal number of m-invariants, so that, finally, we have, as the total number of semi-invariants of order $2n - 1$, the expression $2n(n-1)$. The total number of aberration coefficients, appearing in the homogeneous function $\Phi_{2n}(\alpha, \beta, \gamma)$, is $(2n+1)(2n+2)/2$. Remembering now that there is always one term, namely that one in the variable γ alone, which is annihilated by the operators, we see that the number of unrestricted aberrations, of the third type, is $4n$.

The preceding paragraph deals with the distribution of the aberrations of an odd order, namely, of order $2n - 1$. For the aberrations of an even order, for example of order $2n$, we consider the function $\Phi_{2n+1}(\alpha, \beta, \gamma)$, to which we apply operators of the same general form. Then, repeating the argument, for a given value of q we have $p + r + t = 2n + 1 - 2q$, and therefore, if $t = 0$, there are $2n + 2 - 2q$ sets of values of p and r satisfying this condition, so that the total number of s-invariants, obtained in this manner, is given by

$$\sum_{q=1}^{n} (2n - 2q + 2) = 2\sum_{q=1}^{n} (n - q + 1) = n(n+1);$$

but, of these, n are invariants of the first type, for which $r = 0$ and $t = 0$. Thus the number of s-invariants is $n(n+1) - n$, or n^2: and the total number of invariants, of both kinds, is $2n^2$. In this case the number of unrestricted aberrations is

$$(2n+2)(2n+3)/2 - n - 2n^2 - 1,$$

that is, $4n + 2$.

Finally, we may summarise our results concerning the distribution of the aberrations, of all orders, amongst the three types, as follows, namely,

order of aberration:	number of invariant aberrations:	number of semi-invariant aberrations:	number of unrestricted aberrations:	total number of aberrations:
$2n-1$	n	$2n(n-1)$	$4n$	$n(2n+3)$.
$2n$	n	$2n^2$	$4n+2$	$(2n+1)(n+2)$.

In particular, for the aberrations of the first few orders, we have the following scheme, namely,

1	1[1]	0	4	5.
2	1	2	6	9.
3	2	4	8	14.
4	2	8	10	20.

5. We may give here a simple illustration of the preceding investigation; thus we may write the function $\Phi_2(\alpha, \beta, \gamma)$ in the form

$$\begin{aligned}8\,\Phi_2(\alpha, \beta, \gamma) &= \sigma_1\alpha^2 - 4\sigma_2\alpha\beta + 2\sigma_3\alpha\gamma + 4\sigma_4\beta^2 - 4\sigma_5\beta\gamma + \sigma_6\gamma^2\\ &= \sigma_1\alpha^2 - 4\sigma_2\alpha\beta + 2(\sigma_3 + 2\sigma_4)(\gamma\alpha + 2\beta^2)/3 - 4\sigma_5\beta\gamma + \sigma_6\gamma^2 + 4(\sigma_3 - \sigma_4)(\gamma\alpha - \beta^2)/3.\end{aligned}$$

The numerical coefficients appearing in the first line of the right-hand side of this equation are those arising from the expansion of $(a - 2b + c)^2$, a function of importance in connection with the focal-eikonal. Hence,

$$\Pi\,\Phi_2(\alpha, \beta, \gamma) = \sigma_3 - \sigma_4,$$

and for a system of co-axial spherical surfaces, the end media having optical indices unity, we have,

$$(\sigma_3 - \sigma_4)/d^2 = \varpi = \frac{1}{K}\Sigma(\varkappa_\lambda/\mu_{\lambda-1}\mu_\lambda),$$

where K is the power of the whole system, and $\varkappa_\lambda$ is the power of the surface separating media of optical indices $\mu_{\lambda-1}$, μ_λ: and, of course, ϖ is the usual 'Petzval-sum'.

Or, again, we may write, for the aberrations of the second order,

$$\begin{aligned}\Phi_3(\alpha, \beta, \gamma) = \cdots\; & 3(\tau_3 + 4\tau_4)(\gamma\alpha + 4\beta^2)\alpha/5 - 4(3\tau_5 + 2\tau_7)(3\gamma\alpha + 2\beta^2)\beta/5\\ & + 3(\tau_6 + 4\tau_8)(\gamma\alpha + 4\beta^2)\gamma/5 + \{12(\tau_3 - \tau_4)\alpha - 24(\tau_5 - \tau_7)\beta\\ & + 12(\tau_6 - \tau_8)\gamma\}(\gamma\alpha - \beta^2)/5 + \cdots.\end{aligned}$$

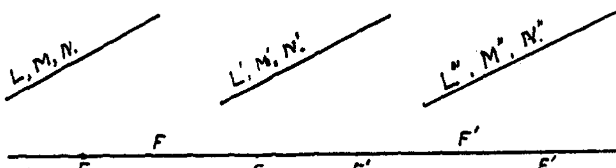
Here the expressions $\tau_3 - \tau_4$, $\tau_5 - \tau_7$, $\tau_6 - \tau_8$ are semi-invariants, while the expression $\tau_3 - \tau_4 - 2(\tau_5 - \tau_7) + \tau_6 - \tau_8$ is an invariant of the first type.

[1] This invariant relation, the simplest of its type, is, of course, the 'Petzval-sum'.

Part II.

1. Our next step must be the calculation, for the general symmetrical optical system, of the 'invariant' and of the 'semi-invariant' functions which have emerged from the purely qualitative investigation of Part I; and this is readily effected by the use of the operators introduced there. We regard such an optical system as composed of co-axial spherical surfaces, and we observe, in passing, that we have not hitherto supposed the component surfaces to be spherical, but only that they are surfaces of revolution about the axis of the system. In proceeding, however, to evaluate the various functions which we have obtained we limit ourselves here to the consideration of spherical surfaces since, in practice, these are most commonly used. We have to evaluate the expressions for a single spherical surface, and then to investigate the 'addition' of these expressions corresponding to the 'addition' of the various single surfaces, which together form the composite optical system. It will be convenient to address ourselves immediately to the second investigation.

2. Let there be two symmetrical optical systems, having the same axis of symmetry, and let F_1 and F_2, F_1' and F_2', respectively, be their principal foci:



let F and F' be the principal foci of the combined system. Let L, M, N; L', M', N'; and L'', M'', N'', be the direction cosines of the three portions of a ray of light, incident, intermediate, and emergent respectively, where the axis of the system is taken as the common (x) axis of reference, and the remaining axes are parallel in threes, and rectangular. Let the modified powers of the component systems be J_1 and J_2, and that of the combined system be J; and let the modified and reduced eikonals be U_1 and U_2, and U respectively.

Then we have the following relations, namely,

$$F_2 F_1' = -J/J_1 J_2, \quad F' F_2' = +J_1/J_2 J, \text{ and } F_1 F = +J_2/J J_1;$$

in each case these distances are 'reduced', are multiplied, that is to say, by the optical indices of the media in which severally they are measured. Further, we have,

$$U_1 = -(MM' + NN'),\ U_2 = -(M'M'' + N'N''),\ \text{and}\ U = -(MM'' + NN'').$$

These are the first terms in the expansions of the focal-eikonal functions, and are correct therefore to the second order in the quantities M, N, M', N', M'', and N''.

The partial differential coefficient of an eikonal gives a co-ordinate of the point of intersection of the ray with the normal plane through the corresponding base point; and we have, then,

$$\frac{\partial}{\partial M}(U/J) = \frac{\partial}{\partial M}(U_1/J_1) + M \,.\, F_1 F,$$

the second term being the correction needed on account of the different base points associated with the functions U and U_1. Thus,

$$-M''/J = -M'/J_1 + MJ_2/JJ_1,$$

or,

$$JM' = J_2 M + J_1 M'',$$

and similarly,

$$JN' = J_2 N + J_1 N''.$$

We write now,

$$\begin{array}{lll} a = M^2 + N^2, & a_1 = M^2 + N^2, & a_2 = M'^2 + N'^2, \\ b = MM'' + NN'', & b_1 = MM' + NN', & b_2 = M'M'' + N'N'', \\ c = M''^2 + N''^2, & c_1 = M'^2 + N'^2, & c_2 = M''^2 + N''^2. \end{array} \qquad (1)$$

Then from relations (1), on multiplication by the appropriate factors, we have,

$$\begin{array}{ll} a_1 = a & J^2 a_2 = J_2^2 a + 2 J_2 J_1 b + J_1^2 c, \\ Jb_1 = J_2 a + J_1 b, & Jb_2 = J_2 b + J_1 c, \\ J^2 c_1 = J_2^2 a + 2 J_2 J_1 b + J_1^2 c, & c_2 = c. \end{array}$$

These relations give the values of our fundamental variables a, b, and c for the two component systems, in terms of the similar variables for the combined system.

3. Now we may regard the first system as composed of $\lambda - 1$ sub-systems, $1, 2, 3, \ldots \lambda - 1$; and the second system as comprising a single system λ, together with a block of $n - \lambda$ sub-systems, $\lambda + 1, \ldots n$. These systems are not,

of necessity, single spherical surfaces, but may themselves be general symmetrical systems. The combined system is then composed of n sub-systems.

It follows at once, from the preceding paragraph, that

$$J^2 a_\lambda = J^2_{\lambda,n}\, a \quad + 2 J_{\lambda,n} J_{1,\lambda-1}\, b + J^2_{1,\lambda-1}\, c,$$

$$J^2 c_\lambda = J^2_{\lambda+1,n}\, a + 2 J_{\lambda+1,n} J_{1,\lambda}\, b + J^2_{1,\lambda}\, c;$$

where, for example, a_λ is the a-variable associated with the sub-system λ, and $J_{\lambda,n}$ denotes the modified power of the system comprising sub-systems λ to n inclusive.

In the notation of the preceding paragraph we have,

$$J a_2 = J_2 b_1 + J_1 b_2,$$

and, if we apply this to the second block of sub-systems, we have

$$J^2_{\lambda,n}\, c_\lambda = J_{\lambda+1,n} J_{\lambda,n}\, b_\lambda + J_\lambda J_{\lambda,n}\, b_{\lambda+1,n}.$$

Whence, substituting for $b_{\lambda+1,n}$, and remembering that, on account of the continued fraction definition of the modified power J,

$$J_\lambda J_{1,n} + J_{1,\lambda-1} J_{\lambda+1,n} = J_{1,\lambda} J_{\lambda,n},$$

we have the following expression for b_λ, namely,

$$J^2 b_\lambda = J_{\lambda,n} J_{\lambda+1,n}\, a + (J_{1,\lambda-1} J_{\lambda+1,n} + J_{1,\lambda} J_{\lambda,n})\, b + J_{1,\lambda-1} J_{1,\lambda}\, c.$$

We may collect these results as follows,

$$J^2 a_\lambda = J^2_{\lambda,n}\, a + 2 J_{\lambda,n} J_{1,\lambda-1}\, b + J^2_{1,\lambda-1}\, c,$$

$$J^2 b_\lambda = J_{\lambda,n} J_{\lambda+1,n}\, a + (J_{\lambda,n} J_{1,\lambda} + J_{\lambda+1,n} J_{1,\lambda-1})\, b + J_{1,\lambda-1} J_{1,\lambda}\, c,$$

$$J^2 c_\lambda = J^2_{\lambda+1,n}\, c + 2 J_{\lambda+1,n} J_{1,\lambda}\, b + J^2_{1,\lambda}\, c.$$

These then are the generalisations of paragraph 2, and they tell of the state of the ray at any intermediate stage of its progress through the combined optical system. It will be noticed that J has been written in place of $J_{1,n}$, the modified power of the composite system.

Further we have,

$$J^2 \frac{\partial}{\partial a} = J^2_{\lambda, n} \frac{\partial}{\partial a_\lambda} + J_{\lambda, n} J_{\lambda+1, n} \frac{\partial}{\partial b_\lambda} + J^2_{\lambda+1, n} \frac{\partial}{\partial c_\lambda},$$

$$J^2 \frac{\partial}{\partial b} = 2 J_{\lambda, n} J_{1, \lambda-1} \frac{\partial}{\partial a_\lambda} + (J_{\lambda, n} J_{1, \lambda} + J_{\lambda+1, n} J_{1, \lambda-1}) \frac{\partial}{\partial b_\lambda} + 2 J_{\lambda+1, n} J_{1, \lambda} \frac{\partial}{\partial c_\lambda},$$

$$J^2 \frac{\partial}{\partial c} = J^2_{1, \lambda-1} \frac{\partial}{\partial a_\lambda} + J_{1, \lambda-1} J_{1, \lambda} \frac{\partial}{\partial b_\lambda} + J^2_{1, \lambda} \frac{\partial}{\partial c_\lambda};$$

the operators being applied in each case to the variables indicated.

4. Our fundamental operators, involving the variables a, b, and c, are the following, namely,

$$P \equiv 4 \frac{\partial^2}{\partial c \partial a} - \frac{\partial^2}{\partial b^2}, \quad \text{and} \quad O \equiv \frac{\partial}{\partial a};$$

these for the 'invariant' functions: and for the 'semi-invariant' functions, we have the operators

$$\frac{\partial}{\partial \gamma} \equiv s^2 \frac{\partial}{\partial a} + s \frac{\partial}{\partial b} + \frac{\partial}{\partial c},$$

and

$$\frac{\partial}{\partial \alpha} \equiv m^2 \frac{\partial}{\partial a} + m \frac{\partial}{\partial b} + \frac{\partial}{\partial c}.$$

Here s and m are reduced magnifications associated respectively with the pupil-planes and the object-image planes. We may use a suffix notation to indicate operations upon the variables associated with the sub-system λ, and then we have immediately, from the relations of the preceding paragraph,

$$J^2 P = (J^2 P)_\lambda,$$

and

$$J^2 O = J^2_{\lambda, n} \frac{\partial}{\partial a_\lambda} + J_{\lambda, n} J_{\lambda+1, n} \frac{\partial}{\partial b_\lambda} + J^2_{\lambda+1, n} \frac{\partial}{\partial c_\lambda}.$$

These are the 'addition operators' for the 'addition' of the sub-systems, as far as the 'invariant' relations are concerned. The similar operators, for the addition of the 'semi-invariant' relations, are

$$J^2 \frac{\partial}{\partial \alpha} \equiv (J_{1, \lambda-1} + m J_{\lambda, n})^2 \frac{\partial}{\partial a_\lambda} + (J_{1, \lambda-1} + m J_{\lambda, n})(J_{1, \lambda} + m J_{\lambda+1, n}) \frac{\partial}{\partial b_\lambda} +$$
$$+ (J_{1, \lambda} + m J_{\lambda+1, n})^2 \frac{\partial}{\partial c_\lambda};$$

and a similar expression involving s in place of m, to give $J^2 \frac{\partial}{\partial \gamma}$.

5. Since we wish to find the 'invariant' and the 'semi-invariant' relations in terms of the optical constants of the system as a whole, we concentrate upon the focal-eikonals, $U(a, b, c)$ for the whole system, and $U_\lambda(a_\lambda, b_\lambda, c_\lambda)$ for the several sub-systems. Now, between these quantities there is a relation

$$(U/J) = \sum_{\lambda=1}^{n} (U/J)_\lambda + \cdots \qquad (1)$$

where the terms omitted arise from the adjustment of the various base-points, and depend, therefore, each separate term, upon *one* only of the variables a_λ, and c_λ, and not upon the variable b_λ at all. These terms then are each annihilated by the operator P, or P_λ, which appears, in every case, at least once. In our application of the operators then we may omit these terms, as playing the part of 'constants'. Moreover, we have divided the U-functions by the modified power J, or J_λ, since these functions have previously been 'modified', that is, multiplied by the quantity J, or J_λ. And therefore (1) reduces to a direct geometrical relationship.

In the previous paragraph we have found relations between operators applied to the system as a whole and corresponding operators involving, in each case, only the variables associated with a particular sub-system. Accordingly, we apply these operators, the one set to the left-hand side of (1), and the other set to the right-hand side of (1). As perhaps the simplest example we have

$$(J^2 P)^r (U/J) = \sum_{\lambda=1}^{n} (J^2 P)_\lambda^r (U/J)_\lambda,$$

where the operator P has been applied r times. We proceed to other examples later.

6. Hitherto, the sub-system λ has been any optical system whatever. We proceed now to take, as our unit sub-system, the single spherical surface separating media of optical indices μ and μ'. And, for the simplest class of 'invariants', we have merely to apply the operator P_λ, repeatedly, to U_λ, the focal-eikonal for this single spherical surface. Moreover, the base-points may be moved, if necessary, in any manner along the axis of the surface, for the terms introduced thereby contain, each one of them, only one of the variables a_λ and c_λ, and the variable b_λ not at all; and so these terms are annihilated by P_λ.

Now, we have,

$$U_\lambda = (1 + v\chi)^{1/2}/v + L + L' - 1/v - 2,$$

where $v = \mu\mu'/(\mu'-\mu)^2$, and $\chi + 2(LL' + b - 1) = 0$. Effectively, we may write, omitting the suffix λ,

$$vU = \theta + \ldots,^{1}$$

the remaining terms being annihilated by the operator P, where

$$\theta^2 = 1 + v\chi.$$

Now, U is a function of a, b, and c; but, if we write $\varepsilon = LL'$, we may use the variables ε and θ, and then we have, by direct differentiation,

$$P \equiv \frac{1}{\varepsilon}\frac{\partial}{\partial\varepsilon}\,\varepsilon\,\frac{\partial}{\partial\varepsilon} - \left(2\frac{\partial}{\partial\varepsilon} + \frac{1}{\varepsilon}\right)\frac{v}{\theta}\frac{\partial}{\partial\theta}.$$

Actually, the result of a few applications of this operator to the function U may be found by direct methods. Thus we have,

$$\begin{aligned}
PU &= -1/\varepsilon\theta \\
P^2U &= -(1/\varepsilon^3\theta - v/\varepsilon^2\theta^3), \\
P^3U &= -3^2(1/\varepsilon^5\theta - v/\varepsilon^4\theta^3 + v^2/\varepsilon^3\theta^5), \\
P^4U &= -3^2 5^2(1/\varepsilon^7\theta - v/\varepsilon^6\theta^3 + 6v^2/5\,\varepsilon^5\theta^5 - v^3/\varepsilon^4\theta^7), \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

And, since we need only the coefficients of the appropriate terms of U, we write in these expressions $a = b = c = 0$, that is, we write $\varepsilon = 1$, and $\theta = 1$; and then we have,

$$\begin{aligned}
PU &= -1, \\
P^2U &= -(1 - v), \\
P^3U &= -3^2(1 - v + v^2), \\
P^4U &= -3^2 5^2(1 - v + 6v^2/5 - v^3), \\
&\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

The corresponding conditions are found by writing $r = 1, 2, 3, \ldots$ in the formula of paragraph 5; and we have,

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[1] θ/v is in fact the modified and reduced eikonal for a single spherical surface, separating media of optical indices μ and μ', the base-points being coincident at the centre of curvature of the surface.

$$\text{for } r = 1,\ \Sigma(J^2 P)\ (U/J) = \Sigma\ JPU\ = -\Sigma J = -\Sigma(\varkappa/\mu\mu') = -\varpi_1,$$
$$\text{for } r = 2,\ \Sigma(J^2 P)^2 (U/J) = \Sigma J^3 P^2 U = -\Sigma(\varkappa/\mu\mu')^3 (1 - v) = -\varpi_3,$$
$$\text{for } r = 3,\ \Sigma(J^2 P)^3 (U/J) = \Sigma J^5 P^3 U = -\Sigma(\varkappa/\mu\mu')^5 (1 - v + v^2) = -\varpi_5,$$

etc.

It will be remembered that $v = \mu\mu'/(\mu' - \mu)^2$, and it is seen that these conditions are precisely analogous to the 'Petzval-condition', and that, indeed, ϖ_1 is the 'Petzval-sum'.

7. But it is of interest to examine the general case. To this end we notice that

$$P(1/\varepsilon^q \theta^p) = q^2/\varepsilon^{q+2}\theta^p + (1 - 2q)pv/\varepsilon^{q+1}\theta^{p+2},$$

and so we assume, as covering the general case,

$$P^m U = A_{m,1}/\varepsilon^{2m-1}\theta + A_{m,2}/\varepsilon^{2m-2}\theta^3 + \cdots + A_{m,2m-1}/\varepsilon^m \theta^{2m-1},$$

and then, by an application of the operator P to each side of this relation, we have, for the A-coefficients, the partial difference equation

$$A_{m+1,2p-1} = (2m - p)^2 A_{m,2p-1} + (2p - 4m - 1)(2p - 3)\, v\, A_{m,2p-3}.$$

Let us write further

$$A_{m+1,2p-1} = \psi(m, p)\, v^{p-1} A_{m+1,1}$$

and then the partial difference equation for the function $\psi(m, p)$ is

$$(2m - 1)^2 \psi(m, p) = (2m - p)^2 \psi(m - 1, p) + (2p - 4m - 1)(2p - 3)\psi(m - 1, p - 1).$$

Now clearly we have

$$\psi(m, 1) = 1, \quad \text{and} \quad \psi(m, 2) = -1,$$

and the appropriate solution of the difference equation in $\psi(m, p)$, subject to these conditions, is

$$\psi(m, p) = \frac{(-1)^{p-1}}{2} \frac{\Gamma(2m - p + 2)}{\Gamma(2m)} \frac{\Gamma(m)}{\Gamma(m - p + 2)} \frac{\Gamma(2p - 1)}{[\Gamma(p)]^2}$$

expressed in terms of Gamma-functions.

If now, in the expression for $P^m U$, we write $\varepsilon = 1$, $\theta = 1$, we have

$$P^m U = A_{m,1} \sum_{p=1}^{m} \psi(m-1, p)\, v^{p-1} = A_{m,1}\, \Omega_m(v), \text{ say,}$$

and the corresponding 'invariant' condition is,

$$\sum (\varkappa/\mu\mu')^{2m-1}\, \Omega_m(v) \equiv \varpi_{2m-1} = 0.$$

We have here then a complete generalisation of the 'Petzval-condition': indeed, writing $m = 1, \varpi_1$ is the usual 'Petzval-sum'. We have written the generalised expression ϖ_{2m-1} to indicate that the condition applies to aberrations of order $2m-1$; that is, we have a generalised 'Petzval-sum' for each set of aberrations of odd order. And it will be noticed that each of these conditions depends upon the powers of the optical surfaces and upon the indices of the media separated by these surfaces, and upon no other quantity at all: and that the conditions are very simple in form, and easy of application.

8. *On the invariant relations of the second class.* We have considered, in the preceding paragraph, only those invariant relations which involve the operator P alone; but invariant relations are obtainable also from the joint application of the operators O and P, where,

$$O = \frac{\partial}{\partial a}, \quad \text{and} \quad J^2 O = J_{\lambda,n}^2 \frac{\partial}{\partial a_\lambda} + J_{\lambda,n} J_{\lambda+1,n} \frac{\partial}{\partial b_\lambda} + J_{\lambda+1,n}^2 \frac{\partial}{\partial c_\lambda}.$$

These operators have to be applied to functions of ε and θ; and we have

$$\frac{\partial}{\partial a_\lambda} = \frac{\partial \varepsilon}{\partial a_\lambda}\left(\frac{\partial}{\partial \varepsilon} - \frac{v_\lambda}{\theta}\frac{\partial}{\partial \theta}\right) = \frac{L_\lambda}{2L_{\lambda-1}}\left(\frac{v_\lambda}{\theta}\frac{\partial}{\partial \theta} - \frac{\partial}{\partial \varepsilon}\right),$$

$$\frac{\partial}{\partial b_\lambda} = \qquad -\frac{v_\lambda}{\theta}\frac{\partial}{\partial \theta},$$

$$\frac{\partial}{\partial c_\lambda} = \frac{\partial \varepsilon}{\partial c_\lambda}\left(\frac{\partial}{\partial \varepsilon} - \frac{v_\lambda}{\theta}\frac{\partial}{\partial \theta}\right) = \frac{L_{\lambda-1}}{2L_\lambda}\left(\frac{v_\lambda}{\theta}\frac{\partial}{\partial \theta} - \frac{\partial}{\partial \varepsilon}\right),$$

where L_λ denotes the direction cosine of the ray *after* incidence upon the surface λ. It follows then that

$$J^2 O = \frac{v_\lambda}{2\varepsilon\theta}\left(L_\lambda J_{\lambda,n} - L_{\lambda-1} J_{\lambda+1,n}\right)^2 \frac{\partial}{\partial \theta} - \frac{1}{2\varepsilon}\left(L_\lambda^2 J_{\lambda,n}^2 + L_{\lambda-1}^2 J_{\lambda+1,n}^2\right)\frac{\partial}{\partial \varepsilon}.$$

Now, for a single application of the operator O, we have

$$J^2\,O = \frac{v_\lambda}{2}(J_{\lambda,n} - J_{\lambda+1,n})^2 \frac{\partial}{\partial\theta} - \frac{1}{2}(J^2_{\lambda,n} + J^2_{\lambda+1,n})\frac{\partial}{\partial\varepsilon}\,.$$

And therefore,

$$\begin{aligned} J^2\,O\,J^2\,P(U/J) &= \sum_{\lambda=1}^{n} J_\lambda \left[v_\lambda \left(J_\lambda \frac{\partial J_{\lambda,n}}{\partial J_\lambda}\right)^2 - \left(J_\lambda \frac{\partial J_{\lambda,n}}{\partial J_\lambda}\right)^2 - 2\,J_{\lambda,n}\,J_{\lambda+1,n}\right]/2 \\ &= \sum_{\lambda=1}^{n} J_\lambda \left[(v_\lambda - 1) \left(J_\lambda \frac{\partial J_{\lambda,n}}{\partial J_\lambda}\right)^2 - 2\,J_{\lambda,n}\,J_{\lambda+1,n}\right]/2\,. \end{aligned}$$

This then is an 'invariant' relation, the only one of the second order: and others follow immediately in the same manner.

9. *On the semi-invariant relations.* The m-invariants arise from an application of the operator

$$O^p\,P^q \left(\frac{\partial}{\partial\gamma}\right)^r$$

to the general eikonal, where p, q, and r, are any integers such that $q \nless 1$; and

$$O \equiv \frac{\partial}{\partial a}, \quad P \equiv 4\frac{\partial^2}{\partial c\,\partial a} - \frac{\partial^2}{\partial b^2}, \quad \text{and} \quad \frac{\partial}{\partial\gamma} \equiv s^2\frac{\partial}{\partial a} + s\frac{\partial}{\partial b} + \frac{\partial}{\partial c};$$

these variables and operators refer to the system as a whole. The s-invariants arise similarly by replacing $\frac{\partial}{\partial\gamma}$ by $\frac{\partial}{\partial\alpha}$, where

$$\frac{\partial}{\partial\alpha} \equiv m^2\frac{\partial}{\partial a} + m\frac{\partial}{\partial b} + \frac{\partial}{\partial c}\,.$$

We have also

$$J^2\,P = (J^2\,P)_\lambda$$

and

$$\begin{aligned} J^2\frac{\partial}{\partial\gamma} = (J_{1,\lambda-1} + s\,J_{\lambda,n})^2\frac{\partial}{\partial a_\lambda} + (J_{1,\lambda-1} + s\,J_{\lambda,n})(J_{1,\lambda} + s\,J_{\lambda+1,n})\frac{\partial}{\partial b_\lambda} + \\ + (J_{1,\lambda} + s\,J_{\lambda+1,n})^2\frac{\partial}{\partial c_\lambda} \end{aligned}$$

and, as previously, we apply these operators to the relation

$$U/J = \sum_{\lambda=1}^{n}(U/J)_\lambda + \cdots$$

As an example, we may consider the special case obtained by writing $p = 0$, $q = 1$, and $r = 1$; that is to say, we deal with aberrations of the second order. Since, in our usual notation, we have

$$P\,U = -1/\varepsilon\theta$$

for a single spherical surface, we have, by a direct application of the preceding operator $\frac{\partial}{\partial \gamma}$, the expression

$$\frac{1}{2}\sum_{\lambda=1}^{n} J_\lambda \left[v_\lambda \left\{ \frac{\partial J_{1,\lambda}}{\partial J_\lambda} - s\,\frac{\partial J_{\lambda,n}}{\partial J_\lambda} \right\}^2 J_\lambda^2 - \{(J_{1,\lambda-1} + s\,J_{\lambda,n})^2 + (J_{1,\lambda} + s J_{\lambda+1,n})^2\} \right]$$

the summation extending throughout the optical system. The vanishing of this expression then is an example of a 'semi-invariant' relation; since s alone is involved, and not m. Other relations may be found readily in the same way.

As another special case let us write $p = 0$, q and r being unrestricted, save only that $q \not< 1$: further, let us assume that $s = 1$, and that the optical system is *thin*, so that

$$J_{\mu,\nu} = \sum_{\lambda=\mu}^{\nu} J_\lambda .$$

Then

$$\frac{\partial}{\partial \gamma} \equiv \frac{\partial}{\partial a_\lambda} + \frac{\partial}{\partial b_\lambda} + \frac{\partial}{\partial c_\lambda} = \delta_\lambda \quad \text{(say)}$$

and

$$(J^2 P)^q \left(\frac{\partial}{\partial \gamma}\right)^r (U/J) = \sum_{\lambda=1}^{n} (J^2 P)_\lambda^q\, \delta_\lambda^r\, (U/J)_\lambda .$$

The operators P and δ are commutative, and we may shew that, if $q = 1$, the right-hand side of this relation is

$$\sum_{\lambda=1}^{n} f(r)\, J_\lambda = f(r) \sum_{\lambda=1}^{n} J_\lambda ,$$

where $f(r)$ is a certain function of r. The vanishing of this then for *any* value of r leads to the 'Petzval-condition', namely,

$$\sum_{\lambda=1}^{n} J_\lambda \equiv \sum \varkappa/\mu\mu' = 0 .$$

The more general case, in which $q \neq 1$, leads to

$$\sum_{\lambda=1}^{n} f(r, v_\lambda) J_\lambda,$$

where here $v = \mu\mu'/(\mu' - \mu)^2$, and the vanishing of this leads to a generalisation, of a quite different kind, of the 'Petzval-condition'.

Appendices to Part II.

1. We have to evaluate the operator $P^q \delta^r$, applied to the eikonal for a single spherical surface. The operators P and δ, involving the variables a, b, and c, are commutative; and we know that P^q, operating upon the eikonal for a single spherical surface, leads to a series of terms such as $\varepsilon^{-\nu}\theta^{-\omega}$, where ν and ω are positive integers, and

$$\varepsilon = LL', \quad \theta^2 = 1 + v\chi, \quad \text{and} \quad \chi + 2(LL' + b - 1) = 0.$$

Since δ is a linear operator, with constant coefficients, we have

$$\delta^r \varepsilon^{-\nu} \theta^{-\omega} = \theta^{-\omega} \delta^r \varepsilon^{-\nu} + {}_rC_1\, \delta\theta^{-\omega} \cdot \delta^{r-1} \varepsilon^{-\nu} + \cdots$$

Now, with our usual notation,

$$L^2 = 1 - a = l \text{ (say), and } L'^2 = 1 - c = l' \text{ (say);}$$

so that

$$\delta = \frac{\partial}{\partial a} + \frac{\partial}{\partial b} + \frac{\partial}{\partial c} = -\frac{\partial}{\partial l} + \frac{\partial}{\partial b} - \frac{\partial}{\partial l'},$$

and

$$\varepsilon = \sqrt{ll'}, \quad \theta^2 = 1 - 2v(\sqrt{ll'} + b - 1).$$

Also, if $t > 1$,

$$\left(\frac{\partial}{\partial l} + \frac{\partial}{\partial l'}\right)^t \sqrt{ll'} = 0,$$

where we write $l = l'$, *after* differentiation.

Now, after performing the various operations indicated above, we have, as usual, to write $a = b = c = 0$, that is, $\varepsilon = 1$, and $\theta = 1$. Then, denoting this by the suffix zero, we have,

$$(\delta\varepsilon)_0 = -1, \quad (\delta^n \varepsilon)_0 = 0 \qquad \text{for } n > 1,$$

and

$$(\delta^n \theta)_0 = 0, \qquad \text{for } n > 0.$$

Further, we have,

$$(\delta^r \varepsilon^{-\nu})_0 = \nu(\nu + 1) \cdots (\nu + r - 1)$$

$$(\delta^r \theta^{-\omega})_0 = 0$$

and

$$(\delta^r \varepsilon^{-\nu} \theta^{-\omega})_0 = \nu(\nu + 1) \cdots (\nu + r - 1);$$

and these are the results used in paragraph 9.

2. A certain ambiguity is apt to arise from the use of the phrase »the geometrical aberrations of order n», in connection with the symmetrical optical system, whenever n is greater than unity. Either of two meanings may be assigned to this phrase.

(a) We write the aberration-function $\Phi(\alpha, \beta, \gamma)$ in the form

$$\Phi(\alpha, \beta, \gamma) = \sum_{n=1}^{\infty} \Phi_{n+1}(\alpha, \beta, \gamma),$$

where the function $\Phi_n(\alpha, \beta, \gamma)$ is homogeneous, and of degree n, in the variables α, β, and γ. Now, we may concentrate attention upon the function $\Phi_{n+1}(\alpha, \beta, \gamma)$; corresponding to the appearance of this function *alone* there is a displacement from the Gaussian focus, upon the paraxial image plane, which we may denote by Δ_n, and which comprises a *finite* series of terms, homogeneous and of degree $2n + 1$ in ϱ (the radius of the exit-pupil) and Y_1 (the distance of the Gaussian focus from the axis of the optical system). Then we may speak of this displacement, either as a single group, or else with regard to its several terms, as »the aberration, or aberrations, of order n», and this is the method which we have followed in the text.

(b) We may take the general, and complete, aberration-function

$$\Phi(\alpha, \beta, \gamma) \left[= \sum_{n=1}^{\infty} \Phi_{n+1}(\alpha, \beta, \gamma) \right],$$

and operate upon this in the manner indicated in the text. Corresponding to the appearance of this *complete* function, consisting of an infinite series of groups of terms, there is a displacement from the Gaussian focus, upon the paraxial image plane, which we may write $\sum_{n=1}^{\infty} \Delta'_n$. Here each Δ'_n comprises a *finite* series of terms, homogeneous and of degree $2n + 1$ in ϱ and Y_1. And we may speak of the group Δ'_n as »the aberrations of order n».

The group of terms Δ'_n will not differ qualitatively, either by excess or defect, from the group of terms Δ_n; the aberration curves derived from Δ'_n and Δ_n are of the same type and number of types. But there is a quantitative difference; for the coefficients of corresponding terms, and groups of terms, will not be the same. The reason is, of course, that the expression Δ'_n allows for the effect, upon »the aberrations of order n», of the presence of the aberrations of lower orders: hence, indeed, the possibility of the balancing of the aberrations amongst the various orders.

These considerations do not affect the analysis of Part I of this paper; but in Part II we have used certain first approximations. For example, we derived the result

$$J^2 a_\lambda = J^2_{\lambda, n} a + 2 J_{\lambda, n} J_{1, \lambda - 1} b + J^2_{1, \lambda - 1} c.$$

Here a closer approximation would exhibit a_λ as a series of terms containing powers of a, b, and c, involving also the aberrations of the system. But if we consider aberrations of any given order, in the absence of those of lower orders, then we may legitimately use the first approximations: and accordingly we have adopted, in Part II, the alternative (a) above.

Part III.

1. We proceed now to seek the geometrical implications of the 'invariant' and of the 'semi-invariant' relations which have emerged from the investigation undertaken in Part I, and which have been evaluated quantitatively, for the general symmetrical system, in Part II, — for a system, that is to say, the media, surfaces and separations of which are supposed given. And, in the first place, we change our variables slightly; we write

$$\alpha = \psi, \quad 2\beta = \varphi, \quad \text{and} \quad \gamma = \theta;$$

for these have been used in a detailed examination of the geometrical aberrations of the symmetrical optical system, to which it is convenient here to make reference.[1] In this notation our 'invariant' operators, save for an irrelevant multiplying factor, are

$$\Pi \equiv \frac{\partial^2}{\partial \psi \, \partial \theta} - \frac{\partial^2}{\partial \varphi^2}, \quad \text{and} \quad \Omega \equiv \frac{\partial}{\partial \theta} + 2 \frac{\partial}{\partial \varphi} + \frac{\partial}{\partial \psi};$$

[1] *The Aberrations of a Symmetrical Optical System:* Trans. Camb. Phil. Soc. XXIII. No IX.

while the 'semi-invariant' operators are

$$\frac{\partial}{\partial\theta} \text{ and } \frac{\partial}{\partial\psi}.$$

In the paper indicated the aberrations are grouped in 'orders', depending successively upon orders of small quantities; and their geometrical meanings, and corresponding aberration curves, are fully investigated there. It is shewn that each 'aberration', of each order, falls into one or other of two categories: for each aberration belongs either to the S-(spherical)-type, or else to the C-(coma)-type. Here then we have a quite different grouping of the aberrations, and the two types are sharply differentiated by their possession of various properties. For our present purpose we may mention only one such property: namely, that for members of the C-type change of focus, from the paraxial image plane, is of no advantage, — indeed, the aberration displacements are the same upon planes equidistant from the paraxial, or Gaussian, image plane. The C-type may be named then, in this sense, the 'symmetrical' type. But with aberrations of the S-type the matter stands quite otherwise, for, with them, change of focus is of advantage, and they may therefore be said to belong to the 'unsymmetrical' type. A smaller aberration curve, that is to say, may be obtained by change of focus. Indeed, in the absence of astigmatism, *and for a given annulus of the exit-pupil*, a point image may be obtained by a suitable change in the position of the receiving plane: but, for varying annuli of the exit-pupil, these images are distributed along a 'central line', joining the centre of the exit-pupil to the Gaussian, or non-aberration, image point. We have then for these higher order aberrations, of the S-type, something in some ways akin to the astigmatism and curvature of the field of the first order — already well-known.

If, now, we use the Characteristic-function, in place of the Eikonal, we have a corresponding Aberration-function, depending upon the variables θ, φ, and ψ: and any general term of this function may be written

$$A_{p,q,r}\,\theta^p\,\varphi^q\,\psi^r, \tag{1}$$

while the corresponding aberration displacement, upon the paraxial image plane, is given by[1]

$$\Delta Y = 2^q A_{p,q,r}(Y_1/d)^{2p+q}(\varrho/d)^{q+2r-1}\cos^{q-1}\varphi_1\,(2\,r\cos^2\varphi_1 + q),$$
$$\Delta Z = 2^q A_{p,q,r}(Y_1/d)^{2p+q}(\varrho/d)^{q+2r-1}\,2\,r\cos^q\varphi_1\sin\varphi_1,$$

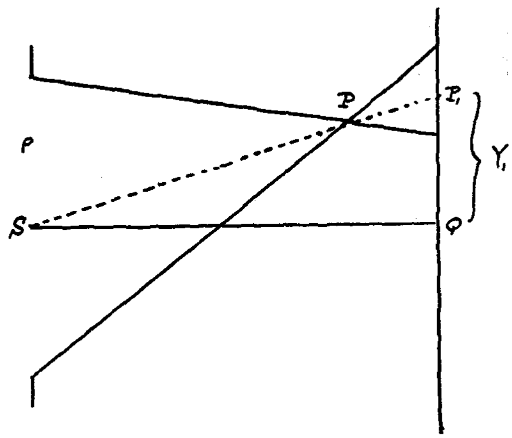
[1] Trans. Camb. Phil. Soc. XXIII. No. IX. § 3.

where the origin of coordinates is the paraxial image point, at distance Y_1 from the axis of the system, and ϱ, φ_1 are the polar coordinates of the point of intersection of the ray with the exit-pupil.

The A-coefficient here is an aberration coefficient of order $p + q + r - 1$, and it is to a series of terms such as (1) that we apply our operators.

We consider, in the first place, successive applications of the operator Π. We notice that this operator *alone* is to be applied only to aberration terms of an *odd* order, for which then $p + q + r$ is equal to an even integer: and it follows that all terms for which q is odd are annihilated by the operator Π. Now, even values of q indicate that we are dealing with aberrations of the S-type, for which therefore change of focus is beneficial. Those of our invariant relations then which arise from applications of the operator Π alone — and do not involve the operator Ω — deal with the S-type aberrations. And these appear, in the sequel, to be the simpler ones.

2. Let S be the centre of the exit-pupil, of radius ϱ, and Q the point of insersection of the image plane with the axis SQ of the optical system: let P_1



be the Gaussian image, of coordinates Y_1, Z_1, in the plane QP_1, referred to Q as origin; so that QP_1 is equal to Y_1, where we have assumed $Z_1 = 0$; and this is secured by a proper choice of axes. Let a ray intersect the pupil-plane in a point at distance ϱ from S, the angular coordinate of this point in the pupil-plane being φ_1, referred to the radius of the exit-pupil parallel to QP_1: and let this ray intersect the image plane in a point of coordinates ΔY, ΔZ, referred to P_1 as origin. Then, if we assume the presence of the S-type of aberration alone, we have

$$\varDelta Y = k_1 \cos \varphi_1,$$

$$\varDelta Z = k_2 \sin \varphi_1.$$

Here k_1 and k_2 depend upon powers of ϱ, of Y_1, and of $\cos^2 \varphi_1$, and also upon the aberration coefficients; but, if we confine ourselves to suitable groups of terms, k_1 and k_2 will be homogeneous in ϱ, and also in Y_1.

Two rays for which $\varphi_1 = 0$, or π, which therefore are axial rays, intersect in a point upon the central line SP_1: and also two rays for which $\varphi_1 = \pm \pi/2$ intersect in a (different) point upon this central line. Other rays do not, in general, intersect this central line at all, and we have then something akin to the usual and well-known first order astigmatism. But if $k_1 = k_2 = k$ then all rays *from a given annulus of the exit-pupil* intersect in a point upon this central line; but the position of this point depends upon the value of ϱ. We have then, corresponding to the whole of the exit-pupil, a series of such points distributed along the central line SP_1, something after the nature of the elementary spherical aberration. Indeed the distance x, from the plane QP_1, of the point corresponding to a given value of ϱ is $x = kd/\varrho$, where d is equal to the (reduced, and modified) distance SQ. If, in addition, $k = 0$, all such points coincide with P_1, and we have a flat field, as far as this group of terms is concerned.

3. Let us, in the first place, consider the first order aberrations. We deal then with a homogeneous quadratic function of the variables θ, φ, and ψ, and a single application of our operator Π annihilates all terms except

$$\cdots + a_3 \psi \theta + a_4 \varphi^2 + \cdots$$

and, for these, gives the expression

$$a_3 - 2 a_4.$$

The corresponding aberration displacements are

$$d^3 \varDelta Y = 2 Y_1^2 \varrho (a_3 + 4 a_4) \cos \varphi_1,$$

$$d^3 \varDelta Z = 2 Y_1^2 \varrho (a_3) \sin \varphi_1,$$

or,

$$d^3 \varDelta Y = 2 Y_1^2 \varrho (a_3 - 2 a_4 + 6 a_4) \cos \varphi_1,$$

$$d^3 \varDelta Z = 2 Y_1^2 \varrho (a_3 - 2 a_4 + 2 a_4) \sin \varphi_1.$$

Astigmatism, of this order, is absent if $a_4 = 0$, and then we have

$$d^3 \Delta Y = 2\, Y_1^2\, \varrho\, a_3 \cos \varphi_1,$$

$$d^3 \Delta Z = 2\, Y_1^2\, \varrho\, a_3 \sin \varphi_1.$$

All rays therefore pass through a point at distance x from the Gaussian image plane, where

$$x = 2\, a_3 (Y_1/d)^2.$$

The aggregate of all such points, for varying object points, gives a surface of revolution about the axis of the system, that is to say, a curved field. Moreover, the condition for the flatness of this field is $a_3 = 0$, or, since astigmatism is assumed absent, $a_3 - 2\, a_4 = 0$. That is, from paragraph 6, Part II, the condition is

$$\varpi_1 \equiv \Sigma\, \varkappa/\mu\mu' = 0,$$

in the usual notation. This is the usual 'Petzval-condition', and is well-known: we give it here because a single infinity of exactly analogus results follow in precisely the same manner.

4. We proceed now to seek a double application of the operator Π, and we deal therefore with terms of the fourth order in our variables θ, φ, and ψ: that is, we deal with the *third* order aberrations. The appropriate terms of the aberration-function are the following, namely,

$$\cdots + c_7\, \theta^2 \psi^2 + c_8\, \theta\, \varphi^2 \psi + c_9\, \varphi^4 + \cdots$$

An application of the operator $\left(\dfrac{\partial^2}{\partial \psi\, \partial \theta} - \dfrac{\partial^2}{\partial \varphi^2}\right)^2$ to this expression leads to

$$c_7 - c_8 + 6\, c_9,$$

apart from an irrelevant multiplying factor. This then is our particular 'invariant' function, of the aberration coefficients, of the third order.

Now, the corresponding displacement upon the Gaussian image[1] plane is given by

$$d^7 \Delta Y = 4\, \varrho^3\, Y_1^4 \{(2\, c_8 + 8\, c_9) \cos^2 \varphi_1 + 2\, c_8 + c_7\} \cos \varphi_1,$$

$$d^7 \Delta Z = 4\, \varrho^3\, Y_1^4 \{2\, c_8 \cos^2 \varphi_1 + c_7\} \sin \varphi_1.$$

If the astigmatic effects, for this group of terms, be absent we must have $c_8 = 0$, and $c_9 = 0$; and then,

[1] Trans. Camb. Phil. Soc. XXIII. No. IX. § 17.

$$d^7 \Delta Y = 4\varrho^3\, Y_1^4\, c_7 \cos\varphi_1,$$

$$d^7 \Delta Z = 4\varrho^3\, Y_1^4\, c_7 \sin\varphi_1.$$

Rays therefore, from a given annulus of the exit-pupil, meet in a point at distance x from the Gaussian image plane, given by

$$x d^6 = 4\varrho^2\, Y_1^4\, c_7.$$

This relation gives the position of the point P, upon the 'central-line', and it will be noticed that the position of P depends upon the annulus of the exit-pupil chosen. Corresponding, then, to the complete exit-pupil we have a series of points distributed along the 'central-line', that is to say, we have a spherical aberration effect. And these points will all coincide with the Gaussian image P_1, giving a flat field for this order and group of terms, only if $c_7 = 0$; that is, in the absence of astigmatism, if

$$c_7 - c_8 + 6\,c_9 = 0.$$

But this, arising from a double application of the operator Π, is from paragraph 6, Part II, the condition $\varpi_3 = 0$: namely, we have

$$\varpi_3 \equiv \Sigma\,(\varkappa/\mu\mu')^3\,(1 - v) = 0,$$

where $v = \mu\mu'/(\mu' - \mu)^2$, in the usual notation.

5. It is evident now that we have a series of conditions $\varpi_1, \varpi_3, \varpi_5, \ldots \varpi_{2n-1}, \ldots$, corresponding to aberrations of orders $1, 3, 5, \ldots (2n - 1), \ldots$, given by

$$\begin{aligned}
\varpi_1 &\equiv \Sigma\,(\varkappa/\mu\mu') = 0,\\
\varpi_3 &\equiv \Sigma\,(\varkappa/\mu\mu')^3\,(1 - v) = 0,\\
\varpi_5 &\equiv \Sigma\,(\varkappa/\mu\mu')^5\,(1 - v + v^2) = 0,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
\varpi_{2n-1} &\equiv \Sigma\,(\varkappa/\mu\mu')^{2n-1}\,\Omega_n(v) = 0,
\end{aligned}$$

where $v = \mu\mu'/(\mu' - \mu)^2$, and $\Omega_n(v)$ is the function of paragraph 7, Part II.

These conditions are in every way analogous to the 'Petzval-condition', and, indeed, the 'Petzval-condition' is the first and simplest of them. They form, then, in the aggregate, a complete generalisation of this well-known 'Petzval-condition'. And the geometrical implication of any one of them is that its satisfaction, in the absence of astigmatism (of a generalised type), involves »flatness of field» for the particular order of aberration indicated. Moreover, as with the 'Petzval-condition', these conditions are,

1. independent of the positions of the object and of the image planes;
2. similarly independent of the positions of the pupil-planes of the optical system;
3. independent of the separations of the component surfaces of the optical system;
4. particularly simple in form;
5. dependent only upon the powers of the separate surfaces, and upon the indices of the media between these surfaces.

They form then the complete class of optical conditions to which the 'Petzval-sum' belongs; of which, indeed, this sum is the simplest member and the only one hitherto known.

6. We proceed to consider the 'invariant' relations of the second class, those, namely, which arise from joint applications of the two operators Π and Ω: and we know that there are many more of these than of the simpler type arising from applications of Π alone. Moreover, since

$$\Omega \equiv \frac{\partial}{\partial \theta} + 2\frac{\partial}{\partial \varphi} + \frac{\partial}{\partial \psi},$$

the resulting relations between the aberration-coefficients involve coefficients both of the C-(coma)-type, and also those of the S-(spherical)-type: and, in consequence, the aberration displacements, ΔY and ΔZ, upon the Gaussian image plane, are homogeneous, for any given order of aberration, in Y_1 and ϱ together, instead of being homogeneous in Y_1 and ϱ separately — as with the relations arising from applications of the operator Π alone. We deal here, then, with aberrations which do not naturally fall together into a group, in the ordinary investigations. Further, the resulting expressions, or conditions for freedom from these aberrations, while being independent of the positions of the pupil-planes and of the object-image planes, yet involve the *separations* of the optical surfaces, in addition to the powers of these surfaces, and the indices of the media between them. We derive optical conditions, then, which are not so simple in form as those of the preceding paragraph, while yet being very simple when compared with the usual conditions.

As the simplest example we consider the single 'invariant' relation associated with the aberrations of the second order, which arises from an application of the operator $\Omega \Pi$. That is to say, we apply these operators to the relevant terms of the homogeneous cubic in θ, φ, and ψ, which appears in the aberration-function. These terms are

$$\cdots + b_3 \theta^2 \psi + b_4 \theta \varphi^2 + b_5 \theta \varphi \psi + b_6 \theta \psi^2 + b_7 \varphi^3 + b_8 \varphi^2 \psi + \cdots$$

and the operator is

$$\left(\frac{\partial}{\partial \theta} + 2 \frac{\partial}{\partial \varphi} + \frac{\partial}{\partial \psi}\right) \left(\frac{\partial^2}{\partial \psi \partial \theta} - \frac{\partial^2}{\partial \varphi^2}\right),$$

leading to

$$2(b_3 - b_4) + 2(b_5 - 6 b_7) + 2(b_6 - b_8).$$

The first and last brackets involve aberrations of the S-type and the second bracket an aberration of the C-type.

The resulting displacements upon the paraxial image plane is given by[1]

$$d^5 \varDelta Y = 4 \varrho^3 Y_1^2 (2 b_8 \cos^2 \varphi_1 + 2 b_3 + b_6) \cos \varphi_1 + 2 \varrho^2 Y_1^3 \{b_5 (2 \cos^2 \varphi_1 + 1) + \\ + 12 b_7 \cos^2 \varphi_1\} + 2 \varrho Y_1^4 (b_8 + 4 b_4) \cos \varphi_1,$$

$$d^5 \varDelta Z = 4 \varrho^3 Y_1^2 (2 b_8 \cos^2 \varphi_1 + b_6) \sin \varphi_1 + 4 \varrho^2 Y_1^3 b_5 \cos \varphi_1 \sin \varphi_1 + 2 \varrho Y_1^4 b_8 \sin \varphi_1.$$

For a given annulus of the exit-pupil a point image is obtained only if $b_4 = b_5 = b_7 = b_8 = 0$. And then this image is at distance x from the paraxial image plane, given by

$$x d^4 = 2 (b_3 Y_1^2 + 2 b_6 \varrho^2) Y_1^2,$$

that is to say, we have for varying annuli of the exit-pupil, a point image moving along the central-line. If now we apply our condition, which is

$$b_3 + b_6 = 0,$$

or,

$$\sum_{\lambda=1}^{n} J_\lambda \left\{ (v_\lambda - 1) \left(J_\lambda \frac{\partial J_{\lambda, n}}{\partial J_\lambda} \right)^2 - 2 J_{\lambda, n} J_{\lambda+1, n} \right\} = 0,$$

the average range of the foci is

[1] Trans. Camb. Phil. Soc. XXIII. No IX. §§ 7, 8, 9.

$$\frac{1}{\pi \varrho^2} \int_0^\varrho 2 (b_3 Y_1^2 + 2 b_6 \varrho^2) Y_1^2 \, 2 \pi \varrho \, d \varrho = 2 (b_3 Y_1^2 + b_6 \varrho^2) Y_1^2 = 2 b_3 (Y_1^2 - \varrho^2) Y_1^2 ;$$

and this appears to be the best that can be done with this aberration, as regards flatness of field.

7. *On the semi-invariant relations.* The semi-invariant aberration groups arise from applications of the operators Π and Ω, together with *either* $\frac{\partial}{\partial \theta}$, *or* $\frac{\partial}{\partial \psi}$; of which, as usual, the operator Π must be applied at least once. The m-invariants, independent of the positions of the object-image planes, arise from $\frac{\partial}{\partial \theta}$. It is evident therefore that there is here a wide field for investigation; we deal here however only with a few simple cases.

As perhaps the simplest and most immediately interesting example we apply the operator

$$\Pi \left(\frac{\partial}{\partial \theta} \right)^{n-1}$$

to the general aberration-function for aberrations of order n; namely, to terms such as

$$A_{pqr} \, \theta^p \varphi^q \psi^r ,$$

where, as is clear, we must suppose that $p + q + r = n + 1$.

If $n = 1$ we are led to the invariant relation of the first order, the usual 'Petzval-sum', concerned therefore with field curvature, in the absence of astigmatism. In the general case, in which $n > 1$, we have a semi-invariant relation, the geometrical implications of which, however, are similar to those of the simpler case.

In general, the displacement upon the Gaussian image plane is given by[1]

$$d^{2n+1} \varDelta Z = 2 \varrho \, Y_1^{2n} (A_{n,0,1} + 4 A_{n-1,2,0}) \cos \varphi_1 ,$$

$$d^{2n+1} \varDelta Z = 2 \varrho \, Y_1^{2n} A_{n,0,1} \sin \varphi_1 .$$

It is evident then that we are dealing with something precisely similar to the curvature of the field and astigmatism of the first order, — as in paragraph 3, —

[1] Trans. Camb. Phil. Soc. XXIII. No IX. § 27.

depending however upon Y_1^{2n} instead of upon Y_1^2. If $A_{n-1,2,0} = 0$ all rays, from all annuli of the exit-pupil, pass through a point upon the 'central-line', and we have then, for the aggregate of all such points, a curved field, depending upon the coefficient $A_{n,0,1}$, which becomes flat if $A_{n,0,1} = 0$. All this is exactly similar to the well-known first order case. If $n = 1$, and in the absence of astigmatism, $A_{1,0,1}$ is the usual 'Petzval-sum', ϖ; while, if $n > 1$, this coefficient depends upon the positions of the pupil planes alone, and not upon the positions of the object-image planes. The aberrations of higher orders, then, which resemble exactly the curvature of the field of the first order, differ from this latter in that they depend, as regards their vanishing, upon the position of the pupil planes.

If $n = 2$, we are dealing with pure curvature of the field of the *second* order, and the condition for the absence of this, in the absence of astigmatism, the condition, that is to say, that in the absence of pure astigmatism of the second order a flat field should be reproduced as flat, as far as these second order terms are considered, is the condition of paragraph 9, Part II: namely, the vanishing of the expression

$$\sum_{\lambda=1}^{n} J_\lambda \left[v_\lambda \left\{ \frac{\partial J_{1,\lambda}}{\partial J_\lambda} - s \frac{\partial J_{\lambda,n}}{\partial J_\lambda} \right\}^2 J_\lambda^2 - \left\{ (J_{1,\lambda-1} + s J_{\lambda,n})^2 + (J_{1,\lambda} + s J_{\lambda+1,n})^2 \right\} \right],$$

where $v_\lambda = \mu_\lambda \mu_{\lambda-1}/(\mu_\lambda - \mu_{\lambda-1})^2$.

But a particular case of some importance arises. Let us write $s = 1$, that is, let us assume the pupil planes of the system to have associated with them the magnification $+1$: further, let us assume the optical system to be *thin*. Then, as shewn in paragraph 9, Part II, all the semi-invariant conditions, for aberrations of *all* orders, reduce to the usual 'Petzval-condition' $\varpi_1 \equiv \Sigma \varkappa/\mu\mu' = 0$.

8. *Conclusion.* Each geometrical aberration of the symmetrical optical system, of whatever 'order', is found to fall into one or other of three categories, and the classification here is altogether different from that commonly adopted, depending, as this latter does, upon the idea of small quantities of successive orders. There is first the 'invariant' category. The conditions for the vanishing of the aberrations belonging to this category are entirely independent of the positions of the object-image planes, and of the pupil-planes chosen; and, for the first sub-class of the category, independent also of the separations of the optical surfaces composing the system. This sub-class resembles exactly the well-known 'Petzval-sum'. The second sub-class forms a new type of condition. The

second category is that of the 'semi-invariant' aberrations. Here the conditions for the freedom from the aberrations involve either the positions of the object-image planes, or the positions of the pupil-planes of the system; but not both of these quantities. And finally, there is the third category, to which belong entirely unrestricted aberrations — the vanishing of which depends upon all the quantities mentioned above.

The number of the aberrations, of each order, falling in each of these categories is found; and the precise conditions associated with each, for any given general symmetrical optical system, are investigated. Thus, the general condition for freedom from aberrations of the first sub-class of the 'invariant' category is

$$\varpi_{2\nu-1} = \sum_{\lambda=1}^{n} (\varkappa/\mu\,\mu')^{2\nu-1}\,\Omega_\nu(v) = 0,$$

where $\varkappa$ is the power of the optical surface separating media of indices μ and μ', and $v = \mu\,\mu'/(\mu' - \mu)^2$: the summation is taken throughout the system. $\Omega_\nu(v)$ is a function the general form of which is found. In particular, the first few values are given by

$$\varpi_1 = \Sigma(\varkappa/\mu\,\mu'),\ \varpi_3 = \Sigma(\varkappa/\mu\,\mu')^3(1 - v),$$

$$\varpi_5 = \Sigma(\varkappa/\mu\,\mu')^5(1 - v + v^2).$$

Thus ϖ_1 is the usual form of the 'Petzval-condition'. And, in the aggregate, these conditions form a complete generalisation of the 'Petzval-condition'. It will be seen that there is just one condition associated with every set of aberrations of odd order.

The precise forms of the conditions associated with the second sub-class of the 'invariant' category are found, and also those associated with the 'semi-invariant' category; and this for the general symmetrical optical system.

The satisfaction of the well-known 'Petzval-condition' is associated with a certain geometrical simplicity; for there is thereby ensured that, in the absence of astigmatism, the optical system shall reproduce a flat field. But this applies only to aberrations of the first 'order', as commonly presented. In this paper, the satisfaction of the conditions associated with the first sub-class of the 'invariant' category is shewn to have a similar implication with regard to the aberrations of higher 'orders'; for the satisfaction of each of these implies flatness of field, in the absence of (generalised) astigmatism, for the particular 'order' contemplated. This sub-class of the 'invariant' relations is, then, a complete generalisation of the 'Petzval-condition', alike with regard to the form of the condition, and with regard to the geometrical implications of the condition.

The geometrical meanings associated with the second sub-class of the 'invariant' category, and with the 'semi-invariant' category, are also investigated. And, in particular, it appears that for thin systems, the pupil-planes of which are at magnification $+1$, the usual form of the 'Petzval-condition' emerges.