THEORY OF LINEAR DIFFERENTIAL EQUATIONS CONTAINING A PARAMETER.

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Index.

1. **Introduction.** Our present object is to establish the *asymptotic properties of the solutions* of a linear differential equation of order n

$$(A_1) \qquad L_n(x, \lambda; y) \equiv \sum_{k=0}^{n} {}_1a_{n-k}(x, \lambda)\, y^{(k)} = 0$$

$$[{}_1a_0(x, \lambda) \not\equiv 0; \quad {}_1a_n(x, \lambda) \not\equiv 0]^1,$$

in so far as the parameter λ is concerned. The theory will be given for the complex plane of λ; moreover, *no restrictions will be made concerning the λ-formal series solutions of* (A_1). The coefficients in (A_1) will be assumed to be indefinitely

[1] $f^{(k)}$ $(k \geqq 0)$ here and in the sequel denotes $\dfrac{\partial^k f}{\partial x^k}$.

differentiable in x $(a \leqq x \leqq b)$ and analytic in λ for $|\lambda| \geqq \varrho > 0 (\lambda \neq \infty)$, being representable by convergent series of the form

$$(1) \qquad {}_1a(x, \lambda) = \sum_{\nu=0}^{\infty} {}_1a_\nu(x)\, \lambda^{m-\nu} \qquad \text{(integer } m).$$

Here the ${}_1a_\nu(x)$ are indefinitely differentiable on the closed interval (a, b). More generally, the coefficients in (A_1) will be allowed to be merely asymptotic in certain regions to such possibly divergent series. In the latter case the obtained results will be valid in correspondingly restricted regions of the λ-plane. The interval (a, b) will be taken sufficiently small so that the formal series solutions will maintain essentially the same characteristic features for all x in the interval. The main results are formulated in the Fundamental Existence Theorem of § 6. Applications of this Theorem will be made to non-homogeneous and integro-differential equations, as well as to some boundary value problems.

The precise notion of asymptotic relationship, employed in this work, is as follows. Let R be region, extending to infinity, bounded by regular curves and situated in the λ-plane. Let $f(x, \lambda)$ be defined for λ in R and $a \leqq x \leqq b$. Suppose now that a series (convergent or divergent)

$$s(x, \lambda) = \sum_{\nu=-H}^{\infty} s_\nu(x)\, \lambda^{-\frac{\nu}{p}}$$

$$\text{(integers } H \text{ and } p;\ p \geqq 1)$$

be given, whose coefficients $s_\nu(x)$ are defined, each being bounded for $a \leqq x \leqq b$. We shall say that $f(x, \lambda)$ *is asymptotic to* $s(x, \lambda)$ *in* λ, *at* $\lambda = \infty$, *in the region* R *and for* x *in the interval* (a, b) *provided for every* m $(m = 1, 2, \ldots)$

$$(2) \qquad f(x, \lambda) = \sum_{\nu=-H}^{m-1} s_\nu(x) \lambda^{-\frac{\nu}{p}} + f_m(x, \lambda)\, \lambda^{-\frac{m}{p}},$$

where $f_m(x, \lambda)$ *is defined for* x *in* (a, b) *and for* λ *in* R *and*

$$(2\text{ a}) \qquad |f_m(x, \lambda)| \leqq f_m \qquad (a \leqq x \leqq b;\ \lambda \text{ in } R).$$

Here the numbers f_m $(m = 1, 2, \ldots)$ *are independent of* x *and* λ. Such an asymptotic relationship will be said to be in the »*ordinary sense*» or »*to infinitely many terms*». Whenever a function $f(x, \lambda)$ has all the properties stated above, except that in (2) the number m cannot be taken greater than m_0 (a fixed integer), *the asymptotic relationship* (2) *will be said to hold to* m_0 *terms,*

$$f(x,\lambda) \underset{m_0}{\sim} s(x,\lambda) \qquad (\lambda \text{ in } R;\ a \leqq x \leqq b). \tag{2 b}$$

According to this notation,

$$f(x,\lambda) \underset{\infty}{\sim} s(x,\lambda) \tag{2 c}$$

would signify that $f(x,\lambda)$ is asymptotic to $s(x,\lambda)$ in the ordinary sense. For convenience in place of $\underset{\infty}{\sim}$ we shall write $\sim$. Since the f_m $(m = 1, 2, \ldots)$ in (2 a) are independent of x the asymptotic relationships, specified above, will be said to be *uniform* in x for x in the closed interval (a, b).

Let c be a point of the interval (a, b). Suppose $f(x,\lambda)$ is a function defined for λ in R and for all x of (a, b) distinct from c. Let functions $s_\nu(x)$ $(\nu = 0, 1, \ldots)$ be defined for all x of (a, b) distinct from c. If (2), (2 a) hold with the f_m $(m = 1, 2, \ldots)$ denoting some positive functions of x, defined on (a, b) except possibly at $x = c$, then we shall write

$$f(x,\lambda) \sim s(x,\lambda) \qquad (\lambda \text{ in } R;\ x \text{ in } (a,b);\ x \neq c) \tag{3}$$

(to m_0 or to infinitely many terms — as the case may be). Such an asymptotic relationship will be termed *non-uniform* in x.

The literature in the field under consideration is very extensive. No effort will be made to give extensive references. Of the contributions of the earlier writers those, which from the point of view of the present paper are of outstanding importance, are due to G. D. Birkhoff[1], J. D. Tamarkin[2], R. E. Langer[3] and P. Noaillon.[4] The first two of these authors assume that the

[1] G. D. Birkhoff, *On the asymptotic character of solutions of certain linear differential equations*, Trans. Am. Math. Soc., vol. 9 (1908), pp. 219—231; also *Boundary value and expansion problems …*, Trans. Am. Math. Soc., vol. 9 (1908), pp. 373—395. G. D. Birkhoff and R. E. Langer, *The boundary problems and developments …*, Proc. Am. Acad. Arts and Sciences, vol. 58 (1923), pp. 51—128. G. D. Birkhoff, *Quantum mechanics and asymptotic series* (an address), Bull. Am. Math. Soc., vol. 39 (1933), pp. 681—700.

[2] J. D. Tamarkin, A work published in 1917 (in Russian), Petrograd. J. D. Tamarkin, *Some general problems of the theory of ordinary linear differential equations and expansions of an arbitrary function in series of fundamental functions*, Math. Zeitschrift, vol. 27 (1927), pp. 1—54.

[3] R. E. Langer, cf. Birkhoff and Langer in a preceding reference. R. E. Langer, *The asymptotic solutions of ordinary linear differential equations …* (a symposium lecture), Bull. Am. Math. Soc., vol. 40 (1934), pp. 545—582. Also cf. a series of papers by the same author, concerning some special problems, in volumes 25 (1923), 31 (1929), 32 (1930), 33 (1931), 34 (1932), 36 (1934) of the Trans. Am. Math. Soc.,

[4] P. Noaillon, *Développements asymptotiques dans les équations différentielles linéaires à paramètre variable*, Mémoires de la Soc. des Sc. de Liege, Troisième Série, Tome IX (1912), 197 pages.

roots of the characteristic equation of (A_1) are distinct for all values of x under consideration. Under this hypothesis the formal series solutions are, of course, of a restricted type. In the work of TAMARKIN the coefficients of the equation (or system) are allowed to possess a suitable finite number of derivatives. As seen from his work, such a lightening of the conditions upon the coefficients results in the solutions being asymptotic to the formal series to a finite number of terms only. The results of the present paper could also be suitably extended so as to apply to equations whose coefficients possess merely a limited number of derivatives. However, for convenience of demonstration it will be assumed throughout that the coefficients in (A_1) are all indefinitely differentiable with respect to x. The work of TAMARKIN contains also a very substantial treatment of boundary value problems.

An elegant treatment had been given by BIRKHOFF and LANGER for the case of a linear system, with coefficients linear in the parameter.

In NOAILLON's work the general case is considered and it is proved that there always exists a full set of linearly independent formal series solutions.[1] On the basis of the latter NOAILLON obtains actual solutions, which

1° *are asymptotic to the corresponding formal series to a finite number of terms only. On the other hand,*

2° *the asymptotic relations are proved only along a fixed ray in the plane of the parameter.*

Some of the solutions obtained in the present paper will have the following properties.

A. *They will be asymptotic to the corresponding formal series to infinitely many terms (that is, in the ordinary sense).*

B. *The asymptotic relations will be valid in certain regions, extending to infinity in the complex plane of the parameter.*

The distinction between 2° and B will be appreciated in view of the following considerations. Let R be a sector

$$\alpha_1 \leqq \text{angle}\, \lambda \leqq \alpha_2; \quad \alpha_1 < \alpha_2; \quad |\lambda| \geqq \varrho > 0.$$

If it is known that along every ray in R a function is asymptotic to a series

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[1] Formal solutions of the type found by NOAILLON were known to exist a number of years earlier. Cf., for instance, L. SCHLESINGER, *Über asymptotische Darstellung der Lösungen ...*, Math. Annalen, vol. 63 (1907), pp. 277—300 (in particular, p. 282). However, previous to NOAILLON's work existence of a full set of such solutions had not been proved.

(say, of the form $s(x, \lambda)$), *it does not necessarily follow that the asymptotic relationship holds throughout* R.

Inasmuch as investigation of the asymptotic properties of the solutions of (A) is concerned, the present paper has a significance analogous to that which certain papers by TRJITZINSKY have in the fields of ordinary linear difference (with BIRKHOFF)[1], q-difference[2] and differential equations[3] (not containing a parameter). The three papers, just referred to, together with the work at hand present a certain aspect of unity. *These papers derive their significance from the fact that, when a class of analytic functions is at hand, the problem of central importance is to investigate the nature of the functions in the vicinity of their singular points.*

2. **Some Preliminary Facts.** It is convenient to write the equation (A_1) in the form

$$(A) \qquad L(x, \lambda; y) \equiv \sum_{k=0}^{n} \lambda^{H(n-k)} a_{n-k}(x, \lambda) y^{(k)} = 0,$$

$$(1) \qquad a_{n-k}(x, \lambda) \sim \sum_{\nu=0}^{\infty} \alpha_{n-k,\nu}(x) \lambda^{-\nu} = \alpha_{n-k}(x, \lambda) \quad (k = 0, \ldots n;\ x \text{ in } (a, b)).$$

Without any loss of generality it will be assumed that $a_0(x, \lambda) \equiv 1$. Moreover, it will be supposed that the series $\alpha_n(x, \lambda)$ is not formally zero. The involved integer H will be taken the smallest possible. If $H \leqq 0$, while $a_{n-k}(x, \lambda) = \alpha_{n-k}(x, \lambda)$ $(k = 0, 1, \ldots n)$, the latter series being convergent, we have an analogue to the Fuchsian Theory. In fact, as H. POINCARÉ proved, in this case there exists a full set of solutions analytic in λ at $\lambda = \infty$ (provided, of course, that the initial conditions are of the same character).

The characteristic equation associated with (A) is

$$(2) \qquad E(x; \varrho) \equiv \sum_{k=0}^{n} \alpha_{n-k,0}(x) \varrho^k = 0.$$

[1] G. D. BIRKHOFF and W. J. TRJITZINSKY, *Analytic theory of singular difference equations,* Acta mathematica, vol. 60 (1932), pp. 1—89.

[2] W. J. TRJITZINSKY, *Analytic theory of linear q-difference equations,* Acta mathematica, vol. 61 (1933), pp. 1—38.

[3] W. J. TRJITZINSKY, *Analytic theory of linear differential equations,* Acta mathematica, vol. 62 (1934), pp. 167—226.

Let $\varrho_i = \varrho_i(x)$ $(i = 1, \ldots n)$ be its roots. The interval (a, b) will be suitably chosen so that the following is true for every pair of functions $\varrho_i(x)$, $\varrho_j(x)$ $(i \neq j)$. Either $\varrho_i(x) = \varrho_j(x)$ $(a \leqq x \leqq b)$ or $\varrho_i(x) \neq \varrho_j(x)$ for every x in the interval (a, b). Moreover, this interval can be so chosen that, as a consequence of NOAILLON's work, there exists a full set of formal (in general divergent) series solutions satisfying

$$(A^*) \qquad L^*(x, \lambda; s) \equiv \sum_{k=0}^{n} \lambda^{H(n-k)} \alpha_{n-k}(x, \lambda)\, s^{(k)} = 0$$

and of the form

$$(3) \qquad s_i(x, \lambda) = e^{Q_i(x, \lambda)} \sigma_i(x, \lambda) \qquad (i = 1, \ldots n),$$

where

$$(3\,a) \qquad Q_i(x, \lambda) = \sum_{\alpha=0}^{k_i H-1} q_{i,\alpha}(x)\, \lambda^{\frac{k_i H-\alpha}{k_i}} \quad \text{(the } k_i \text{ positive integers)},$$

$$(3\,b) \qquad \sigma_i(x, \lambda) = \sum_{r=0}^{\infty} \sigma_{i,r}(x)\, \lambda^{-\frac{r}{k_i}} \qquad (i = 1, \ldots n).$$

Here the functions $q_{i,\alpha}(x)$, $\sigma_{i,r}(x)$ are all indefinitely differentiable and are finite in the interval (a, b).

The integer H, in a sense, has a significance in the theory of equations (A) analogous to that which the rank of a singular point has in the theory of ordinary linear differential equations without a parameter.

The leading coefficients in the $Q_i(x, \lambda)$ are connected with the roots of the characteristic equation by means of the relations

$$(4) \qquad q_{i,0}^{(1)}(x) \equiv \varrho_i(x) \qquad (i = 1, \ldots n).$$

When the roots are all distinct (for all x in (a, b)) the integers k_i in the formal series (3), (3 a), (3 b) are each equal to unity. The corresponding theory has been developed in the essential particulars by BIRKHOFF (when $H = 1$) and by TAMARKIN (when H is any integer). When multiple roots are admitted some of the k_i may exceed unity. If $\varrho_1 = \varrho_1(x)$ is a root of multiplicity m there will be ν $(1 \leqq \nu \leqq m)$ corresponding series $s_i(x, \lambda)$ such that, if ${}_1k, {}_2k, \ldots {}_\nu k$ are the associated values of the k_i, on one hand we shall have

$${}_1k + {}_2k + \cdots + {}_\nu k = m$$

and, on the other hand, it will be possible to obtain a set of m linearly independent formal solutions (associated with the root ϱ_1) by forming all possible determinations of the mentioned ν series. Any series (3) which satisfies (A) has k_i distinct determinations, obtained by letting λ describe closed circuits (1, 2, ..., $k_i - 1$ times, say, in the positive sense) around $\lambda = \infty$. Each of these determinations will satisfy (A). This is a consequence of the fact that for each circuit the series $\alpha_{n-k}(x, \lambda)$ remain unaltered. Accordingly, the series (3) can be grouped so that the elements within the same group are given by the totality of all determinations of a certain one series; on the other hand, no particular series from one group will be a determination of a series from another group. According as a series (3) contains or does not contain fractional powers of λ it will be termed *anormal* or *normal*.[1]

Associated with the equation (A) is the system, which in matrix notation will be written as

$$\text{(B)} \qquad Y^{(1)}(x, \lambda) = Y(x, \lambda)\, D(x, \lambda), \quad Y(x, \lambda) = (y_{i,j}(x, \lambda))^{2},$$

where

$$D(x, \lambda) = \begin{pmatrix} 0, & 0, & \cdots, & , & -\lambda^{Hn} a_n(x, \lambda) \\ 1, & 0, & \cdots, & , & -\lambda^{H(n-1)} a_{n-1}(x, \lambda) \\ 0, & 1, & \cdots, & , & \cdots \\ \cdots, & \cdots, & , & & \cdots \\ 0, & 0, & \cdots, & 1, & -\lambda^{H} a_1(x, \lambda) \end{pmatrix} = (d_{i,j}(x, \lambda)).$$

Here $(y_{i,j}(x, \lambda))$, for instance, denotes a matrix which in the i-th row and in the j-th column contains the element $y_{i,j}(x, \lambda)$ $(i, j = 1, \ldots n)$. If $(y_{i,j}(x, \lambda))$ is a matrix solution of (B) then

$$\text{(5)} \qquad (y_{i,j}(x, \lambda)) = (y_{i,1}^{(j-1)}(x, \lambda))$$

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[1] This definition is analogous to that employed for linear differential and linear difference equations without a parameter. By analogy to the results of the paper by W. J. Trjitzinsky, *Laplace integrals and factorial series in the theory of linear differential and linear difference equations*, Trans. Am. Math. Soc., vol. 37 (1935), pp. 80—146, one might expect that the method contained therein would lead to convergent factorial series developments whenever the series (3), corresponding to a multiple root of (2), are all normal and have the same exponential factor exp. $Q_i(x)$. This, however, is not the case.

[2] $Y^{(1)}(x, \lambda) = (y_{i,j}^{(1)}(x, \lambda)) = \left(\frac{\partial}{\partial x} y_{i,j}(x, \lambda)\right).$

and the $y_{i,1}(x, \lambda)$ $(i = 1, \ldots n)$ will constitute a fundamental set of solutions of (A). On the other hand, if functions $y_i(x, \lambda)$ $(= y_{i,1}(x, \lambda))$ $i = 1, \ldots n$ form a full set of solutions of (A), the matrix

$$Y(x, \lambda) = (y_{i,j}(x, \lambda)) = (y_i^{(j-1)}(x, \lambda)) \tag{6}$$

will satisfy (B).

In the sequel use will be made of the formula

$$y(x, \lambda) = \sum_{\tau=1}^{n} y_{\tau,1}(x, \lambda) \int z(x, \lambda)\, \bar{y}_{n,\tau}(x, \lambda)\, dx \tag{7}$$

which, under appropriate conditions, represents a solution of the non-homogeneous equation

$$L(x, \lambda; y) = z(x, \lambda). \tag{8}$$

In (7) the $y_{\tau,1}(x, \lambda)$ $(\tau = 1, \ldots n)$ are elements of a fundamental set of solutions of (A) and the $\bar{y}_{n,j}(x, \lambda)$ $(j = 1, \ldots n)$ are the elements in the n-th row of the inverse of the matrix $(y_{i,1}^{(j-1)}(x, \lambda))$.

Analogous facts can be stated for a system

$$\begin{gathered} Y^{(1)}(x, \lambda) = Y(x, \lambda)\, B(x, \lambda), \quad Y(x, \lambda) = (y_{i,j}(x, \lambda)), \\ B(x, \lambda) = (b_{i,j}(x, \lambda)),\ |B(x, \lambda)| \neq 0 \quad (a \leqq x \leqq b;\ |\lambda| \geqq \varrho > 0), \end{gathered} \tag{C}$$

where the coefficients $b_{i,j}(x, \lambda)$ are of the same type as those in (A). The elements in the i-th row $(i = 1, 2, \ldots n)$ will constitute a solution of the system. It is not difficult to relate to (C) a single differential equation of order n and of type (A). It can be shown that (C) is formally satisfied by a matrix

$$S(x, \lambda) = (s_{i,j}(x, \lambda)) = (e^{Q_i(x, \lambda)}\, \sigma_{i,j}(x, \lambda)), \tag{9}$$

where the $Q_i(x, \lambda)$ and the $\sigma_{i,j}(x, \lambda)$ are expressions of the form of (3 a) and (3 b), respectively. Such a matrix can be found so that formally $|S(x, \lambda)|$ does not vanish. This is of course under the supposition, which we shall make, that the determinant of the formal matrix, corresponding to $B(x, \lambda)$, is not zero.

It will be convenient to introduce the definition

Definition. *A series of the form*

$$\sum_{\nu=-m}^{\infty} \sigma_\nu(x)\,\lambda^{-\frac{\nu}{k}} \qquad (k \text{ a positive integer}),$$

whose coefficients are defined on a closed interval (α, β), *will be termed a σ-series.*

In the sequel, whenever we write

$$(10) \qquad a(x, \lambda) \sim \sum_r e^{G_r(x, \lambda)}\,\sigma_r(x, \lambda),$$

where the $\sigma_r(x, \lambda)$ are σ-series and the $G_r(x, \lambda)$ are functions defined for the involved values of the variables, the implication will be that

$$a(x, \lambda) = \sum_r e^{G_r(x, \lambda)}\,a_r(x, \lambda)$$

where the functions $a_r(x, \lambda)$ satisfy asymptotic relations

$$(10\text{a}) \qquad a_r(x, \lambda) \sim \sigma_r(x, \lambda).$$

The above statement refers also to the case when $\sim$ is replaced by $\underset{w}{\sim}$.

3. **Formal Integration.** We shall now solve the formal equation

$$(1) \qquad y^{(1)}(x, \lambda) = e^{Q(x, \lambda)}\,\sigma(x, \lambda),$$

where

$$(1\text{a}) \qquad Q(x, \lambda) = \sum_{\alpha=m}^{kH-1} q_\alpha(x)\,\lambda^{\frac{kH-\alpha}{k}}$$

(positive integers k and H)

$$(1\text{b}) \qquad \sigma(x, \lambda) = \sum_{r=0}^{\infty} \sigma_r(x)\,\lambda^{-\frac{r}{k}}$$

and where the coefficients $q_\alpha(x)$, $\sigma_r(x)$ are indefinitely differentiable and finite on (a, b). The series (1 b) may be divergent. When $Q(x, \lambda) \equiv 0$ then a formal solution of (1) will be

$$(2) \qquad y(x, \lambda) = \sum_{r=0}^{\infty} \eta_r(x)\,\lambda^{-\frac{r}{k}}$$

where $\eta_r^{(1)}(x) = \sigma_r(x)$ $(r = 0, 1, \ldots)$. When $Q(x, \lambda) \not\equiv 0$ the function $q_m(x)$ $(0 \leqq m \leqq kH - 1)$ will be not identically zero. It will be assumed that $q_m(x)$ is not a constant. *Let Δ_ε denote a perfect subset of the interval (a, b) such that in Δ_ε.*

$$|q_m^{(1)}(x)| \geqq \varepsilon > 0.$$

On letting

$$y(x, \lambda) = e^{Q(x, \lambda)} \eta(x, \lambda) \tag{3}$$

it is observed that $\eta(x, \lambda)$ satisfies

$$\eta^{(1)}(x, \lambda) = \sigma(x, \lambda) - Q^{(1)}(x, \lambda)\, \eta(x, \lambda). \tag{4}$$

It will be shown that (4) is formally satisfied by a series

$$\eta(x, \lambda) = \sum_{r=0}^{\infty} \eta_r(x)\, \lambda^{-\frac{r}{k}}. \tag{5}$$

In fact, on substituting (5) in (4) we obtain

$$\begin{aligned} \sum_{r=0}^{\infty} (\eta_r^{(1)}(x) - \sigma_r(x))\, \lambda^{-\frac{r}{k}} &= -\lambda^{H} \sum_{\alpha=m}^{kH-1} q_\alpha^{(1)}(x) \sum_{r=0}^{\infty} \eta_r(x)\, \lambda^{-\left(\frac{r+\alpha}{k}\right)} \\ &= -\lambda^{H} \sum_{\nu=0}^{\infty} \lambda^{-\frac{\nu}{k}} f_\nu(x), \end{aligned} \tag{6}$$

where

$$f_\nu(x) = \sum_{\alpha=m}^{\alpha_\nu} q_\alpha^{(1)}(x)\, \eta_{\nu-\alpha}(x). \tag{7}$$

Here α_ν is the smaller one of the numbers $kH - 1, \nu$.[1] It is noted that

$$f_\nu(x) = 0 \qquad (\nu = 0, 1, \ldots m - 1),$$

so that (6) can be written in the form

$$\sum_{r=-kH+m}^{-1} \lambda^{-\frac{r}{k}} f_{r+kH}(x) + \sum_{r=0}^{\infty} \lambda^{-\frac{r}{k}} g_r(x) = 0, \tag{8}$$

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[1] Here and in the sequel $\sum_{\alpha}^{\beta} = 0$, whenever $\beta < \alpha$.

(8 a) $$g_r(x) = f_{r+kH}(x) + \eta_r^{(1)}(x) - \sigma_r(x).$$

The equations

$$f_{r+kH}(x) = 0 \qquad (r = -kH + m, \ldots -1)$$

are satisfied if

(9) $$\eta_0(x) = \eta_1(x) = \cdots = \eta_{kH-m-1}(x) = 0.$$

On using (9) and (7) it is observed that (8) is equivalent to the set of equations

(10) $$g_r(x) \equiv \sum_{\alpha=m}^{\alpha'} q_\alpha^{(1)}(x)\,\eta_{r+kH-\alpha}(x) + \eta_r^{(1)}(x) - \sigma_r(x) = 0 \qquad (r \geqq 0).$$

In (10) α' is the smaller one of the numbers $kH - 1$, $r + m$. Thus, for $r \leqq kH - m - 1$, we have $\alpha' = r + m$ and, for $r > kH - m - 1$, $\alpha' = kH - 1$. Accordingly, in view of (9), relations (10) may be written as follows

(10 a) $$q_m^{(1)}(x)\,\eta_{r+kH-m}(x) = \sigma_r(x) - \sum_{\alpha=m+1}^{r+m} q_m^{(1)}(x)\,\eta_{r+kH-\alpha}(x)$$

$$(r = 0, 1, \ldots kH - m - 1),$$

(10 b) $$q_m^{(1)}(x)\,\eta_{r+kH-m}(x) = \sigma_r(x) - \eta_r^{(1)}(x) - \sum_{\alpha=m+1}^{kH-1} q_\alpha^{(1)}(x)\,\eta_{r+kH-\alpha}(x)$$

$$(r = kH - m,\ kH - m + 1, \ldots).$$

It is seen that equations (10 a) determine uniquely functions

(11) $$\eta_{kH-m}(x) = \frac{\sigma_0(x)}{q_m^{(1)}(x)},\ \eta_{kH-m+1}(x), \ldots, \eta_{2kH-2m-1}.$$

On the other hand, equations (10 b) determine in succession functions

(11 a) $$\eta_{2kH-2m}(x),\ \eta_{2kH-2m+1}(x), \ldots.$$

Functions (11), (11 a) are all indefinitely differentiable and finite in Δ_ε.

We note that if $y(x, \lambda)$ is a solution of

(12) $$y^{(1)}(x, \lambda) = e^{Q(x,\lambda)}\,\sigma(x, \lambda),$$

then $\mathfrak{z}(x, \lambda) = h(\lambda)\,y(x, \lambda)$ ($h(\lambda)$ a function of λ only) satisfies the equation

(13) $$\mathfrak{z}^{(1)}(x, \lambda) = h(\lambda)\,e^{Q(x,\lambda)}\,\sigma(x, \lambda).$$

Lemma 1. *Given a formal equation* (13), *where* $Q(x, \lambda)$ *and* $\sigma(x, \lambda)$ *are of the form* (1 a), (1 b), *there exists a formal solution* $\mathfrak{z}(x, \lambda) = h(\lambda) y(x, \lambda)$. *Here* $y(x, \lambda)$ *satisfies* (1). *When* $Q(x, \lambda) \equiv 0$, *then* $y(x, \lambda)$ *is given by* (2), *the involved coefficients being indefinitely differentiable and finite on* (a, b). *When* $Q(x, \lambda) \not\equiv 0$, $y(x, \lambda)$ *is of the form* (3), (5). *In this case the involved coefficients satisfy* (9), (10 a), (10 b) *and they are indefinitely differentiable and finite in* $\varDelta_\varepsilon$ (*cf. the statement in italics following* (2)).

Note. When $Q(x, \lambda) \not\equiv 0$ the coefficients, involved in the solution referred to in the above Lemma, are uniquely defined at *every* point of (a, b) for which $q^{(1)}(x)$ does not vanish. They are indefinitely differentiable (that is, possess a unique derivative) at each such point. In the neighbourhood of a point where $q^{(1)}(x) = 0$ these coefficients may become infinite. The order of infinitude may be estimated with the aid of (10 a) and (10 b).

4. **Analytic Integration.** Consider the equation

$$y^{(1)}(x, \lambda) = e^{Q(x, \lambda)} a(x, \lambda) \tag{1}$$

where $Q(x, \lambda)$, if not identically zero, is the function so denoted in § 3. As a function of x, let $a(x, \lambda)$ be indefinitely differentiable on (a, b), when λ is in a certain region R extending to infinity. Such a region R ($|\lambda| \geqq \lambda_0 > 0$), satisfying either one of the two conditions

$$\Re\, Q^{(1)}(x, \lambda) \geqq 0 \qquad (x \text{ in } (a, b);\ \lambda \text{ in } R), \tag{1 a}$$

$$\Re\, Q^{(1)}(x, \lambda) \leqq 0 \qquad (x \text{ in } (a, b);\ \lambda \text{ in } R), \tag{1 b}$$

will be supposed to exist. Moreover, it will be assumed that $a(x, \lambda)$ is analytic in λ for λ in R (when x is in (a, b)) and that

$$a(x, \lambda) \underset{w}{\sim} \sigma(x, \lambda) \quad (x \text{ in } \varDelta_\varepsilon;\ \lambda \text{ in } R;\quad w \geqq kH - m + 2). \tag{2}$$

Here $\sigma(x, \lambda)$ is given by (1 b; § 3) and $\varDelta_\varepsilon$ is defined in the italicized statement following (2; § 3). We shall first consider the case when $Q(x, \lambda) \not\equiv 0$.

According to Lemma 1 (§ 3) the formal equation

$$s^{(1)}(x, \lambda) = e^{Q(x, \lambda)} \sigma(x, \lambda) \tag{3}$$

possesses a formal solution

(3 a)
$$s(x, \lambda) = e^{Q(x,\lambda)} \lambda^{-\left(\frac{kH-m}{k}\right)} \sum_{r=0}^{\infty} s_r(x) \lambda^{-\frac{r}{k}} = e^{Q(x,\lambda)} \bar{s}(x, \lambda)$$
$$(s_r(x) = \eta_{kH-m+r}(x)),$$

where the coefficients are indefinitely differentiable and finite for x in $\varDelta_\varepsilon$. Let t be an integer such that $w \geqq t \geqq kH - m - 2$. Form the function

(4)
$$t(x, \lambda) = e^{Q(x,\lambda)} \lambda^{-\left(\frac{kH-m}{k}\right)} \sum_{r=0}^{t-1} s_r(x) \lambda^{-\frac{r}{k}} = e^{Q(x,\lambda)} \bar{t}(x, \lambda).$$

An application of the transformation

(5)
$$y(x, \lambda) = t(x, \lambda) + z(x, \lambda)$$

will yield

(6)
$$z^{(1)}(x, \lambda) = e^{Q(x,\lambda)} a(x, \lambda) - t^{(1)}(x, \lambda) = e^{Q(x,\lambda)} \varphi(x, \lambda).$$

In view of (2), (4) and (1 a; § 3) it follows that

(7)
$$\varphi(x, \lambda) = \left[b_w(x) \lambda^{-\frac{w}{k}} + \sum_{r=0}^{w-1} \sigma_r(x) \lambda^{-\frac{r}{k}} \right] - \lambda^{-\left(\frac{kH-m}{k}\right)} \sum_{r=0}^{t-1} s_r^{(1)}(x) \lambda^{-\frac{r}{k}}$$
$$- \left(\sum_{\alpha=m}^{kH-1} q_\alpha^{(1)}(x) \lambda^{\frac{kH-\alpha}{k}} \right) \lambda^{-\left(\frac{kH-m}{k}\right)} \sum_{r=0}^{t-1} s_r(x) \lambda^{-\frac{r}{k}}.$$

Here $b_w(x)$ is defined on $\varDelta_\varepsilon$ and

(7 a)
$$| b_w(x) | < b_w \quad (x \text{ in } \varDelta_\varepsilon).$$

Denoting the product of the two summations in the second member of (7) by $\psi(x, \lambda)$, we have

(8)
$$\psi(x, \lambda) = \sum_{\nu=0}^{t'} h_\nu(x) \lambda^{-\frac{\nu}{k}} \qquad (t' = t + kH - m - 2)$$

where

(8 a)
$$h_\nu(x) = \sum_{\sigma=\sigma_1}^{\sigma_2} q_{m+\sigma}^{(1)}(x) s_{\nu-\sigma}(x).$$

Here σ_1 is the greater one of the numbers $0, \nu - t + 1$ and σ_2 is the smaller one of the numbers $\nu, kH - m - 1$. Thus, in particular, (8 b) $$h_\nu(x) = \sum_{\sigma=0}^{\nu} q^{(1)}_{m+\sigma}(x)\, s_{\nu-\sigma}(x) \qquad (\nu = 0, 1, \ldots kH - m - 1),$$

(8 c) $$h_\nu(x) = \sum_{\sigma=0}^{kH-m-1} q^{(1)}_{m+\sigma}(x)\, s_{\nu-\sigma}(x)$$
$$(\nu = kH - m, kH - m + 1, \ldots t - 2).$$

Accordingly, it is noted that for $\nu \leqq t - 2$ the $h_\nu(x)$ are independent of t. With this fact in view we write (7) in the form

(9) $$\varphi(x, \lambda) = \varphi_1(x, \lambda) + \varphi_2(x, \lambda),$$

(9 a) $$\varphi_1(x, \lambda) = b_w(x)\lambda^{-\frac{w}{k}} + \sum_{\nu=t-1}^{w-1} \sigma_\nu(x)\lambda^{-\frac{\nu}{k}} - \sum_{\nu=t-1}^{t'+1} s^{(1)}_{\nu-kH+m}(x)\lambda^{-\frac{\nu}{k}}$$
$$- \sum_{\nu=t-1}^{t'} h_\nu(x)\lambda^{-\frac{\nu}{k}},$$

(9 b) $$\varphi_2(x, \lambda) = \sum_{\nu=0}^{t-2} \left[\sigma_\nu(x) - s^{(1)}_{\nu-kH+m}(x) - h_\nu(x)\right]\lambda^{-\frac{\nu}{k}}$$
$$(s^{(1)}_{\nu-kH+m}(x) = 0 \text{ for } \nu < kH - m).$$

In (9 b) the coefficients of $\lambda^{-\frac{\nu}{k}}$ $(\nu = 0, \ldots, t-2)$ are independent of t; in view of (8 b), (8 c) and since the $s_r(x) = \eta_{kH-m-r}(x)$ satisfy relations (9), (10 a), (10 b) of § 3, it follows that these coefficients are all zero. Thus $\varphi(x, \lambda) = \varphi_1(x, \lambda)$.

Let, as is possible, $t = w$. In consequence of (9 a) and (6) we then have

(10) $$z^{(1)}(x, \lambda) = e^{Q(x, \lambda)}\lambda^{-\left(\frac{w-1}{k}\right)} g_w(x, \lambda),$$

(10 a) $$|g_w(x, \lambda)| \leqq g_w \qquad (x \text{ in } \Delta_\varepsilon;\ \lambda \text{ in } R),$$

where $g_w(x, \lambda)$ is defined and finite in Δ_ε and g_w is independent of x and λ. *With* (c, d) *denoting a closed interval contained in* Δ_ε $(a \leqq c < d \leqq b)$ we shall write

(11) $$\lambda^{\frac{w-1}{k}} z(x, \lambda) = \int_\gamma^x e^{Q(u, \lambda)} g_w(u, \lambda)\, du \qquad (x \text{ in } (c, d)),$$

where $\gamma = c$ when (1 a) holds and $\gamma = d$ when (1 b) holds. Consider the first case. We have

$$Q(u, \lambda) - Q(x, \lambda) = -\int_{v=u}^{x} Q^{(1)}(v, \lambda)\, dv \tag{12}$$

so that

$$\Re\,[Q(u, \lambda) - Q(x, \lambda)] \leqq \int_{v=u}^{x} - \Re\, Q^{(1)}(v, \lambda)\, dv.$$

In view of (1 a), the integrand in the last member is equal to or is less than zero for λ in R, provided $c \leqq u \leqq x \leqq d$. Accordingly, in the first case

$$\Re\,[Q(u, \lambda) - Q(x, \lambda)] \leqq 0 \qquad (c \leqq u \leqq x \leqq d;\ \lambda \text{ in } R). \tag{13}$$

When (1 b) holds, on writing (12) in the form

$$Q(u, \lambda) - Q(x, \lambda) = \int_{v=x}^{u} Q^{(1)}(v, \lambda)\, dv$$

we obtain

$$\Re\,[Q(u, \lambda) - Q(x, \lambda)] \leqq \int_{v=x}^{u} \Re\, Q^{(1)}(v, \lambda)\, dv \quad (c \leqq x \leqq u \leqq d).$$

Since the integrand in the last member is equal to or is less than zero, (1 b) is seen to imply the inequality

$$\Re\,[Q(u, \lambda) - Q(x, \lambda)] \leqq 0 \qquad (c \leqq x \leqq u \leqq d;\ \lambda \text{ in } R). \tag{13 a}$$

In consequence of (11), (10 a) and by virtue of (13) or (13 a) (as the case may be) it follows that

$$\begin{aligned} \left| \lambda^{\frac{w-1}{k}} z(x, \lambda)\, e^{-Q(x, \lambda)} \right| &\leqq \int_{\gamma}^{x} e^{\Re\,[Q(u, \lambda) - Q(x, \lambda)]} \left| g_w(u, \lambda) \right| \left| du \right| \\ &< g_w \int_{\gamma}^{x} \left| du \right| \leqq (d - c)\, g_w \qquad (c \leqq x \leqq d;\ \lambda \text{ in } R). \end{aligned} \tag{14}$$

Thus

$$(15)\qquad z(x,\lambda)=e^{Q(x,\lambda)}\lambda^{-\left(\frac{w-1}{k}\right)}z_w(x,\lambda),$$

$$|z_w(x,\lambda)|\leqq z_w \qquad (x \text{ in } (c,d);\ \lambda \text{ in } R),$$

the function $z_w(x,\lambda)$ being defined for x in (c,d) and λ in R.

By (5), (4) (with $t=w$) and (15)

$$(16)\qquad y(x,\lambda)=e^{Q(x,\lambda)}\lambda^{-\left(\frac{kH-m}{k}\right)}\left[\sum_{r=0}^{w_1-1}s_r(x)\lambda^{-\frac{r}{k}}+\lambda^{-\frac{w_1}{k}}c_{w_1}(x,\lambda)\right]$$

$$(w_1=w-kH+m-1\geqq 1),$$

where $c_{w_1}(x,\lambda)$ is defined for x in (c,d) and λ in R. Moreover,

$$(16\text{a})\qquad |c_{w_1}(x,\lambda)|\leqq c_{w_1} \qquad (x \text{ in } (c,d);\ \lambda \text{ in } R);$$

here c_{w_1} is independent of x and λ. The relations (16), (16a) can be written in the form

$$(17)\qquad y(x,\lambda)\underset{w_1}{\sim}s(x,\lambda) \qquad (x \text{ in } (c,d);\ \lambda \text{ in } R),$$

where $s(x,\lambda)$ is the formal solution of the formal equation (3).

When $Q(x,\lambda)\equiv 0$, an analogous solution $y(x,\lambda)$ can be obtained so that (17) holds with w_1 replaced by w.

Assume now that R is defined by (1a) and replace (2) by an asymptotic relationship in the ordinary sense. In view of the preceding the problem (1) will then possess an infinity of solutions, in general distinct, such that

$$(18)\qquad {}_iy(x,\lambda)\underset{w_i}{\sim}e^{Q(x,\lambda)}\bar{s}(x,\lambda)$$

$$(x \text{ in } (c,d);\ \lambda \text{ in } R;\ i=1,2,\ldots;\ w_1<w_2<\cdots).$$

Each of the functions

$$(18\text{a})\qquad {}_iz(x,\lambda)={}_iy(x,\lambda)-{}_iy(c,\lambda) \qquad (i=1,2,\ldots)$$

will constitute a solution. Since ${}_iz(c,\lambda)=0$ $(i=1,2,\ldots)$, these functions are independent of i and are seen to represent a single solution, say $z(x,\lambda)$. In view of (18) and (18a) we have, for x in (c,d) and for λ in R,

$$(19)\qquad z(x,\lambda)\sim e^{Q(x,\lambda)}\bar{s}(x,\lambda)-e^{Q(c,\lambda)}\bar{s}(c,\lambda),$$

where the asymptotic relation is to infinitely many terms. For λ in R and x in (c, d), by (13)

$$\left| e^{Q(c,\lambda) - Q(x,\lambda)} \right| \leqq 1.$$

If there exists a positive number ε and a region R_1, extending to infinity and forming part of R (R defined by (1 a)), such that

$$\left| e^{Q(c,\lambda) - Q(x,\lambda)} \right| \sim 0 \qquad (c + \varepsilon \leqq x \leqq d;\ \lambda \text{ in } R_1),$$

then (19) is seen to imply

$$z(x, \lambda) \sim s(x, \lambda) \qquad (c + \varepsilon \leqq x \leqq d;\ \lambda \text{ in } R_1). \tag{20}$$

When R is specified as in (1 b) an analogous result can be obtained.

In view of (5), (6), (10) and (11) a solution $y(x, \lambda)$, for which (17) had been stated, can be represented in the form

$$\begin{aligned} y(x, \lambda) &= t(x, \lambda) + \int_{\gamma}^{x} [e^{Q(u,\lambda)} a(u, \lambda) - t^{(1)}(u, \lambda)]\, du \\ &= t(\gamma, \lambda) + \int_{\gamma}^{x} e^{Q(u,\lambda)} a(u, \lambda)\, du \end{aligned} \tag{21}$$

where $\gamma = c$ or $\gamma = d$, as the case may be.

Lemma 2. *Consider the equation* (1) *where $a(x, \lambda)$ is a known function satisfying* (2). *Suppose there exists a region R so that either* (1 a) *or* (1 b) *holds throughout the region. Let $s(x, \lambda)$ be a formal solution of the formal equation* (3). *Define Δ_ε as in* § 3 *and let (c, d) $(a \leqq c < d \leqq b)$ be a closed interval contained in Δ_ε. The equation* (1) *will then possess a solution $y(x, \lambda)$, defined for x in (c, d) and for λ in R and satisfying for these values of the variables the asymptotic relation* (17) (*with $w_1 = w - kH + m - 1$, when $Q(x, \lambda) \not\equiv 0$, and $w_1 = w$, when $Q(x, \lambda) \equiv 0$;* cf. (16)).

If the asymptotic relationship (2) *is in the ordinary sense and R_1 exists as defined in the italics preceding* (20), *then there exists a solution $z(x, \lambda)$ satisfying in the ordinary sense the asymptotic relation* (20). *A similar statement can be made when R is defined by* (1 b).

5. **Iterations.** We shall now consider a system

$$\text{(C)} \qquad Y^{(1)}(x, \lambda) = Y(x, \lambda) B(x, \lambda) \qquad \text{(cf. (C) of § 2)},$$

$$\text{(1)} \qquad B(x, \lambda) = (b_{i,j}(x, \lambda)) \underset{w}{\sim} (\beta_{i,j}(x, \lambda))$$
$$(x \text{ in } (a, b);\ a < b;\ \lambda \text{ in } R').$$ [1]

In the series

$$\text{(1 a)} \qquad \beta_{i,j}(x, \lambda) = \sum_{\nu=0}^{\infty} \beta_{i,j:\nu}(x) \lambda^{\frac{r-\nu}{k}} \qquad (r, k \text{ positive integers})$$

the coefficients are finite and differentiable for x in (a, b); these series may be divergent for all λ. It will be assumed that formally the determinant $|(\beta_{i,j}(x, \lambda))|$ is not identically zero. The formal system associated with (C),

$$\text{(C}_1\text{)} \qquad S^{(1)}(x, \lambda) = S(x, \lambda)\ (\beta_{i,j}(x, \lambda)),$$

possesses a matrix solution, whose elements are certain possibly divergent series,

$$\text{(2)} \qquad S(x, \lambda) = (s_{i,j}(x, \lambda)) = (e^{Q_i(x, \lambda)} \sigma_{i,j}(x, \lambda)),$$

$$\text{(2 a)} \qquad \sigma_{i,j}(x, \lambda) = \sum_{\nu=0}^{\infty} \sigma_{i,j:\nu}(x) \lambda^{-\frac{\nu}{k}}.$$

Here the $Q_i(x, \lambda)$ $(i = 1, \ldots n)$ are of the form (3 a; § 2).[2] The coefficients of various powers of λ in (2 a) and the $Q_i(x, \lambda)$ are finite and differentiable on (a, b).

Form a matrix

$$\text{(3)} \qquad T(x, \lambda) = (t_{i,j}(x, \lambda)) = (e^{Q_i(x, \lambda)} \tau_{i,j}(x, \lambda)),$$

where the functions $\tau_{i,j}(x, \lambda)$ are obtained from the $\sigma_{i,j}(x, \lambda)$ by deleting in the latter series the powers of λ,

$$\text{(3 a)} \qquad \lambda^{-\left(\frac{t+\nu}{k}\right)} \qquad (\nu = 0, 1, \ldots),$$

t being a suitable positive integer. If we define $E(x, \lambda) = (e_{i,j}(x, \lambda))$ by the equation

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[1] To start with, w is assumed suitably great. R' contains at least a region R, to be defined below.

[2] Throughout, k is taken as the lowest common multiple of all the involved k_i. The interval (a, b) is taken suitably small.

$$(\mathrm{C}_2) \qquad T^{(1)}(x, \lambda) = T(x, \lambda)\, E(x, \lambda),$$

it is observed that the functions $e_{i,j}(x, \lambda)$ are expressible as convergent series in powers of $\lambda^{\frac{1}{k}}$ with coefficients defined for x on (a, b). These series are computed as elements of $T^{-1}(x, \lambda)\, T^{(1)}(x, \lambda)$. In each element $e_{i,j}(x, \lambda)$ a certain initial number g_t of the involved coefficients is obviously independent of t. These coefficients are correspondingly the same as those obtained by formally calculating the elements in the matrix $S^{-1}(x, \lambda)\, S^{(1)}(x, \lambda)$; that is, in view of (C_1), they are correspondingly the same as the initial g_t coefficients of the various powers of $\lambda^{\frac{1}{k}}$ in the series $\beta_{i,j}(x, \lambda)$. The precise nature of the dependence of g_t on t is immaterial for our purposes. Of importance is the evident property $\lim\limits_{t \to \infty} g_t = \infty$.

Thus, t can be chosen depending on w so that, in view of (1) and in view of the stated facts, it is possible to assert that

$$(4) \qquad B(x, \lambda) - E(x, \lambda) \equiv H(x, \lambda) \underset{\sigma'}{\sim} 0 \qquad (\sigma' = \sigma'(w))$$
$$(x \text{ in } (a, b);\ \lambda \text{ in } R')$$

where $\sigma'(w)$ can be made arbitrarily great, whenever w can be made to approach infinity. Accordingly,

$$(5) \qquad H(x, \lambda) = (\lambda^{-\frac{\sigma'}{k}}\, h_{i,j}(x, \lambda)),$$
$$|h_{i,j}(x, \lambda)| \leqq h(\sigma') \qquad (i, j = 1, \ldots n),$$

where $h(\sigma')$ is independent of x and λ for x in (a, b) and λ in R.

The transformation

$$(6) \qquad Y(x, \lambda) = Z(x, \lambda)\, T(x, \lambda), \quad Z(x, \lambda) = (z_{i,j}(x, \lambda)),$$

applied to (C), will result in the system

$$(7) \qquad Z^{(1)}(x, \lambda) = Z(x, \lambda)\, C(x, \lambda), \quad C(x, \lambda) = (c_{i,j}(x, \lambda)),$$

where

$$(7\,\mathrm{a}) \qquad C(x, \lambda) = T(x, \lambda)\, B(x, \lambda)\, T^{-1}(x, \lambda) - T^{(1)}(x, \lambda)\, T^{-1}(x, \lambda).$$

On replacing $T^{(1)}(x, \lambda)$ and $B(x, \lambda)$ with the aid of (C_2) and (4) it follows that

$$(7\,\mathrm{b}) \qquad C(x, \lambda) = T(x, \lambda)\, H(x, \lambda)\, T^{-1}(x, \lambda).$$

Now

$$(8) \qquad T^{-1}(x, \lambda) = (e^{-Q_j(x,\lambda)} \bar{\tau}_{i,j}(x, \lambda)).$$

Here the functions $\bar{\tau}_{i,j}(x, \lambda)\lambda^{-\varrho}$ (ϱ independent of t; $\varrho \geqq 0$) have bounded absolute values for x in (a, b) and $|\lambda| \geqq \lambda_0 > 0$. By (3), (5) and (8) from (7 b) we then have

$$(9) \qquad (c_{i,j}(x, \lambda)) = (e^{Q_{i,j}(x,\lambda)} \lambda^{-\frac{\sigma}{k}} h^{\sigma}_{i,j}(x, \lambda)),$$

$$(9\,\mathrm{a}) \qquad |h^{\sigma}_{i,j}(x, \lambda)| \leqq h_\sigma \qquad (x \text{ in } (a, b);\ \lambda \text{ in } R'),$$

where $\sigma = \sigma(w)$ and $\sigma(w) \to \infty$, whenever w can be indefinitely increased. *Here and in the sequel* $Q_{\alpha,\beta}(x, \lambda) = Q_\alpha(x, \lambda) - Q_\beta(x, \lambda)$.

We shall suppose that in the λ-plane there exists a region R such that, for all x in (a, b), either

$$(10) \qquad \Re\, Q_1^{(1)}(x, \lambda) = \Re\, Q_2^{(1)}(x, \lambda) = \cdots = \Re\, Q_\tau^{(1)}(x, \lambda) > \Re\, Q_{\tau+1}^{(1)}(x, \lambda) \geqq \cdots \cdots \geqq \Re\, Q_n^{(1)}(x, \lambda) \qquad (1 \leqq \tau \leqq n)$$

or

$$(11) \qquad \Re\, Q_1^{(1)}(x, \lambda) = \Re\, Q_2^{(1)}(x, \lambda) = \cdots = \Re\, Q_\tau^{(1)}(x, \lambda) < \Re\, Q_{\tau+1}^{(1)}(x, \lambda) \leqq \cdots \leqq \Re\, Q_n^{(1)}(x, \lambda).$$

If in (10) and (11), for some values of the variables, $>$ or $<$ is replaced by $=$, the results obtained below will continue to remain valid.

First it will be assumed that (10) is the case.

With x in (a, b), let $a = x_0 \leqq x_1 \leqq x_2 \leqq \cdots \leqq x_{m-1} \leqq x_m = x$. Write $v_k = x_k - x_{k-1}$ $(k = 1, \ldots m)$ and choose the x_i $(i = 0, 1, \ldots m)$ so that max. $v_k \to 0$, as $m \to \infty$. Let $y_1, \ldots y_m$ be some numbers such that

$$x_{k-1} \leqq y_k \leqq x_k \qquad (k = 1, \ldots m).$$

Consider the matrix

$$(12) \qquad Z_m(x, \lambda) = (z_{i,j:m}) = (I + v_1 C(y_1, \lambda))(I + v_2 C(y_2, \lambda)) \cdots (I + v_m C(y_m, \lambda))$$
$$(I = (\delta_{i,j}), \text{ the identity matrix}).$$

By the classical theory of Product-integrals the limiting matrix

$$(12\,\mathrm{a}) \qquad Z(x, \lambda) = \lim_{m \to \infty} Z_m(x, \lambda) = \widehat{\int_a^x} (C(x, \lambda)\, dx + I)$$

exists and constitutes a matrix solution of (7). It will be found possible to investigate the asymptotic properties of the elements in the first τ rows of $Z(x, \lambda)$. We have

$$(13)\qquad Z_m(x,\lambda) = I + \sum_{k_1=1}^{m} v_{k_1} C(y_{k_1},\lambda) + \sum_{k_1<k_2}^{m} v_{k_1} v_{k_2} C(y_{k_1},\lambda)\, C(y_{k_2},\lambda) + \cdots$$
$$+ \sum_{k_1<\cdots<k_s}^{m} v_{k_1} v_{k_2}\cdots v_{k_s} C(y_{k_1},\lambda)\, C(y_{k_2},\lambda)\cdots C(y_{k_s},\lambda) + \cdots$$
$$+ v_1\cdots v_m C(y_1,\lambda)\cdots C(y_m,\lambda) = I + \sum_{s=1}^{m} L_s, \qquad L_s = (l_{i,j:s}).$$

On using (9) and on writing $h_{i,j}(x,\lambda)$ in place of $h^{\sigma}_{i,j}(x,\lambda)$, the $l_{i,j:s}$ are seen to be of the form

$$(13\,\mathrm{a})\qquad l_{i,j:s} = \sum_{k_1<\cdots<k_s}^{m} \Bigg\{ \left(v_{k_1}\lambda^{-\frac{\sigma}{k}}\right)\left(v_{k_2}\lambda^{-\frac{\sigma}{k}}\right)\cdots\left(v_{k_s}\lambda^{-\frac{\sigma}{k}}\right)\cdot$$
$$\sum_{r_1,\ldots r_{s-1}=1}^{n} e^{W_{i,j}} h_{i,r_1}(y_{k_1},\lambda)\, h_{r_1,r_2}(y_{k_2},\lambda)\cdots h_{r_{s-1},j}(y_{k_s},\lambda)\Bigg\},$$

where

$$W_{i,j} = Q_{i,r_1}(y_{k_1},\lambda) + Q_{r_1,r_2}(y_{k_2},\lambda) + \cdots + Q_{r_{s-2},r_{s-1}}(y_{k_{s-1}},\lambda)$$
$$+ Q_{r_{s-1},j}(y_{k_s},\lambda).$$

Regrouping terms,

$$(14)\qquad W_{i,j} = Q_{i,j}(x,\lambda) + [Q_{j,i}(x,\lambda) - Q_{j,i}(y_{k_s},\lambda)] + [Q_{r_{s-1},i}(y_{k_s},\lambda) - Q_{r_{s-1},i}(y_{k_{s-1}},\lambda)]$$
$$\cdots + [Q_{r_2,i}(y_{k_3},\lambda) - Q_{r_2,i}(y_{k_2},\lambda)] + [Q_{r_1,i}(y_{k_2},\lambda) - Q_{r_1,i}(y_{k_1},\lambda)].$$

By (10)

$$(15)\qquad \Re\, Q^{(1)}_{j,i}(x,\lambda) \leqq 0 \qquad (x \text{ in } (a,b);\ \lambda \text{ in } R)$$

for $i = 1, 2, \ldots \tau$ and $j = 1, \ldots n$. A condition (15) is of the form (1 b; § 4). Thus, in view of (13 a; § 4), (15) is seen to imply

$$(16)\qquad \Re\,[Q_{j,i}(u,\lambda) - Q_{j,i}(y,\lambda)] \leqq 0 \qquad (i = 1, \ldots \tau;\ j = 1, \ldots n)$$

for $a \leqq y \leqq u \leqq b$ and for λ in R. Since in (14) $k_1 < k_2 < \cdots < k_s$ it follows that

$$a \leqq y_{k_1} \leqq y_{k_2} \leqq \cdots \leqq y_{k_s} \leqq x \ (\leqq b).$$

Accordingly, by (16), the real parts of the functions contained in the square brackets of the second member of (14) are all equal or are less than zero for $i = 1, \dots \tau$ and $j = 1, \dots n$, provided x is in (a, b) and λ is in R. Whence it follows that

$$\Re W_{i,j} \leqq \Re Q_{i,j}(x, \lambda) \tag{17}$$
$$(i = 1, \dots \tau;\ j = 1, \dots n;\ x \text{ in } (a, b);\ \lambda \text{ in } R).$$

Thus, by (13), (13 a) and (17), it follows that, for $i = 1, \dots \tau$,

$$|z_{i,j:m} - \delta_{i,j}| \leqq \sum_{s=1}^{m} |l_{i,j:s}| \leqq |e^{Q_{i,j}(x,\lambda)}| \cdot \tag{18}$$
$$\cdot \sum_{s=1}^{m} \sum_{k_1 < \dots < k_s}^{m} \left\{ |v_{k_1} \dots v_{k_s}| \, |\lambda|^{-\frac{s\sigma}{k}} \sum_{r_1, \dots r_{s-1}=1}^{n} |h_{i,r_1}(y_{k_1}, \lambda) \dots h_{r_{s-1},j}(y_{k_s}, \lambda)| \right\}.$$

Take the subdivisions equal,

$$0 \leqq v_{k_1} = v_{k_2} = \cdots = v_{k_s} = \frac{x-a}{m} \leqq \frac{b-a}{m}.$$

Using (9a) from (18) we then obtain, for $i \leqq \tau$, for x in (a, b) and for λ in R,

$$|z_{i,j:m} - \delta_{i,j}| \leqq |e^{Q_{i,j}(x,\lambda)}| \sum_{s=1}^{m} \sum_{k_1 < \dots < k_s}^{m} \left\{ \left(\frac{b-a}{m}\right)^s |\lambda|^{-\frac{s\sigma}{k}} \cdot n^{s-1} (h_\sigma)^s \right\} \tag{18 a}$$
$$= \frac{1}{n} |e^{Q_{i,j}(x,\lambda)}| \sum_{s=1}^{m} \sum_{k_1 < \dots < k_s}^{m} \left(\frac{h}{m} |\lambda|^{-\frac{\sigma}{k}}\right)^s$$
$$= \frac{1}{n} |e^{Q_{i,j}(x,\lambda)}| \left[-1 + \left(1 + \frac{h}{m} |\lambda|^{-\frac{\sigma}{k}}\right)^m \right] \qquad (h = (b-a) n h_\sigma).$$

Now

$$1 + \frac{h}{m} |\lambda|^{-\frac{\sigma}{k}} < e^{\frac{h}{m} |\lambda|^{-\frac{\sigma}{k}}},$$

so that

$$f(m, \lambda) = \left(1 + \frac{h}{m} |\lambda|^{-\frac{\sigma}{k}}\right)^m < e^{h |\lambda|^{-\frac{\sigma}{k}}}$$

and

$$-1 + f(m, \lambda) < |\lambda|^{-\frac{\sigma}{k}} n f(h),$$

where $nf(h)$ is independent of m and λ, but depends on $\sigma(=\sigma(w))$. In view of (18 a) the latter inequality implies that

$$z_{i,j:m} = \delta_{i,j} + \lambda^{-\frac{\sigma}{k}} e^{Q_{i,j}(x,\lambda)} \mathfrak{z}_{i,j:m} \tag{19}$$

where

$$|\mathfrak{z}_{i,j:m}| < f(h) \tag{19 a}$$

$$(i = 1, \ldots \tau;\ j = 1, \ldots n;\ x \text{ in } (a,b);\ \lambda \text{ in } R).$$

For the limiting functions $z_{i,j}(x,\lambda)$ we have, for x in (a,b) and λ in R,

$$z_{i,j}(x,\lambda) = \delta_{i,j} + \lambda^{-\frac{\sigma}{k}} e^{Q_{i,j}(x,\lambda)} \mathfrak{z}_{i,j}(x,\lambda), \tag{20}$$
$$|\mathfrak{z}_{i,j}(x,\lambda)| < f(h) \qquad (i = 1, \ldots \tau;\ j = 1, \ldots n).$$

By (6), (20) and (3) we accordingly have

$$y_{i,j}(x,\lambda) = \sum_{r=1}^{n} z_{i,r}(x,\lambda)\, t_{r,j}(x,\lambda) = \sum_{r=1}^{n} [\delta_{i,r} + e^{Q_{i,r}(x,\lambda)} \cdot \tag{21}$$
$$\cdot \lambda^{-\frac{\sigma}{k}} \mathfrak{z}_{i,r}(x,\lambda)]\, e^{Q_r(x,\lambda)} \tau_{r,j}(x,\lambda) = e^{Q_i(x,\lambda)} [\tau_{i,j}(x,\lambda) + \lambda^{-\frac{\sigma}{k}} {}_{\sigma}\bar{\eta}_{i,j}(x,\lambda)]$$

where

$$|{}_{\sigma}\bar{\eta}_{i,j}(x,\lambda)| = \left|\sum_{r=1}^{n} \mathfrak{z}_{i,r}(x,\lambda)\, \tau_{r,j}(x,\lambda)\right| < nf(h)\, \tau_1 = c(w) \tag{21 a}$$

$$(i = 1, \ldots \tau;\ j = 1, \ldots n;\ x \text{ in } (a,b);\ \lambda \text{ in } R).^{1}$$

On taking, as is possible, $\sigma = \sigma(w) \leqq t\ (= t(w))$ relations (21) are seen to imply that, for $i \leqq \tau$ and $j = 1, \ldots n$,

$$y_{i,j}(x,\lambda) \underset{\sigma}{\sim} s_{i,j}(x,\lambda) \qquad (x \text{ in } (a,b);\ \lambda \text{ in } R;\ \sigma = \sigma(w)). \tag{22}$$

In the case when there exists a region R, for which (11) holds, we obtain an analogous result, the corresponding Product-integrals being taken along the path extending from b to x.

The following Lemma can be now stated.

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[1] $|\sigma_{r,j}(x,\lambda)| < \tau_1 \qquad (x \text{ in } (a,b);\ \lambda \text{ in } R)$.

Lemma 3. *Suppose a system* (C), (1) *is given. Let* $S(x, \lambda)$ *be a corresponding formal matrix solution* (2). *Let* $T(x, \lambda) = (t_{i,j}(x, \lambda))$ *be the matrix formed by deleting in* $S(x, \lambda)$ *the powers* (3 a) ($t = t(w)$ *a suitably great integer*). *Define* $C(x, \lambda)$ *by* (7 a).

Case I. *There exists a region* R *so that* (10) *holds. In this case determine the matrix* $Z(x, \lambda) = (z_{i,j}(x, \lambda))$ *by the Product-integral*

$$Z(x, \lambda) = \widehat{\int_a^x} (C(x, \lambda)\, dx + I).$$

Case II. *There exists a region* R *such that* (10) *holds. In this case we define a matrix* $Z(x, \lambda)$ *by*

$$Z(x, \lambda) = \widehat{\int_b^x} (C(x, \lambda)\, dx + I).$$

The matrix $Y(x, \lambda) = Z(x, \lambda)\, T(x, \lambda)$ *will satisfy the system* (C). *If there exists a region* R *such that* (10) *or* (11) *holds, we have*

$$(23) \qquad Y(x, \lambda) = {}_wY(x, \lambda) \underset{\sigma(w)}{\sim} S(x, \lambda) \qquad (x \text{ in } (a, b);\ \lambda \text{ in } R)$$

in the first τ *rows. Here* $\sigma(w) \to \infty$, *whenever it is possible to increase* w *indefinitely.*

As the elements in $Y(x, \lambda)$ depend on w, the above Lemma does not necessarily imply that if the asymptotic relations (1) are in the ordinary sense those in (23) would also be in the ordinary sense.

6. **The Fundamental Existence Theorem.** Consider the equation (A; § 2). Suppose that the asymptotic relations (1; § 2) are valid in the ordinary sense for x in (a, b) and for λ in a region R' coincident with or containing the region R, defined below. The interval (a, b) will be assumed such that the formal equation (A*; § 2) has a full set of formal solutions

$$(1) \qquad s_i(x, \lambda) = e^{Q_i(x, \lambda)} \sigma_i(x, \lambda)$$
$$(i = 1, \ldots n;\ \text{cf. } (3\text{ a}; \S\ 2),\ (3\text{ b}; \S\ 2))$$

of the character specified in § 2.

Consider all possible differences $Q_{i,j}(x, \lambda) \equiv Q_i(x, \lambda) - Q_j(x, \lambda)$ *which are such that in the polynomials*

$$Q_{i,j}^{(1)}(x, \lambda) = \sum_{\alpha=m} q_{i,j:\alpha}^{(1)}(x) \lambda^{\frac{kH-\alpha}{k}} \qquad (m = m_{i,j}) \tag{2}$$

the leading coefficients $q_{i,j:m}^{(1)}(x) \equiv q_{i,m}^{(1)}(x) - q_{j,m}^{(1)}(x)$ *are not identically zero. Assume that there exists a closed interval* (c, d) *so that for all functions* (2)

$$|q_{i,j:m}^{(1)}(x)| \geqq \varepsilon > 0 \qquad (x \text{ in } (c, d)). \tag{2 a}$$

It will be assumed that in the λ-plane there exists a region R (extending to infinity) such that, for all λ in R and for all x in (c, d), we have either one of the following two cases.

Case I.

$$\begin{gathered} \Re Q_1^{(1)}(x, \lambda) = \cdots = \Re Q_{\tau_1}^{(1)}(x, \lambda) > \Re Q_{\tau_1+1}^{(1)}(x, \lambda) = \cdots = \Re Q_{\tau_2}^{(1)}(x, \lambda) \\ > \Re Q_{\tau_2+1}^{(1)}(x, \lambda) = \cdots = \Re Q_{\tau_3}^{(1)}(x, \lambda) > \cdots > \Re Q_{\tau_{\nu-1}+1}^{(1)}(x, \lambda) = \cdots = \Re Q_{\tau_\nu}^{(1)}(x, \lambda) \\ (1 \leqq \tau_1 < \tau_2 < \cdots < \tau_{\nu-1} < \tau_\nu = n). \end{gathered} \tag{3}$$

Case II. *Inequalities* (3) *with* $>$ *replaced by* $<$. In connection with these inequalities a remark is made similar to that following (11; § 5).

Case I will be discussed first.

By Lemma 3 (§ 5) the system (B; § 2), associated with the equation (A), has τ_1 solutions,

$$_1y_i^{(j-1)}(x, \lambda) \qquad (i = 1, \ldots \tau_1;\ j = 1, \ldots n) \tag{4}$$

such that

$$\begin{gathered} _1y_i^{(j-1)}(x, \lambda) \underset{w_1}{\sim} s_i^{(j-1)}(x, \lambda) \\ (i = 1, \ldots \tau_1;\ j = 1, \ldots n;\ x \text{ in } (c, d);\ \lambda \text{ in } R).^1 \end{gathered} \tag{4 a}$$

The elements $_1y_i(x, \lambda)$ $(i = 1, \ldots \tau_1)$ will form a linearly independent set of τ_1 solutions of (A). The number w_1 in (4 a) and the numbers in the sequel can be made arbitrarily great by suitable choice of the matrix $T(x, \lambda)$, used in Lemma 3.

[1] Lemma 3 continues to hold when the power series factors, involved in the formal solutions, are allowed to contain a finite number of positive powers of $\lambda^{1/k}$. To start with, w_1 will be supposed to be sufficiently great.

We form the equation

$$(5)\qquad L_{\tau_1}(x,\lambda;\,{}_1y)\equiv$$

$$\begin{vmatrix} {}_1y_1^{(\tau_1-1)}(x,\lambda), & \ldots, & {}_1y_1(x,\lambda) \\ \cdots & \cdots & \cdots \\ {}_1y_{\tau_1}^{(\tau_1-1)}(x,\lambda), & \ldots, & {}_1y_{\tau_1}(x,\lambda) \end{vmatrix}^{-1} \begin{vmatrix} {}_1y^{(\tau_1)}(x,\lambda), & {}_1y^{(\tau_1-1)}(x,\lambda), & \ldots, & {}_1y(x,\lambda) \\ {}_1y_1^{(\tau_1)}(x,\lambda), & {}_1y_1^{(\tau_1-1)}(x,\lambda), & \ldots, & {}_1y_1(x,\lambda) \\ \cdots & \cdots & \cdots & \cdots \\ {}_1y_{\tau_1}^{(\tau_1)}(x,\lambda), & {}_1y_{\tau_1}^{(\tau_1-1)}(x,\lambda), & \ldots, & {}_1y_{\tau_1}(x,\lambda) \end{vmatrix}$$

$$\equiv \sum_{k=0}^{\tau_1} {}_1a_{\tau_1-k}(x,\lambda)\,{}_1y^{(k)}=0 \qquad ({}_1a_0(x,\lambda)\equiv 1).$$

By replacing in (5) the ${}_1y_i(x,\lambda)$ $(i=1,\ldots\tau_1)$ by the series $s_i(x,\lambda)$ and the various derivatives of the ${}_1y_i(x,\lambda)$ by the corresponding derivatives of the $s_i(x,\lambda)$ we obtain a formal equation

$$(6)\qquad L^*_{\tau_1}(x,\lambda;\,{}_1y)\equiv \sum_{k=0}^{\tau_1} {}_1\alpha_{\tau_1-k}(x,\lambda)\,{}_1y^{(k)}=0 \qquad ({}_1\alpha_0(x,\lambda)\equiv 1).$$

The coefficients here are σ-series, in general divergent. In view of (4 a), there exists a number $v_1=v_1(w_1)\leqq w_1$ so that

$$(6\,\mathrm{a})\qquad {}_1a_{\tau_1-k}(x,\lambda)\underset{v_1}{\sim}{}_1\alpha_{\tau_1-k}(x,\lambda) \qquad (k=0,\ldots\tau_1)$$

for x in (c,d) and λ in R. The set of functions $(4;\,j=1)$ being linearly independent, every solution of (5) is a linear combination, with coefficients independent of x, of these functions; accordingly, every solution of (5) is a solution of (A; § 2). Hence there exists an analytic factorization of (A),

$$(7)\qquad L\equiv L_n(x,\lambda;\,{}_1y)\equiv L_{n-\tau_1}L_{\tau_1}(x,\lambda;\,{}_1y)=0,$$

$$(7\,\mathrm{a})\qquad L_{n-\tau_1}(x,\lambda;\,{}_1z)\equiv\sum_{k=0}^{n-\tau_1}{}_1b_{n-\tau_1-k}(x,\lambda)\,{}_1z^{(k)}(x,\lambda) \qquad ({}_1b_0(x,\lambda)\equiv 1).$$

Since the determinant $|(s_i^{(j-1)}(x,\lambda))|$ $(i,j=1,\ldots\tau_1)$ formally is not identically zero, by a reasoning analogous to that just employed a formal factorization of $L^*_n=0$ is found,

$$(8)\qquad L^*_n(x,\lambda;\,{}_1y)\equiv L^*_{n-\tau_1}L^*_{\tau_1}(x,\lambda;\,{}_1y)=0,$$

$$(8\,\mathrm{a})\qquad L^*_{n-\tau_1}(x,\lambda;\,{}_1z)\equiv\sum_{k=0}^{n-\tau_1}{}_1\beta_{n-\tau_1-k}(x,\lambda)\,{}_1z^{(k)}(x,\lambda) \qquad ({}_1\beta_0(x,\lambda)\equiv 1).$$

Here the coefficients are σ-series.[1] By (8) and in view of the fact that formally

$$L_n^*(s_{\tau_1+i}(x, \lambda)) = 0 \qquad (i = 1, \ldots n - \tau_1),$$

it follows that the equation

$$L_{n-\tau_1}^*(x, \lambda; {}_1z) = 0 \tag{9}$$

has a full set of formal solutions

$$\text{(9 a)} \qquad {}_1\mathfrak{z}_{\tau_1+i}(x, \lambda) = L_{\tau_1}^*(s_{\tau_1+i}(x, \lambda)) = e^{Q_{\tau_1+i}(x, \lambda)}\, {}_1\varphi_{\tau_1+i}(x, \lambda)$$
$$(i = 1, \ldots n - \tau_1)$$

where the ${}_1\varphi_{\tau_1+i}(x, \lambda)$ are σ-series with, possibly, a few positive powers of $\lambda^{\frac{1}{k}}$ present.

Also, in view of the involved asymptotic properties,

$$\text{(10)} \qquad {}_1b_i(x, \lambda) \underset{r_1}{\sim} {}_1\beta_i(x, \lambda)$$
$$(i = 0, \ldots n - \tau_1;\ x \text{ in } (c, d);\ \lambda \text{ in } R).$$

Here $r_1 = r_1(w_1)$ and $\lim\limits_{w_1 \to \infty} r_1(w_1) = \infty$. Accordingly, it is observed that the equation $L_{n-\tau_1}(x, \lambda; {}_1z) = 0$ is of the type to which Lemma 3 can be applied. This can be effected by associating with this equation a system of order $n - \tau_1$ and of the type of (B; § 2). On taking account of (9 a) and of (3) it is noted that Lemma 3 enables determination of the asymptotic properties of those solutions which correspond to the series (9 a) for $i = 1, \ldots \tau_2 - \tau_1$. We have $\tau_2 - \tau_1$ distinct solutions of $L_{n-\tau_1}({}_1z) = 0$, ${}_1z_{\tau_1+i}(x, \lambda)$ $(i = 1, \ldots \tau_2 - \tau_1)$, such that

$$\text{(11)} \qquad {}_1z_{\tau_1+i}^{(j-1)}(x, \lambda) \underset{q_1}{\sim} {}_1\mathfrak{z}_{\tau_1+i}^{(j-1)}(x, \lambda)$$
$$(i = 1, \ldots \tau_2 - \tau_1;\ j = 1, \ldots n - \tau_1)$$

for x in (c, d) and λ in R. Here q_1 can be made arbitrarily great by suitable choice of $T(x, \lambda)$ in each of the previously involved applications of Lemma 3. The same refers to the q_i involved in the subsequent asymptotic relations. On taking account of (7) it is observed that besides the τ_1 solutions ${}_1y_i(x, \lambda)$ $(i = 1, \ldots \tau_1)$, previously obtained, the equation (A) has $\tau_2 - \tau_1$ other solutions,

[1] The ${}_1\beta_{n-\tau_1-k}(x)$ are in integral powers of a rational power of λ; the latter power may be distinct from that involved in the coefficient of L_n^*. A similar remark holds for subsequent factorizations.

$$(12) \qquad {}_2y_{\tau_1+i}(x, \lambda) \qquad (i = 1, \ldots \tau_2 - \tau_1),$$

each satisfying the non-homogeneous equation

$$(13) \qquad L_{\tau_1}({}_2y) = {}_1z_{\tau_1+i}(x, \lambda) \qquad (1 \leqq i \leqq \tau_2 - \tau_1).$$

This fact is a consequence of the factorization (7).

Now, by (8), (7) of § 2, a solution of (13) is given by

$$(14) \qquad {}_2y_{\tau_1+i}(x, \lambda) = \sum_{r=1}^{\tau_1} {}_1y_r(x, \lambda) \int {}_1z_{\tau_1+i}(x, \lambda)\, {}_1\bar{y}_{\tau_1, r}(x, \lambda)\, dx,$$

where

$$(14\text{a}) \qquad ({}_1\bar{y}_{i,j}(x, \lambda)) = ({}_1y_i^{(j-1)}(x, \lambda))^{-1} \qquad (i, j = 1, \ldots \tau_1).$$

On formally computing the elements $\bar{s}_{i,j}(x, \lambda)$, defined by the relation

$$(14\text{b}) \qquad (\bar{s}_{i,j}(x, \lambda)) = (s_i^{(j-1)}(x, \lambda))^{-1} \qquad (i, j = 1, \ldots \tau_1),$$

(14 a) and (4) are seen to imply

$$(15) \qquad ({}_1\bar{y}_{i,j}(x, \lambda)) \underset{q'}{\sim} (\bar{s}_{i,j}(x, \lambda)) = (e^{-Q_j(x, \lambda)} \bar{\sigma}_{i,j}(x, \lambda))$$
$$(i, j = 1, \ldots \tau_1;\ x \text{ in } (c, d);\ \lambda \text{ in } R),$$

where $\bar{\sigma}_{i,j}(x, \lambda)$ are series of the same description as the ${}_1\varphi_{\tau_1+i}(x, \lambda)$ in (9 a) and q' can be made as great as desired by suitable previous applications of Lemma 3.

By (15), (11) and (9 a) the integrand displayed in (14) satisfies the relation

$$(16) \qquad {}_1z_{\tau_1+i}(x, \lambda)\, {}_1\bar{y}_{\tau_1, r}(x, \lambda) \underset{q(1)}{\sim} e^{Q_{\tau_1+i, r}(x, \lambda)}\, {}_1\varphi_{\tau_1+i}(x, \lambda)\, \bar{\sigma}_{\tau_1, r}(x, \lambda) = e^{Q(x, \lambda)} \psi(x, \lambda)$$
$$(i = 1, \ldots \tau_2 - \tau_1;\ r = 1, \ldots \tau_1;\ x \text{ in } (c, d);\ \lambda \text{ in } R).$$

By (3)

$$\Re\, Q^{(1)}_{\tau_1+i, r}(x, \lambda) < 0 \qquad (x \text{ in } (c, d);\ \lambda \text{ in } R);$$

moreover, by definition of the interval (c, d), the derivative of the highest power of λ in $Q_{\tau_1+i, r}(x, \lambda)$ is distinct from zero throughout (c, d). Thus, on one hand, application of Lemma 1 is possible in order to find a formal solution of the formal equation

$$(16\text{a}) \qquad y^{(1)}(x, \lambda) = e^{Q(x, \lambda)} \psi(x, \lambda)$$

(the coefficients of the solution are here defined on (c, d)). On the other hand, on the basis of the stated fact and in view of the inequality satisfied by $\Re\, Q^{(1)}_{\tau_1+i,r}(x, \lambda)$ (cf. 1 b; § 4), by Lemma 2 (§ 4) it follows that the integrals displayed in (14) can be so evaluated that, to a number of terms,

$$\int \sim e^{Q_{\tau_1+i,r}(x,\lambda)}\, \eta_{i,r}(x, \lambda) \qquad (x \text{ in } (c, d);\ \lambda \text{ in } R). \tag{16 b}$$

The formal series in the last member of (16 b) is a solution of (16 a). By (4 a; $j = 1$), (16 b) and (14) we have

$$\begin{gathered} {}_2y_{\tau_1+i}(x, \lambda) \underset{w_2}{\sim} e^{Q_{\tau_1+i}(x,\lambda)}\, {}_2\sigma_{\tau_1+i}(x, \lambda) = {}_2s_{\tau_1+i}(x, \lambda) \\ (i = 1, \dots \tau_2 - \tau_1;\ x \text{ in } (c, d);\ \lambda \text{ in } R). \end{gathered} \tag{17}$$

Analogous to the manner in which (5) had been established we now construct the equation

$$L_{\tau_2}({}_2y) \equiv \sum_{k=0}^{\tau_2} {}_2a_{\tau_2-k}(x, \lambda)\, {}_2y^{(k)} = 0, \tag{18}$$

which is satisfied by the τ_2 solutions

$${}_2y_i(x, \lambda) = {}_1y_i(x, \lambda) \qquad (i = 1, \dots \tau_1), \tag{18 a}$$

$${}_2y_{\tau_1+i}(x, \lambda) \qquad (i = 1, \dots \tau_2 - \tau_1). \tag{18 b}$$

In view of (14) the functions (18 a), (18 b) are seen to satisfy all the needed differentiability conditions. Analogous to (6) we now have a formal equation

$$L^*_{\tau_2}(x, \lambda;\, {}_2y) \equiv \sum_{k=0}^{\tau_2} {}_2\alpha_{\tau_2-k}(x, \lambda)\, {}_2y^{(k)} = 0 \tag{19}$$

which is satisfied by τ_2 distinct formal solutions

$$\begin{aligned} {}_2s_i(x, \lambda) &= {}_1s_i(x, \lambda) \qquad (i = 1, \dots \tau_1), \\ {}_2s_{\tau_1+i}&(x, \lambda) \qquad (i = 1, \dots \tau_2 - \tau_1). \end{aligned} \tag{19 a}$$

Moreover, for x in (c, d) and for λ in R,

$${}_2a_i(x, \lambda) \underset{v_2}{\sim} {}_2\alpha_i(x, \lambda) \qquad (i = 1, \dots \tau_2). \tag{20}$$

Equation (A) will be analytically factorable as follows

$$(21)\qquad L_n(x, \lambda; {}_2y) \equiv L_{n-\tau_2} L_{\tau_2}(x, \lambda; {}_2y) = 0,$$

$$(21\,a)\qquad L_{n-\tau_2}(x, \lambda; {}_2z) \equiv \sum_{k=0}^{n-\tau_2} {}_2b_{n-\tau_2-k}(x, \lambda)\, {}_2z^{(k)}(x, \lambda) \quad ({}_2b_0(x, \lambda) \equiv 1).$$

A formal factorization, similar to (8), (8 a), will also take place,

$$(22)\qquad L_n^*(x, \lambda; {}_2y) \equiv L_{n-\tau_2}^* L_{\tau_2}^*(x, \lambda; {}_2y) = 0,$$

$$(22\,a)\qquad L_{n-\tau_2}^*(x, \lambda; {}_2z) \equiv \sum_{k=0}^{n-\tau_2} {}_2\beta_{n-\tau_2-k}(x, \lambda)\, {}_2z^{(k)}(x, \lambda) \quad ({}_2\beta_0(x, \lambda) \equiv 1).$$

Moreover,

$$(23)\qquad {}_2b_i(x, \lambda) \underset{r_2}{\sim} {}_2\beta_i(x, \lambda)$$
$$(i = 1, \ldots n - \tau_2;\ x \text{ in } (c, d);\ \lambda \text{ in } R).$$

The equation $L_{n-\tau_2}^*(x, \lambda; {}_2z) = 0$ will possess $n - \tau_2$ distinct formal solutions

$$(24)\qquad {}_2\mathfrak{z}_{\tau_2+i}(x, \lambda) = L_{\tau_2}^*(s_{\tau_2+i}(x, \lambda)) = e^{Q_{\tau_2+i}(x, \lambda)}\, {}_2\varphi_{\tau_2+i}(x, \lambda)$$
$$(i = 1, \ldots n - \tau_2),$$

where the ${}_2\varphi_{\tau_2+i}(x, \lambda)$ are σ-series.

On making use of (3) and applying Lemma (3) to the system (B; § 2), associated with the equation $L_{n-\tau_2}(x, \lambda; {}_2z) = 0$, the latter equation is seen to possess $\tau_3 - \tau_2$ solutions, ${}_2z_{\tau_2+i}(x, \lambda)$ $(i = 1, \ldots \tau_3 - \tau_2)$, satisfying relations

$$(25)\qquad {}_2z_{\tau_2+i}^{(j-1)}(x, \lambda) \underset{q_2}{\sim} {}_2\mathfrak{z}_{\tau_2+i}^{(j-1)}(x, \lambda)$$
$$(i = 1, \ldots \tau_3 - \tau_2;\ j = 1, \ldots n - \tau_2)$$

for x in (c, d) and λ in R. An equation

$$(26)\qquad L_{\tau_2}({}_3y) = {}_2z_{\tau_2+i}(x, \lambda) \qquad (1 \leqq i \leqq \tau_3 - \tau_2)$$

has a solution

$$(26\,a)\qquad {}_3y_{\tau_2+i}(x, \lambda) = \sum_{r=1}^{\tau_2} {}_2y_r(x, \lambda) \int {}_2z_{\tau_2+i}(x, \lambda)\, {}_2\bar{y}_{\tau_2, r}(x, \lambda)\, dx,$$

where

$$(26\,\mathrm{b}) \qquad {}_2z_{\tau_2+i}(x,\lambda)\,{}_2\bar{y}_{\tau_2,r}(x,\lambda) \underset{q^{(2)}}{\sim} e^{Q_{\tau_2+i,r}(x,\lambda)}\,{}_2\varphi_{\tau_2+i}(x,\lambda)$$
$$(i = 1, \ldots \tau_3 - \tau_2;\ r = 1, \ldots \tau_2;\ x \text{ in } (c,d);\ \lambda \text{ in } R).$$

Here the ${}_2\varphi_{\tau_2+i}(x,\lambda)$ are σ-series.

With the inequality $\Re\, Q^{(1)}_{\tau_2+i,r}(x,\lambda) < 0$ (x in (c,d); λ in R) in view, Lemma 2 is seen to be applicable. In (26 a) evaluating the integrals according to this Lemma we get

$$(27) \qquad {}_3y_{\tau_2+i}(x,\lambda) \underset{w_3}{\sim} e^{Q_{\tau_2+i}(x,\lambda)}\,{}_3\sigma_{\tau_2+i}(x,\lambda) = {}_3s_{\tau_2+i}(x,\lambda)$$
$$(i = 1, \ldots \tau_3 - \tau_2;\ x \text{ in } (c,d);\ \lambda \text{ in } R).$$

Thus we have obtained the asymptotic form of τ_3 solutions of (A), ${}_1y_i(x,\lambda)$ $(i = 1, \ldots \tau_1)$, ${}_2y_{\tau_1+i}(x,\lambda)$ $(i = 1, \ldots \tau_2 - \tau_1)$, ${}_3y_{\tau_2+i}(x,\lambda)$ $(i = 1, \ldots \tau_3 - \tau_2)$. All of these could be denoted by ${}_3y_i(x,\lambda)$ $(i = 1, \ldots \tau_3)$.

Beginning with Iterations (Lemma 3) we follow in succession by $\nu - 1$ triple operations, each consisting of (1°) a Factorization, (2°) Iterations and (3°) Integrations (according to Lemma 2). We thus obtain a full set of solutions, $y_i(x,\lambda)$ $(i = 1, \ldots n)$, satisfying (A) and such that for x in (c,d) and λ in R

$$(28) \qquad y_i(x,\lambda) = {}_\nu y_i(x,\lambda) \underset{w}{\sim} e^{Q_i(x,\lambda)}\,{}_\nu\sigma_i(x,\lambda) = s_i(x,\lambda) \qquad (i = 1, \ldots n).$$

Moreover,

$$(28\,\mathrm{a}) \qquad y_i^{(j-1)}(x,\lambda) \underset{w}{\sim} s_i^{(j-1)}(x,\lambda) \qquad (j = 2, \ldots n).$$

Such solutions can be constructed so that w has as great a value as desired. However, these solutions may depend on w and thus we are not in the position to assert the asymptotic relations (28), (28 a) in the ordinary sense (cf. § 1).

Precisely similar conclusions are reached when the region is specified as in Case II.

Analogous results hold for a system (C) (§ 2).

We shall now obtain solutions possessing asymptotic properties in the ordinary sense. According to the preceding developments the equation (A) has an infinity of fundamental sets of solutions,

$${}_ry_1(x,\lambda),\ {}_ry_2(x,\lambda), \ldots,\ {}_ry_n(x,\lambda) \qquad (r = 1, 2, \ldots)$$

such that, on writing

$$(29) \qquad {}_rY(x,\lambda) = ({}_ry_i^{(j-1)}(x,\lambda)) = ({}_ry_{i,j}(x,\lambda)) \qquad (i,j = 1,\ldots n),$$

we have

$$(29\,\text{a}) \qquad {}_rY(x,\lambda) \underset{\alpha_r}{\sim} S(x,\lambda), \quad {}_rY^{-1}(x,\lambda) = ({}_r\bar{y}_{i,j}(x,\lambda)) \sim S^{-1}(x,\lambda)$$
$$(r = 1, 2, \ldots; \ \alpha_1 < \alpha_2 < \cdots; \ x \text{ in } (c,d); \ \lambda \text{ in } R)$$

where

$$(30) \qquad S(x,\lambda) = (e^{Q_i(x,\lambda)}\sigma_{i,j}(x,\lambda)), \quad S^{-1}(x,\lambda) = (e^{-Q_j(x,\lambda)}\bar{\sigma}_{i,j}(x,\lambda))$$
$$(\sigma_{i,j}(x,\lambda), \ \bar{\sigma}_{i,j}(x,\lambda) \ \sigma\text{-series}; \ i,j = 1,\ldots n; \ |S(x,\lambda)| \neq 0, \text{ formally}).$$

That is,

$$(31) \qquad {}_ry_{i,j}(x,\lambda) = e^{Q_i(x,\lambda)}\left[\sum_{\nu=-m}^{\alpha_r-1}\sigma_{i,j:\nu}(x)\lambda^{-\frac{\nu}{k}} + \sigma_r^{i,j}(x,\lambda)\lambda^{-\alpha_r/k}\right],$$

$$(31\,\text{a}) \qquad {}_r\bar{y}_{i,j}(x,\lambda) = e^{-Q_j(x,\lambda)}\left[\sum_{\nu=-m}^{\alpha_r-1}\bar{\sigma}_{i,j:\nu}(x)\lambda^{-\frac{\nu}{k}} + \bar{\sigma}_r^{i,j}(x,\lambda)\lambda^{-\alpha_r/k}\right],$$

$$(31\,\text{b}) \qquad |\sigma_r^{i,j}(x,\lambda)|, \ |\bar{\sigma}_r^{i,j}(x,\lambda)| < \sigma_r \quad (r = 1,2,\ldots; \ x \text{ in } (c,d); \ \lambda \text{ in } R).$$

Each of the infinitude of matrices

$$(31\,\text{c}) \qquad {}_rZ(x,\lambda) \equiv ({}_rz_{i,j}(x,\lambda)) = {}_rY^{-1}(c,\lambda)\,{}_rY(x,\lambda) \qquad (r = 1, 2, \ldots)$$

constitutes a matrix solution of the system (B; § 2), associated with (A) [the ${}_rz_{i,1}(x,\lambda)$ $(i = 1,\ldots n)$ will form a set of solutions of (A)]. Since

$$(32) \qquad {}_rZ(c,\lambda) = I \qquad (r = 1, 2, \ldots)$$

it follows that the matrices ${}_rZ(x,\lambda)$ are all identical and represent the same matrix, say, $Z(x,\lambda)$. Thus

$$(33) \qquad Z(x,\lambda) = {}_rY^{-1}(c,\lambda)\,{}_rY(x,\lambda) \qquad (r = 1, 2, \ldots)$$

and $Z(x,\lambda)$ is independent of r. By (33) and by (31), (31 a),

$$(34) \qquad z_{i,j}(x,\lambda) = \sum_{s=1}^{n} e^{Q_s(x,\lambda) - Q_s(c,\lambda)}\,\mathfrak{z}_{i,j:s}(x,\lambda),$$

(34 a) $$\mathfrak{z}_{i,j:s}(x,\lambda) = \sum_{\nu=-m}^{\alpha_r-1} \sum_{\sigma=-m}^{\alpha_r-1} \bar{\sigma}_{i,s:\nu}(c)\, \sigma_{s,j:\sigma}(x)\, \lambda^{-\left(\frac{\nu+\sigma}{k}\right)} + \lambda^{-\left(\frac{\alpha_r-m}{k}\right)}\, {}_r b_{i,j:s}(x,\lambda),$$

where, in view of (31 b),

$$|{}_r b_{i,j:s}(x,\lambda)| = \Big|\sum_{\nu=-m}^{\alpha_r-1} \lambda^{-\left(\frac{\nu+m}{k}\right)} [\sigma_r^{s,j}(x,\lambda)\, \bar{\sigma}_{i,s:\nu}(c) + \bar{\sigma}_r^{i,s}(c,\lambda)\, \sigma_{s,j:\nu}(x)] + \lambda^{-\left(\frac{\alpha_r+m}{k}\right)} \bar{\sigma}_r^{i,s}(c,\lambda)\, \sigma_r^{s,j}(x,\lambda)\Big| < b_r$$

$$(x \text{ in } (c,d);\ \lambda \text{ in } R;\ r = 1, 2, \ldots;\ i, j, s = 1, \ldots n).$$

Here b_r is independent of x and λ. Whence (34 a) is seen to imply the relationship

(34 b) $$\mathfrak{z}_{i,j:s}(x,\lambda) \underset{\alpha_r-m}{\simeq} \mathfrak{z}^*_{i,j:s}(x,\lambda) = \sum_{\nu=-\mu}^{\infty} \mathfrak{z}^*_{i,j,s:\nu}(x)\, \lambda^{-\frac{\nu}{k}} = \sum_{\nu=-m}^{\infty} \sum_{\sigma=-m}^{\infty} \bar{\sigma}_{i,s:\nu}(c)\, \sigma_{s,j:\sigma}(x)\, \lambda^{-(\nu+\sigma)/k} \quad (x \text{ in } (c,d);\ \lambda \text{ in } R).$$

In the first and in the last member, above, r does not enter; moreover, α_r can be indefinitely increased. Accordingly,

(35) $$\mathfrak{z}_{i,j:s}(x,\lambda) \sim \mathfrak{z}^*_{i,j:s}(x,\lambda) \qquad (x \text{ in } (c,d);\ \lambda \text{ in } R).$$

In the sense that the elements of $Z(x,\lambda)$ are of the form (34), *while the asymptotic relations* (35) *hold for the involved functions, we may write*

(36) $$Z(x,\lambda) \sim \left(\sum_{s=1}^{n} e^{Q_s(x,\lambda)-Q_s(c,\lambda)}\, \mathfrak{z}^*_{i,j:s}(x,\lambda)\right)$$

$$(x \text{ in } (c,d);\ \lambda \text{ in } R).$$

When $x = c$ the formal matrix in the second member of (3 b) reduces to I and the symbol $\sim$ is replaced by $=$. *A similar result is obtained for the solution* $Z(x,\lambda) = {}_rY^{-1}(d,\lambda)\, {}_rY(x,\lambda)$ $(r = 1, 2, \ldots)$.

When R is specified as in Case I we have inequalities, valid for λ in R,

$$(\mathrm{H}_1)\qquad \Re\,[Q_{s,1}(x,\lambda) - Q_{s,1}(c,\lambda)] = \int_c^x \Re\, Q_{s,1}^{(1)}(u,\lambda)\,du < 0$$

$$(c < x \leqq d;\ s \geqq \tau_1 + 1),$$

$$(\mathrm{H}_2)\qquad \Re\,[Q_{s,n}(x,\lambda) - Q_{s,n}(d,\lambda)] = \int_d^x \Re\, Q_{s,n}^{(1)}(u,\lambda)\,du < 0$$

$$(c \leqq x < d;\ s \leqq \tau_{\nu-1}).$$

On the other hand, in the Case II

$$(\mathrm{H}_3)\qquad \Re\,[Q_{s,n}(x,\lambda) - Q_{s,n}(c,\lambda)] = \int_c^x \Re\, Q_{s,n}^{(1)}(u,\lambda)\,du < 0$$

$$(c < x \leqq d;\ s \leqq \tau_{\nu-1}),$$

$$(\mathrm{H}_4)\qquad \Re\,[Q_{s,1}(x,\lambda) - Q_{s,1}(d,\lambda)] = \int_d^x \Re\, Q_{s,1}^{(1)}(u,\lambda)\,du < 0$$

$$(c \leqq x < d;\ s \geqq \tau_1 + 1).$$

There may exist a subregion R_0 of R, which extends to infinity and throughout which one of the following four sets of inequalities holds, for all positive σ and for some fixed positive ε,

$$(\mathrm{G}_1)\qquad |e^{Q_{s,1}(x,\lambda) - Q_{s,1}(c,\lambda)}| < |\lambda|^{-\sigma} h(\sigma) \quad (s \geqq \tau_1 + 1;\ c + \varepsilon \leqq x \leqq d),$$

$$(\mathrm{G}_2)\qquad |e^{Q_{s,n}(x,\lambda) - Q_{s,n}(d,\lambda)}| < |\lambda|^{-\sigma} h(\sigma) \quad (s \leqq \tau_{\nu-1};\ c \leqq x \leqq d - \varepsilon),$$

$$(\mathrm{G}_3)\qquad |e^{Q_{s,n}(x,\lambda) - Q_{s,n}(c,\lambda)}| < |\lambda|^{-\sigma} h(\sigma) \quad (s \leqq \tau_{\nu-1};\ c + \varepsilon \leqq x \leqq d),$$

$$(\mathrm{G}_4)\qquad |e^{Q_{s,1}(x,\lambda) - Q_{s,1}(d,\lambda)}| < |\lambda|^{-\sigma} h(\sigma) \quad (s \geqq \tau_1 + 1;\ c \leqq x \leqq d - \varepsilon).$$

Here $h(\sigma)$ is defined and finite for every finite $\sigma (> 0)$. The inequalities (G_i) correspond to the cases (H_i) $(i = 1, \ldots 4)$, respectively. *When R_0 exists its boundaries have at infinity the limiting directions of the corresponding boundaries of R.* On the other hand, *when the limiting directions of the boundaries of R are distinct existence of a region R_0 is assured.*

Consider the solution $Z(x, \lambda)$, satisfying (36). The properties of $Z(x, \lambda)$

may be investigated further if, as will be supposed to be the case, a region R_0 satisfying (G_1) exists. We write

$$z_{i,j}(x,\lambda) = e^{Q_1(x,\lambda)-Q_1(c,\lambda)}\mathfrak{z}_{i,j}(x,\lambda), \tag{37}$$

$$\mathfrak{z}_{i,j}(x,\lambda) = \eta_{i,j}(x,\lambda) + r_{i,j}(x,\lambda), \tag{37 a}$$

$$\eta_{i,j}(x,\lambda) = \sum_{s=1}^{\tau_1}\mathfrak{z}_{i,j:s}(x,\lambda), \tag{37 b}$$

$$r_{i,j}(x,\lambda) = \sum_{s=\tau_1+1}^{n} e^{Q_{s,1}(x,\lambda)-Q_{s,1}(c,\lambda)}\mathfrak{z}_{i,j:s}(x,\lambda). \tag{37 c}$$

By (35) and (37 b)

$$\eta_{i,j}(x,\lambda) \sim \mathfrak{z}^*_{i,j}(x,\lambda) = \sum_{\nu=-\mu}^{\infty}\mathfrak{z}^*_{i,j:\nu}(x)\lambda^{-\frac{\nu}{k}} = \sum_{s=1}^{\tau_1}\mathfrak{z}^*_{i,j:s}(x,\lambda) \tag{38}$$

$$(i,j = 1,\ldots n;\ x \text{ in } (c,d);\ \lambda \text{ in } R).$$

Since, by (35),

$$|\mathfrak{z}_{i,j}(x,\lambda)| < \mathfrak{z}\,|\lambda|^{\frac{\mu}{k}}$$

$$(i,j = 1,\ldots n;\ x \text{ in } (c,d);\ \lambda \text{ in } R)$$

on taking account of (G_1) from (37 c) we obtain

$$r_{i,j}(x,\lambda) \sim 0 \tag{38 a}$$

$$(i,j = 1,\ldots n;\ c+\varepsilon \leqq x \leqq d;\ \lambda \text{ in } R_0).$$

Thus, in view of (38), (38 a) and (37 a) we have $(\mathfrak{z}_{i,j}(x,\lambda)) \sim (\mathfrak{z}^*_{i,j}(x,\lambda))$. Accordingly, we shall write

$$Z(x,\lambda) \sim (e^{Q_1(x,\lambda)-Q_1(c,\lambda)}\mathfrak{z}^*_{i,j}(x,\lambda)) \tag{39}$$

$$(c+\varepsilon \leqq x \leqq d;\ \lambda \text{ in } R_0).$$

The solution $Y_1(x,\lambda) = (\delta_{i,j}e^{Q_1(c,\lambda)})\,Z(x,\lambda)$ will have the property

$$Y_1(x,\lambda) \sim (e^{Q_1(x,\lambda)}\mathfrak{z}^*_{i,j}(x,\lambda)) \qquad (c+\varepsilon \leqq x \leqq d;\ \lambda \text{ in } R_0). \tag{40}$$

When R exists as defined in Case I and R_0 is specified by (G_2) a matrix solution $Y_2(x,\lambda)$ may be constructed so that

$$Y_2(x,\lambda) \sim (e^{Q_n(x,\lambda)}\mathfrak{z}^*_{i,j}(x,\lambda)) \qquad (c \leqq x \leqq d-\varepsilon;\ \lambda \text{ in } R_0). \tag{40 a}$$

When R is characterised as in Case II, solutions $Y_3(x,\lambda)$, $Y_4(x,\lambda)$ exist such that, depending on whether R_0 is specified by (G_3) or by (G_4), we have either

$$Y_3(x,\lambda) \sim (e^{Q_n(x,\lambda)}\, \mathfrak{z}^*_{i,j}(x,\lambda)) \quad (c+\varepsilon \leqq x \leqq d;\ \lambda \text{ in } R_0) \tag{40 b}$$

or

$$Y_4(x,\lambda) \sim (e^{Q_1(x,\lambda)}\, \mathfrak{z}^*_{i,j}(x,\lambda)) \quad (c \leqq x \leqq d-\varepsilon;\ \lambda \text{ in } R_0). \tag{40 c}$$

In (40), (40 a), (40 b), (40 c) *the formal series* $\mathfrak{z}^*_{i,j}(x,\lambda)$ *are* σ*-series.* The construction of $Y_2(x,\lambda)$, $Y_3(x,\lambda)$, $Y_4(x,\lambda)$ is along lines similar to those employed in the construction of $Y_1(x,\lambda)$.

Precisely analogous results will hold for a system (C) (§ 2).

The results obtained above will be summed in the following Theorem, stated as relating to a differential system (C) (§ 2).

Fundamental Existence Theorem. *Let* (c, d) $(c < d)$ *be an interval as specified in the italics in connection with* (2) *and* (2 a). *Assume that there exists a region* R, *as specified in Case I or in Case II (above). The following can be stated concerning solutions of the system* (C) (§ 2), *whose coefficients satisfy the stated asymptotic relations in the ordinary sense (that is, to infinitely many terms) for* x *in* (c, d) *and for* λ *in* R *(at least).*

I. *There exists an infinity of matrix solutions* ${}_rY(x,\lambda)$ $(r = 1, 2, \ldots)$ *such that*

$$ {}_rY(x,\lambda) \underset{\alpha_r}{\sim} S(x,\lambda) = (e^{Q_i(x,\lambda)}\, \sigma_{i,j}(x,\lambda)) \tag{41}$$
$$(i, j = 1, \ldots n;\ c \leqq x \leqq d;\ \lambda \text{ in } R;\ r = 1, 2, \ldots).$$

Here $\alpha_1 < \alpha_2 < \cdots$ $(\lim\limits_{r\to\infty} \alpha_r = \infty)$. *The asymptotic relations* (41) *are uniform in* x. $S(x,\lambda)$ *is a formal matrix solution of the formal differential system corresponding to* (C). *Formally* $|S(x,\lambda)| \neq 0$.

II. *The matrix solution* $Z(x,\lambda) = {}_rY^{-1}(c,\lambda)\, {}_rY(x,\lambda)$ *is independent of* r $(r = 1, 2, \ldots)$ *and it satisfies the asymptotic relation* (36) *in the ordinary sense for* x *in* (c, d) *and for* λ *in* R. *The matrix solution* ${}_rY^{-1}(d,\lambda)\, {}_rY(x,\lambda)$ *will be also independent of* r $(r = 1, 2, \ldots)$ *and it will have properties analogous to those of* $Z(x,\lambda)$.

III. *If there exists a subregion* R_0 *of* R, *as defined in the italics in connection with* (G_1), (G_2), (G_3) *and* (G_4), *we shall have at least one matrix solution which is of the form of one of the following matrices*

$$Y_1(x,\lambda),\ Y_2(x,\lambda),\ Y_3(x,\lambda),\ Y_4(x,\lambda).$$

Accordingly, this solution satisfies in the ordinary and uniform sense one of the asymptotic relationships (40), (40 a), (40 b), (40 c).

Note. The determinant of a formal matrix occurring in (40), (40 a), (40 b), (40 c) may formally be equal to zero even though the determinant of the corresponding actual matrix solution will be, of course, distinct from zero. It is also to be noted that inasmush as the coefficients of the system (or single equation of order n) are analytic in λ ($\lambda \neq \infty$) the same will be true of the solutions (when the initial conditions are analytic).

7. **Non-homogeneous Equations.** In the theorem of § 6 the elements in the first row of any particular matrix solution will constitute a fundamental set of solutions of the differential equation (A) (§ 2), provided the theorem is applied to the system (B) (§ 2), associated with (A). Consider now the non-homogeneous equation

$$L(x, \lambda; y(x, \lambda)) = a(x, \lambda) \tag{1}$$

where L is the differential operator involved in the left member of (A) *and*

$$a(x, \lambda) \sim \sum_{\nu=-\alpha}^{\infty} \alpha_\nu(x) \lambda^{-\frac{\nu}{k}} = \alpha_0(x, \lambda), \tag{1 a}$$

the asymptotic relations being valid, in the ordinary sense, for λ in R and for x in (c, d) $(c < d)$. The region R will be specified, say, as in Case I (§ 6) and it will be assumed to exist. If $y_1(x, \lambda), y_2(x, \lambda), \ldots, y_n(x, \lambda)$ denote a full set of solutions of the equation $L = 0$ such that, on writing

$$Y(x, \lambda) = (y_i^{(j-1)}(x, \lambda)), \quad Y^{-1}(x, \lambda) = (\bar{y}_{i,j}(x, \lambda)) \tag{2}$$

we have

$$Y(x, \lambda) \sim S(x, \lambda), \quad Y^{-1}(x, \lambda) \sim S^{-1}(x, \lambda), \tag{2 a}$$

where the asymptotic relations are valid in a certain sense[1] for x in (c, d) and for λ in R, then a solution of (1) can be given in the form

[1] Necessarily, in the problem now at hand, we have to deal with solutions of the homogeneous problem for which $|S(x, \lambda)|$ does not formally vanish.

(3) $$y(x, \lambda) = \sum_{r=1}^{n} y_r(x, \lambda) \int^{x} a(u, \lambda)\, \bar{y}_{n, r}(u, \lambda)\, du.$$

If the involved integrals are evaluated with the aid of Lemma 2, the asymptotic properties of the solution $y(x, \lambda)$ can be investigated.

We shall be looking for a solution whose asymptotic properties are in the ordinary sense. *Assume that in R not only inequalities of Case I (§ 6) hold but that also*

(4) $$0 > \Re Q_1^{(1)}(x, \lambda) \qquad (c \leqq x \leqq d;\ \lambda \text{ in } R).$$

It follows than that

(5) $$\left| e^{Q_1(x, \lambda) - Q_1(c, \lambda)} \right| < 1, \quad \left| e^{Q_{\tau_1+1, 1}(x, \lambda) - Q_{\tau_1+1, 1}(c, \lambda)} \right| < 1, \quad \left| e^{Q_{n, \tau_{\nu}-1}(x, \lambda) - Q_{n, \tau_{\nu}-1}(c, \lambda)} \right| < 1$$

$$(c + \varepsilon \leqq x \leqq d;\ \lambda \text{ in } R;\ \varepsilon > 0).$$

Suppose there exists an infinite subregion R_1 of R such that

(5 a) $$\left| e^{Q_1(x, \lambda) - Q_1(c, \lambda)} \right| \sim 0, \quad \left| e^{Q_{\tau_1+1, 1}(x, \lambda) - Q_{\tau_1+1, 1}(c, \lambda)} \right| \sim 0, \quad \left| e^{Q_{n, \tau_{\nu}-1}(x, \lambda) - Q_{n, \tau_{\nu}-1}(c, \lambda)} \right| \sim 0$$

$$(c + \varepsilon \leqq x \leqq d;\ \lambda \text{ in } R_1).$$

Consider a matrix solution of the homogeneous problem,

(6) $$Y(x, \lambda) = (y_i^{(j-1)}(x, c, \lambda)) = {}_rY^{-1}(c, \lambda)\, {}_rY(x, \lambda) \qquad (r = 1, 2, \ldots),$$

as given by (31 c; § 6). Since

$$Y^{-1}(x, \lambda) = (\bar{y}_{i, j}(x, c, \lambda)) = {}_rY^{-1}(x, \lambda)\, {}_rY(c, \lambda)$$

it is observed at once that

(6 a) $$(\bar{y}_{i, j}(x, c, \lambda)) = y_i^{(j-1)}(c, x, \lambda));$$

that is, the elements of $Y^{-1}(x, \lambda)$ are obtained from those of $Y(x, \lambda)$ by interchanging x and c. In view of (36; § 6), we accordingly have

(7) $$Y(x, \lambda) \sim \left(\sum_{s=1}^{n} e^{Q_s(x, \lambda) - Q_s(c, \lambda)}\, \mathfrak{z}^{*}_{i, j: s}(x, c, \lambda) \right) = S(x, \lambda),$$

(7 a) $$(\bar{y}_{i,j}(x, c, \lambda)) \sim \left(\sum_{s=1}^{n} e^{Q_s(c,\lambda) - Q_s(x,\lambda)} \mathfrak{z}^*_{i,j:s}(c, x, \lambda) \right) = S^{-1}(x, \lambda)$$

$(x$ in (c, d); λ in R; $\mathfrak{z}^*_{i,j:s}(x, c, \lambda)$, $\mathfrak{z}^*_{i,j:s}(c, x, \lambda)$ σ-series).

By (3), (7 a) and by Lemma 2 it follows that the integral displayed in (3) can be evaluated so that

(8) $$\int^{x} a(u, \lambda)\, \bar{y}_{n,r}(u, \lambda) \sim \sum_{\sigma=1}^{n} e^{Q_\sigma(c,\lambda) - Q_\sigma(x,\lambda)} \eta_{n,r:\sigma}(c, x, \lambda)$$

$(r = 1, \ldots n;\ c + \varepsilon \leqq x \leqq d;\ \lambda$ in R_1; $\eta_{n,r:\sigma}(c, x, \lambda)$ σ-series).

Now, (8), (7) and (3) will imply that

(9) $$y(x, \lambda) \sim \sum_{s=1}^{n} \sum_{\sigma=1}^{n} e^{G_{s,\sigma}} \eta_{s,\sigma}(x, \lambda) \qquad (c + \varepsilon \leqq x \leqq d;\ \lambda \text{ in } R_1).$$

Here

(9 a) $$G_{s,\sigma} = Q_{s,\sigma}(x, \lambda) - Q_{s,\sigma}(c, \lambda)$$

and $\eta_{s,\sigma}(x, \lambda)$ is a σ-series defined by

(9 b) $$\eta_{s,\sigma}(x, \lambda) = \sum_{r=1}^{n} \mathfrak{z}^*_{r,1:s}(x, c, \lambda)\, \eta_{n,r:s}(c, x, \lambda).$$

In view of (5 a) it is concluded that

(10) $$y(x, \lambda) \sim e^{Q_{1,n}(x,\lambda) - Q_{1,n}(c,\lambda)} \eta(x, \lambda) \qquad (c + \varepsilon \leqq x \leqq d;\ \lambda \text{ in } R_1)$$

where $\eta(x, \lambda)$,

(10 a) $$\eta(x, \lambda) = \sum_{s=1}^{\tau_1} \sum_{\sigma=\tau_{\nu-1}+1}^{n} \eta_{s,\sigma}(x, \lambda),$$

is a σ-series. This is a consequence of the fact that, in view of (5 a) and in view of the inequalities of Case I (§ 6), we have

$$\left| e^{G_{s,\sigma} - Q_{1,n}(x,\lambda) + Q_{1,n}(c,\lambda)} \right| \sim 0 \qquad (c + \varepsilon \leqq x \leqq d;\ \lambda \text{ in } R_1).$$

Theorem I. *Consider the non-homogeneous equation* (1). *Assume that there exists a region R, as specified in the italics preceding* (4), *and assume that there* *exists a subregion* R_1 *of* R *so that* (5 a) *holds. The equation* (1) *will then possess a solution* $y(x, \lambda)$ *satisfying to infinitely many terms the asymptotic relation* (10) (*where* $\varepsilon > 0$).

An analogous theorem can be stated when R is defined as in Case II (§ 6).

8. **Integro-differential Equations.** We shall now apply the Fundamental Existence Theorem for the solution of the integro-differential equation

$$(1)\qquad L(x, \lambda; y(x, \lambda)) = a(x, \lambda) + \int_c^x b(u, x, \lambda)\, y(u, \lambda)\, du \,(\equiv V(x, \lambda; y)).$$

Here $a(x, \lambda) \not\equiv 0$ *and* L *is the differential operator of the left member of* (A) (§ 2) *and*

$$(1\text{ a})\qquad a(x, \lambda) \sim \sum_{\nu=-\alpha}^{\infty} \alpha_\nu(x)\, \lambda^{-\frac{\nu}{k}} = \alpha_0(x, \lambda),$$

$$(1\text{ b})\qquad b(u, x, \lambda) \sim \sum_{\nu=-\beta}^{\infty} \beta_\nu(u, x)\, \lambda^{-\frac{\nu}{k}} = \beta(u, x, \lambda).$$

The asymptotic relations are assumed valid for λ *in* R (a region specified, say, as in Case I (§ 6)) *and for* x *and* u *in the interval* (c, d). The latter interval will be assumed to be the one for which, according to the Fundamental Existence Theorem, the asymptotic properties of solutions of (A) are stated. Moreover, it will be assumed that the asymptotic relations (1 a) and (1 b) are uniform in u and x $(c \leqq u, x \leqq d)$.[1]

Equation (A) will possess a set of solutions $y_i(x, \lambda)$ $(i = 1, \ldots n)$,

$$(2)\qquad (y_i^{(j-1)}(x, \lambda)) \underset{w}{\sim} (e^{Q_i(x, \lambda)}\, \sigma_{i,j}(x, \lambda)),$$

$$(2\text{ a})\qquad (y_i^{(j-1)}(x, \lambda))^{-1} = (\bar{y}_{i,j}(x, \lambda)) \underset{w}{\sim} (e^{-Q_j(x, \lambda)}\, \bar{\sigma}_{i,j}(x, \lambda))$$
$$(i, j = 1, \ldots n;\ x \text{ in } (c, d);\ \lambda \text{ in } R);$$

here the $\sigma_{i,j}(x, \lambda)$ and the $\bar{\sigma}_{i,j}(x, \lambda)$ are σ-series. By (1 a), (1 b), (2) and by (2 a)

$$(3)\qquad |a(x, \lambda)| \leqq |\lambda|^{\frac{\alpha}{k}} a,\quad |b(u, x, \lambda)| \leqq |\lambda|^{\frac{\beta}{k}} b,$$

[1] Extension of the notion of uniformity of an asymptotic relationship (cf. § 1) from one variable to several is quite obvious.

$$(3\,a)\qquad |y_i^{(j-1)}(x,\lambda)| < |e^{Q_i(x,\lambda)}|\,y,\quad |\bar{y}_{i,j}(x,\lambda)| < |e^{-Q_j(x,\lambda)}|\,|\lambda|^{\frac{\eta}{k}}y$$
$$(i, j = 1, \ldots n;\ x \text{ and } u \text{ in } (c, d);\ \lambda \text{ in } R).$$

On writing

$$(4)\qquad F(z) \equiv \sum_{\tau=1}^{n} y_\tau(x) \int_c^x z(u,\lambda)\,\bar{y}_{n,\tau}(u,\lambda)\,du,$$

in view of (3 a) it follows that, for x in (c, d) and λ in R,

$$(5)\qquad |F(z)| \leqq y^2 |\lambda|^{\frac{\eta}{k}} \sum_{\tau=1}^{n} \int_c^x |e^{Q_\tau(x,\lambda)-Q_\tau(u,\lambda)}|\,|z(u,\lambda)|\,du.$$

Suppose there exists *a region* R_1, *extending to infinity and forming part of* R, *such that*

$$(6)\qquad \Re\, Q_1^{(1)}(x,\lambda) \leqq 0 \qquad (x \text{ in } (c, d);\ \lambda \text{ in } R_1).$$

Since R is defined as in Case I (§ 6) the condition (6) implies

$$(6\,a)\qquad \Re\, Q_\tau^{(1)}(x,\lambda) \leqq 0 \quad (x \text{ in } (c, d);\ \lambda \text{ in } R_1;\ \tau = 1, \ldots n).$$

Thus

$$(6\,b)\qquad \Re\,[Q_\tau(x,\lambda) - Q_\tau(u,\lambda)] \leqq 0$$
$$(\tau = 1, \ldots n;\ c \leqq u \leqq x \leqq d;\ \lambda \text{ in } R_1).$$

If, for x in (c, d) and λ in R_1,

$$(7)\qquad |z(x,\lambda)| \leqq |\lambda|^{\delta/k} z\,.\,(x-c)^r \qquad (r \geqq 0)$$

then, in view of (6 b), from (5) it will follow that

$$(8)\qquad |F(z)| \leqq nzy^2 |\lambda|^{(\eta+\delta)/k}(x-c)^{r+1}/(r+1)$$
$$(x \text{ in } (c, d);\ \lambda \text{ in } R_1).$$

On the other hand, by (3), it will follow that (7) implies

$$(9)\qquad \left|\int_c^x b(u,x,\lambda)\,z(u,\lambda)\,du\right| \leqq bz\,|\lambda|^{(\beta+\delta)/k}(x-c)^{r+1}/(r+1)$$
$$(x \text{ in } (c, d);\ \lambda \text{ in } R_1).$$

Functions $z_i(x, \lambda)$, $c_i(x, \lambda)$ $(i = 0, 1, \ldots)$ will be defined in succession by the relations

$$L(x, \lambda; z_0) = a(x, \lambda) \equiv c_0(x, \lambda), \tag{10}$$

$$L(x, \lambda; z_i) = c_i(x, \lambda) \equiv \int_c^x b(u, x, \lambda) z_{i-1}(u, \lambda)\, du. \tag{10 a}$$

By (7; § 2) and in view of (10) and (10 a), on using notation (4) we may write

$$z_i(x, \lambda) = F(c_i) \qquad (i = 0, 1, \ldots). \tag{11}$$

Unless stated otherwise the following inequalities will be for x in (c, d) and for λ in R_1. From (10) it follows that

$$|c_0(x, \lambda)| \leqq a_0 |\lambda|^{\alpha_0/k} \qquad (a_0 = a;\ \alpha_0 = \alpha). \tag{12}$$

By virtue of (11; $i = 1$) and since (7) implies (8),

$$|z_0(x, \lambda)| \leqq z_0 |\lambda|^{\mathfrak{z}_0/k} (x - c) \qquad (z_0 = n y^2 a_0;\ \mathfrak{z}_0 = \eta + \alpha). \tag{12 a}$$

Thus, by (10; $i = 1$) and since (7) implies (9),

$$|c_1(x, \lambda)| \leqq a_1 |\lambda|^{\alpha_1/k} (x - c)^2 \quad \left(a_1 = \frac{b z_0}{2};\ \alpha_1 = \beta + \mathfrak{z}_0\right).$$

Continuing, the following inequalities are obtained

$$|z_1(x, \lambda)| \leqq z_1 |\lambda|^{\mathfrak{z}_1/k} (x - c)^3 \quad \left(z_1 = \frac{n y^2 a_1}{3};\ \mathfrak{z}_1 = \eta + \alpha_1\right),$$

$$|c_2(x, \lambda)| \leqq a_2 |\lambda|^{\alpha_2/k} (x - c)^4 \quad \left(a_2 = \frac{b z_1}{4};\ \alpha_2 = \beta + \mathfrak{z}_1\right),$$

$$\ldots \qquad \ldots \qquad \ldots \qquad \ldots.$$

By induction it can be shown that

$$|c_i(x, \lambda)| \leqq a_i |\lambda|^{\alpha_i/k} (x - c)^{2i}, \tag{13}$$

$$|z_i(x, \lambda)| \leqq z_i |\lambda|^{\mathfrak{z}_i/k} (x - c)^{2i+1} \qquad (i = 0, 1, \ldots), \tag{13 a}$$

where

$$(14)\qquad a_i = \frac{b z_{i-1}}{2i}, \qquad z_i = n y^2 \frac{a}{2i+1},$$

$$(14\text{a})\qquad \mathfrak{z}_i = \eta + \alpha_i, \quad \alpha_i = \beta + \mathfrak{z}_{i-1} \qquad (i = 1, 2, \ldots).$$

Now from (14) and (14a)

$$(15)\qquad \frac{z_i}{z_{i-1}} = \frac{n y^2 b}{(2i)(2i+1)}, \quad \mathfrak{z}_i - \mathfrak{z}_{i-1} = \eta + \beta \qquad (i = 1, 2, \ldots).$$

Thus

$$(16)\qquad z_i = \frac{z_0 h^{2i}}{i!\, 3 \cdot 5 \ldots (2i+1)} \qquad \left(h = \frac{n y^2 b}{2}\right),$$

$$(16\text{a})\qquad \mathfrak{z}_i = \mathfrak{z}_0 + i(\eta + \beta) \qquad (i = 1, 2, \ldots).$$

The series

$$(17)\qquad y(x, \lambda) = \sum_{i=0}^{\infty} z_i(x, \lambda)$$

is absolutely and uniformly convergent for x in (c, d) and for λ in any finite part of R_1. In fact, by (13a), (16) and (16a), for these values of the variables we shall have

$$(17\text{a})\qquad |y(x, \lambda)| \leqq \sum_{i=0}^{\infty} |z_i(x, \lambda)| \leqq (x-c) z_0 |\lambda|^{\frac{\eta+\alpha}{k}} f\left((x-c)|\lambda|^{\frac{\eta+\beta}{2}}\right),$$

where $f(u)$ is the function, entire in u, defined by

$$(17\text{b})\qquad f(u) = 1 + \sum_{i=1}^{\infty} \frac{h^i u^{2i}}{i!\, 3 \cdot 5 \ldots (2i+1)}.$$

In view of (10) and (10a), the series (17) represents a function, defined for x in (c, d) and for λ in R_1 and satisfying (1). Such a solution will be unique in every case when zero is the only solution of the equation

$$L(x, \lambda; y(x, \lambda)) = \int_c^x b(u, x, \lambda)\, y(u, \lambda)\, du.$$

Theorem II. *Consider the integro-differential equation* (1), *specified in the italics following* (1). *Assume that there exists a region* R_1, *as defined by* (6). *The* *equation will then possess a solution* $y(x, \lambda)$, *defined for* x *in* (c, d) *and for* λ *in* R_1 *and satisfying for these values of the variables* (17 a), (17 b). *Moreover, this solution will be analytic in* λ *for* λ *in* R_1 ($\lambda \neq \infty$; x in (c, d)), *provided the involved coefficients have the same property.*

An analogous result can be stated when R is defined as in Case II (§ 6) and also when d is made to play the role of c.

9. **Concerning Boundary Value Problems.** Let $L(x, \lambda; y)$ be the differential polynomial involved in the left member of (A; § 2). Of various possible formulations of Boundary Problems, associated with the operator L, of special significance is the following.

To determine a function $y(x, \lambda)$ *which satisfies*

$$L(x, \lambda; y(x, \lambda)) = f(x) \tag{1}$$

and the boundary conditions

$$M_i(y) \equiv \sum_{k=1}^{n} \int_c^d y^{(k-1)}(t, \lambda)\, d\alpha_{i,k}(t) = 0 \qquad (i = 1, \ldots n). \tag{1 a}$$

Here the operators M_i are linearly independent, the involved integrals are in the sense of Stieltjes and the $\alpha_{i,k}(x)$ are functions of bounded variation. When $f(x) = 0$ the problem is termed homogeneous; otherwise it is called non-homogeneous. An extensive treatment of the problem (1), (1 a) has been given by Tamarkin[1], who presents developments under the assumption that the roots of the characteristic equation of (A) are distinct and that in (A) $H = 1$ (there are also some other hypotheses). In the case when H is allowed to exceed unity and the roots of the characteristic equation of (A) are not required to be distinct, development of an adequate Boundary Value Theory (leading to expansions of arbitrary functions) necessitates some restrictions on the nature of the polynomials $Q_i(x, \lambda)$ $(i = 1, 2, \ldots n)$. Thus, for instance, we would have to assume that the various regions, for which (A) has solutions of known asymptotic form (as implied by the Fundamental Existence Theorem), abut on each other. A requirement of this type would mean that the functions

[1] J. D. Tamarkin, Math. Zeit., *loc. cit.*, pp. 1—54. Concerning the possibility of the homogeneous and non-homogeneous problem and concerning the Green's function, enabling representation of the solution of the non-homogeneous problem (1), (1 a) in particular see Tamarkin, *loc. cit.*, pp. 5—10. It will be assumed that the reader is acquainted with these facts.

$$\Re\left[Q_i^{(1)}(x, \lambda) - Q_j^{(1)}(x, \lambda)\right] \qquad (i \neq j;\ i, j = 1, \ldots n) \tag{2}$$

would have to be all independent of x at least after a function of x (only) has been divided out. However, it is not the aim of the present paper to present developments based on such a hypothesis.

Of particular importance is the special problem (1), (1 a)

$$L(x, \lambda; y(x, \lambda)) = f(x), \tag{3}$$

$$M_j(y) \equiv C_j(y) + D_j(y) = 0, \tag{3 a}$$

$$C_j(y) = \sum_{k=1}^{n} y^{(k-1)}(c, \lambda)\, c_{k,j}, \quad D_j(y) = \sum_{k=1}^{n} y^{(k-1)}(d, \lambda)\, d_{k,j} \tag{3 b}$$

$$[c_{i,j},\ d_{i,j} \text{ constants; } |c_{i,j}|, |d_{i,j}| \neq 0;\ i, j = 1, \ldots n].$$ [1]

We shall be concerned merely with the following Problem.

For what values of λ is the non-homogeneous problem (3), (3 a), (3 b) *possible, when L is the unrestricted operator of* (A) (§ 2) *and* (c, d) $(c < d)$ *is the interval for which the Fundamental Existence Theorem had been stated?*

The stated problem is possible for those and only those values λ for which the determinant

$$\Delta(\lambda) \equiv |(M_j(y_i))| \qquad (i, j = 1, \ldots n) \tag{4}$$

does not vanish. Here $y_1(x, \lambda), \ldots, y_n(x, \lambda)$ denotes a set of n distinct solutions of (A). The values for which $\Delta(\lambda) = 0$ are called characteristic values.

Let the elements $y_i(x, \lambda)$ $(i = 1, \ldots n)$ be those in the first row of the matrix $Y(x, \lambda) = {}_rY^{-1}(c, \lambda)\, {}_rY(x, \lambda)$, referred to in Part II of the Fundamental Existence Theorem, as applied to the system B (§ 2) (associated with the equation (A)). On writing

$$C = (c_{i,j}), \quad D = (d_{i,j}), \quad Y(x, \lambda) = (y_{i,j}(c, x, \lambda)) = (y_i^{(j-1)}(c, x, \lambda)) \tag{5}$$

it follows that

$$\Delta(\lambda) \equiv |Y(c, \lambda)\, C + Y(d, \lambda)\, D| \equiv |C + Y(d, \lambda)\, D|. \tag{6}$$

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[1] Cf. Tamarkin, *loc. cit.*, p. 10. Also cf. Birkhoff, *loc. cit.*, and Birkhoff and Langer, *loc. cit.* There are, of course developments due to a number of other writers. All these developments are for a restricted operator L.

By taking the interval (c, d) sufficiently small it is observed that the neighborhood of $\lambda = \infty$ is divided in a finite number of regions (each bounded by regular curves extending to infinity and nowhere intersecting for $|\lambda| > \lambda_0 > 0$, where λ_0 is sufficiently great),

$$(7) \qquad R_1,\, R_{1,2},\, R_2,\, R_{2,3},\, R_3,\, \ldots,\, R_{N-1,N},\, R_N,\, R_{N,N+1}\ (\text{or } R_{N,1})^1,$$

such that

1°. *Interior any particular region R_ν $(1 \leqq \nu \leqq n)$ there exists no curve along which, for some i and some j $(i \neq j)$ and for some x in (c, d),*

$$(8) \qquad \Re\, Q_i^{(1)}(x, \lambda) = \Re\, Q_j^{(1)}(x, \lambda),$$

unless $\Re\, Q_i^{(1)}(x, \lambda) \equiv \Re\, Q_j^{(1)}(x, \lambda)$.

2°. *Every region $R_{\nu,\nu+1}$ $(1 \leqq \nu \leqq N)$ can be included in a sector whose angle is as small as desired.*

The last part of the above statement is a consequence of the fact that every function $Q_i(x, \lambda) - Q_j(x, \lambda)$, which is not identically zero, is of the form (2; § 6), (2 a; § 6). In fact, as seen from (2; § 6), we have for every function of this type

$$(9) \qquad \Re\, Q_{i,j}^{(1)}(x, \lambda) = |q_{i,j:m}^{(1)}(x)|\, |\lambda|^{\frac{kH-m}{k}} \left[\cos\left(\varphi_{i,j}(x) + \frac{kH-m}{k}\theta\right) + h_{i,j}(x, \lambda)\right]$$

where, for $\lambda \to \infty$, $h_{i,j}(x, \lambda) \to 0$ (uniformly in x for x in (c, d)). Here $\varphi_{i,j}(x)$ is the angle of $q_{i,j:m}^{(1)}(x)$ and θ is the angle of λ. *Let us define the $B_{i,j:x}^1$ curve as a branch, extending in the λ-plane to infinity and satisfying* (8). Such a curve is defined for every x in (c, d). Moreover, in view of (9) it follows that any particular curve $B_{i,j:x}^1$ has a limiting direction $\theta = \theta_{i,j}(x)$, where θ satisfies the equation

$$(9\text{a}) \qquad \cos\left(\varphi_{i,j}(x) + \frac{kH-m}{k}\theta\right) = 0.$$

Since the angle of $q_{i,j:m}^{(1)}(x)$ is continuous in consequence of previously made hypotheses, the variation of $\theta_{i,j}(x)$ can be made as small as desired by choosing the interval (c, d) sufficiently small. This, however, implies 2°. From the relation (9) it also follows that, if for a fixed i and j the angle of $q_{i,j}^{(1)}(x)$ is in-

[1] Here $R_{N,N+1}$ is adjacent on one side with R_N and on the other with R_1.

dependent of x, all the corresponding curves $B^1_{i,j:x}$, formed by varying x from c to d, will have limiting directions independent of x. Thus

3°. *If all the functions* $q^{(1)}_{i,j}(x)$, *involved in* (2 a; § 6), *have angles independent of* x *then every region* $R_{\nu,\nu+1}$ $(1 \leqq \nu \leqq N)$ *can be included in a curvilinear sector bounded by two curves with the same limiting direction at infinity.*

In every region R_ν the real parts of the $Q^{(1)}_i(x, \lambda)$ $(i = 1, \ldots n)$ can be ordered for x in (c, d), for instance, as in Case I. (§ 6). According to Part II of the Fundamental Existence Theorem, associated with R_ν there will be a matrix solution ${}^\nu Y(x, \lambda)$, such that

$$(10) \qquad {}^\nu Y(x, \lambda) \equiv ({}^\nu y_{i,j}(c, x, \lambda)) \sim \left(\sum_{s=1}^{n} e^{Q_s(x,\lambda) - Q_s(c,\lambda)} \mathfrak{z}^*_{i,j:s}(c, x, \lambda) \right) = \Gamma(c, x, \lambda)$$

$$(x \text{ in } (c, d);\ \lambda \text{ in } R_\nu;\ \text{cf. } (36;\ \S\ 6)).$$

The formal matrix in the last member of (10) is independent of ν. In general, of course, inequalities of Case I (§ 6) will be for a set of subscripts distinct from that displayed. For a given ν the ${}^\nu y(c, x, \lambda)$ possess analytic continuations in the complete vicinity $(|\lambda| \geqq \lambda_0 > 0)$ of $\lambda = \infty$; this being true for every x in (c, d).[1]

Since

$${}^\nu Y(c, \lambda) = I \quad (\nu = 1, 2, \ldots N)$$

it will follow that

$${}^1 Y(x, \lambda) = {}^2 Y(x, \lambda) = \cdots = {}^N Y(x, \lambda) = Y(x, \lambda).$$

In other words, *the matrix* $Y(x, \lambda)$ *involved in* (6) *has the asymptotic property* (10) *for* x *in* (c, d) *and for* λ *in* R_ν $(\nu = 1, \ldots N)$.

There will be no loss of generality to assume, as we shall, that

$$(11) \qquad \mathfrak{z}^*_{i,j:s}(c, d, \lambda) = \sum_{\nu=0}^{\infty} \mathfrak{z}^*_{i,j,s:\nu}(c, d)\, \lambda^{-\frac{\nu}{k}},$$

where

$$(11\text{ a}) \qquad \mathfrak{z}^*_{i,j,s:0}(c, d) = \bar{\sigma}_{i,s:0}(c)\, \sigma_{s,j:0}(d)$$

[1] We assume now that the coefficients in (A) are analytic in the complete vicinity of $\lambda = \infty$ $(\lambda \neq \infty;\ |\lambda| \geqq \lambda_0 > 0)$.

(cf. (30; § 6) and (34 b; § 6)).[1] It will be convenient to introduce the following definition.

Definition. *Consider the regions* R_ν $(\nu = 1, \ldots N)$ *of* (7). *Let the totality (if any) of identically vanishing functions* $Q_s(d, \lambda) - Q_s(c, \lambda)$ *be denoted by*

$$Q_{s_w}(d, \lambda) - Q_{s_w}(c, \lambda) \qquad (w = 1, \ldots p). \tag{12}$$

Any particular region R_1 *may contain a finite set of non-overlapping closed subregions,*

$$R^1_{\nu, i} \qquad (i = 1, \ldots N_\nu), \tag{12 a}$$

each extending to infinity and such that

$$\lim_{\lambda} \left| e^{Q_s(d, \lambda) - Q_s(c, \lambda)} \right| = 0 \tag{12 b}$$

for every $s \neq s_w$ $(w = 1, \ldots \varrho)$. *Here the limit is taken when* $\lambda \to \infty$ *in a region* $R'_{\nu, i}$ $(i = 1, \ldots N_\nu)$. *We shall define* R' *as a particular set of regions* $R'_{\nu, i}$ $(i = 1, \ldots N_\nu;\ \nu = 1, \ldots N)$. *Replacing* (12 b) *by the condition*

$$\lim_{\lambda} \left| e^{Q_s(c, \lambda) - Q_s(d, \lambda)} \right| = 0 \qquad [s \neq s_w\,(w = 1, \ldots \varrho)] \tag{12 c}$$

we similarly define a totality of regions R''.

The functions displayed in (12 b) and (12 c) will be assumed to approach the limit uniformly.

Define a matrix

$$M^{c, d} = \left(c_{i, j} + \sum_{w=1}^{\varrho} \sum_{k=1}^{n} \mathfrak{z}^*_{i, k, s_w : 0}(c, d)\, d_{k, j} \right) \qquad (i, j = 1, \ldots n). \tag{13}$$

The matrix obtained by interchanging c, $c_{i,j}$ $(i, j = 1, \ldots n)$ with d, $d_{i,j}$ $(i, j = 1, \ldots n)$ will be denoted as $M^{d, c}$.

Since

$$Y^{-1}(x, \lambda) \equiv (y_{i,j}(x, c, \lambda)) \sim \Gamma(x, c, \lambda)$$
$$(x \text{ in } (c, d);\ \lambda \text{ in } R_\nu;\ \nu = 1, \ldots n),$$

in view of (6) and (10) it will follow that

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[1] Referring to § 6, we have taken $m = \mu = 0$ and $|(\sigma_{i, j:0}(x))| \neq 0$ $(c \leqq x \leqq d)$.

(14) $$\Delta(\lambda) \sim | C + \Gamma(c, d, \lambda) D |,$$

(14 a) $$| Y^{-1}(d, \lambda) | \Delta(\lambda) \sim | \Gamma(d, c, \lambda) C + D |$$
$$(\lambda \text{ in } R_\nu;\ \nu = 1, \ldots n).$$[1]

We have

$$\Delta(\lambda) \sim \left| \left\{ c_{i,j} + \sum_{s=1}^{n} e^{Q_s(d,\lambda) - Q_s(c,\lambda)} \sum_{r=1}^{n} \mathfrak{z}^*_{i,r:s}(c, d, \lambda)\, d_{r,j} \right\} \right|$$
$$= | \{ M + q^*_{i,j}(c, d, \lambda) \} | \qquad (\lambda \text{ in } R_\nu;\ \nu = 1, \ldots n),$$

where

$$q^*_{i,j}(c, d, \lambda) = \sum_{w=1}^{\varrho} \sum_{r=1}^{n} \sum_{\nu=1}^{\infty} \mathfrak{z}^*_{i,r,s_w:\nu}(c, d)\, \lambda^{-\frac{\nu}{k}} + \sum_{s \neq s_w}^{n} e^{Q_s(d,\lambda) - Q_s(c,\lambda)} \sum_{r=1}^{n} \mathfrak{z}^*_{i,r:s}(c, d, \lambda)\, d_{r,j}.$$

Thus, on writing

$$\Delta(\lambda) = | \{ M^{c,d} + q_{i,j}(c, d, \lambda) \} |,$$

in view of (12 b) it follows that (uniformly)

$$\lim_{\lambda \to \infty} q_{i,j}(c, d, \lambda) = 0 \qquad (i, j = 1, \ldots;\ \lambda \text{ in } R').$$

Accordingly, one may write

(15) $$\Delta(\lambda) = | M^{c,d} | + \xi(c, d, \lambda),$$

where

(15 a) $$\lim_{\lambda \to \infty} \xi(c, d, \lambda) = 0 \qquad (\lambda \text{ in } R').$$

Similarly, with the aid of (12 c) it may be demonstrated that

(16) $$| Y^{-1}(d, \lambda) | \Delta(\lambda) = | M^{d,c} | + \xi_1(d, c, \lambda),$$

where

(16 a) $$\lim_{\lambda \to \infty} \xi_1(d, c, \lambda) = 0 \qquad (\lambda \text{ in } R'').$$

The following theorem can be now stated.

[1] (14), for instance, means that $\Delta(\lambda)$ can be obtained by replacing the formal σ-series, involved in the second member, by certain functions asymptotic, as stated, to these series.

Theorem III. *Consider the non-homogeneous boundary problem* (3), (3 a), (3 b). *Let* (c, d) $(c < d)$ *be a suitable interval. Let a particular set of regions* R' *and a particular set of regions* R'' *be specified as in the Definition above. Let matrices* $M^{c,d}$, $M^{d,c}$ *be defined by* (13), (11 a). *Suppose the determinants* $|M^{c,d}|$, $|M^{d,c}|$ *are distinct from zero. The non-homogeneous problem will then be possible for every* λ *in* R' *and for every* λ *in* R'', *provided* $|\lambda| \geqq \lambda_0 > 0$. *Here* λ_0 *is a fixed number, depending on the choice of* R' *and* R''.

The determinants $|M^{c,d}|$, $|M^{d,c}|$ will certainly be distinct from zero when there exist no identically vanishing functions (12).