LINEAR HYPERBOLIC PARTIAL DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS.

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Contents.

Introduction.

Let $C(\infty)$ be the class of complex functions $f(x)$ of n real variables $x_1, \ldots, x_n$ which are defined and infinitely differentiable for all x. Let $q(\zeta) = q(\zeta_1, \ldots, \zeta_n)$ be a polynomial in $\zeta_1, \ldots, \zeta_n$ with complex coefficients and let $A(q)$ be the class of all functions f in $C(\infty)$ which satisfy

$$q(\partial/\partial x) f(x) = 0 \tag{1}$$

for all x.

Every polynomial q can be written in the form $p+r$ where p is homogeneous and, if q is not a constant, the degree of p is greater than the degree of r. If q is a constant we put $p = q$. We call the polynomial p thus defined the principal part of q.

Let $\xi = (\xi_1, \ldots, \xi_n) \neq 0$ be a real vector. We say that q is hyperbolic with respect to ξ if $p(\xi) \neq 0$ and if there exists a real number t_0 such that $q(t\xi + i\eta) \neq 0$ when $t > t_0$ and η is any real vector. If q is hyperbolic with respect to ξ, it is clearly hyperbolic with respect to any positive multiple of ξ and we will show that it is hyperbolic also with respect to any negative multiple of ξ. We say that (1) is a hyperbolic differential equation if q is hyperbolic with respect to at least one ξ.

Let $f_1, f_2, \ldots, f_k, \ldots$ be a sequence of elements in $C(\infty)$. If f_k and every derivative of f_k tends to zero with $1/k$ uniformly on every compact[1] set in the plane $(y, \xi) = y_1\xi_1 + \cdots + y_n\xi_n = 0$ or in the entire space we say that f_k tends to zero in the plane $(y, \xi) = 0$ or in the entire space and write

$$f_k \to 0\,(\xi) \tag{a}$$

and

$$f_k \to 0 \tag{b}$$

respectively. It is clear that (b) implies (a) but the converse is not true. In Chapter 1 the following theorem is proved[2]

[1] A set S whose elements are real vectors $x = (x_1, \ldots, x_n)$ is called bounded if $|x| = \max_k |x_k|$ is bounded when x is in S, and closed, if it together with the elements of a sequence $x^{(k)}$ also contains every x such that $\underline{\lim}\, |x^{(k)} - x| = 0$. It is compact if it is both bounded and closed.

[2] The theorems I and III were announced in GÅRDING [3], an outline of the construction of the Riesz kernel and the solution of Cauchy's problem in GÅRDING [2]. See, however, the first footnote to Chapter 5.

Theorem I. *Let* $f_k \in A(q)$, $(k = 1, 2, \ldots)$. *If there exists a point* x *such that* $(x, \xi) \neq 0$ *and* $f_k(x)$ *tends to zero with* $1/k$ *whenever* f_k *tends strongly to zero in the plane* $(y, \xi) = 0$, *then* q *is hyperbolic with respect to* ξ.

Put $f(\zeta, y) = e^{(\zeta, y-x)}$ where ζ is a complex vector such that $q(\zeta) = 0$. Then $f(\zeta, \cdot)$ is in $A(q)$, it equals 1 at the point x, and the proof, whose origin was a remark by Hadamard ([4] p. 40), uses the fact that if q is not hyperbolic with respect to ξ, then we can always find a sequence of vectors $\zeta^{(1)}, \ldots, \zeta^{(k)}, \ldots$ such that $q(\zeta^{(k)}) = 0$ for all k and $f(\zeta^{(k)}, \cdot) \to 0\,(\xi)$.[1]

The main object of the rest of the paper is the following theorem which is a strong converse of Theorem I.

Theorem II. *Let* $f_k \in A(q)$, $(k = 1, 2, \ldots)$ *and let* q *be hyperbolic with respect to* ξ. *Then if* f_k *tends strongly to zero in the plane* $(y, \xi) = 0$ *it tends strongly to zero in the entire space.*

Combining the two theorems we have the following concise theorem.

Theorem III. *Let* $f_k \in A(q)$, $(k = 1, 2, \ldots)$. *Then a necessary and sufficient condition that* (a) *implies* (b) *is that* q *is hyperbolic with respect to* ξ.

The simplest not trivial hyperbolic equation is the wave equation in two variables which corresponds to the case $n = 2$ and $q = \zeta_1^2 - \zeta_2^2$. Then q is hyperbolic with respect to $\xi = (1, 0)$. In fact, $p(\xi) = q(\xi) = 1 \neq 0$ and $q(t\xi + i\eta) = (t + i(\eta_1 + \eta_2))(t + i(\eta_1 - \eta_2)) \neq 0$ when $t > 0$ (or $t < 0$). Also if $f \in A(q)$ one has the elementary formula

$$(2) \qquad f(x) = \tfrac{1}{2}(f(0, x_2 + x_1) + f(0, x_2 - x_1)) + \tfrac{1}{2}\int_{x_2 - x_1}^{x_2 + x_1} f'(0, t)\, dt$$

where $f'(x) = \partial f(x)/\partial x_1$. Hence if $A(q) \ni f_k$ and f_k tends strongly to zero in the plane $(x, \xi) = x_1 = 0$, i.e. on the x_2-axis, it follows that $f_k(x)$ tends to zero for all x and, more generally, that f_k tends strongly to zero in the entire space. This proves Theorem II in our special case. The proof in the general case is similar. In fact, if q is not constant and hyperbolic with respect to ξ, it is possible to construct a linear functional $K(f) = K(\xi, x, f)$, in the case just considered given by the right side of (2), with the following properties.

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[1] The proof rests mainly on a lemma on the rate of growth of certain algebraic functions. This lemma is perhaps of interest in itself and it proves a conjecture by PETROWSKY ([9] footnote p. 24).

It is a projection of $C(\infty)$ upon $A(q)$, i.e. it is defined for all f in $C(\infty)$ and is itself an element of $A(q)$, and it reproduces the elements of $A(q)$ so that

$$f(x) = K(\xi, x, f)$$

for all x if f belongs to $A(q)$. If f_k tends strongly to zero in the plane $(y, \xi) = 0$ then $K(\xi, \cdot, f_k)$ tends strongly to zero in the entire space. Moreover, if the derivatives of order $< m$ of f vanish in a certain compact part $B(x)$ of the plane $(y, \xi) = 0$, then $K(\xi, x, f)$ vanishes, m being the degree of q. Finally, the derivatives of order $< m$ of $f - K(\xi, \cdot, f)$ vanish on the plane $(y, \xi) = 0$. At least when q is homogeneous one can write K explicitly in a form similar to (2) as a sum of certain integrals over $B(x)$.

The functional K also gives the solution of the problem of Cauchy to which we give the following seemingly sophisticated, but in fact simple and convenient form. Given an element $g \in C(\infty)$, find an element u in $A(q)$ such that the derivatives of $u - g$ of order $< m$ vanish on the plane $(y, \xi) = 0$. In fact, one solution is simply

$$u(x) = K(\xi, x, g)$$

and because the difference v of any two solutions is an element in $A(q)$ whose derivatives of order $< m$ vanish on the plane $(y, \xi) = 0$ it follows from the properties of K that $v(x) = K(\xi, x, v) = 0$ for all x and hence the solution is unique.

Conversely, assume that for a given $\xi \neq 0$ and not constant q and an arbitrary $g \in C(\infty)$ the problem of Cauchy has a unique solution $H(\xi, x, g)$ with the property that $H(\xi, x, g_k)$ tends to zero with $1/k$ for at least one x with $(x, \xi) \neq 0$ whenever $g_k \to 0(\xi)$. Then if $A(q)$ contains every element of the sequence $f_1, \ldots, f_k, \ldots$ and $f_k \to 0(\xi)$ we get that $f_k(x) = H(\xi, x, f_k)$ tends to zero with $1/k$. Hence the requirements of Theorem I are satisfied and it follows that q is hyperbolic with respect to ξ. It then follows that $H(\xi, x, g) = K(\xi, x, g)$ for all x and all $g \in C(\infty)$.

The continuity property of H used above is a variant of Hadamard's classical condition that the problem of Cauchy should be correctly set ([5] pp. 40—41). Another variant was given by Petrowsky [9] who, however, restricts the behaviour of the function g at infinity in the plane $(y, \xi) = 0$. The consequence is that in his case there are other than hyperbolic equations, e.g. the heat equation, for which one can find a suitable correctly set Cauchy problem.

Most equations (1) which so far have been classified as hyperbolic are hyperbolic in our sense, in particular the equations considered by Herglotz [6] and

Petrowsky [8]. Slightly modified, Petrowsky's definition runs as follows.[1] A homogeneous polynomial p of positive degree is called hyperbolic with respect to $\xi \neq 0$ if $p(\xi) \neq 0$ and the zeros of the equation $p(t\xi + \eta) = 0$ are all real and different if η is real and not proportional to ξ. If m is the degree of p it then follows that $p(t\xi + i\eta) = i^m p(-it\xi + \eta) \neq 0$ when $t > 0$ (or $t < 0$) and η is real, so that p is hyperbolic in our sense. More generally, one can show that if p is a homogeneous polynomial of degree $m > 0$ which in the sense of Petrowsky is hyperbolic with respect to ξ, and r' is any polynomial of degree less than m, then $q' = p + r'$ is (in our sense) hyperbolic with respect to ξ. If p is hyperbolic merely in our sense, this need not be true. A rather trivial example is given by $q' = \zeta_1^2 + \zeta_2$, a less trivial one by $q' = \zeta_1^2(\zeta_1^2 - \zeta_2^2) + \zeta_2^3$. In both cases the principal parts are hyperbolic with respect to $(1, 0)$, but the polynomials are not. Hence the hyperbolic character of a polynomial is in general not determined by its principal part alone. It is, however, true that if a polynomial is hyperbolic with respect to a vector ξ, then also its principal part is.

We study in the first section of Chapter 2 the effect of a linear transformation, $x' = x\tilde{M}$, where $\tilde{M}$ is the transpose of a real, square and not singular matrix M, upon (1). It is transformed into

$$q'(\partial / \partial x') f'(x') = 0,$$

where $f'(x') = f(x' \tilde{M}^{-1}) = f(x)$ and $q'(\zeta') = q(\zeta)$. We call the polynomial q reduced if there is no M such that q' is a polynomial in $\zeta_1', \ldots, \zeta_l'$ alone where $l < n$. The fact that a polynomial is not always reduced introduces some complications in the proof of Theorem II. Let $\Omega(q)$ be the linear manifold of all real vectors η such that $q(t\eta + \eta') = q(\eta')$ for all real t and η'. Then q is reduced if and only if $\Omega(q)$ contains only the element $\eta = 0$.

Later in Chapter 2 we collect some facts concerning not constant hyperbolic polynomials. Let the polynomial q be hyperbolic with respect to ξ. Then the same is true of its principal part p. Let the common degree m of q and p

[1] In the paper [8] Petrowsky considers only homogeneous equations with constant coefficients, in [9] and [10], however, he extends what is substantially the definition given above to very general systems of differential equations which need not even be linear. For them he solves the problem of Cauchy. — The wellknown textbook *Methoden der Math. Physik* by R. COURANT and D. HILBERT (Berlin 1937), has a terminology which differs from ours. There the equation (1) is called hyperbolic unless it is elliptic and it is elliptic if the principal part of q is a definite polynomial. An equation which is hyperbolic in the sense of Petrowsky is called totally hyperbolic (l.c. II p. 373—374).

be positive. Because $p(\xi) \neq 0$, the degree of $p(t\xi + \eta)$ with respect to t is m and we can write it in the form $p(\xi) \prod_1^m (t + u_\nu)$ where $u_\nu = u_\nu(\xi, \eta)$ are certain complex numbers. It turns out that if η is real then also the numbers $u_\nu(\xi, \eta)$ are real. They need not all be different when η is not proportional to ξ, but if they are, we have the case considered by Petrowsky. Let $\Gamma(q, \xi)$ be the set of all real η such that $\min_\nu u_\nu(\xi, \eta) > 0$ or briefly,

$$\Gamma(q, \xi) = (\eta;\ \min_\nu u_\nu(\xi, \eta) > 0).$$

It turns out that $\Gamma(q, \xi) = \Gamma(p, \xi)$ is the interior of a convex cone containing ξ. Also, if $\xi' \in \Gamma(q, \xi)$ then q is hyperbolic with respect to both ξ' and $-\xi'$, and we have $\Gamma(q, \xi') = \Gamma(q, \xi)$. We also consider the dual cone $C = C(q, \xi)$ of $\Gamma = \Gamma(q, \xi)$ defined as the set of all real vectors x such that $(x, \eta) \geq 0$ for all $\eta \in \Gamma$, or briefly

$$C(q, \xi) = (x; (x, \eta) \geq 0,\ \eta \in \Gamma(q, \xi)).$$

It is convex and orthogonal to $\Omega(q)$. Its interior is not empty if q is reduced, and the part of C where $(x, \xi') \leq b$ is closed and bounded if $\xi' \in \Gamma$.

The central question in the Chapters 3 and 4 is the effective determination of the linear functional $K(\xi, x, f)$. We use a method of fractional integration developed by M. Riesz [11] for the wave equation. Again, let the polynomial q be not constant and hyperbolic with respect to ξ. Let $\Gamma_1 = \Gamma_1(q, \xi)$ be the set of vectors ξ' in $\Gamma = \Gamma(q, \xi)$ for which there exists a $t_0 < 1$ such that $q(t\xi' + i\eta) \neq 0$ when η is real and $t > t_0$. If $\bar{\xi} \in \Gamma$, then a suitable positive multiple of $\bar{\xi}$ is in Γ_1. Let η be real, let ξ' be in Γ_1, put $\zeta = \xi' + i\eta$ and define $q(\zeta)^{-\alpha}$ as $e^{-\alpha(\log q + i \arg q)}$. Then it turns out that $\arg q$ and hence also $q(\zeta)^{-\alpha}$ is if locally continuous also singlevalued when $\xi' \in \Gamma_1$ and η is real. Different choices of $\arg q$ at a point will affect $q(\zeta)^{-\alpha}$ only by a factor $e^{-2\pi i k \alpha}$ where k is an integer. Assume for a moment that q is reduced and that $\Re \alpha > n$.[1] Then $q(\zeta)^{-\alpha}$ is the Fourier-Laplace transform of a continuous function $Q(\alpha, x)$ which vanishes outside C, and we have the reciprocal formulas

$$q(\zeta)^{-\alpha} = \int Q(\alpha, x)\, e^{-(\zeta, x)}\, dx$$

$$Q(\alpha, x) = (2\pi)^{-n} \int q(\zeta)^{-\alpha}\, e^{(\zeta, x)}\, d\eta$$

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[1] $\Re \alpha$ means the real part of α.

where the integrals are taken over the whole space. When $q = \zeta_1^2 - \zeta_2^2 - \cdots - \zeta_n^2$, in which case (1) becomes the wave equation, then q is hyperbolic with respect to any ξ such that $q(\xi) > 0$. One finds that $\Gamma = (\eta;\ \xi_1 \eta_1 > 0,\ q(\eta) > 0)$, that $C = (x;\ x_1 \xi_1 \geq 0,\ q(x) \geq 0)$ and that with a suitable choice of arg q,

$$Q(a, x) = q(x)^{a - \frac{1}{2} n} / \pi^{\frac{1}{2}(n-2)}\, 2^{2a-1}\, \Gamma(a)\, \Gamma(a - \tfrac{1}{2}(n-2))$$

when $x \in C$ and zero elsewhere. This, with a changed to $\frac{1}{2} a$, is the kernel of M. Riesz.

Returning to the general case, we proceed as follows. Let $S = S(\xi)$ be the plane $(y, \xi) = 0$ and $T = T(\xi)$ the region $(y, \xi) > 0$. When $h \in C(\infty)$, $x \in T$ and $\Re\, a > n$ we define the Riesz operator I^a by the formula

$$I^a h(x) = \int_T Q(a, x-y)\, h(y)\, dy.$$

All y such that $x - y \in C$ and $y \in T + S$, i.e. such that $(x - y, \xi) \leq (x, \xi)$, constitute a compact set $C(x)$, and the integrand vanishes outside $C(x)$. Hence the integral always exists. Let $a = a_x \in C(\infty)$, let $a_x(y) = 1$ when $y \in C(x)$ and let $a_x(y) = 0$ when $|y| = \max_k |y_k|$ is large enough. Let $\xi' \in \Gamma_1(q, \xi)$, put $\zeta = \xi' + i\eta$ and

$$H_x(\zeta) = \int_T h(y)\, a_x(y)\, e^{-(\zeta, y)}\, dy.$$

Then by virtue of Parseval's theorem, another form of $I^a h(x)$ is

$$I^a h(x) = (2\pi)^{-n} \int H_x(\zeta)\, q(\zeta)^{-a}\, e^{(\zeta, x)}\, d\eta.$$

When q is not necessarily reduced, we define $I^a h(x)$ by this formula. Then it is independent of ξ' and a_x as long as $\xi' \in \Gamma_1$ and a_x equals one on $C(x)$; and the formula is valid as long as the integral is absolutely convergent, i.e. when $\Re\, a > 0$. It is shown in Chapter 4 that when $x \in T$, $I^a h(x)$ is an entire function of a, that for all values of a all its derivatives with respect to x are continuous in T and at the same time entire functions of a, that $q(\partial/\partial x)\, I^{a+1} h(x) = I^a h(x)$ and that $I^{-k} h(x) = q(\partial/\partial x)^k h(x)$ when $k = 0, 1, 2, \ldots$ If h vanishes on $C(x)$ then $I^a h(x) = 0$ for all a. Further, all the derivatives of $I h(x) = I^1 h(x)$ are continuous in $T + S$ and those of order $< m$ vanish on S.

Let I_-^a be the Riesz operator constructed as above but with ξ changed to $-\xi$. Then $I_-^a h(x)$ is defined when $x \in T^- = T(-\xi)$ and it vanishes if h vanishes on the compact set $C_-(x) = (y;\ x - y \in C(q, -\xi) = -C,\ (y, \xi) \geq (x, \xi))$. It turns out that all the derivatives of $I_-^1 h(x) - I h(x)$ vanish when $x \in S$. We put $I h(x) = I_-^1 h(x)$

when $x \in T^-$. Then $Ih \in C(\infty)$ and one can prove that if $h_k \to 0$, then also $Ih_k \to 0$. Also, if all the derivatives of $h(x)$ of order $< m$ vanish when $x \in S$, or briefly, if $h(x) \overset{(m)}{=} 0$, $(x \in S)$, then

$$h(x) = Iq(\partial/\partial x)\, h(x). \tag{3}$$

In terms of the operator I, the linear functional $K(\xi, x, f)$ is given by the formula

$$K(\xi, x, f) = f(x) - Iq(\partial/\partial x) f(x).$$

It follows from (3) that if f and g are in $C(\infty)$ and $f(x) - g(x) \overset{(m)}{=} 0$, $(x \in S)$, the right side of this formula does not change if we change f to g. Put[1]

$$P_\xi f(x) = \sum_0^{m-1} (a, \xi)^{-k} (x, \xi)^k f^{(k)}(x - (a, \xi)^{-1} (x, \xi)\, a)/k!$$

where a is a vector such that $(a, \xi) \neq 0$ and $f^{(k)}(x) = (a, \partial/\partial x)^k f(x)$. Then $P_\xi f \in C(\infty)$ and $f(x) - P_\xi f(x) \overset{(m)}{=} 0$, $(x \in S)$, and consequently another form of $K(\xi, x, f)$ is

$$K(\xi, x, f) = P_\xi f(x) - Iq(\partial/\partial x)\, P_\xi f(x).$$

Now $P_\xi f$ depends only on the values of f and its derivatives of order $< m$ in the plane $(y, \xi) = 0$ and it is easy to see that if $f_k \to 0(\xi)$ then $P_\xi f_k \to 0$. Hence if $f_k \to 0(\xi)$ it follows that $K(\xi, \cdot, f_k) \to 0$. Moreover, if $f \in A(q)$ then

$$f(x) = K(\xi, x, f).$$

This proves Theorem II when q is not a constant. If q is a constant, it is not zero so that $A(q)$ contains only the element $f = 0$ and the theorem is trivially true.

It is clear that $K(\xi, x, f)$ depends only on the values of f and its derivatives in the pointset $B(x) = (y; x - y \in C_1, (y, \xi) = 0)$, where $C_1 = \pm C$ according as $x \in T$ or $x \in T^-$, and $B(x) = x$ when $x \in S$. Moreover, $B(x)$ is bounded and closed, i.e. compact, and $K(\xi, x, f)$ vanishes if the derivatives of f of order $< m$ vanish on $B(x)$.

In Chapter 5 we consider the problem of Cauchy when a suitable surface plays the part of the plane $S = S(\xi)$, but only for the case that q is homogeneous and reduced.[2]

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[1] $P_\xi f$ is the beginning of a Taylor series for f with respect to the variable (x, ξ).

[2] In GÅRDING [2] the results of this chapter were announced for arbitrary hyperbolic and reduced q, not necessarily homogeneous. See the first footnote to Chapter 5. The third footnote to the same chapter contains a correction to GÅRDING [4].

In Chapter 6, finally, we give some remarks concerning the domain of dependence of the operator I and the operator J defined by $Jf(x) = f(x) - - Iq(\partial/\partial x) f(x)$. It summarizes the important progress in the theory of phenomena connected with Huygens' principle that has been made recently in a paper by Petrowsky [8] and also in a paper by the author [4]. — I want to thank here C. Hyltén-Cavallius, who proved Lemma 2.2, and H. Jacobinski for a critical reading of parts of the manuscript.

Chapter 1.

Proof of Theorem I.

Let q be an arbitrary polynomial in n variables with complex coefficients, let $\xi = (\xi_1, \ldots, \xi_n) \neq 0$ be an arbitrary real vector and define $A(q)$ as in the introduction. What is meant by $f_k \to 0$ and $f_k \to 0(\xi)$ when f_k, $(k = 1, 2, \ldots)$, is a sequence of elements in $C(\infty)$ is explained in the introduction. It is assumed that there is a real point $x = (x_1, \ldots, x_n)$ such that $(x, \xi) = x_1 \xi_1 + \cdots + x_n \xi_n \neq 0$ and $f_k(x) \to 0$ with $1/k$ whenever $A(q) \ni f_k \to 0(\xi)$ and we have to show that in this case q is hyperbolic with respect to ξ. It is shown in Lemma 2.2 in the next chapter that if q is hyperbolic with respect to ξ it is also hyperbolic with respect to $-\xi$. Hence changing if necessary ξ to $-\xi$ we may suppose without loss of generality that $(x, \xi) > 0$.

Let ζ be a complex vector and t a complex number (if any) such that

$$q(t\xi + \zeta) = 0. \tag{1}$$

Then $A(q)$ contains the function

$$f(t, \zeta, y) = e^{(y-x, t\xi+\zeta)}.$$

It is clear that $f(t, \zeta, x) = 1$ and when $(y, \xi) = 0$ one has

$$f(t, \zeta, y) = e^{-t(x, \xi)} e^{(y-x, \zeta)}. \tag{2}$$

Clearly our assumption implies that we cannot find a sequence $t^{(k)}$, $\zeta^{(k)}$ satisfying (1) such that $f(t^{(k)}, \zeta^{(k)}, \cdot) \to 0(\xi)$. Let us first assume that there exists a vector $\zeta = \zeta'$ such that (1) is satisfied for all t. Let Df be a fixed derivative of f with respect to y and B a compact set in the plane $(y, \xi) = 0$. Then if t is real, performing the differentiation and putting $(y, \xi) = 0$ afterwards we get as in (2)

$$Df_t = Df(t, \zeta', y) = O(t^M) e^{-t(x, \xi)}, \ (M \geq 0),$$

uniformly in B. Letting $t \to \infty$ it follows that $f_t \to 0(\xi)$. Hence there can be no vector ζ such that (1) is satisfied for all t.

Let s be a complex number and put $\zeta = s\zeta'$ with arbitrary but fixed ζ'. Then the polynomial $q(\tau, \sigma) = q(\tau\xi + \sigma\zeta')$ in the indeterminates τ and σ is not zero for any complex value s of σ.[1] Let the degree of $q(\tau, \sigma)$ with respect to both indeterminates and the indeterminate τ be m' and m respectively. We are going to show that $m = m'$. Write $q(\tau, \sigma)$ according to descending powers of τ,

$$q(\tau, \sigma) = q_m(\sigma)\tau^m + \cdots.$$

If $q_m(\sigma)$ is not a constant then $m' > m$ and there exists a complex number s_0 such that $q_m(s_0) = 0$. In a certain neighborhood of $s = s_0$, every zero $t = t(s)$ of $q(t, s) = 0$ is of the form $t(s) = 0$ or

$$t(s) = a(s - s_0)^b (1 + o(1)), \tag{3}$$

where $a \neq 0$, b is rational and $o(1) \to 0$ as $s \to s_0$. Not all $t(s)$ are bounded when $s \to s_0$, because then $q(t', s_0) = \lim_{s \to s_0} q_m(s) \Pi (t' - t(s)) = 0$ for every complex number t'. Hence we may assume that $b < 0$ in (3). We also choose $\arg(s - s_0)$ so that $a(s - s_0)^b$ is real and positive. Then $\Re\, t(s) = |a|\, |s - s_0|^b (1 + o(1))$ and it is easy to see that

$$Df_s = Df(t(s), s\zeta', y) = O(|s - s_0|^{-M})\, e^{-(x, \xi)\Re t(s)}, \quad (M > 0),$$

uniformly in B so that $f_s \to 0(\xi)$ as $s \to s_0$. Next assume that q_m is a constant but that $m' > m$, in which case m is necessarily positive. In a certain neighborhood of $s = \infty$, every zero $t = t(s)$ of $q(t, s) = 0$ is of the form $t(s) = 0$ or

$$t(s) = a s^b (1 + o(1)), \tag{4}$$

where $a \neq 0$, b is rational and $o(1) \to 0$ as $s \to \infty$. Not every b is ≤ 1 because otherwise $q(t's, s) = q_m \Pi (t's - t(s)) = O(s^m)$ for every complex number t' which contradicts the assumption $m' > m$. Let $b > 1$ in (4) and choose $\arg s$ so that as^b is real and positive and consequently $\Re\, t(s) = |a|\, |s|^b (1 + o(1))$. Then one gets

$$Df_s = Df(t(s), s\zeta', y) = O(|s|^{M_1})\, e^{M_2 |s|}\, e^{-(x, \xi)\Re t(s)}, \quad (M_1, M_2 > 0),$$

uniformly in B so that $f_s \to 0(\xi)$ as $s \to \infty$.

Now let p be the principal part of q, so that $q = p + r$ where p is homo-

[1] To say that $q(\tau, s)$ is zero means because τ is an indeterminate, that it is identically zero, considered as a polynomial in τ.

geneous and the degree of p is greater than the degree of r, or if q is a constant, $r = 0$. Because q is not identically zero, p is not identically zero. Hence we can choose ζ' so that $p(\zeta') \neq 0$. If $p(\xi) = 0$ then for $q(\tau\xi + \sigma\zeta')$ one would have $m' > m$ which is impossible. Hence $p(\xi) \neq 0$, and m is the common degree of p and q. If $m = 0$ then $q(t\xi + i\eta) = p(\xi) \neq 0$. Consider the case $m > 0$, let η be real and consider the zeros $t = t(i\eta)$ of the equation $q(t\xi + i\eta) = 0$. Let $\Re t(i\eta)$ attain its maximum $t'(s)$ in the domain $\max_k |\eta_k| \leq s$ when $\eta = \eta(s)$ and $t(i\eta) = t(s)$. By virtue of the lemma proved next in this chapter, for sufficiently large s one has $t'(s) = 0$ or

$$t'(s) = a s^b (1 + o(1)) \tag{5}$$

where a is real and not zero, b is rational and $o(1) \to 0$ as $s \to \infty$. It is clear that $t(s) = O(s^{b'})$ for some $b' > 0$, (actually $b' = 1$). If $t'(s)$ were not bounded from above when $s \to \infty$, one would have $a > 0$ and $b > 0$ in (5) and then

$$Df_s = Df(t(s), \eta(s), y) = O(s^M)\, e^{-(x,\xi)\, t'(s)}, \quad (M > 0),$$

uniformly in B so that $f_s \to 0(\xi)$ as $s \to \infty$.

If $t'(s) \leq t_0$ one has $q(t\xi + i\eta) \neq 0$ when $t > t_0$ and η is real. This reduces the proof of Theorem I to the proof of the following lemma.

Lemma. *Let $q(\tau, \sigma_1, \ldots, \sigma_n)$ be a complex polynomial in the indeterminates $\tau, \sigma_1, \ldots, \sigma_n$ such that when $s_1, \ldots, s_n$ are real, the degree with respect to τ of the polynomial $q(\tau) = q(\tau, s_1, \ldots, s_n)$ is positive and independent of $s_1, \ldots, s_n$.*[1] *Let $M(s)$ be the maximum of the real parts of the zeros of the equation $q(\tau) = 0$ when $\max_k |s_k| \leq s$. Then for sufficiently large s, either $M(s) = 0$ or*

$$M(s) = a s^b (1 + o(1))$$

where a is real and not zero, b is rational and $o(1) \to 0$ as $s \to \infty$.

Proof. Let $M_n = M_n(s_1, \ldots, s_n)$ be the maximum of the real parts of the zeros of the equation $q(\tau) = 0$. It is clearly a continuous function of $s_1, \ldots, s_n$. Let $|s_1, \ldots, s_n|$ be the greatest of the numbers $|s_1|, \ldots, |s_n|$. Let $M_k = M_k(s_1, \ldots, s_k, s)$ be the maximum of M_n when $s_1, \ldots, s_k$ are fixed and $s_{k+1}, \ldots, s_n$ vary so that $|s_{k+1}, \ldots, s_n| \leq s$. It is also a continuous function of $s_1, \ldots, s_k, s$. This is almost evident, but we give here a formal proof. Put $M'_k = M_k(s'_1, \ldots, s'_k, s')$

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[1] If we write q in the form $\sum_0^m q_k(\sigma_1, \ldots, \sigma_n)\tau^k$ where $q_m(\sigma_1, \ldots, \sigma_m)$ is not (identically) zero this means that $m > 0$ and that $q_m(s_1, \ldots, s_n)$ is never zero.

and suppose that $|s_1 - s_1', \ldots, s_k - s_k', s - s'| \le \delta$. Choose $s_{k+1}, \ldots, s_n$ such that $M_n = M_n(s_1, \ldots, s_n) = M_k$ and put $s_j' = c s_j$, $(j > k)$, where $c = 1$ when $s' \ge s$ and $c = s'/s$ when $s' < s$. Then $|s - s_1', \ldots, s_n - s_n'| \le |\delta, (1 - c)s| \le |\delta, s - s'| \le \delta$ and also $|s_{k+1}', \ldots, s_n'| \le cs \le s'$. Hence by the definition of M_k' we get $M_k' \ge M_n' = M_n(s_1', \ldots, s_n')$ so that $M_k' \ge M_k - |M_n - M_n'|$. Now when $|t - t_1', \ldots, t_n - t_n'| \le \delta$ and $|t_1, \ldots, t_n, t_1', \ldots, t_n'|$ is less than some constant greater than $|s_1, \ldots, s_n, s_1', \ldots, s_n'|$ then by uniform continuity, $|M_n(t_1, \ldots, t_n) - M_n(t', \ldots, t')| \le \varepsilon(\delta)$ where $\varepsilon(\delta) \to 0$ as $\delta \to 0$. Hence $M_k' \ge M_k - \varepsilon(\delta)$ and by symmetry, $M_k \ge M_k' - \varepsilon(\delta)$ so that $|M_k - M_k'| \le \varepsilon(\delta)$.

Let $C = C[u_1, \ldots, u_l]$ be the ring of all real polynomials in the indeterminates $u_1, \ldots, u_l$. An element $q \in C$ is called a proper factor of $p \in C$ if $p = q q'$ where $q' \in C$ and q and q' are not real numbers. An element p is called primitive with respect to u_1 if it contains no proper factor independent of u_1.

Let A_k be the class of all real polynomials $P = P(\tau, \sigma_1, \ldots, \sigma_k, \sigma) \neq 0$[1] satisfying

$$P(M_k, s_1, \ldots, s_k, s) = 0 \tag{6}$$

for all real $s_1, \ldots, s_k, s$ such that $s \ge 0$ and having no proper factor with the same property. It then follows trivially that P has no proper multiple factors but also that it is primitive with respect to τ. In fact, let $P = P_1 P_2$ where P_2 is primitive and P_1 is independent of τ. The formula (6) shows that $P_2(M_k, s_1, \ldots, s_k, s) = 0$ at every point where $P_1(s_1, \ldots, s_k, s) \neq 0$. But these points are dense in the region $s \ge 0$ and M_k is continuous there so that (6) follows for P_2 and consequently P_1 is a constant so that P is primitive.

That A_n has at least one element can be seen as follows. There is certainly a real polynomial $Q'(\tau, \sigma_1, \ldots, \sigma_n) \neq 0$ which vanishes when $\sigma_j = s_j$, $(j = 1, \ldots, n)$, and $\tau = \frac{1}{2}(t_j + \bar{t}_k)$, $(j, k = 1, \ldots, m)$, where $t_1, \ldots, t_m$ are the $m > 0$ zeros of the equation $q(\tau, s_1, \ldots, s_n) = 0$. Consequently it has at least one factor Q in A_n.

Assume now that $k > 0$ and that $P \in A_k$. We are going to construct an element $P' \in A_{k-1}$. If $P_k = \partial P / \partial \sigma_k = 0$, then P is independent of σ_k, so that because M_k is continuous it is independent of s_k for such $s_1, \ldots, s_k, s$ that $P(\tau, s_1, \ldots, s_k, s) \neq 0$. But these are dense in the region where M_k is defined so that because M_k is continuous it is independent of s_k for all values of the other arguments. Hence $P' = P \in A_{k-1}$. Assume that $P_k \neq 0$ and let

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[1] From now on in this chapter, small Greek letters indicate indeterminates. That $P = 0$ then means that all the coefficients of P vanish.

$$M_{k-1} = M_k(s_1, \ldots, s_{k-1}, s'_k, s)$$

for fixed $s_1, \ldots, s_{k-1}$ and s. If $|s'_k| = s$ we have one of the equalities

$$P(M_{k-1}, s_1, \ldots, s_{k-1}, \pm s, s) = 0. \tag{7}$$

Assume next that $|s'_k| < s$. Let the values of $\partial P/\partial \tau$ and P_k be c_1 and c_2 respectively when $\tau = M_{k-1}$, $\sigma_1 = s_1, \ldots, \sigma_k = s'_k$ and $\sigma = s$. If not $c_1 = c_2 = 0$, the plane curve whose points are (s_k, M_k), $(|s_k| < s)$, has a tangent at the point (s'_k, M_{k-1}) and because $M_k \leq M_{k-1}$ it follows from elementary considerations that this tangent must be parallell to the s_k-axis and this again implies that $c_2 = 0$. Hence we get

$$\begin{aligned} &P(M_{k-1}, s_1, \ldots, s_{k-1}, s'_k, s) = 0 \\ &P_k(M_{k-1}, s_1, \ldots, s_{k-1}, s'_k, s) = 0, \end{aligned} \tag{8}$$

and these equations are also true if $c_1 = c_2 = 0$.

Consider the discriminant R of P with respect to σ_k. It belongs to $C = C[\tau, \sigma_1, \ldots, \sigma_{k-1}, \sigma]$ and we want to prove that it does not vanish. Put $C_1 = C[\tau, \sigma_1, \ldots, \sigma_k, \sigma]$, let C' be the quotient field of C and let $C'[\sigma_k]$ be the ring of all real polynomials in σ_k with coefficients in C'. It is clear that an element in $C'[\sigma_k]$ whose derivative with respect to σ_k vanishes is independent of σ_k. Hence because P depends on σ_k, it follows[1] that if $R = 0$ then P is of the form $P_1^2 P_2$ where P_1 and P_2 are in $C'[\sigma_k]$ and P_1 depends on σ_k. But then[2] we can also write P as $\bar{P}_1^2 \bar{P}_2$ where $\bar{P}_1 = p_1 P_1$ and $\bar{P}_2 = p_2 P_2$ are in C_1 and p_1 and p_2 are suitable elements of C'. Hence P has the proper multiple factor $\bar{P}_1$ so that $P \bar{\in} A_k$ against the assumption. Consequently $R \neq 0$.

It follows from (8) that

$$R(M_{k-1}, s_1, \ldots, s_{k-1}, s) = 0. \tag{9}$$

Moreover, $P^{\pm} = P(\tau, \sigma_1, \ldots, \pm \sigma, \sigma) \neq 0$ because otherwise P has the factor $\sigma_k \mp \sigma$ which implies that P is not primitive with respect to τ against the assumption that $P \in A_k$. Hence if $P_1 = P^+ P^- R$ we have $P_1 \neq 0$ and by virtue of (7) and (9)

$$P_1(M_{k-1}, s_1, \ldots, s_{k-1}, s) = 0$$

when $s \geq 0$. Hence P has at least one factor P' in A_{k-1}. Starting from Q in A_n we can thus construct an element $Q_{n-1} \in A_{n-1}$ and, continuing, finally an

[1] VAN DER WAERDEN [12] I p. 93.
[2] l.c. p. 75—77.

element G in A_0. Because $M(s) = M_0(s)$ we get $G(M(s), s) = 0$, $(s \geq 0)$. Now in a neighborhood N of $s = \infty$, every solution t of $G(t, s) = 0$ is equal to one of a finite number of different convergent series of certain real fractional descending powers of s, one of which may vanish identically while the others have the form

$$(10) \qquad a s^b + \cdots = a s^b (1 + o(1)),$$

where $a \neq 0$, b is rational and s^b is the highest power of s that occurs in the series, so that $o(1) \to 0$ as $s \to \infty$. All these series assume different values in a suitable N and because $M(s)$ is continuous it is identical with one of them there and we assume that it is (10). Then a is real because it is the limit of $M(s) s^{-b}$ as $s \to \infty$, and this proves the lemma, which of course also is true if we by $M(s)$ mean the minimum of the real parts of the zeros of $q(\tau) = 0$ in the region $\max_k |s_k| \leq s$.

Chapter 2.

Hyperbolic Polynomials.

Reduced polynomials. Let $q(\zeta) = q(\zeta_1, \ldots, \zeta_n)$ be a polynomial in $\zeta_1, \ldots, \zeta_n$ with complex coefficients and consider the differential equation

$$(1) \qquad q(\partial / \partial x) f(x) = 0,$$

where $f(x) = f(x_1, \ldots, x_n)$ is a complex and infinitely differentiable function of n real variables $x_1, \ldots, x_n$. Write $x = (x_1, \ldots, x_n)$ and consider a real linear transformation

$$x' = x \tilde{M},$$

where $\tilde{M}$ is the transpose of a real quadratic non-singular matrix M. It then follows that $\partial / \partial x = (\partial / \partial x') M$ so that (1) becomes

$$(2) \qquad q(\partial / \partial x' M) f(x' \tilde{M}^{-1}) = 0.$$

Let us put $q'(\zeta') = q(\zeta' M)$ and $f'(x') = f(x' \tilde{M}^{-1})$. It is clear from (2) that $f \rightleftarrows f'$ is a linear one-to-one mapping of the solutions of (1) upon the solutions of

$$(3) \qquad q'(\partial / \partial x') f'(x') = 0.$$

Because the argument $\partial / \partial x$ of q in (1) and the argument x of f transform differently and we have to consider arguments of q which like x are vectors with numerical components, it is convenient to do as follows. Consider two vector spaces E and E^* where E consists of all vectors with n real components and

E^* consists of all vectors with n complex components. We denote the elements of E by Latin letters $x, y, \ldots$ and those of E^* by Greek letters $\zeta, \xi, \eta, \ldots$ If E^* is subjected to the linear transformation $\zeta \to \zeta' M$, the elements of E should be transformed according to the formula $x \to x' \tilde{M}^{-1}$. In such a way the scalar product

$$(x, \zeta) = x_1 \zeta_1 + \cdots + x_n \zeta_n$$

remains invariant if we substitute x' for x and ζ' for ζ. When it is a complex or real vector, the argument of q should always be thought of as an element of E^*, while the argument of a solution of (1) ought to be considered an element of E. We have tacitly stuck to this convention in the preceding chapter.

A suitable choice of M may make (3) easier to handle than (1). Let l be the least of all integers l' for which there exists a matrix M' such that $q(\zeta' M')$ is a polynomial in $\zeta'_1, \ldots, \zeta'_{l'}$ only when $\zeta'_1, \ldots, \zeta'_n$ are considered as indeterminates. If M is a matrix corresponding to l it is clear that $x'_{l+1}, \ldots, x'_n$ enter into (3) only as parameters. A polynomial for which $l = n$ will be called reduced.

Let q be an arbitrary polynomial in n variables with complex coefficients. The following concept is useful.

Definition. Let $\Omega(q)$ be the set of all real vectors η' in E^* such that

$$q(\eta + t\eta') = q(\eta)$$

for all real numbers t and real vectors η in E^*.

Lemma 2.1. *The set $\Omega(q)$ is linear over the real numbers. A polynomial q is reduced if and only if $\Omega(q) = 0$. If the real vectors $\theta^{(1)}, \ldots, \theta^{(n)}$ are linearly independent and $\theta^{(l+1)}, \ldots, \theta^{(n)}$ constitute a basis of $\Omega(q)$ then $q'(\zeta'_1, \ldots, \zeta'_l) = q(\zeta'_1 \theta^{(1)} + \cdots + \zeta'_n \theta^{(n)})$ is reduced.*

Proof. If η' and η'' are in $\Omega(q)$ then

$$q(\eta + t'\eta' + t''\eta'') = q(\eta + t'\eta') = q(\eta)$$

for all real η and real t' and t''. Hence $\Omega(q)$ is linear. Assume that q is not reduced and let $\mu^{(1)}, \ldots, \mu^{(n)}$ be the columns of such a matrix M that $q(\zeta' M) = q(\zeta'_1 \mu^{(1)} + \cdots + \zeta'_n \mu^{(n)})$ is independent of ζ'_n. Then $q(\eta + t\mu^{(n)}) = q(\eta)$ for all real η and t. Hence $\Omega(q)$ contains the element $\mu^{(n)} \neq 0$. Conversely, suppose that $0 \neq \eta' \in \Omega(q)$ and let $\mu^{(1)}, \ldots, \mu^{(n-1)}$ and $\mu^{(n)} = \eta'$ be a basis for all real vectors η. Then

$$q(\zeta'_1 \mu^{(1)} + \cdots + \zeta'_n \mu^{(n)}) = q(\zeta'_1 \mu^{(1)} + \cdots + \zeta'_{n-1} \mu^{(n-1)})$$

for all real $\zeta_1', \ldots, \zeta_n'$. But then the same equality holds for indeterminate $\zeta_1', \ldots, \zeta_n'$ and consequently q is not reduced. As to the last assertion of the lemma, the same argument shows that $q(\zeta_1'\theta^{(1)} + \cdots + \zeta_n'\theta^{(n)})$ is a polynomial in $\zeta_1', \ldots, \zeta_l'$ alone, say $q'(\zeta_1', \ldots, \zeta_l')$. If q' were not reduced, then one could find real numbers $\eta_1', \ldots, \eta_l'$ not all zero such that

$$q'(\eta_1 + t\eta_1', \ldots, \eta_l + t\eta_l') = q(\eta_1, \ldots, \eta_l)$$

for all real t and $\eta_1, \ldots, \eta_l$. But then $q(\eta) = q(\eta + t\eta'')$ for all real t and η if $\eta'' = \eta_1'\theta^{(1)} + \cdots + \eta_l'\theta^{(l)}$. Hence $0 \neq \eta'' \in \Omega(q)$ and $\theta^{(l+1)}, \ldots, \theta^{(n)}$ is not a basis of $\Omega(q)$ against assumption.

Hyperbolic polynomials. Let E consist of all real elements in E*, i.e. of all vectors with n real components. Let q be a polynomial in n variables with complex coefficients, let it be hyperbolic[1] with respect to $\xi \in \mathrm{E}$ and let the degree m of q be positive. If p is the principal part of q and $\eta \in \mathrm{E}$, then because $p(\xi) \neq 0$, the degree of $q(t\xi + i\eta) = p(\xi)t^m + \cdots$ with respect to t is m. Hence there are complex numbers $v_\nu(\xi, i\eta)$, $(\nu = 1, \ldots, m)$, such that

$$q(t\xi + i\eta) = p(\xi)\prod_1^m (t + v_\nu(\xi, i\eta)) \tag{4}$$

for any complex t. Let $\Re t$ be the real and $\Im t$ the imaginary part of t. Because q is hyperbolic with respect to ξ we get

$$q(t\xi + i\eta) = q(\Re t\xi + i(\Im t\xi + \eta)) \neq 0$$

if $\Re t > t_0$. Now $q(t\xi + i\eta)$ vanishes when $t = -v_\nu(\xi, i\eta)$. Hence $\max_\nu -\Re v_\nu(\xi, i\eta) \leq t_0$ so that

$$\min_\nu \Re v_\nu(\xi, i\eta) \geq -t_0 \tag{5}$$

for all real η. Conversely, if $p(\xi) \neq 0$ and (5) is satisfied, it follows from (4) that $q(t\xi + i\eta) \neq 0$ when $\Re t > t_0$ so that q is hyperbolic with respect to ξ.

It follows directly from the definition that if q is hyperbolic with respect to ξ, it is also hyperbolic with respect to any positive multiple of ξ. The same conclusion is, however, also true for the negative multiples of ξ. In order to prove this it is sufficient to prove the following lemma.

Lemma 2.2. *If a polynomial q is hyperbolic with respect to ξ, it is also hyperbolic with respect to $-\xi$.*

[1] See the definition given in the beginning of the introduction.

Proof.[1] If q is a constant it is not zero, and it follows that q is hyperbolic with respect to all real vectors, in particular $-\xi$. If q is not a constant we can use (4). The sum $R = \Re v_1(\xi, i\eta) + \cdots + \Re v_m(\xi, i\eta)$ is a polynomial in $\eta_1, \ldots, \eta_n$ of degree ≤ 1 and by virtue of (5) bounded from below and hence it must be constant. But then with $r_\nu = \Re v_\nu(\xi, i\eta)$ we get

$$r_\nu = R - r_1 - \cdots - r_{\nu-1} - r_{\nu+1} - \cdots - r_m \leq R + (m-1)t_0$$

for all ν. But then it follows from (4) that $q(-t\xi + i\eta) \neq 0$ when $t > R + (m-1)t_0$. Also $p(-\xi) = (-1)^m p(\xi) \neq 0$. Hence the lemma is proved.

The degree of $p(t\xi + \eta) = p(\xi)t^m + \cdots$ with respect to t is m. Hence there are m complex numbers $u_\nu(\xi, \eta)$, $(\nu = 1, \ldots, m)$, such that for any complex t,

$$p(t\xi + \eta) = p(\xi)\prod_1^m (t + u_\nu(\xi, \eta)) \tag{6}$$

and in particular when $t = 0$,

$$p(\eta) = p(\xi)\prod_1^m u_\nu(\xi, \eta). \tag{7}$$

The following identities in which $a \neq 0$ is a complex number, ξ' a vector in E^* such that $p(\xi') \neq 0$, η an arbitrary element in E^* and a suitable labelling of the numbers $u_\nu(\xi, \eta)$ is understood are immediate consequences of (6) and the homogeneity of p,

$$\begin{gathered} u_\nu(\xi, \xi) = 1, \qquad u_\nu(\xi, a\eta) = a\,u_\nu(\xi, \eta), \\ u_\nu(a\xi, \eta) = a^{-1}u_\nu(\xi, \eta), \qquad u_\nu(\xi, \xi + a\eta) = 1 + a\,u_\nu(\xi, \eta), \\ u_\nu(\xi', \xi) = u_\nu^{-1}(\xi, \xi'). \end{gathered} \tag{8}$$

It is clear that (6), (7) and (8) are valid when p is any homogeneous polynomial of degree m and $p(\xi) \neq 0$.

Lemma 2.3. *A necessary and sufficient condition that a homogeneous polynomial p of positive degree is hyperbolic with respect to ξ is that $p(\xi) \neq 0$ and that the numbers $u_\nu(\xi, \eta)$ defined by (6) are all real when η is real.*

Proof. Let p be hyperbolic with respect to ξ. Applying (8) we get if a and η are real

$$\Re u_\nu(\xi, a i\eta) = -a\,\Im u_\nu(\xi, \eta).$$

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[1] I owe the proof to C. Hyltén-Cavallius.

By virtue of (5), the left side is bounded from below for all ν and real a. But this clearly implies that $\Im u_\nu(\xi, \eta) = 0$ for all ν.

Conversely, if $p(\xi) \neq 0$ and the numbers $u_\nu(\xi, \eta)$ are real when η is, applying (8) we get

$$p(t\xi + i\eta) = p(\xi) \prod_1^m (t + i u_\nu(\xi, \eta)) \neq 0$$

when $t > 0$ (or $t < 0$), so that p is hyperbolic with respect to ξ.

Remark. Multiplying both sides of (7) by $p(\xi)^{-1}$ we get

$$p(\xi)^{-1} p(\eta) = \prod_1^m u_\nu(\xi, \eta).$$

Here the right side is real so that $p(\xi)^{-1} p(\eta)$ is a real polynomial in η.

Our next lemma is classical.

Lemma 2.4. *Let*

$$t^m + a_1 t^{m-1} + \cdots + a_m = \prod_1^m (t - t_\nu)$$

and

$$t^m + b_1 t^{m-1} + \cdots + b_m = \prod_1^m (t - s_\nu)$$

be two polynomials with complex coefficients. Then there exists a labelling of the numbers $s_1, \ldots, s_m$ *such that* $\max_\nu |t_\nu - s_\nu|$ *tends to zero when* $a_1, \ldots, a_m$ *are fixed and* $\max_\nu |a_\nu - b_\nu|$ *tends to zero.*

Ostrowski[1] proved the more precise result that if

$$c(a, b) = 4m \max_\nu (1, |a_\nu|^{1/\nu}, |b_\nu|^{1/\nu}) \Big(\sum_\mu |a_\mu - b_\mu|^2\Big)^{1/2m},$$

then there exists a labelling of the numbers $s_1, \ldots, s_m$ such that

$$\max_\nu |t_\nu - s_\nu| \leq c(a, b).$$

Lemma 2.5. *If a polynomial* q *is hyperbolic with respect to* ξ, *then also its principal part is.*

Proof. If q is a constant, then $p = q$ and the lemma is trivial. Hence assume that the degree m of q is positive, let (6), (7) and (8) refer to the prin-

[1] [7] p. 209—212.

cipal part p of q, let η and $s \neq 0$ be real and put $q_1(t) = s^{-m} p(\xi)^{-1} q(st\xi + is\eta)$. Then as $s \to \infty$, $q_1(t) = t^m + \cdots$ considered as a polynomial in t tends to $p_1(t) = p(\xi)^{-1} p(t\xi + i\eta)$. Now by virtue of (4) and (6) the zeros of $q_1(t) = 0$ and $p_1(t) = 0$ are $t = -s^{-1} v_\nu(\xi, is\eta)$ and $t = -u_\nu(\xi, i\eta)$ respectively, $(\nu = 1, \ldots, m)$. Hence the preceding lemma combined with (8) shows that

$$\min_\nu s^{-1} \Re v_\nu(\xi, is\eta) \to \min_\nu \Re i u_\nu(\xi, \eta)$$

as $s \to \infty$. Here by virtue of (5), the limit of the left side is ≥ 0 so that $\min_\nu \Re i u_\nu(\xi, \eta) \geq 0$. Hence changing η to $-\eta$ and using (8) we get

$$0 \leq \min \Re i u_\nu(\xi, -\eta) = \min_\nu - \Re i u_\nu(\xi, \eta) = - \max_\nu \Re i u_\nu(\xi, \eta),$$

so that $\max_\nu \Re i u_\nu(\xi, \eta) \leq 0$. Hence all the numbers $u_\nu(\xi, \eta)$ are real when η is. Hence Lemma 2.3 shows that p is hyperbolic with respect to ξ.

The converse of this lemma is not true. In fact, $p = \zeta_1^2$ is hyperbolic with respect to $\xi = (1, 0)$ but putting $q = \zeta_1^2 + \zeta_2$ we have

$$q(t\xi + i\eta) = (t + i\eta_1)^2 + i\eta_2 = (t + i\eta_1 + \sqrt{i\eta_2})(t + i\eta_1 - \sqrt{i\eta_2})$$

so that

$$\min_\nu \Re v_\nu(\xi, i\eta) = \min(\Re\sqrt{i\eta_2}, -\Re\sqrt{i\eta_2}) = -\sqrt{\tfrac{1}{2}|\eta_2|}$$

which is not bounded from below. A less trivial example is given by

$$q = \zeta_1^2(\zeta_1^2 - \zeta_2^2) + \zeta_2^3.$$

Now there is one important case when the converse of our last lemma is true,[8] namely when q is not degenerate. Let (6) refer to the principal part p of q. We say that q is not degenerate if $u_\nu(\xi, \eta) \neq u_\mu(\xi, \eta)$ when $\nu \neq \mu$ and η is real and not proportional to ξ. To prove our assertion write $q(s) = q(s\xi + i\eta)$, $p(s) = p(s\xi + i\eta)$ and $r(s) = r(s\xi + i\eta)$ where $r = q - p$ and resolve qp^{-1} into partial fractions as follows

$$q(s)p(s)^{-1} = 1 + r(s)p(s)^{-1} = 1 + \sum_{\nu=1}^{m} \frac{r(-iu_\nu)}{p'(-iu_\nu)}(s + iu_\nu)^{-1} \tag{9}$$

where $p'(s) = dp/ds$ and $u_\nu = u_\nu(\xi, \eta)$ and where we have used (8). Let E_ξ be a linear subspace of the space E of all vectors with n real components such that E_ξ does not contain ξ and ξ and E_ξ together span E. By assumption, $\min_\nu |p'(-iu_\nu)| = \min_\nu |p(\xi)| \prod_{\mu=1}^{m} |-u_\nu + u_\mu|$ has a positive minimum M_1 when

$\eta \in E_\xi$ and $|\eta| = \max_k |\eta_k| = 1$. Now according to (8) every u_ν is homogeneous of order 1 in η. Hence

$$\min_\nu |p'(-iu_\nu)| \geq M_1 |\eta|^{m-1}$$

when $\eta \in E_\xi$. Because r is of degree $< m$ then $\max_\nu |r(-iu_\nu)| = O(|\eta|^{1-m})$ when $|\eta| \geq 1$. Hence there exists an M_2 such that $\max_\nu |r(-iu_\nu)/p'(-iu_\nu)| \leq M_2$ when $|\eta| \geq 1$ and $\eta \in E_\xi$ so that by (9)

$$|q(s)| \geq |p(s)| (1 - (\Re s)^{-1} m M_2) > 0$$

if $\Re s > M_3 = m M_2$ and $\eta \in E_\xi$ and $|\eta| \geq 1$. When $\eta \in E_\xi$ and $|\eta| \leq 1$ then $\max_\nu - \Re v_\nu(\xi, i\eta)$ has a maximum M_4. Hence $q(s) = q(\Re s \xi + i \Im s \xi + i\eta) \neq 0$ when $\Re s > \max(M_3, M_4)$ and $\eta \in E_\xi$ so that q is hyperbolic with respect to ξ.

Let the degree m of the polynomial q be positive, let (6) refer to its principal part p so that

$$p(t\xi + \eta) = p(\xi) \prod_1^m (t + u_\nu(\xi, \eta)).$$

It follows from Lemma 2.5 and Lemma 2.3 that the numbers $u_\nu(\xi, \eta)$ are all real when η is real.

Definition. Let $\Gamma(q, \xi)$ be the set of all real vectors ξ' for which

$$\min_\nu u_\nu(\xi, \xi') > 0.$$[1]

It follows from (8) that $\Gamma(q, \xi)$ contains ξ. Because $\min_\nu u_\nu(\xi, \xi')$ is a continuous function of ξ', $\Gamma(q, \xi)$ is open and hence it also contains all ξ' which are sufficiently close to ξ. By virtue of (7) we get

$$p(\xi') = p(\xi) \prod_1^m u_\nu(\xi, \xi') \neq 0 \tag{10}$$

if $\xi' \in \Gamma(q, \xi)$. More detailed information is given in Lemma 2.8. For the moment we want to prove

Lemma 2.6. *Let the polynomial q be not constant and let it be hyperbolic with respect to ξ so that $q(t\xi + i\eta) \neq 0$ when $\Re t > t_0$ and η is real. Let $\xi' \in \Gamma(q, \xi)$ and let $\Re s \geq 0$. Then also $q(t\xi + s\xi' + i\eta) \neq 0$ if $\Re t > t_0$ and η is real.*

Proof. Let $m > 0$ be the degree of q and p its principal part and put with complex t and s

$$q_1(t, s) = q(t\xi + s\xi' + i\eta) = p(\xi) t^m + p(\xi') s^m + \cdots.$$

[1] The set $\Gamma(q, \xi)$ is also the largest set in E which is connected with ξ and only contains vectors ξ' such that $p(\xi') \neq 0$. This is perhaps the simplest definition.

By assumption, $p(\xi) \neq 0$ and it follows from (10) that $p(\xi') \neq 0$. Hence the equation $q_1(t, s) = 0$ defines a $(1, m)$-correspondence between s and t and also between t and s. Let $s_1, \ldots, s_m$ be the correspondents of t and $t_1, \ldots, t_m$ those of s. Consider

$$a(s) = \max_\nu \Re t_\nu.$$

Because q is assumed to be hyperbolic with respect to ξ, it follows that

$$q_1(t, s) = q(\Re t \xi + s \xi' + i(\Im t \xi + \eta)) \neq 0$$

when $\Re t > t_0$ and $\Re s = 0$. Hence we know that $a(s) \leq t_0$ when $\Re s = 0$. In order to prove the lemma it is obviously sufficient to prove that the same inequality is true when $\Re s > 0$.

Consider also the function

$$b(t) = \max_\nu \Re s_\nu.$$

It is continuous and we want to study it when t is large. Put

$$h(s) = t^{-m} p(\xi')^{-1} q_1(t, ts) = t^{-m} p(\xi')^{-1} q(t\xi + ts\xi' + i\eta).$$

It is clear that the zeros of $h(s)$ are $t^{-1} s_\nu$, $(\nu = 1, \ldots, m)$. When $|t| \to \infty$ then $h(s) = s^m + \cdots$ considered as a polynomial in s tends to

$$h_1(s) = p(\xi')^{-1} p(\xi + s\xi') = s^m + \cdots.$$

Using (6) and (8) we may write $h_1(s)$ as

$$p(\xi')^{-1} \prod_1^m (1 + s u_\nu(\xi, \xi')).$$

Because $\xi' \in \Gamma(q, \xi)$, the numbers $u_\nu(\xi, \xi')$ are all positive. Hence the zeros of $h_1(s)$ are $-u_\nu^{-1}(\xi, \xi')$ and consequently it follows from Lemma 2.4 that

$$\max_\nu \Re t^{-1} s_\nu \to -\min_\nu u_\nu^{-1}(\xi, \xi') \tag{11}$$

as $|t| \to \infty$.

Assume now that there exists an s' such that $\Re s' > 0$ and $a(s') \geq t_0$. Then s' has at least one correspondent t' such that $\Re t' \geq t_0$. Now s' is also one of the correspondents of t' so that it follows that $b(t') > 0$. Let $c(\tau)$ be a complex continuous function of the real variable τ such that $c(0) = t'$, $\Re c(\tau)$ is strictly increasing and $c(\tau) = \tau$ when τ is large, and put $t = c(\tau)$. Then as τ goes from 0 to $+\infty$, $b(t)$ goes from $b(t') > 0$ to $-\infty$. In fact, when τ is large then $t = \tau$ is large, real and positive so that $t^{-1} b(t)$ equals the left side of (11). Because

$\xi' \in \Gamma(q, \xi)$, the right side is negative and hence $b(t) \to -\infty$ as $\tau \to +\infty$. Hence there exists a t'' such that $\Re t'' > \Re t' \geq t_0$ and $b(t'') = 0$. But then t'' has at least one purely imaginary correspondent s'' and hence $p(\xi')^{-1} q(t''\xi + s''\xi' + i\eta) = 0$ which is impossible. Consequently $a(s) < t_0$ when $\Re s > 0$ and this proves the lemma.

We can now prove the following important lemma.

Lemma 2.7. *If a polynomial q of positive degree is hyperbolic with respect to ξ, it is also hyperbolic with respect to any ξ' such that $\xi' \in \Gamma(q, \xi)$ or $-\xi' \in \Gamma(q, \xi)$.*

Proof. Let $\xi' \in \Gamma = \Gamma(q, \xi)$. The formula (10) shows that if p is the principal part of q, then $p(\xi') \neq 0$. Because Γ is open there exists a positive number a such that $\xi' - a\xi \in \Gamma$. Then by virtue of the preceding lemma one has

$$q(t\xi' + i\eta) = q(ta\xi + t(\xi' - a\xi) + i\eta) \neq 0$$

when η is real and $\Re t > \max(0, a^{-1} t_0)$. Hence q is hyperbolic with respect to ξ'. If $-\xi' \in \Gamma$ then q is hyperbolic with respect to $-\xi'$ and hence also, by virtue of Lemma 2.2, with respect to ξ'.

Lemma 2.8. *Let q be a not constant polynomial and let it be hyperbolic with respect to ξ. Then $\Gamma = \Gamma(q, \xi)$ is open, convex and not empty. If p is the principal part of q, then $\Gamma(p, \xi) = \Gamma(q, \xi)$. If $a > 0$, $\xi' \in \Gamma$ and $\xi'' \in \Omega(p)$ then also $a\xi'$ and $\xi' + \xi''$ are in Γ and $\Gamma(q, \xi') = \Gamma(q, \xi)$. There are real vectors x such that $(x, \eta) = x_1\eta_1 + \cdots + x_n\eta_n > 0$ when $\eta \in \Gamma$.*

Proof. We know already that Γ is open and that it contains ξ. It follows from Lemma 2.5 that p is hyperbolic with respect to ξ. Hence we can form $\Gamma(p, \xi)$ and it is clear that $\Gamma(p, \xi) = \Gamma(q, \xi)$. If $a > 0$ and $\xi' \in \Gamma$ then by virtue of (8), $\min_\nu u_\nu(\xi, a\xi') = a \min_\nu u_\nu(\xi, \xi') > 0$ so that $a\xi' \in \Gamma$. If $\xi'' \in \Omega(p)$ then $p(\xi + t\xi' + t\xi'') = p(\xi + t\xi')$ for all real t so that because p is homogeneous we get that $p(t\xi + \xi' + \xi'') = p(t\xi + \xi')$ for all real t. Hence (6) shows that $\min_\nu u_\nu(\xi, \xi' + \xi'') = \min_\nu u_\nu(\xi, \xi') > 0$ so that $\xi' + \xi'' \in \Gamma$. Because $\Omega(p)$ is linear it also follows that $\xi' + b\xi'' \in \Gamma$ for all real b.

Let η and the number s be real, let $\xi' \in \Gamma$ and consider

$$a = a(s) = \min_\nu u_\nu(\xi, s\xi' + \eta).$$

We want to prove that it is a strictly increasing function of s.

Put $q_1(t, s) = p(-t\xi + s\xi' + \eta)$ with real s and t. Then the coefficients of t^m and s^m in $q_1(t, s)$ are $(-1)^m p(\xi)$ and $p(\xi')$ respectively. By assumption, $p(\xi) \neq 0$

and it follows from (10) that $p(\xi') \neq 0$. Hence the equation $q_1(t, s) = 0$ defines a $(1, m)$-correspondence between t and s and also between s and t. Let the correspondents of s be $t_1, \ldots, t_m$ and $s_1, \ldots, s_m$ those of t. By virtue of Lemma 2.5 and Lemma 2.7, p is hyperbolic with respect to both ξ and ξ'. Hence it follows from Lemma 2.3 that if s and t are real, then also $t_1, \ldots, t_m$ and $s_1, \ldots, s_m$ are real. It follows from (6) that

$$q_1(t, s) = p(-t\xi + s\xi' + \eta) = p(\xi) \prod_1^m (-t + u_\nu(\xi, s\xi' + \eta))$$

and hence that

$$a(s) = \min_\nu t_\nu.$$

Consider also

$$b(t) = \max_\nu s_\nu.$$

It is a continuous function of t. If $|t| \to \infty$, then

$$h(s) = p(\xi')^{-1} t^{-m} q_1(t, ts) = p(\xi')^{-1} t^{-m} p(-t\xi + st\xi' + \eta)$$

considered as a polynomial in s tends to

$$h_1(s) = p(\xi')^{-1} p(-\xi + s\xi').$$

Now $t^{-1}s_\nu$ are the zeros of $h(s)$ and by virtue of (6) and (8), those of $h_1(s)$ are $u_\nu(\xi', \xi) = u_\nu^{-1}(\xi, \xi')$, $(\nu = 1, \ldots, m)$. The coefficient of s^m in $h(s)$ is 1. Hence using Lemma 2.4 we get

$$\min_\nu t^{-1} s_\nu \to \min_\nu u_\nu^{-1}(\xi, \xi')$$

when $|t| \to \infty$. Because $\xi' \in \Gamma$, $\min_\nu u_\nu^{-1}(\xi, \xi') > 0$ and hence the right side is positive. Hence $b(t) = \max_\nu s_\nu = t \min_\nu t^{-1} s_\nu \to -\infty$ if $0 > t \to -\infty$.

We can now prove that $a(s)$ is a strictly increasing function of s. Assume that $s < s'$. Because s' is one of the correspondents of $t' = a(s')$ it then follows that $b(t') > s'$. Now $b = b(t)$ is continuous and tends to $-\infty$ when t does. Hence there exists a $t < t'$ such that $b(t) = s$. Because t is one of the correspondents of s it then follows that $a(s) \leq t < t' = a(s')$. Hence $a(s) = \min_\nu u_\nu(\xi, s\xi' + \eta)$ is strictly increasing. A slight modification of the above proof shows that also $\max_\nu u_\nu(\xi, s\xi' + \eta)$ is a strictly increasing function of s.

Assume now that ξ' and ξ'' both are in Γ. Then we get

$$0 < \min_\nu u_\nu(\xi, \xi') < \min_\nu u_\nu(\xi, \xi' + s\xi'')$$

when $s > 0$. Hence in particular $\xi' + \xi'' \in \Gamma$ so that Γ is convex. According to Lemma 2.7, $\Gamma(q, \xi')$ has a sense. Our last formula combined with (7) and (6) shows that

$$0 \neq p(\xi)\prod_1^m u_\nu(\xi, s\xi' + \xi'') = p(s\xi' + \xi'') = p(\xi')\prod_1^m (s + u_\nu(\xi', \xi''))$$

when $s \geq 0$. Hence the real numbers $u_\nu(\xi', \xi'')$ are all positive and this proves that if $\xi'' \in \Gamma(q, \xi)$ then $\xi'' \in \Gamma(q, \xi')$. In particular, $\xi \in \Gamma(q, \xi')$. Interchanging ξ and ξ' we get $\Gamma(q, \xi') = \Gamma(q, \xi)$. If $\xi' \in \Gamma$ the sum $\sum_1^m u_\nu(\xi', \eta)$ is linear and homogeneous in η and real when η is real and hence of the form $(x, \eta) = x_1\eta_1 + \cdots + x_n\eta_n$ with real x. It is clear that $(x, \eta) > 0$ when $\eta \in \Gamma$. This completes the proof of the lemma.

To the geometrical intuition, $\Gamma(q, \xi)$ appears in the general case as the interior of a convex infinite ditch situated entirely on one side of any plane $(x, \eta) = 0$. The edge of the ditch is the linear manifold $\Omega(p)$. It will be shown later that $\Omega(p) = \Omega(q)$. Hence if q is reduced, $\Omega(p)$ contains only the element zero, the edge reduces to a point and $\Gamma(q, \xi)$ is a proper convex cone.

Let the polynomial q of positive degree be hyperbolic with respect to ξ, let $\xi' \in \Gamma(q, \xi)$ and let $B(\xi')$ be the set of all real numbers t_0 with the property that $q(t\xi' + i\eta) \neq 0$ when $t > t_0$ and η is real. Then $B(\xi')$ has a least element $b(\xi')$ and consists of all $t_0 \geq b(\xi')$. In fact, $B(\xi')$ is closed and by Lemma 2.7 not empty. Since the degree of $q(t\xi' + i\eta)$ with respect to t is positive, $B(\xi')$ does not consists of all real numbers, which implies that it is bounded from below. Hence $B(\xi')$ has a least element $b(\xi')$ and it obviously consists of all numbers $t_0 \geq b(\xi')$.

Definition. Let $\Gamma_1(q, \xi)$ be the set of all $\xi' \in \Gamma(q, \xi)$ for which there exists a number $t_0 < 1$ such that $q(t\xi' + i\eta) \neq 0$ for all real η when $t > t_0$.

It follows from this definition that if $\xi' \in \Gamma_1(q, \xi)$ then $b(\xi') < 1$ and conversely. If q is homogeneous then $b(\xi') = 0$ for all ξ' and hence $\Gamma_1(q, \xi) = \Gamma(q, \xi)$. In the general case we have

Lemma 2.9. *Let the polynomial q be of positive degree and let it be hyperbolic with respect to ξ. Then $\Gamma_1(q, \xi)$ is open and connected. If $\xi' \in \Gamma(q, \xi)$ then a suitable positive multiple of ξ' is in $\Gamma_1(q, \xi)$. If $\xi' \in \Gamma_1(q, \xi)$ and $\xi'' \in \Gamma(q, \xi)$ then $\xi' + \xi'' \in \Gamma_1(q, \xi)$. If Γ' is a compact subset of $\Gamma_1(q, \xi)$ then there exists a number $b < 1$ such that $b(\xi') \leq b$ when $\xi' \in \Gamma'$.*

Proof. Let $s > 0$. If $q(t\xi' + i\xi) \neq 0$ for all real η when $t > t_0$ then $q(ts\xi' + i\eta) \neq 0$ for all real η when $t > s^{-1}t_0$ and conversely. Consequently

$b(s\xi') = s^{-1} b(\xi')$. Hence if $\xi' \in \Gamma = \Gamma(q, \xi)$ and $s > \max(0, b(\xi'))$ then $s\xi' \in \Gamma$ and $b(s\xi') = s^{-1} b(\xi') < 1$ so that $s\xi' \in \Gamma_1 = \Gamma_1(q, \xi)$. If $\xi' \in \Gamma_1$ and $\xi'' \in \Gamma$, it follows from Lemma 2.6 that $q(t(\xi' + \xi'') + i\eta) \neq 0$ for all real η when $t > \max(0, b(\xi'))$. Hence

$$b(\xi' + \xi'') \leq \max(0, b(\xi'))$$

and because the right side is less than 1 it follows that $\xi' + \xi'' \in \Gamma_1$. If ξ' and ξ'' are in Γ_1, then because Γ is open we can choose $a > 0$ so small that $a < 1$ and $\xi' - a\xi'' \in \Gamma$. But then $s\xi'$ and $\xi'' + s(\xi' - a\xi'')$ are both in Γ_1 when $s \geq 1$ and $s \geq 0$ respectively. The two expressions are equal when $s = a^{-1}$. Hence Γ_1 is connected.

To prove that Γ_1 is open we do as follows. Let $\xi' \in \Gamma$. Then we can choose a number $s > 1$ such that $s b(\xi') < 1$. Because $(s-1)\xi' \in \Gamma$ and Γ is open we can also choose a number $\delta = \delta(\xi', s) > 0$ which is so small that $\xi'' = (s-1)\xi' + s\eta' \in \Gamma$ whenever η' is real and $|\eta'| = \max_k |\eta'_k| \leq \delta$. Then also $\xi' + \eta' = s^{-1}\xi' + (1 - s^{-1})\xi' + \eta' \in \Gamma$, and

$$\begin{aligned} b(\xi' + \eta') = s b(s\xi' + s\eta') = s b(\xi' + (s-1)\xi' + s\eta') = \\ = s b(\xi' + \xi'') \leq s \max(0, b(\xi')) < 1. \end{aligned}$$

Hence $\xi' + \eta' \in \Gamma_1$ so that Γ_1 is open, and if Γ' equals the neighborhood of ξ' which consists of all $\xi' + \eta'$ where η' is real and $|\eta'| \leq \delta$, then the last assertion of the lemma follows. Now any compact set Γ' can be covered by a finite number of such neighborhoods and hence the lemma is proved.

As an illustration of this chapter we shall consider two important homogeneous hyperbolic polynomials.

Example 1. Put $q(\eta) = p(\eta) = \eta_1^2 - \eta_2^2 - \cdots - \eta_n^2$ in which case (2. 1)[1] becomes the wave equation. Let $\bar{\xi} = (1, 0, \ldots, 0)$. Then $u_1(\bar{\xi}, \eta) = \eta_1 + (\eta_2^2 + \cdots + \eta_n^2)^{1/2}$ and $u_2(\bar{\xi}, \eta) = \eta_1 - (\eta_2^2 + \cdots + \eta_n^2)^{1/2}$ are both real if η is so that q is hyperbolic with respect to $\bar{\xi}$. If $u_1(\bar{\xi}, \xi)$ and $u_2(\bar{\xi}, \xi)$ are both positive then $\xi_1 > 0$ and $p(\xi) > 0$ and conversely. Hence $\Gamma(q, \bar{\xi})$ consists of all real vectors ξ such that $\xi_1 > 0$ and $p(\xi) > 0$. It is easy to see that if q is hyperbolic with respect to ξ, then either ξ or $-\xi$ is in $\Gamma(q, \bar{\xi})$.

Example 2.[2] Let η_{jk} be complex numbers such that $\eta_{jk} = \eta_{kj}^*$ [3] $(j, k = 1, \ldots, \bar{n})$. Then the matrix $\eta = (\eta_{jk})$ is hermitian. It is determined by the $n = \bar{n}^2$ real

[1] The formula (1) in Chapter 2.

[2] See GÅRDING [4].

[3] When a is a complex number, a^* denotes its conjugate.

numbers η_{jj}, $\frac{1}{2}(\eta_{jk}+\eta_{kj}^*)$, $-\frac{1}{2}i(\eta_{jk}-\eta_{jk}^*)$, $(j<k)$. Let $\eta_1, \ldots, \eta_n$ be these numbers taken in some order and put

$$q(\eta)=p(\eta)=\det \eta.$$

If $\bar{\xi}$ and η are hermitian and $\bar{\xi}$ is positive definite, then $p(\bar{\xi})>0$ and the equation $p(s\bar{\xi}+\eta)=0$ has only real zeros so that q is hyperbolic with respect to $\bar{\xi}$. One finds that $\Gamma(q,\bar{\xi})$ consists of all positive definite matrices. Similarly if $\bar{\xi}$ is negative definite.

Three lemmas. We now come to three lemmas connecting the notions of hyperbolic and reduced polynomial.

Lemma 2.10. *Let the polynomial p be not constant, homogeneous and hyperbolic with respect to ξ. Then p is reduced if and only if the equality*

$$p(\xi+t\eta)=p(\xi)$$

for some real η and all real numbers t implies that $\eta=0$.

Proof. Let η be real and let $p(\xi+t\eta)=p(\xi)$ for every real t. Then if $m>0$ is the degree of p, also $p(t\xi+\eta)=t^m p(\xi)$ for every real t, so that if (6) refers to p we get $u_\nu(\xi,\eta)=0$, $(\nu=1,\ldots,m)$. Let $\xi'\in\Gamma(p,\xi)$. It was shown in the proof of Lemma 2.9 that in this case $\min_\nu u_\nu(\xi, t\xi'+\eta)$ and $\max_\nu u_\nu(\xi, t\xi'+\eta)$ are both strictly increasing functions of t. Now both vanish when $t=0$. Hence using (7) and (6) we get

$$0\neq p(\xi)\prod_1^m u_\nu(\xi, t\xi'+\eta)=p(t\xi'+\eta)=p(\xi')\prod_1^m(t+u_\nu(\xi',\eta))$$

when $t>0$ or $t<0$. It follows from Lemma 2.7 and Lemma 2.3 that the numbers $u_\nu(\xi',\eta)$ are all real, and hence they all vanish. But then $p(t\xi'+\eta)= = t^m p(\xi')$ and consequently also $p(\xi'+t\eta)=p(\xi')$ for all real t if only $\xi'\in\Gamma(p,\xi)$. Now $\Gamma(p,\xi)$ is open and hence the last equality follows for all real ξ' and all real t. Hence if p is reduced it follows that $\eta=0$. If p is not reduced there exists a $\eta\neq 0$ such that $p(\xi'+t\eta)=p(\xi')$ for all real t and real ξ' and hence also for $\xi'=\xi$. This proves the lemma.

Lemma 2.11. *If a polynomial q of positive degree is hyperbolic with respect to ξ, p is the principal part of q, η is real and $p(\xi+t\eta)$ is independent of t then also $q(\xi+t\eta)$ is independent of t.*

Proof. Consider the polynomial $q_1(s, t) = q(s\xi + t\eta)$. By assumption, $p(s\xi + t\eta) = s^m p(\xi + ts^{-1}\eta) = s^m p(\xi)$ where m is the common degree of p and q and $s \neq 0$ is real, and hence $p(s\xi + t\eta) = s^m p(\xi)$ for all real s and t. Hence q_1 is of the form

$$p(\xi) s^m + r_1(s, t) \tag{12}$$

where the degree of r_1 is less than m. Because q is hyperbolic with respect to ξ it follows that if p_1 is the principal part of q_1, then $p_1(1, 0) = p(\xi) \neq 0$. Moreover, $q_1(s, it) = q(s\xi + it\eta) \neq 0$ when t is real and $\Re s$ is greater than some fixed number and hence q_1 is hyperbolic with respect to $(1, 0)$. Let $v_1(t), \ldots, v_m(t$ be complex numbers such that

$$q_1(s, t) = p(\xi) \prod_1^m (s - v_\nu(t))$$

for all complex s and t. It follows from (12) that if $|t| \to \infty$, then $p(\xi)^{-1} t^{-m} q_1(st, t)$ considered as a polynomial in s tends to s^m. The zeros of the two polynomials are $t^{-1} v_1(t), \ldots, t^{-1} v_m(t)$ and $0, \ldots, 0$ respectively and the coefficient of s^m is one in both. Hence Lemma 2.4 shows that

$$\max_\nu |v_\nu(t)| = o(|t|) \tag{13}$$

as $|t| \to \infty$. Because q_1 is hyperbolic with respect to $(1,0)$ it follows that $\max_\nu \Re v_\nu(it)$ is bounded from above when t is real. Now by the classical theory of algebraic functions there exist m descending powerseries in $t^{1/m}$, $\pi_\nu(t)$, each containing only a finite number of positive powers and convergent in a suitable neighborhood N of $t = \infty$, such that when $t \in N$ one can label $v_1(t), \ldots, v_m(t)$ in such a fashion that $\pi_\nu(t) = v_\nu(t)$ for all ν. If we vary arg t, then the series $\pi_1(t), \ldots, \pi_m(t)$ are permuted among each other. It follows from (13) that

$$\pi_\nu(t) = \sum_{m-1}^{-\infty} (t^{1/m})^k a_{\nu k}.$$

Now if k and m are integers and $m > k > 0$ and $a \neq 0$, then one can choose arg t such that t is real and $\Re a(it)^{k/m} = a'|t|^{k/m}$ where $a' > 0$. Hence because $\max_\nu \Re v_\nu(it)$ is bounded from above in N it follows that $a_{\nu k} = 0$ when $k > 0$ and hence that all $v_\nu(t)$ are bounded. But then $q_1(s, t)$ is bounded for all complex t when s is fixed and hence q_1 is a polynomial in s alone, i.e. $q_1(1, t) = q(\xi + t\eta)$ is independent of t. This proves the lemma.

Lemma 2.12. *Let the polynomial q be not constant and hyperbolic with respect to ξ and let p be its principal part. Then $\Omega(p) = \Omega(q)$. In particular, if q is reduced, then p is reduced.*

Proof. If $q(\eta + t\eta') = q(\eta)$ for every real η and t and some real η', then $q(s\eta + st\eta') = q(s\eta)$ is true under the same conditions if s is real. Hence identifying the coefficients of s^m we get that $p(\eta + t\eta') = p(\eta)$ for all real η and t. Hence $\Omega(p) > \Omega(q)$. Conversely, let $\eta' \in \Omega(p)$ so that $p(\eta + t\eta') = p(\eta)$ for all real η and t. Now if $\eta \in \Gamma = \Gamma(q, \xi)$, then q is hyperbolic with respect to η and by virtue of the preceeding lemma we have $q(\eta + t\eta') = q(\eta)$ for all real t and $\eta \in \Gamma$. But Γ is open so that the same is true for any real η. Hence $\eta' \in \Omega(q)$ so that $\Omega(p) < \Omega(q)$. This proves the lemma.

The dual cone. Let $\Gamma(q, \xi)$ be the cone associated with a polynomial q of positive degree which is hyperbolic with respect to ξ. Following the convention introduced in the beginning of this chapter we shall consider it as a subset of the vector space E which consists of all real elements in E*.

Definition. Let $C(q, \xi)$ be the set of all real vectors x such that

$$(x, \eta) = x_1\eta_1 + \cdots + x_n\eta_n \geq 0$$

for all η in $\Gamma(q, \xi)$.

We shall prove that $C(q, \xi)$ is a cone and we shall call it the dual cone of $\Gamma(q, \xi)$. It is to be considered as a subset of the vector space E defined in the beginning of this chapter.

Lemma 2.13. *$C = C(q, \xi)$ contains elements $\neq 0$, it is convex and closed and if $x \in C$ and $b \geq 0$ then $bx \in C$. The part of C where $(x, \xi') \leq b$ is closed and bounded if $\xi' \in \Gamma(q, \xi)$. All elements of C are orthogonal to $\Omega(q)$. If q is reduced, the interior of C is not empty.*

Proof. The first statement follows from Lemma 2.9, the three following are immediate. Put $|x| = \max_k |x_k|$. Let there be a sequence $x^{(k)}$, $(k = 1, 2, \ldots)$, such that $x^{(k)} \in C$ and $(x^{(k)}, \xi') \leq b$ and $\overline{\lim}\, |x^{(k)}| = \infty$. Then there exists a η with $|\eta| = \max |\eta_k| = 1$ such that $\overline{\lim}\, (x^{(k)}, \eta) = \infty$. Because $\Gamma = \Gamma(q, \xi)$ is open we can choose $a > 0$ so small that $\xi' - a\eta \in \Gamma$. But then $\underline{\lim}\, (x^{(k)}, \xi' - a\eta) = -\infty$, which contradicts the assumption that all $x^{(k)}$ are in C. Hence the part of C where $(x, \xi') \leq b$ is bounded and it is clear that it is closed. Combining the Lemmas 2.1 and 2.12 we see that $\Omega(q) = \Omega(p)$ is linear and hence it follows from

Lemma 2.8 that if $\xi' \in \Gamma$ and $\xi'' \in \Omega(q)$ and t is real then $\xi' + t\xi'' \in \Gamma$. Hence if $x \in C$ it follows that $(x, \xi' + t\xi'') \geq 0$ for all t which implies that $(x, \xi'') = 0$.

Let $\xi' \in \Gamma$. Then $\sum_1^m u_\nu(\xi', \eta)$ is of the form (x, η) where $x \in E$ and it is clear that $x \in C$. Let q be reduced so that $\Omega(p) = \Omega(q) = 0$. Then if $u_\nu(\xi', \eta) = 0$ for all ν it follows from (6) that $p(\xi' + t\eta) = p(\xi')$ for all real t and hence from Lemma 2.10 that $\eta = 0$. Hence the minimum of (x, η) when $\eta \in \Gamma$ and $|\eta| = 1$ is positive. Hence the same is true of the minimum of $(x + y, \eta)$ when $\eta \in \Gamma$ and $|\eta| = 1$ provided that $|y| > 0$ is small enough. But this means that $x + y$ belongs to C if $|y| > 0$ is small enough. This proves the lemma.

Consider the two examples on p. 25.

Example 1. Let $\xi \in \Gamma(p, \bar{\xi})$ and let $x_1 \geq 0$ and $p(x) = x_1^2 - x_2^2 - \cdots - x_n^2 \geq 0$. Then it is easily verified that $(x, \xi) \geq 0$. On the other hand, if $x_1 < 0$ or $p(x) < 0$ then one can find a $\xi \in \Gamma(p, \bar{\xi})$ such that $(x, \xi) < 0$. It follows that $C(p, \bar{\xi})$ consists of all x such that $x_1 \geq 0$ and $p(x) \geq 0$.

Example 2. Let $x = (x_{jk})$, $(j, k = 1, \ldots, \bar{n})$, be a hermitian matrix so that $x_{jk} = x_{kj}^*$ and let $x_1, \ldots, x_n$ be the numbers x_{jj}, $(x_{jk} + x_{jk}^*)$, $i(x_{jk} - x_{jk}^*)$, $(j < k)$, in some order. With a suitable choice of this order we have

$$(x, \xi) = \sum_j x_{jj}\,\xi_{jj} + \sum_{j<k} \tfrac{1}{2}(x_{jk} + x_{jk}^*)(\xi_{jk} + \xi_{jk}^*) + \sum_{j<k} \tfrac{1}{2}(x_{jk} - x_{jk}^*)(\xi_{jk} - \xi_{jk}^*) =$$
$$= \sum_{j,k} x_{jk}\,\xi_{jk} = \sigma(x\,\xi^*)$$

where ξ^* is the conjugate of ξ and $\sigma(x\,\xi^*)$ is the trace of the matrix $x\,\xi^*$. It is wellknown that if $\xi \in \Gamma(p, \bar{\xi})$, i.e. if ξ is positive definite, and all the characteristic roots of x are not negative then $\sigma(x\,\xi^*) \geq 0$. But if x has at least one negative root then we can find a $\xi \in \Gamma(p, \bar{\xi})$ such $\sigma(x\,\xi^*) < 0$. Hence $C(p, \bar{\xi})$ consists of all matrices x with not negative roots.

Chapter 3.

The Riesz Kernel.

A lemma. In this chapter we are going to construct a Riesz kernel for the differential equation

$$q(\partial/\partial x) f(x) = 0, \tag{1}$$

where q is a polynomial in n variables with complex coefficients which is assumed to be of positive degree m, reduced, and hyperbolic with respect to a real vector $\bar{\xi}$. It then follows from Lemma 2.3 and Lemma 2.12 that the same is true of its principal part p.

From the definition of $\Gamma_1(q, \bar{\xi})$ it follows that $q(s\xi + i\eta) \neq 0$ when $\xi \in \Gamma_1(q, \bar{\xi})$, $s \geq 1$ and η is real. Our next lemma gives us more precise information. Because the degree of $q(s\xi + i\eta) = p(\xi) s^m + \cdots$ with respect to s is m there exist complex numbers $v_\nu(\xi, i\eta)$, $(\nu = 1, \ldots, m)$, such that for any complex s we have

$$q(s\xi + i\eta) = p(\xi) \prod_1^m (s + v_\nu(\xi, i\eta)). \tag{2}$$

Let $b(\xi)$ be the function defined in connection with Lemma 2.9. Because $q(t\xi + i\eta) \neq 0$ when $\Re t > b(\xi)$ and η is real it follows that $-\Re v_\nu(\xi, i\eta) \leq b(\xi)$ for all ν and η and hence that

$$\min_\nu \Re v_\nu(\xi, i\eta) + b(\xi) \geq 0 \tag{3}$$

for all real η. Put $v_\nu = v_\nu(\xi, i\eta)$ and let $s > b(\xi)$. Then by virtue of (3)

$$\min_\nu |s + v_\nu| \geq s + \min_\nu \Re v_\nu \geq s - b(\xi). \tag{4}$$

Let $u_\nu = u_\nu(\xi, i\eta)$, $(\nu = 1, \ldots, m)$, be defined by (2.6). According to Lemma 2.3 and the formulas (2.8) the numbers $u_\nu = i u_\nu(\xi, \eta)$ are all purely imaginary. Hence because s is real we get

$$|s + v_\nu| \geq |s + u_\nu| - |u_\nu - v_\nu| \geq |u_\nu| - |u_\nu - v_\nu|. \tag{5}$$

Let $|\eta| = \max_k |\eta_k| > 0$ and put $q = p + r$. Using (2) and (2.6) we get

$$\prod_1^m (s + v_\nu |\eta|^{-1}) = \prod_1^m (s + u_\nu |\eta|^{-1}) + |\eta|^{-1} p(\xi)^{-1} |\eta|^{1-m} r(s|\eta|\xi + i\eta)$$

for all complex s. Here both sides are polynomials in s of order m where the coefficients of s^m are 1 and the other coefficients are continuous functions of $(\eta_1, \ldots, \eta_n, \xi_1, \ldots, \xi_n)$ when $|\eta| > 0$ and $\xi \in \Gamma(q, \bar{\xi})$. Hence we can identify the left side and the first term of the right side with the two polynomials of Lemma 2.4. Then the last term of the right side equals $\sum_1^m (a_\nu - b_\nu) s^{m-\nu}$ and Ostrowski's function $c(a, b)$ is continuous in $(\eta_1, \ldots, \xi_1, \ldots)$. Let $c_0(t, \xi)$ be its maximum when $|\eta|$ is real and $|\eta|^{-1} = t$. Then $c_0(t, \xi)$ is continuous in (t, ξ) when $t > 0$ and because $\max_\nu |a_\nu, b_\nu|$ is uniformly bounded and $\max_\nu |a_\nu - b_\nu|$ tends uniformly to zero when $|\eta| \to \infty$ and ξ belongs to a compact subset of Γ it follows that if we put $c_0(0, \xi) = 0$, then $c_0(t, \xi)$ is continuous when $t \geq 0$. Moreover, we can label the numbers $u_1, \ldots, u_m$ so that

$$\max_\nu |v_\nu - u_\nu| < |\eta| \, c_0(|\eta|^{-1}, \xi). \tag{6}$$

We suppose that this labelling is used also in (5).

If η is such that $u_1 = \cdots = u_m = 0$, then as in the proof of Lemma 2.10, the fact that p is reduced implies that $\eta = 0$. Consequently the continuous function $\max_\nu |u_\nu(\xi, \eta)|$ has a positive minimum $c_1(\xi)$ when $|\eta| = 1$. This function is clearly continuous when $\xi \in \Gamma(q, \bar{\xi})$. It follows from (5) and (6) that if $|\eta| > 0$ then

$$\min_\nu |s + v_\nu| \geq (c_1(\xi) - c_0(|\eta|^{-1}, \xi)) \, |\eta|.$$

Estimating one of the factors in (2) by means of this formula and (4) and the others by means of (4) only we get

$$|q(s\xi + i\eta)| \geq |p(\xi)| \, (s - b(\xi))^{m-1} \max(s - b(\xi), (c_1(\xi) - c_0(|\eta|^{-1}, \xi)) \, |\eta|) \tag{7}$$

if $|\eta| > 0$ and $s > b(\xi)$. The formula is true for all real η if we agree to interpret the right side as $|p(\xi)| \, (s - b(\xi))^m$ when $|\eta| = 0$.

Let $\Gamma' < \Gamma_1$ be compact, let c_1 and c_2 be the minima of $c_1(\xi)$ and $|p(\xi)|$ when $\xi \in \Gamma'$. Further, let $c_0(t)$ be the maximum of $c_0(t, \xi)$ and b the least upper bound of $b(\xi)$ when $\xi \in \Gamma'$. Then by virtue of (7)

$$|q(s\xi + i\eta)| \geq c_2 (s - b)^{m-1} \max(s - b, (c_1 - c_0(|\eta|^{-1})) \, |\eta|)$$

if $s > b$ and we interpret the right side as $c_2(s - b)^m$ when $|\eta| = 0$. It follows from Lemma 2.9 that $b < 1$ and we know that c_1 and c_2 are positive and that $c_0(t)$ tends to zero with t. Hence it follows that $|q(s\xi + i\eta)| \, (1 + |\eta|)^{-1} (s - b)^{-m}$ has a positive lower bound when $s \geq 1$. Hence we have proved

Lemma 3.1. *To every compact $\Gamma' < \Gamma_1(q, \bar{\xi})$ there exist numbers $B > 0$ and $b < 1$ such that*

$$|q(s\xi + i\eta)| \geq B(s - b)^m (1 + |\eta|)$$

when $\xi \in \Gamma'$ and $s \geq 1$.

Construction of the kernel. Let $\xi \in \Gamma_1$ so that $q(s\xi + i\eta) \neq 0$ when $s \geq 1$ and η is real. Then if the numbers $v_\nu = v_\nu(\xi, i\eta)$ are defined by (2) we have $-\Re v_\nu < 1$ so that if $s \geq 1$ then

$$\min_\nu \Re(s + v_\nu) > 0. \tag{8}$$

Let (2.6) refer to p and put $\xi = \bar{\xi}$ and $\eta = \xi$ in (2.7). Then

$$(9) \qquad p(\xi) = p(\bar{\xi}) \prod_1^m u_\nu(\bar{\xi}, \xi).$$

Because $\xi \in \Gamma$ then

$$(10) \qquad \min_\nu u_\nu(\bar{\xi}, \xi) > 0.$$

It follows from (2) and (9) that

$$q(s\xi + i\eta) = p(\bar{\xi}) \prod_1^m u_\nu(\bar{\xi}, \xi) \prod_1^m (s + v_\nu).$$

Hence if we define

$$\arg q(s\xi + i\eta) = \arg p(\bar{\xi}) + \sum_1^m \arg(s + v_\nu)$$

where $\arg p(\bar{\xi})$ is fixed once for all and $\max_\nu |\arg(s + v_\nu)| < \frac{1}{2}\pi$, then it follows from (8) and (10) that $\arg q(s\xi + i\eta)$ is a continuous function of s, ξ and η when $\xi \in \Gamma_1$, $s \geq 1$ and η is real. When α is a complex number we put with $q = q(s\xi + i\eta)$,

$$(11) \qquad q^{-\alpha} = e^{-\alpha(\log|q| + i \arg q)}.$$

In this way different choices of $\arg p(\bar{\xi})$ will affect $q^{-\alpha}$ only by a factor $e^{-2\pi k i \alpha}$ where k is either a positive or a negative integer. Now only integral values of α will be used in our final results and hence the particular choice of $\arg p(\bar{\xi})$ is of no real importance to us. If $p(\bar{\xi}) > 0$ then we can choose $\arg p(\xi) = 0$. This simple situation is brought about also in the general case provided that we change q to $p(\bar{\xi})^{-1} q$, a change which does not affect the manifold of solutions of (1).

Let B be an open subset of a real vector space of finite order. We define $C(k, B)$ to be the class of all complexvalued functions whose derivatives of order $\leq k$ exist and are continuous in B. Sometimes we write only $C(k)$, indicating B in another fashion. We let $C(\infty, B)$ be the intersection of all $C(k, B)$ for all possible k. When $\bar{B}$ is a part of the boundary of B we mean by $C(k, B + \bar{B})$ the class of all functions in $C(k, B)$ whose derivatives of order $\leq k$ have continuous extensions to $B + \bar{B}$.

Because M. Riesz [11] was the first to consider it in connection with the wave equation, we shall call the function $Q(\alpha, x)$ defined in our next theorem the Riesz kernel associated with q and $\bar{\xi}$.

Theorem 3.1. *Let q be a polynomial in n variables with complex coefficients which is reduced and hyperbolic with respect to $\bar{\xi}$. Let $\Re\, a > n$, let $\xi \in \Gamma_1(q, \bar{\xi})$, let $q(\zeta)^{-a}$ where $\zeta = \xi + i\eta$ and η is real, be defined by (11) and put*

$$Q(a, x) = (2\pi)^{-n} \int q(\zeta)^{-a} e^{(\zeta, x)}\, d\eta. \tag{12}$$

Then $Q(a, x)$ is independent of ξ and vanishes when $x \bar{\in} C(q, \bar{\xi})$. If $k < \Re\, a - n$, it belongs to $C(k)$ considered as a function of x and all its derivatives of order $\leq k$ are analytic in a when $\Re\, a > n + k$ and one has the inversion formula

$$q(\zeta)^{-a} = \int\limits_{C(q, \bar{\xi})} Q(a, x)\, e^{-(\zeta, x)}\, dx.^{1} \tag{13}$$

Proof. Let Γ' be a compact set in $\Gamma_1 = \Gamma_1(q, \bar{\xi})$. It follows from (11) and Lemma 3.1 that

$$|q(s\xi + i\eta)^{-a}| = |q|^{-\Re a} e^{\arg q\, \Im a} \leq (s - b)^{-m\Re a} B^{-\Re a} e^{|\Im a|(\frac{1}{2} m\pi + |\arg p(\xi)|)} (1 + |\eta|)^{-\Re a} \tag{14}$$

where $s \geq 1$, $b < 1$, $B > 0$ and $\xi \in \Gamma'$. Consequently the right side of (12) is absolutely and uniformly convergent when $\xi \in \Gamma'$, $\Re\, a > n$ and $|x| = \max_k |x_k|$ is bounded and we get

$$Q(a, x) = (2\pi)^{-n} \int\limits_{-\infty}^{+\infty} e^{(\zeta, x) - \zeta_h x_h}\, d\eta_1 \ldots d\eta_{h-1}\, d\eta_{h+1} \ldots d\eta_n \int\limits_{-\infty}^{+\infty} q(\zeta)^{-a} e^{\zeta_h x_h}\, d\eta_h$$

for all h. Because Γ_1 is open, (Lemma 2.9), there exists a $\delta > 0$ such that all ξ' with $|\xi - \xi'| \leq \delta$ are in Γ_1. Hence by an immediate application of Cauchy's theorem, the inner integral does not change if we replace ξ_h by any ξ'_h such that $|\xi_h - \xi'_h| \leq \delta$. Let $|\xi - \xi'| \leq \delta n^{-1}$ and change successively ξ_h to ξ'_h, $(h = 1, \ldots, n)$. Then $Q(a, x)$ does not change. Because Γ_1 is open and connected, (Lemma 2.9), this proves that $Q(a, x)$ is independent of ξ as long as $\xi \in \Gamma_1$.

If $x \bar{\in} C = C(q, \bar{\xi})$ there exists a $\xi' \in \Gamma = \Gamma(q, \bar{\xi})$ such that $(x, \xi') < 0$. But then according to Lemma 2.9 a suitable positive multiple ξ'' of ξ' is in Γ_1, and putting $\xi = s\xi''$ in (12) and letting $s \to \infty$ it follows from (14) and $(x, \xi) < 0$ that the integrand tends uniformly to zero and hence that $Q(a, x)$ vanishes.

Consider the formal derivatives with respect to x of order $< \Re\, a - n$ of the right side of (12). The resulting integrands are continuous in $(x_1, \ldots, x_n, \eta_1, \ldots, \eta_n)$

[1] Integrals of the type (12) and (13) when the integrand of (12) is square integrable occur in a wellknown theorem by R. E. A. C. Paley and N. Wiener (*Fourier transforms in the complex domain*, Amer. Math. Soc. Coll. Publ. XIX (1934) Theorem V p. 8) and in a generalization to several variables by S. Bochner (*Bounded analytic functions in several variables and multiple Laplace integrals*. Amer. Journ. of Math. 59 (1937) p. 732).

and analytic in a, and by virtue of (14) the integrals are absolutely and uniformly convergent when $|x|$ is bounded. Hence $Q(a, x) \in C(k)$ and its derivatives of order $\leq k$ are analytic in a when $\Re a > n + k$.

Let $\xi \in \Gamma_1$. It follows from (14) that $\int |q(\xi + i\eta)^{-a}|^2 \, d\eta < \infty$ and hence by Parseval's theorem also that $\int e^{-2(x, \xi)} |Q(a, x)|^2 \, dx < \infty$. Consider the integral in (13). Because Γ_1 is open there exists a $\xi' \in \Gamma$ such that $\xi - \xi' \in \Gamma_1$. By Schwarz's inequality

$$\Big\{\int_C |e^{-(x, \xi)} Q(a, x)| \, dx\Big\}^2 \leq \int_C e^{-2(x, \xi - \xi')} |Q(a, x)|^2 \, dx \int_C e^{-2(x, \xi')} \, dx.$$

Because Γ is open and C closed, (x, ξ') has a positive minimum δ when $x \in C$ and $|x| = 1$. It follows that $(x, \xi') \geq |x| \delta$ when $x \in C$ and consequently the right side of our last formula is finite. Hence the integral of (13) is absolutely and square convergent and the formula follows from Plancherel's theorem.

Remark. If q is homogeneous then we have $Q(a, tx) = t^{ma - n} Q(a, x)$ when $t > 0$. In fact, multiplying if necessary ξ in (12) by a positive number we may assume that $t\xi \in \Gamma_1$. Then putting $\zeta = t^{-1} \zeta'$ we get

$$Q(a, tx) = (2\pi)^{-n} \int q(t^{-1} \zeta')^{-a} e^{(\zeta', x)} t^{-n} \, d\eta' = t^{ma - n} Q(a, x).$$

Theorem 3.2. *If $\Re a > n + m$ and $\Re \beta > n$ and $Q(a, x)$ is defined by the preceding theorem then*

$$q(\partial / \partial x) \, Q(a, x) = Q(a - 1, x) \tag{15}$$

and

$$\int_{x - y \in C, \, y \in C} Q(a, x - y) \, Q(\beta, y) \, dy = Q(a + \beta, x) \tag{16}$$

where $C = C(q, \xi)$.

Proof. The formula (15) follows by differentiation of (12), and (16) from (13) and Plancherel's theorem.

Two examples. In the two special cases treated on p. 25, 29 we can calculate the Riesz kernel explicitly.[1]

Example 1. In this case $q(\eta) = p(\eta) = \eta_1^2 - \eta_2^2 - \cdots - \eta_n^2$. Let $\Gamma = \Gamma(q, \xi)$ and $C = C(q, \xi)$ be defined as before, let $\xi \in \Gamma$ and let $\Re a > \frac{1}{2}(n - 2)$. Then by a Lorentz transformation and some elementary integrations we get

[1] In most cases one cannot hope to get so simple explicit expressions for the Riesz kernel as in these examples.

$$q(\xi)^{-\alpha} = \int_C \frac{q(x)^{\alpha - \frac{1}{2}n}}{H_n(\alpha)} e^{-(x,\xi)} \, dx \tag{17}$$

where

$$H_n(\alpha) = \pi^{\frac{1}{2}(n-1)} \Gamma(2\alpha) \Gamma(\alpha - \tfrac{1}{2}(n-2)) / \Gamma(\alpha + \tfrac{1}{2}).$$

If we replace ξ by $\zeta = \xi + i\eta$ where η is real, both sides of (17) are analytic in ζ if only $\xi \in \Gamma$. Now they are equal when $\eta = 0$ so that it follows that they are equal for all η. Hence by virtue of (13) we get

$$Q(\alpha, x) = \begin{cases} 0, & x \bar{\in} C \\ q(x)^{\alpha - \frac{1}{2}n} / H_n(\alpha), & x \in C. \end{cases}$$

This kernel was introduced directly by M. Riesz and it is the starting point in his theory of the wave equation.[1]

Example 2. We have with our previous notations $q(\eta) = p(\eta) = \det \eta$. Let $\Gamma = \Gamma(q, \xi)$ and $C = C(q, \xi)$ be defined as before, let $\xi \in \Gamma$, let $\Re \alpha > \bar{n}$. A formula by Bochner ([1] p. 694—696) reads

$$q(\xi)^{-\alpha} = \int_C \frac{q(x)^{\alpha - \bar{n}}}{L_{\bar{n}}(\alpha)} e^{-\sigma(x\xi^*)} \, dx \tag{18}$$

where

$$L_{\bar{n}}(\alpha) = \pi^{\frac{1}{2}\bar{n}(n-1)} \prod_{1}^{\bar{n}} \Gamma(\alpha - k + 1).$$

Arguing as in Example 1 above we see that (18) is still valid if ξ is changed to $\xi + i\eta$ where η is hermitian. Hence we get

$$Q(\alpha, x) = \begin{cases} 0, & x \bar{\in} C \\ q(x)^{\alpha - \bar{n}} / L_{\bar{n}}(\alpha), & x \in C. \end{cases}$$

It is easy to verify that in terms of the variables x_{jk} instead of $x_1, \ldots$ the associated differential operator becomes

$$q(\partial / \partial x) = \det(\partial / \partial x_{jk}).$$

The Riesz kernel and the elementary solution. Hadamard has called $Q(1, x)$ of Example 1 the elementary solution of the wave equation if n is odd.[2] The corresponding function has been used also by others in more general situations. Consider (12) and assume for a moment that $\alpha = 1$ and that $q = p$ is homo-

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[1] [11] p. 31—33.

[2] See Riesz [11] p. 95—99.

geneous and not degenerate. Assume also that $\xi = \bar{\xi} = (1, 0, \ldots, 0)$ and perform the integration with respect to η_1. Formally the result is

$$Q(1, x) = (2\pi)^{1-n} \int e^{i x_2 \eta_2 + \cdots} \Big(\sum_1^m p_1^{-1}(a_\nu)\, e^{i a_\nu x_1}\Big)\, d\eta_2 \ldots d\eta_n,$$

where $a_\nu = a_\nu(\eta_2, \ldots, \eta_n)$ are the necessarily real zeros of $p(t, \eta_2, \ldots, \eta_n) = 0$ and $p_1 = \partial p / \partial \eta_1$. The integral is easily seen to converge when $m > n$ and is the starting point in the work by Herglotz [6] and Petrowsky [8]. Zeilon [13] uses as elementary (fundamental) solution of (1) the formal expression

$$(2\pi)^{-n} \int q^{-1}(i\eta)\, e^{i(x, \eta)}\, d\eta$$

which he sums by various devices, not confining himself to hyperbolic q.

A close study of the kernel $Q(a, x)$ will give much information about the differential equation (1). As is shown by the work of Herglotz and Petrowsky and our examples we must expect the vectorspace E where Q is defined to split into a finite number of open subsets E_k, domains of analyticity for Q and a $n-1$-dimensional part E' such that $Q(a, x)$ is an entire function of a when $x \in E_k$ and satisfies (15) there. On E' the kernel Q is defined by continuity when $\Re a$ is large but might be discontinuous there for other values of a, as is the case in Example 1. In this case E_1 equals the interior C^i of C while $E_2 = E - C$ and $E' = C - C^i$. For any hyperbolic q, $E - C(q, \bar{\xi})$ is a trivial domain of analyticity because $Q(a, x)$ vanishes identically there. In Chapter 6 we will return to these considerations.

Our main object in the next chapter is Theorem 4.1 whose formulation does not involve the Riesz kernel. It is in fact also possible to prove this theorem by means of an immediate generalization of Lemma 3.1 to not necessarily reduced hyperbolic polynomials. The same remark applies to Lemma 4.1. Because only Theorem 4.1 and Lemma 4.1 are used to prove Theorem II of the introduction it would be possible and perhaps also natural to prove this theorem without any use of the Riesz kernel. However, the generalization of Riesz's theory for the wave equation which we give in Theorem 4.2 might be worth giving for its own sake. In Chapter 5 we give a more complete generalization of Riesz's theory but only for homogeneous and reduced polynomials. Here the theory yields very explicit results and eliminates to a great extent the heavy use of Fourier-Laplace transforms which is its only known substitute in this case.

Chapter 4.

The Riesz Operator.

Definition of the Riesz operator. Two theorems. It is assumed in this section that q is a not constant polynomial in n variables which is hyperbolic with respect to ξ. Let $\Gamma = \Gamma(q, \xi)$ and $C = C(q, \xi)$ be the two cones associated with q and defined in Chapter 2. Assume for a moment that q is also reduced. Then we can construct the Riesz kernel $Q(\alpha, x)$ corresponding to q and ξ. Let $S = S(\xi)$ be the plane $(y, \xi) = 0$ and $T = T(\xi)$ the region $(y, \xi) > 0$. Finally, let $f \in C(\infty, T + S)$. We define the Riesz operator I^α by the formula[1]

$$(1) \qquad I^\alpha f(x) = \int_T Q(\alpha, x-y) f(y)\, dy$$

when $\Re\,\alpha > n$ and $x \in T$. The kernel $Q(\alpha, x-y)$ is different from zero only in the set $C(x)$ of points $y \in T + S$ such that $x - y \in C$, i.e. when $(x-y, \xi) = (x, \xi) - (y, \xi) \leq (x, \xi)$, which is bounded and closed by Lemma 2.13. Hence the integral always exists.

Let C^0 be the set of functions in $C(\infty) = C(\infty, E)$ which vanish outside some compact set. Let A_1 be a bounded open set in T and let A be its completion with respect to C, i.e. the closure of the union of all $C(x)$ where $x \in A$. Then A is bounded and closed and its own completion with respect to C. Let $a \in C^0$ and let $a(y) = 1$ when $y \in A$. Then if $x \in TA$[2] it is clear that $I^\alpha f(x) = I^\alpha a(x) f(x)$. Now let $\xi \in \Gamma_1 = \Gamma_1(q, \xi)$ and let F_a be the Fourier-Laplace transform of af,

$$(2) \qquad F_a(\xi + i\eta) = \int_T e^{-(\xi + i\eta, y)} a(y) f(y)\, dy.$$

It follows from this formula, Theorem 3.1, the formula (3.14) and Parseval's formula that

$$(3) \qquad I^\alpha f(x) = (2\pi)^{-n} \int F_a(\xi + i\eta)\, q(\xi + i\eta)^{-\alpha} e^{(\xi + i\eta, x)}\, d\eta$$

when $\Re\,\alpha > n$ and $x \in TA$. It will be shown that also when q is not necessarily reduced, the right side has a sense even when $\Re\,\alpha > 0$, that it is independent of ξ, a and A as long as $\xi \in \Gamma_1$ and $TA \ni x$. The function $q(\xi + i\eta)^{-\alpha}$ is here still assumed to be defined by (3.11) which obviously applies to all not constant hyperbolic polynomials, reduced or not reduced.

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[1] Riesz [11] p. 47.

[2] TA stands for the common part of T and A.

Because q is hyperbolic also with respect to $-\bar{\xi}$, (Lemma 2.2), we may perform the constructions above starting from q and $-\bar{\xi}$. Then we have to consider $\Gamma^- = \Gamma(q, -\bar{\xi})$ which obviously equals $-\Gamma$ and $C^- = C(q, -\bar{\xi}) = -C$ and if q is reduced, the corresponding Riesz kernel $Q^-(\alpha, x)$. The Riesz operator I_-^α is in this case defined by

$$I_-^\alpha f(x) = \int_{T^-} Q^-(\alpha, x-y) f(y)\, dy \tag{4}$$

where $f \in C(\infty, T^- + S)$ and $x \in T^- = T(-\bar{\xi})$, i.e. $(x, \bar{\xi}) < 0$. Proceding as above we define $C^-(x)$ to be the set of all $y \in T^- + S$ such that $x - y \in C^-$. It is bounded and closed. Let A_1^- be an open and bounded set in T^- and A^- its completion with respect to C^- and let $a^- \in C^0$ and let $a^-(y) = 1$ when $y \in A^-$. Then if ξ is chosen so that besides $\xi \in \Gamma_1(q, \bar{\xi})$ also $-\xi \in \Gamma_1(q, -\bar{\xi})$ we get

$$I_-^\alpha f(x) = (2\pi)^{-n} \int F_{a^-}^-(-\xi + i\eta)\, q(-\xi + i\eta)^{-\alpha} e^{(-\xi + i\eta, x)}\, d\eta \tag{5}$$

where $\Re\, \alpha > n$, $x \in T^- A^-$ and

$$F_{a^-}^-(-\xi + i\eta) = \int_{T^-} a^-(y) f(y)\, e^{-(-\xi + i\eta, x)}\, dy. \tag{6}$$

As (3) the formula (5) has a sense also when q is not reduced and its right side is independent of ξ, a^- and A^- as long as $-\xi \in \Gamma_1(q, -\bar{\xi})$ and $T^- A^- \ni x$.

The following theorem lists the most important properties of the Riesz operator

Theorem 4.1. *If q is a not constant polynomial which is hyperbolic with respect to $\bar{\xi}$, if $f \in C(\infty, T + S)$ and $x \in T$, the function $I^\alpha f(x)$ defined by (3) when $\Re\, \alpha > 0$ is independent of ξ, A and a as long as $\xi \in \Gamma_1(q, \bar{\xi})$ and $TA \ni x$. It is entire analytic in α and for all values of α it belongs to $C(\infty, T)$ considered as a function of x and all its derivatives with respect to x are entire analytic in α. For all α it vanishes at a point x if f vanishes in $C(x)$ and one has*

$$q(\partial / \partial x)\, I^{\alpha+1} f(x) = I^\alpha f(x) \tag{7}$$

and if $\Re\, \alpha, \Re\, \beta > n$

$$I^\alpha I^\beta f(x) = I^{\alpha+\beta} f(x), \tag{8}$$

and when k is a positive integer or zero,

$$I^{-k} f(x) = q(\partial / \partial x)^k f(x). \tag{9}$$

Let $f_1, f_2, \ldots, f_k, \ldots$ be a sequence of functions in $C(\infty)$. What is meant by saying that $f_k \to 0$, i.e. that f_k tends strongly to zero with $1/k$, is explained in the introduction. The proof of our next theorem is the main step in the proof of Theorem II.

Theorem 4.2. *If $f \in C(\infty, T+S)$ then $If = I^1 f$ is in $C(\infty, T+S)$. If m is the degree of q, the derivatives of If of order $< m$ vanish on S and one has*

$$q(\partial/\partial x)\, If(x) = f(x). \tag{10}$$

If $f \in C(\infty)$ and $If(x)$ is defined as $I^1_- f(x)$ when $x \in T^-$, i.e. when $(x, \xi) < 0$, then If belongs to $C(\infty)$ and if f_k tends strongly to zero with $1/k$, the same is true of If_k.

Proof of Theorem 4.1. Let M be a real square matrix of order n whose determinant $\det M$ has absolute value 1 and change variables in (2) and (3) according to the formulas $x = x' \tilde{M}^{-1}$ and $\zeta = \zeta' M$ where $\tilde{M}$ is the transpose of M and $\zeta = \xi + i\eta$ and $\zeta' = \xi' + i\eta'$. With $q'(\zeta') = q(\zeta' M) = q(\zeta)$ and $f'(y') = f(y)$ the result is

$$I^\alpha f'(x') = (2\pi)^{-n} \int F'_{a'}(\zeta')\, q'(\zeta')^{-\alpha} e^{(\zeta', x')}\, d\eta' \tag{11}$$

where x' belongs to the image A' of A under the mapping

$$x \to x' = x\tilde{M} \tag{12}$$

and

$$F'_{a'}(\zeta') = \int_{T'} e^{-(\zeta', y')}\, a'(y')\, f'(y')\, dy' \tag{13}$$

where $T' = T(\xi')$, $f'(y') = f(y' \tilde{M}^{-1}) = f(y)$ and $a'(y') = a(y)$.

Now (11) and (13) define together the Riesz operator I^α when it refers to q' and $\xi' = \xi M$. In fact, if $S' = S(\xi')$ then $f' \in C(\infty, T' + S')$, the degree of q' is the same as that of q and hence positive, if p' is the principal part of q' then $p'(\xi') = p(\xi) \neq 0$. Also, $q'(t\xi' + i\eta') = q(t\xi + i\eta) \neq 0$ if η is real and $t > t_0$ where t_0 is large enough, so that q' is hyperbolic with respect to ξ'. It is immediate to verify that $\Gamma' = \Gamma(q', \xi')$ and $\Gamma'_1 = \Gamma_1(q', \xi')$ are the images of $\Gamma = \Gamma(q, \xi)$ and $\Gamma_1 = \Gamma_1(q, \xi)$ under the mapping $\eta \to \eta' = \eta M^{-1}$. Hence ξ' is in Γ'_1 and $C' = C(q', \xi')$ and $C'(x')$, defined as the set of all y' such that $x' - y' \in C'$ and $(y', \xi') \geq 0$, are the images of C and $C(x)$ under (12) and if A'_1 and A' are the images of A_1 and A it is clear that A'_1 is an open bounded set in

T' and that A' is its completion with respect to C'. Finally, $a' \in C^0$ and $a'(y') = 1$ when $y' \in A'$.

Assume for a moment that the theorem is true when it refers to q' and $\bar{\xi}'$. Then we know in particular that $I^\alpha f'(x')$ as defined by (11) and (13) when $\Re \alpha > 0$ and $x' \in T'$ is independent of ξ', a' and A' as long as $\xi' \in \Gamma_1'$ and $T'A' \ni x'$. But then the first sentence of the theorem follows for $I^\alpha f(x) = I^\alpha f(x' \tilde{M}^{-1}) = I^\alpha f'(x')$ and it is a matter of straightforward verification to show that the entire theorem is true for $I^\alpha f(x)$. Hence it is enough to prove the theorem when it refers to q' and $\bar{\xi}'$. The same remark is in a similar fashion seen to apply to Theorem 4.2 and Lemma 4.1 at the end of this chapter.

It is clear that $\bar{\xi} \bar{\in} \Omega(q)$. Hence Lemma 2.1 shows that if we choose the columns $\eta^{(1)}, \ldots, \eta^{(n)}$ of M such that $\eta^{(1)}$ is a positive multiple of $\bar{\xi}$ and $\eta^{(l+1)}, \ldots, \eta^{(n)}$ form a basis of $\Omega(q)$, then $q(\zeta' M) = q'(\zeta_1', \ldots, \zeta_l')$ does not depend on $\zeta_{l+1}', \ldots, \zeta_n'$ and is reduced considered as a polynomial in $\zeta_1', \ldots, \zeta_l'$. Clearly $\bar{\xi}' = (\bar{\xi}_1', 0, \ldots, 0)$ where $\bar{\xi}_1'$ is positive. Clearly we may fulfil these requirements with an M such that the absolute value of its determinant is 1. Hence deleting for simplicity the primes we may suppose from the beginning, without loss of generality, that the polynomial q in Theorem 4.1 is a reduced polynomial in $\zeta_1, \ldots, \zeta_l$ alone and that $\bar{\xi} = (\bar{\xi}_1, 0, \ldots, 0)$ where $\bar{\xi}_1$ is positive.

Integrating by parts in (3) we see that $\zeta_1 \zeta_k^N F_a(\zeta)$ is bounded when N is a not negative integer and $k > 1$. Hence using the notation

$$|c_1, \ldots, c_s| = \max(|c_1|, \ldots, |c_s|)$$

when $c_1, \ldots, c_s$ are any complex numbers we get

$$F_a(\zeta) = O(|1, \eta_1|^{-1}) \, O(|1, \zeta_2, \ldots, \zeta_n|^{-N}) \tag{14}$$

for all N.

Let $\bar{\xi}_{(l)}$ be the vector composed by the first l components of $\bar{\xi}$. Corresponding to $q_l = q(\zeta_1, \ldots, \zeta_l)$ and $\bar{\xi}_{(l)}$ we can construct $\Gamma_l = \Gamma(q_l, \bar{\xi}_{(l)})$, $C_l = C(q_l, \bar{\xi}_{(l)})$ and a Riesz kernel $Q_l(\alpha, x) = Q_l(\alpha, x_1, \ldots, x_l)$. It is clear that $\Omega(q)$ consists of all η such that $\eta_1 = \cdots = \eta_l = 0$. Hence it follows from Lemma 2.8 that $\Gamma = \Gamma(q, \bar{\xi})$ consists of all η such that $(\eta_1, \ldots, \eta_l) \in \Gamma_l$ and from Lemma 2.13 that $C = C(q, \bar{\xi})$ consists of all x such that $(x_1, \ldots, x_l) \in C_l$ and $x_{l+1} = \cdots = x_n = 0$. By virtue of Lemma 3.1,

$$|q(\zeta)| > B |1, \eta_1, \ldots, \eta_l|, \quad (B > 0), \tag{15}$$

so that by (3.11)

$$q(\zeta)^{-\alpha} = O(|1, \eta_1, \ldots, \eta_l|^{-\Re\alpha}) \tag{16}$$

when $\Re\alpha > 0$. If $\Re\alpha < 0$ using the same formula we get

$$q(\zeta)^{-\alpha} = O(|1, \eta_1, \ldots, \eta_l|^{-m\Re\alpha}), \tag{17}$$

where m is the degree of q.

Hence by (16) and (14), the integrand of (3) is absolutely and square integrable and the integral is analytic in α and continuous in x when $\Re\alpha > 0$. Hence from (2) and Plancherel's theorem

$$I^\alpha f(x) = (2\pi)^{-l} \int q(\zeta)^{-\alpha} F_a(\zeta_1, \ldots, \zeta_l, x_{l+1}, \ldots) e^{\zeta_1 x_1 + \cdots + \zeta_l x_l} d\eta_1 \ldots d\eta_l$$

where

$$F_a(\zeta_1, \ldots, \zeta_l, x_{l+1}, \ldots, x_n) =$$
$$= \int_{y_1>0} e^{\zeta_1 y_1 + \cdots + \zeta_l y_l} a(y_1, \ldots, y_l, x_{l+1}, \ldots) f(y_1, \ldots, x_{l+1}, \ldots) dy_1 \ldots dy_l.$$

Hence by Parseval's theorem and Theorem 3.1 we get

$$I^\alpha f(x) = \int_{y_1>0} Q_l(\alpha, x-y) a(y_1, \ldots, y_l, x_{l+1}, \ldots) f(y_1, \ldots, y_l, x_{l+1}, \ldots) dy_1 \ldots dy_l$$

if $\Re\alpha > l$. Here $Q_l(\alpha, x-y)$ vanishes except when $y \in C_l(x)$, i.e. when $(x_1 - y_1, \ldots, x_l - y_l) \in C_l$ and $y_1 > 0$ and in this region $a(y_1, \ldots)$ equals 1. Hence

$$I^\alpha f(x) = \int_{y_1>0} Q_l(\alpha, x-y) f(y_1, \ldots, y_l, x_{l+1}, \ldots, x_n) dy_1 \ldots dy_l, \quad (x \in TA). \tag{18}$$

This proves that if $\Re\alpha > l$, then $I^\alpha f(x)$ is independent of ξ, a and A as long as $\xi \in \Gamma_1(q, \bar{\xi})$ and $TA \ni x$ and, naturally, $a \in C^0$ and $a(y) = 1$ when $y \in A$. Moreover, it vanishes if f vanishes in $C(x)$. The same results follow by analytical continuation for all α if we can show, as we will do next, that $I^\alpha f(x)$ is an entire analytic function of α.

In (2) and (3) choose $\xi \in \Gamma_1$ such that it is a positive multiple of $\bar{\xi}$. This is possible according to Lemma 2.13. Then $\zeta_1 = \xi_1 + i\eta_1$ and $\zeta_k = i\eta_k$ when $k > 1$. Let $g_j(y) = (\partial/\partial y_1)^j g$ where $g = af$ and put

$$\bar{g}_k(y) = g(y) - \sum_0^{k-1} y_1^j g_j(0, y_2, \ldots, y_n)/j!. \tag{19}$$

It then follows from (2) that

$$F_a = F = \sum_0^{k-1} \zeta_1^{-j-1} \int e^{-\zeta_2 y_2 - \cdots - \zeta_n y_n} g_j(0, y_2, \ldots, y_n) dy_2 \ldots dy_n + \int_{y_1>0} e^{-(\zeta, y)} \bar{g}_k(y) dy. \tag{20}$$

Write the last sum as

$$(21) \qquad F = \sum_{0}^{k-1} \zeta_1^{-j-1} F_j(\zeta_2, \ldots, \zeta_n) + \bar{F}_k(\zeta).$$

From the fact that $g_j(y)$ and $\bar{g}_k(y)$ are in $C(\infty, T+S)$ and vanish outside a compact set and that the derivatives of $\bar{g}_k$ of order $\leq k$ with respect to y_1 vanish when $y_1 = 0$ it follows by partial integration that $\zeta_h^N F_j$ and $\zeta_1^{1+k}\zeta_h^N \bar{F}_k$ are bounded when $h = 2, \ldots, n$ and N is any not negative integer. Hence

$$(22) \qquad \begin{aligned} F_j &= O(|1, \eta_2, \ldots, \eta_n|^{-N}) \\ \bar{F}_k &= O(|1, \eta_1|^{-k-1})\, O(|1, \eta_2, \ldots, \eta_n|^{-N}). \end{aligned}$$

By virtue of (2) and (21) we get

$$(23) \qquad I^\alpha f(x) = (2\pi)^{1-n} \int e^{\zeta_2 x_2 + \cdots + \zeta_n x_n} d\eta_2 \ldots d\eta_n \frac{1}{2\pi i} \int_{\xi_1 - i\infty}^{\xi_1 + i\infty} \sum_{0}^{k-1} \frac{\zeta_1^{-j-1} F_j}{q(\zeta)^\alpha} e^{\zeta_1 x_1} d\zeta_1 +$$

$$+ (2\pi)^{-n} \int e^{(\zeta, x)} \bar{F}_k\, q(\zeta)^{-\alpha} d\eta.$$

Consider the inner integral of the first term

$$(24) \qquad U = \frac{1}{2\pi i} \int_{\xi_1 - i\infty}^{\xi_1 + i\infty} q(\zeta)^{-\alpha} \sum_{0}^{k-1} \zeta_1^{-j-1} F_j e^{\zeta_1 x_1} d\zeta_1.$$

The singular points of the integrand are the zeros $\zeta_1' = v_\nu(\eta)$ of the equation $q(\zeta_1', \ldots, \zeta_l) = 0$, $(\nu = 1, \ldots, m)$. Because $\xi \in \Gamma_1$, and $-\xi \in \Gamma_1^-$ it follows that $\max_\nu \Re v_\nu(\eta) \leq b' < \xi_1$ and that $\min_\nu \Re v_\nu(\xi) \geq b'' > -\xi_1$ for some real b' and b'' and hence that

$$(25) \qquad \max_\nu |\Re v_\nu(\eta)| \leq b < \xi_1$$

for $b = |b', b''|$ and all real η. Put

$$c = c(\eta) = \xi_1 - b + |v_1, \ldots, v_m|$$

and let R be the contour $R_1 + R_2 + R_3$ where R_1 is a straight line from $-\infty - ic$ to $\xi_1 - ic$, R_2 one from $\xi_1 - ic$ to $\xi_1 + ic$ and R_3 one from $\xi_1 + ic$ to $-\infty + ic$. On and outside R we can choose $\max_\nu |\arg(\zeta_1' - v_\nu)| < \pi$. Then

$$\arg q(\zeta_1', \zeta_2, \ldots, \zeta_l) = \arg p(1, 0, \ldots, 0) + \sum_{1}^{m} \arg(\zeta_1' - v_\nu)$$

and, by virtue of (3.11), also the integrand of (24) with ζ_1 changed to ζ_1' is a continuous function of ζ_1' in the same region. Obviously the integrand equals $O(|\zeta_1'|^{-1-m\,\Re\alpha}\, e^{x_1\xi_1'})$ there. Hence because x_1 and $\Re\alpha$ are positive in (24) we can change the contour of integration to R and get

$$U = \frac{1}{2\pi i}\int_R q(\zeta_1', \zeta_2, \ldots)^{-\alpha} \sum_0^{k-1} \zeta_1'^{-j-1} F_j e^{\zeta_1' x_1}\, d\zeta_1'. \tag{26}$$

It follows from (25) and the definition of c that $|\zeta_1' - v_\nu| \geq b_1 = \xi_1 - b > 0$, and hence that

$$\left|\frac{\zeta_1'}{\zeta_1' - v_\nu}\right| \leq 1 + \left|\frac{v_\nu}{\zeta_1' - v_\nu}\right| \leq 1 + \frac{|v_\nu|}{b_1}$$

when $\zeta_1' \in R$. Also, $|\zeta_1'| \geq b_1$ on R so that the same inequality is true for $(\zeta' - v_\nu)/\zeta_1'$. Hence

$$\left(1 + \frac{|v_\nu|}{b_1}\right)^{-1} \leq \left|\frac{\zeta_1'}{\zeta_1' - v_\nu}\right| \leq 1 + \frac{|v_\nu|}{b_1}$$

for all ν if $\zeta_1' \in R$. Now

$$q(\zeta_1', \zeta_2, \ldots) = p(1, 0, \ldots) \prod_1^m (\zeta_1' - v_\nu)$$

and c and v_ν are both $O(|1, \eta_2, \ldots, \eta_l|)$. Hence it follows from (3.11) that

$$\zeta_1'^{m\alpha}\, q(\zeta_1', \zeta_2, \ldots)^{-\alpha} = O(|1, \eta_2, \ldots, \eta_l|^{m|\Re\alpha|}) \tag{27}$$

for all α when $\zeta_1' \in R$. Because $c = c(\eta) \geq b_1$ it follows that

$$|t \pm i b_1| \leq |\zeta_1'| \leq \frac{c}{b_1}|t \pm i b_1|$$

when $\zeta_1' = t - ic \in R_1$ or $\zeta_1' = t + ic \in R_3$. Hence if h is an integer, positive or negative or zero,

$$\int_{R_1+R_3} |\zeta_1'^h e^{\zeta_1' x_1}|\, |d\zeta_1'| \leq 2\,|1, c^h/b_1^h| \int_{-\infty}^{\xi_1} |t + i b_1|^h e^{x_1 t}\, dt.$$

The analogous integral over R_2 equals $O(c^{h+2} e^{\xi_1 x_1})$ and hence we get

$$\int_R |\zeta_1'^h e^{\zeta_1' x_1}|\, |d\zeta_1'| = O(|1, \eta_2, \ldots, \eta_l|^{h+2})$$

uniformly when $0 < \delta \leq x_1 \leq \delta^{-1}$. Hence combining this formula with (27) and (22) it follows that all the derivatives with respect to x_1 of $U = U(a, x_1, \zeta_2, \ldots, \zeta_n)$ are continuous in x_1 and analytic in a when $x_1 > 0$ and equal

$$O(|1, \eta_2, \ldots, \eta_n|^{-N})$$

for every $N \geq 0$, uniformly when $0 < \delta \leq x_1 \leq \delta^{-1}$. Hence the first term of (23) which can be written

$$(2\pi)^{1-n} \int e^{\zeta_2 x_2 + \cdots + \zeta_n x_n} U(a, x_1, \zeta_2, \ldots, \zeta_n)\, d\eta_2 \ldots d\eta_n$$

belongs to $C(\infty, T)$ and is together with all its derivatives with respect to x entire analytic in a. Consider the last term. By virtue of (27) which in particular is valid when $\zeta_1' = \zeta_1 = \xi_1 + i\eta_1$, and (22), it is absolutely convergent, continuous in x and analytic in a when $\Re a > -k/m$ and $x_1 \geq 0$, i.e. when $x \in T + S$, and its derivatives with respect to x of order $< k + m\Re a$ which can be computed by formal differentiation under the sign of integration, have the same properties. Now k is arbitrarily large, and that proves the second sentence of the theorem.

It was shown that every derivative of $I^a f(x)$ with respect to x is entire analytic in a. Hence (7) which is an immediate consequence of Theorem 3.2 for large $\Re a$ is true for all a. As to (8) it follows if we apply Theorem 3.2 and use the form (18) of $I^a f(x)$.

To prove (9) we proceed as follows. Let $h \in C(\infty)$ and let it vanish outside some closed set contained in the interior of A and put

$$H(\zeta) = \int e^{(\zeta, x)} h(x)\, dx. \tag{28}$$

Let $q_1(\zeta)$ be any polynomial and $\tilde{q}_1(\zeta) = q_1(-\zeta)$ its adjoint. Then one gets by partial integration

$$q_1(\zeta) H(\zeta) = \int e^{(\zeta, y)} \tilde{q}_1(\partial/\partial x) h(x)\, dx. \tag{29}$$

The right side is bounded. Hence

$$H(\zeta) = O(|\zeta_1, \ldots, \zeta_n|^{-N}) \tag{30}$$

for any positive N. Hence from (28) and Plancherel's theorem we get

$$h(x) = (2\pi)^{-n} \int H(\zeta)\, e^{-(\zeta, x)}\, d\eta.$$

Hence by (3) and Parseval's formula

$$\int h(x) I^a f(x)\, dx = (2\pi)^{-n} \int H(\zeta) F_a(\zeta)\, q(\zeta)^{-a}\, d\eta.$$

Here by virtue of (30), (14), (16) and (17) the right side is an entire function of a, and we know already that the same is true of the left side. Putting $a = -k$, it follows by (2) and (29) and Parseval's formula and partial integration that

$$\int h(x)\, I^{-k} f(x)\, dx = (2\pi)^{-n} \int H(\zeta)\, F_a(\zeta)\, q(\zeta)^k\, d\eta =$$

$$= \int f(x)\, a(x)\, \tilde{q}(\partial/\partial x)^k\, h(x)\, dx =$$

$$= \int f(x)\, \tilde{q}(\partial/\partial x)^k\, h(x)\, dx = \int h(x)\, q(\partial/\partial x)^k f(x)\, dx.$$

Hence because h is arbitrary, (9) follows. This completes the proof of the theorem.

Proof of Theorem 4.2. It is already clear by Theorem 4.1 that $If(x)$ is independent of ξ, a and A as long as $\xi \in \Gamma_1(q, \bar{\xi})$ and $A \ni x$, and that it is in $C(\infty, T)$. Hence we have to consider it when $x \in S(\bar{\xi})$, i.e. with our special choice of $\bar{\xi}$, when $x_1 = 0$.

If $a = 1$, the integrand of (26) equals $O(|\zeta_1'|^{-1-m})$ and is a singlevalued function of ζ_1'. Hence it follows from (25) that we can deform the contour R of (26) to $R' = R_0' + R_1' + R_2' + R_3'$ where R_0' is a straight line from $-\xi_1 + ic$ to $-\xi_1 - ic$, R_1' one from $-\xi_1 - ic$ to $\xi_1 - ic$, $R_2' = R_2$, and R_3' one from $\xi_1 + ic$ to $-\xi_1 + ic$. As when a is arbitrary we get that

$$(31) \qquad \frac{1}{2\pi i} \int_{R'} q(\zeta_1', \ldots, \zeta_l)^{-1} \sum_0^{k-1} \zeta_1'^{-j-1} F_j\, e^{\zeta_1' x_1}\, d\zeta_1'$$

equals $O(|1, \eta_2, \ldots, \eta_n|^{-N})$ for all positive N but now clearly uniformly when $|x_1|$ is bounded. The same is true for its derivatives with respect to x_1 which all can be computed by formal differentiation. Hence the first term of

$$(32) \qquad (2\pi)^{1-n} \int e^{\zeta_2 x_2 + \cdots + \zeta_n x_n}\, d\eta_2 \ldots d\eta_n \frac{1}{2\pi i} \int_{R'} \sum_0^{k-1} \frac{\zeta_1'^{-j-1} F_j}{q(\zeta_1', \ldots)} e^{\zeta_1' x_1}\, d\zeta_1' +$$

$$+ (2\pi)^{-n} \int e^{(\zeta, x)}\, q(\zeta)^{-1} \tilde{F}_k\, d\eta$$

is in $C(\infty, T + S)$. It follows from (27) and (22) that the second term is in $C(k + m - 1, T + S)$. Because k is arbitrary and $If(x)$ is defined by (32) when $x \in A$ which is arbitrarily large in $T + S$, this shows that $If \in C(\infty, T + S)$.

It is important for the following that the derivatives of order $< k + m$ of the last term of (32) vanish when $x_1 = 0$. In fact, any such derivative can be computed formally and the resulting integrand tends uniformly to zero when $x_1 = 0$ and ξ_1 tends to infinity. Hence the integral, which is independent of ξ, vanishes. The same argument applied to (3) with $a = 1$ shows that the derivatives of order $< m$ of $If(x)$ vanish when $x \in S$.

Put $I^- f(x) = I^1_- f(x)$. It is defined by (5) when $a = 1$ and $x \in A^-$. An obvious modification of the arguments above shows that $I^- f \in C(\infty, T^- + S)$ and that its derivatives of order $< m$ vanish on S. It remains to show that If and $I^- f$ have the same derivatives on S. If $b \in C^0$ and $b(y) = 1$ when $y \in A$ or $y \in A^-$ then we can obviously put $a = a^- = b$. Putting $g = bf$ and writing the formulas corresponding to (19), (20) and (21) for $F^- = F^-_{a^-}$ we get after an easy calculation

$$F^- = -\sum_0^{k-1} \zeta_1^{-j-1} F_j^- (\zeta_2, \ldots, \zeta_n) + \bar{F}_k^- (\zeta).$$

Now it is clear that $F_j^- = F_j$ for all j and that in the formula corresponding to (31) we have to integrate along R' but in the opposite direction. Consequently, if $x \in A^-$, then $I^- f(x)$ is still defined by (32) with the only difference that $\bar{F}_k$ is changed to $\bar{F}_k^-$ and ξ to $-\xi$. Now it follows precisely as for (32) that the derivatives of order $< m + k$ of the last term of the modified formula vanish when $x_1 = 0$. Hence the derivatives of order $< m + k$ of $If - I^- f$ vanish on the common part AA^- of A and A^- which is a part of S. Hence because k is arbitrary and AA^- is arbitrarily large in S it follows that if we put $If(x) = I^- f(x)$ when $x \in T^-$, then $If \in C(\infty)$ and its derivatives of order $< m$ vanish on S. As to (10) it follows from (7) and (9).

It remains to prove the continuity part of the theorem. Let $f^{(1)}, \ldots, f^{(h)}, \ldots$ be a sequence of functions in $C(\infty)$ such that $f^{(h)} \to 0$. Let $g^{(h)} = af^{(h)}$, put $g_j^{(h)}(x) = (\partial / \partial x_1)^j g^{(h)}(x)$,

$$\bar{g}_k^{(h)}(x) = g^{(h)}(x) - \sum_0^{k-1} x_1^j g_j^{(h)}(0, x_2, \ldots, x_n)/j!,$$

$$F_j^{(h)} = \int g_j^{(h)}(0, x_2, \ldots, x_n) e^{-\zeta_2 x_2 - \cdots - \zeta_n x_n} dx_2 \ldots dx_n \tag{33}$$

and

$$\bar{F}_k^{(h)} = \int_{x_1 > 0} \bar{g}_k^{(h)}(x) e^{-(\zeta, x)} dx. \tag{34}$$

Then if $x \in A$ we can write $If^{(h)}$ in the form (32).

$$(35)\quad I f^{(h)}(x) = (2\pi)^{1-n} \int e^{\zeta_2 x_2 + \cdots + \zeta_n x_n} d\eta_2 \ldots d\eta_n \int_{K'} \sum_0^{k-1} \frac{\zeta_1'^{-j-1} F_j^{(h)}}{q(\zeta_1', \ldots)} e^{\zeta_1' x_1} d\zeta_1' + $$

$$+ (2\pi)^{-n} \int e^{(\zeta, x)} q(\zeta)^{-1} \bar{F}_k^{(h)} d\eta.$$

Let t_h be the maximum of the derivatives of $f^{(h)}$ of order $\leq m + 2k + n + 1$ on the necessarily closed and bounded set where a is different from zero and let B denote a constant, not always the same, which depends on q, a and k but is independent of h. Then integrating by parts in (33) and (34) we get

$$|F_j^{(h)}| < B t_h |1, \eta_2, \ldots, \eta_n|^{-m-k-n-1}$$

$$|\bar{F}_k^{(h)}| < B t_h |1, \eta_1, \ldots, \eta_n|^{-m-k-n-1}.$$

Finally, let $\partial = (\partial/\partial x_1)^{m_1} \ldots (\partial/\partial x_n)^{m_n}$ where $m_1 + \cdots + m_n \leq m + k$ and consider $\partial I f^{(h)}(x)$ which clearly can be obtained by formal differentiation of (35). Then it is evident that the absolute value of the last term is less than $B t_h$. The resulting inner integral of the first term is

$$\frac{1}{2\pi i} \int_{K'} \sum_0^{k-1} \frac{\zeta_1'^{m_1-j-1} F_j^{(h)}}{q(\zeta_1', \ldots, \zeta_l)} e^{\zeta_1' x_1} d\zeta_1'.$$

Its absolute value is less than $B t_h |1, \eta_2, \ldots, \eta_n|^{m_1-m-k-n}$. Hence the absolute value of the first term is less than

$$(2\pi)^{1-n} \int B t_h |1, \eta_2, \ldots, \eta_n|^{m_1-m-k-n} |\eta_2, \ldots, \eta_n|^{m_2+\cdots+m_n} d\eta_2 \ldots d\eta_n$$

i.e. less than $B t_h$. Hence $|\partial I f^{(h)}(x)| < B t_h$ when $x \in A$ so that $\partial I f^{(h)}$ tends to zero with $1/h$ uniformly on A. A similar argument shows that the same is true when $x \in A^-$. Hence because k is arbitrary and $A + A^-$ is arbitrarily large, it follows that $I f^{(h)}$ tends strongly to zero with $1/h$. This completes the proof of Theorem 4.2.

A lemma. In the next section we shall use

Lemma 4.1. *Let $f \in C(\infty, T + S)$ and let the derivatives of f of order $< m$ vanish on the common part of S and A, where A is a subset of $T + S$ which with a point x also contains $C(x)$. Then*

$$(36)\qquad f(x) = I q(\partial/\partial x) f(x)$$

whenever $x \in A$.

Proof. Let $\Re\alpha > l + m$ in (18), let $x \in A$ and integrate by parts in $I^{\alpha+1} q(\partial/\partial x) f(x)$. Because the derivatives of f of order $< m$ vanish on the common part of S and A and hence in particular also on the common part of $C(x)$ and S, the result is

$$\int_{y_l > 0} f(y_1, \ldots, y_l, x_{l+1}, \ldots, x_n)\, q(-\partial/\partial y)\, Q_l(\alpha + 1, x - y)\, dy_1 \ldots dy_l.$$

It follows from Theorem 3.2 that

$$q(-\partial/\partial y)\, Q_l(\alpha + 1, x - y) = q(\partial/\partial x)\, Q_l(\alpha + 1, x - y) = Q_l(\alpha, x - y).$$

Hence

$$I^{\alpha+1} q(\partial/\partial x) f(x) = I^{\alpha} f(x)$$

when $x \in A$ and $\Re\alpha > l + m$. Now if also $x \in T$, both sides are analytic in α so that by virtue of Theorem 4.1, (36) follows when x is in the common part TA of T and A. Finally, by Theorem 4.2 the right side of (36) is continuous in $T + S$ and hence the equality is true in $TA + SA = A$.

Proof of Theorem II. Using the results of this chapter we can now prove Theorem II of the introduction in a few lines.

Let q be a polynomial in n variables with complex coefficients which is hyperbolic with respect to ξ, and let $A(q)$ be the set of solutions $f \in C(\infty)$ of the differential equation

$$q(\partial/\partial x) f(x) = 0.$$

Let $A(q)$ contain the sequence $f_1, f_2, \ldots, f_k, \ldots$ What is meant by

(a) $$f_k \to 0\,(\xi)$$

and

(b) $$f_k \to 0$$

is explained in the introduction. We have to prove that (a) implies (b). If q is a constant, then because $q(\xi) = p(\xi) \neq 0$ it is not zero and hence $A(q)$ contains only the element $f = 0$ and the assertion is trivial. Assume that the degree m of q is positive. Then we can apply the results of this chapter, in particular Theorem 4.2. Let $f \in C(\infty)$ and put

$$Jf(x) = f(x) - I q(\partial/\partial x) f(x)$$

where I is defined in Theorem 4.2. It follows from (10) that $Jf \in A(q)$. Put

(37) $$P_{\xi} f(x) = \sum_{0}^{m-1} \frac{(x, \xi)^j}{j!\,(a, \xi)^j} f^{(j)}(x - a(x, \xi)(a, \xi)^{-1})$$

where a is a real vector such that $(a, \xi) \neq 0$ and $f^{(j)}(x) = (a, \partial/\partial x)^j f(x)$. Then the derivatives of $f(x) - P_\xi f(x)$ vanish when $x \in S$, i.e. when $(x, \xi) = 0$, and if $f_k \to 0(\xi)$ then $P_\xi f_k \to 0$. In fact, let $a^{(1)} = a,\ a^{(2)}, \ldots, a^{(n)}$ be a basis for all real vectors, let $(a^{(j)}, \xi) = 0$ when $j > 1$ and put $\partial_j = (a^{(j)}, \partial/\partial x)$. Then $\partial_j (x, \xi) = 0$ when $j > 1$ and $\partial_1 f(x - a(x, \xi)(a, \xi)^{-1}) = 0$ for all f. Also, $x - a(x, \xi)(a, \xi)^{-1} \in S$ for all x. Hence any derivative of $P_\xi f$ is a linear combination of the derivatives of f in the plane S where the coefficients are polynomials in (x, ξ). This proves that $P_\xi f_k \to 0$ whenever $f_k \to 0(\xi)$. As to the other announced property of $P_\xi f$ it follows because $\partial_1^h P_\xi f(x) = f^{(h)}(x)$ and $\partial_j^h P_\xi f(x) = \partial_j^h f(x)$ whenever $j > 1$, $h < m$ and $(x, \xi) = 0$. If $\xi = (1, 0, \ldots, 0)$ and we choose $a = (1, 0, \ldots, 0)$, (37) takes the more familiar form

$$P_\xi f(x) = \sum_0^{m-1} x_1^j f^{(j)}(0, x_2, \ldots, x_n)/j!$$

where $f^{(j)}(x) = (\partial/\partial x_1)^j f(x)$.

It is clear that $Jf - JP_\xi f$ is in $A(q)$ and that its derivatives of order $< m$ vanish on S. Hence Lemma 4.1 shows that it vanishes in $T + S$. Now it follows from Lemma 2.2 that q is hyperbolic also with respect to $-\xi$. Hence it follows that $Jf - JP_{-\xi} f$ vanishes in $S + T(-\xi)$ so that because $P_{-\xi} = P_\xi$ it follows that $Jf(x) = JP_\xi f(x)$ for all x. It is clear that $Jf = f$ whenever $f \in A(q)$ so that in this case

$$f(x) = P_\xi f(x) - Iq(\partial/\partial x) P_\xi f(x)$$

for all x. Apply this formula to every element of a sequence $f_1, \ldots, f_k, \ldots$ of functions in $A(q)$ such that $f_k \to 0(\xi)$. Then $P_\xi f_k \to 0$ so that it follows immediately from Theorem 4.2 that $f_k \to 0$. This proves Theorem II.

Chapter 5.

The Problem of Cauchy. Generalizations.

The problem of Cauchy. Let us use for a moment the assumptions, notations and results of the first section of the preceding chapter. Let g and h be in $C(\infty, T + S)$ and put

$$u(x) = g(x) - Iq(\partial/\partial x)\, g(x) + Ih(x)$$

where $x \in T$. When the derivatives of order $< m$ of a function defined in T have continuous extensions to $T + S$ which vanish on S we write briefly $f(x) \overset{(m)}{=} 0, (x \in S)$. It follows from Theorem 4.2 that

$$u \in C(\infty, T+S)$$

$$q(\partial/\partial x)\, u(x) = h(x), \quad (x \in T) \tag{1}$$

$$u(x) - g(x) \overset{(m)}{=} 0, \quad (x \in S).$$

Now u is the only function with these properties. In fact, if there were two, their difference satisfies the requirements of Lemma 4.1 with $A = T + S$ and hence it vanishes in $T + S$. Hence the classical problem of Cauchy, which is the problem of finding a function u satisfying (1) for given g and h in $C(\infty, T+S)$, has a unique solution. It is clear that the solution vanishes if $h = 0$ in T and $g(x) \overset{(m)}{=} 0$, $(x \in S)$. It is not difficult to see that the problem has a unique solution in $C(m, T+S)$ provided that the functions g and h are in $C(2m + l + 1, T + S)$, but we do not give the details.

The surface S. It is possible to generalize Chapter 4 and the things said above to a case when a suitable surface plays the part so far played by the plane $S = S(\xi)$. We will do this here only when $q = p$ is a homogeneous and reduced polynomial. The method is a little different from that of Chapter 4.[1]

Let S be an open $(n-1)$-dimensional infinitely differentiable manifold in the space E with elements $x, y, \ldots$ For simplicity we assume that S admits a parametric representation of the form

$$y_j = s_j(t) = s_j(t_2, \ldots, t_n)$$

where s_j is defined and infinitely differentiable in some open region P of the real t-space and that $t_2, \ldots, t_n$ are uniquely determined by $y_1, \ldots, y_n$. Put $s_{j \cdot k} = \partial s_j / \partial t_k$, let $u_1, \ldots, u_n$ be indeterminates and put

$$J(u) = \det(u_j, s_{j\cdot 2}, \ldots, s_{j\cdot n}) = \sum_1^n u_j \sigma_j(t) \tag{2}$$

---

[1] The results of this chapter were announced in GÅRDING [2] for an arbitrary reduced and hyperbolic equation, not necessarily homogeneous, and it is in fact also possible to prove them in this case, using the method of Chapter 4. The present method, however, is simpler. It applies to a homogeneous reduced polynominal p, hyperbolic with respect to a vector ξ, or more generally by an expansion in series of

$$q^{-\alpha} = p^{-\alpha}(1 + r/p)^{-\alpha}$$

to all $q = p + r$ where r is of degree less than the degree of p and such that when s is real and positive, the least upper bound when η is real of the absolute values of the zeros of the equation $q(t(s\xi + i\eta)) = 0$ tends to zero with $1/s$. It is probable but not proved that any q which is hyperbolic with respect to ξ has this property.

as a definition of $\sigma_j(t)$. Then the vector $\sigma(t) = (\sigma_1(t), \ldots, \sigma_n(t))$ is called the normal of S: Its sign depends on the order of the rows in (2). We suppose that it is possible to choose this order so that for all t in P

$$\sigma(t) \in \Gamma = \Gamma(q, \bar{\xi})$$

where q is assumed to be a homogeneous, reduced and not constant polynomial which is hyperbolic with respect to $\bar{\xi}$. The simplest case of such a surface is a plane $(y, \xi) = 0$ where $\xi \in \Gamma$. Then σ is a positive multiple of ξ.

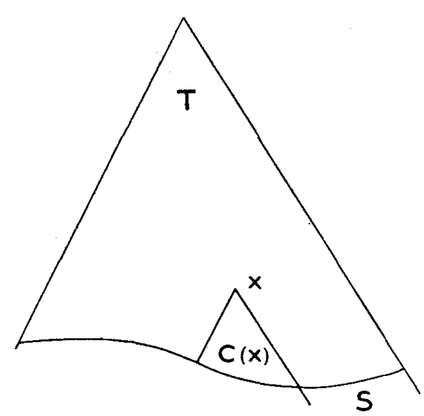

Fig. 1.

Let $\dot{C}$ be $C = C(q, \bar{\xi})$ minus the point $x = 0$. We consider a maximal set T partly bounded by S with the following properties. It is open, it does not contain points of S and if $x \in T$ and $z \in \dot{C}$ then $x - t_1 z \in S$ for some positive $t_1 = b_1(x, z)$ which is continuous in x and z in the product domain $(T + S) \times \dot{C}$, while all points $x - cz$ where $0 \leq c < t_1$ are in T. If $\xi \in \Gamma$ and $z \in \dot{C}$, then $(z, \xi) \geq 0$ and hence because Γ is open and $z \neq 0$ it follows that $(z, \xi) > 0$. Hence if S is the plane $(y, \xi) = 0$ then $x - t_1 z$ is in S if and only if $t_1 = (x, \xi)/(z, \xi)$, so that T consists in this case of all x such that $(x, \xi) > 0$. In the general case, in a suitable neighborhood[1] of one of its points y, S is close to its tangentplane at y, $(y' - y, \sigma) = 0$. Hence it follows that every neighborhood of y contains points in T.

As in the preceding chapter the set of points $x - cz$ where $z \in \dot{C}$ and $0 \leq c \leq b_1(x, z)$ will be called $C(x)$. It is necessarily a subset of $T + S$. If $\xi \in \Gamma$ and $z \in \dot{C}$ we know that $(z, \xi) > 0$. Hence every point in $C(x)$ is of the form

[1] A neighborhood of a point y is an open set containing y.

$x - cz$ where $0 \leq c \leq b_1(x, z)$ and z belongs to the subset C_1 of $\dot{C}$ where $(z, \xi) = 1$. Now Lemma 2.13 shows that C_1 is bounded and closed, i.e. compact. Hence $b_1(x, z)$ which is continuous has a finite maximum on C_1 and it follows that $C(x)$ is bounded. It is clear that it is closed.

By the definition of T, if $x \in T$ and $z \in \dot{C}$ then there are real numbers $t_1, \ldots, t_n$ such that

$$x_j = t_1 z_j - s_j(t_2, \ldots, t_n), \quad (j = 1, \ldots, n), \tag{3}$$

and $t_1 = b_1(x, z)$ is continuous in $(T + S) \times \dot{C}$. It then follows that every t_k, $(k > 1)$, is a continuous function of all $x_j - t_1 z_j$. Hence it is also a function $b_k(x, z)$ of x and z which is continuous in $(T + S) \times \dot{C}$. Let $J(x, t)$ be the absolute value of the Jacobian of x with respect to t. It follows from (3) that $J(x, t) = (\sigma, z) > 0$ in $(T + S) \times \dot{C}$ and with analogous notations that $J(z, t) = J^{-1}(x, z) J(x, t) = (\sigma, z) t_1^{-n} > 0$ in $T \times \dot{C}$. Hence every function $b_k(x, z)$ is for z fixed in $C(\infty, T)$ and for x fixed in $C(\infty, \dot{C})$. Now by differentiation of (3) it follows that any derivative of $b_k(x, z)$ with respect to x and z is a polynomial in the derivatives with respect to t of the right sides of (3) divided by a powerproduct of $J(x, t)$ and $J(z, t)$. Hence $b_k(x, z) \in C(\infty, T \times \dot{C})$. Moreover, if the derivative is taken with respect to x alone, the powerproduct in question consists of a power of $J(x, t)$ alone which is continuous and positive in $(T + S) \times \dot{C}$. Hence the derivative in question is continuous in $(T + S) \times \dot{C}$. Put when $z \in \dot{C}$

$$r(x, z) = b_1(x, z/(z, \xi)) \tag{4}$$

where $\xi \in \Gamma$. Then $z/(z, \xi)$ belongs to C_1 which is compact. Hence we get[1]

Lemma 5.1. *The function $r(x, z)$ defined by (4) is in $C(\infty, T \times \dot{C})$. Its derivatives with respect to x are bounded on every $T' \times \dot{C}$ where T' is a compact subset of $T + S$, and $r(x, z)$ has a positive minimum on $T' \times \dot{C}$ whenever T' is a compact subset of T.*

[1] This lemma corresponds to Lemma 2.3 of section 2 of GÅRDING [4]. This section contains a wrong statement namely Lemma 2.2 and a perhaps dubious definition namely that of E (here called T). However, Lemma 2.2 is never used and the rest of [4] is certainly put in order if in the notations of [4] we defined E (which corresponds to T above) as the maximal open set in the space of all symmetric matrices of order n, which does not contain points of G and has the property that there exists a positive function $u_1 = u_1(x, Z)$ continuous in the product domain $(E + G, \hat{C}^1)$ such that if $x \in E$ then $x - u_1 Z \in G$ and $x - cZ \in E$ whenever $0 \leq c < u_1$. Then because (2.2) is satisfied, every neighborhood of a point in G contains points in E, and Lemma 2.3 is valid. There should be corresponding changes on p. 786 line 7 from the bottom, on p. 789 line 13 from the bottom and on p. 814, line 11 from the bottom.

The Riesz operator. Because q is not constant, reduced and hyperbolic with respect to ξ, we can construct the Riesz kernel $Q(\alpha, x)$ corresponding to q and ξ. We define the Riesz operator by the formula

$$I^\alpha f(x) = \int_{C(x)} Q(\alpha, x-y) f(y)\, dy, \quad (x \in T), \tag{5}$$

where $\Re \alpha > n$. When the derivatives of order $< m$ of a function f defined in T have continuous extensions to $T + S$ which vanish on S we write as before briefly $f(x) \overset{(m)}{=} 0$, $(x \in S)$. We can now prove the following theorem, analogous to Theorem 4.1.

Theorem 5.1. *Let q be a reduced homogeneous and not constant polynomial in n variables which is hyperbolic with respect to ξ and let $f \in C(\infty, T+S)$. Then the function $I^\alpha f(x)$ defined by (5) when $\Re \alpha$ is large is entire analytic in α. For all values of α it is in $C(\infty, T)$ and its derivatives are entire analytic in α. It satisfies (4.7), (4.8) and (4.9) when $x \in T$. Also $If(x) = I^1 f(x) \in C(\infty, T+S)$ and*

$$If(x) \overset{(m)}{=} 0, \quad (x \in S). \tag{6}$$

Proof. Because $C(x)$ is bounded and closed the integral (5) always exists. Let us change the variables y to z defined by

$$y = x - r(x, z)\, z, \quad (y \neq x).$$

Because r is homogeneous of order 0 in z the Jacobian $J(y, z)$ equals r^n. The region $C(x)$ corresponds to the region A of all $z \in C$ for which $(z, \xi) \leq 1$. Hence using the remark in connection with Theorem 3.1 we get when $\Re \alpha > n$

$$I^\alpha f(x) = \int_A Q(\alpha, z) f(x - rz)\, r^{m\alpha}\, dz.$$

Here by virtue of Lemma 5.1 we can differentiate under the sign of integration any number of times and the resulting integrals will be analytic in α and continuous in x when $x \in T$ and $\Re \alpha > n$. When k is an integer and $\Re \alpha$ is large enough it follows from (5) and Theorem 3.2 that

$$I^{\alpha-k} f(x) = q(\partial/\partial x)^k I^\alpha f(x).$$

Hence the second and third sentence of the theorem follows. The fourth is proved precisely as the corresponding part of Theorem 4.1. To prove the fifth we observe that

$$If(x) = \int_A Q(n+1, z)\, q(\partial/\partial x)^n f(x - rz)\, r^{m(n+1)}\, dz.$$

Again it follows from Lemma 5.1 that every derivative of $If(x)$ is in $C(0, T+S)$ and that those of order $< m$ tend to zero with the maximum of $r(x, z)$ when $z \in \dot{C}$. This completes the proof.

The proof of the following lemma is the same as that of Lemma 4.1.

Lemma 5.2. *If $f \in C(\infty, T+S)$ satisfies*

$$f(x) \stackrel{(m)}{=} 0, \quad (x \in S)$$

then

$$f(x) = Iq(\partial/\partial x) f(x), \quad (x \in T).$$

If g and h are in $C(\infty, T+S)$ then Theorem 5.1 shows that

$$u(x) = g(x) - Iq(\partial/\partial x)\, g(x) + Ih(x)$$

is in $C(\infty, T+S)$ and satisfies

$$q(\partial/\partial x)\, u(x) = h(x), \quad (x \in T)$$

$$u(x) - g(x) \stackrel{(m)}{=} 0, \quad (x \in S)$$

and it follows from Lemma 5.2 that a function with these properties is unique. The same conclusion is easily seen to be true if u is in $C(m, T+S)$ and g and h are in $C(m(n+1), T+S)$ and if all the functions s_j defining S are in $C(m(n+1), P)$.

Chapter 6.

The Domain of Dependence.

Introduction. Let q be a not constant complex polynomial in n variables which is hyperbolic with respect to $\bar{\xi}$. Let $\Gamma(q, \bar{\xi})$ and $C(q, \bar{\xi})$ be the associated cones, defined in Chapter 2. Throughout this chapter we shall mean by $\bar{C}(x)$ where $x \in E$, the set of points y such that $x - y \in C = C(q, \bar{\xi})$ or briefly

$$\bar{C}(x) = (y;\ x - y \in C).$$

Let $S = S(\bar{\xi})$ be the plane $(y, \bar{\xi}) = 0$ and $T = T(\bar{\xi})$ the halfspace $(y, \bar{\xi}) > 0$. As usual we put $C(\infty) = C(\infty, E)$ and we let C^0 be the set of all functions in $C(\infty)$ which vanish outside some compact set. The properties of the operator I defined by

$$If(x) = (2\pi)^{-n} \int q(\zeta)^{-1} F(\zeta)\, e^{(\zeta, x)}\, d\eta$$

where $x \in T + S$, $\zeta = \xi + i\eta$, $\xi \in \Gamma_1(q, \bar{\xi})$, $f \in C^0$ and

$$F(\zeta) = \int_T f(y)\, e^{-(\zeta, y)}\, dy$$

are given in the Theorems 4.1 and 4.2. We know that $If \in C(\infty, T + S)$, that $q(\partial/\partial x)\, If(x) = f(x)$ and that $If(x) = 0$ whenever f vanishes on $\bar{C}(x)$.

The solution of Cauchy's problem with respect to the plane S and the halfspace T is composed by means of the operator I and another operator J defined by

$$Jf(x) = f(x) - I\, q(\partial/\partial x) f(x)$$

where $x \in T + S$.

We define the lacunary set $L = L(I, x)$ attached to a point $x \in T$ and the operator I, as follows. It shall consist of all points y with the property that there exists a neighborhood N of y such that $If(x) = 0$ for all f in C^0 which vanish outside N.[1] It is clear that L is open. We define the domain of dependence $D(I, x)$ to be the complement of L in E. In the same way we define $L(J, x)$ and $D(J, x)$.

It is immediate that $D(I, x)$ and $D(J, x)$ are both in $T + S$ and because $\bar{C}(x)$ is closed it follows that $D(I, x)$ is contained in $\bar{C}(x)$. If $y \bar{\in} S$ then there is a neighborhood N of y with no point in common with a suitable neighborhood of S. Then if f vanishes outside N, it vanishes together with its derivatives on S, and it follows from Lemma 4.1 that $Jf(x) = 0$. Hence $D(J, x)$ is contained in S.

In order to get more precise results and in order to eliminate x and S from $D(I, x)$ and $D(J, x)$ it is convenient to do as follows. Let $f \in C^0$, let α be a complex number and put

$$J_1^\alpha f = (2\pi)^{-n} \int q(\zeta)^{-\alpha} F_+(\zeta)\, d\eta \tag{1}$$

where $\zeta = \xi + i\eta$, $\xi \in \Gamma_1(q, \bar{\xi})$, $q(\zeta)^{-\alpha}$ is defined by (3.11) and

$$F_+(\zeta) = \int f(x)\, e^{(\zeta, x)}\, dx. \tag{2}$$

Integrating by parts in the last formula we get that $|\zeta_1, \ldots, \zeta_n|^N F_+(\zeta)$ is bounded for all positive N, and according to (3.2) and (3.4) which apply also to not reduced polynomials, $|q(\zeta)|$ is bounded from below. Hence $J_1^\alpha f$ is an entire function of α. Assume for a moment that q is reduced. Then we can

[1] A neighborhood of a point y is an open set containing y.

construct the Riesz kernel $Q(a, x)$ associated with q and $\bar{\xi}$, and then if $\Re a > n$ it follows from Parseval's formula that

$$J_1^a f = \int Q(a, x) f(x)\, dx.$$

Hence $J_1^a f$ is independent of ξ as long as $\xi \in \Gamma_1(q, \bar{\xi})$ and it vanishes if f vanishes on $C = C(q, \bar{\xi})$. The same results follow by analytical continuation for all a. An application of the arguments used in the beginning of the proof of Theorem 4.1 shows that they are also true if q is not reduced. In particular we shall consider

$$J_1 f = J_1^1 f = (2\pi)^{-n} \int q(\zeta)^{-1} F_+(\zeta)\, d\eta.$$

The operator J_1 has a lacunary set $L(J_1)$ and a domain of dependence $D(J_1)$ which is necessarily a subset of C. In the next section we will prove a lemma that expresses $D(I, x)$ and $D(J, x)$ in terms of $D_1 = D(J_1)$.

Structure of the domains of dependence of I and J.

Lemma 6.1. *The set $D(I, x)$ where $x \in T$ consists of all y in $T + S$ such that $x - y \in D_1$ and the set $D(J, x)$ consists of all y in S such that $x - y \in D_1$.*

Proof. Let $f \in C^0$ and put

$$F(\zeta) = \int e^{-(\zeta, y)} f(y)\, dy \tag{3}$$

where $\zeta = \xi + i\eta$, $\xi \in \Gamma_1(q, \bar{\xi})$ and $-\xi \in \Gamma_1(q, -\bar{\xi})$. Let a be a complex number, let $q(\zeta)^{-a}$ be defined by (3.11) and consider the following slight modification of the Riesz operator

$$I_1^a f(x) = (2\pi)^{-n} \int q(\zeta)^{-a} F(\zeta)\, e^{(\zeta, x)}\, d\eta \tag{4}$$

where x is an arbitrary point in E. It follows from (3.2) and (3.4) which apply also to a not reduced polynomial q that $|q(\zeta)|$ is bounded from below. Integrating by parts in (3) we see that $|\zeta_1, \ldots, \zeta_n|^N F(\zeta)$ is bounded for all positive N. Hence $I_1^a f(x)$ is an entire function of a, for all values of a it is in $C(\infty)$ and it satisfies $q(\partial/\partial x) I_1^{a+1} f(x) = I_1^a f(x)$. In particular when $a = 1$ we get with $I_1 = I_1^1$, using Plancherel's theorem,

$$q(\partial/\partial x) I_1 f(x) = f(x).$$

The arguments in the beginning of the proof of Theorem 4.1 apply without change to I_1^a, and they show that $I_1^a f(x)$ does not depend on ξ as long as

$\xi \in \Gamma_1(q, \bar{\xi})$ and that $I_1^\alpha f(x)$ vanishes whenever f vanishes in $\bar{C}(x)$. If q is reduced this follows from the formula

$$I^\alpha f(x) = \int Q(\alpha, x - y) f(y)\, dy$$

which is valid when $\Re \alpha > n$ and where $Q(\alpha, x)$ is the Riesz kernel associated with q and $\bar{\xi}$. Because $\bar{C}(x)$ is closed it follows from the above that

$$D(I_1, x) < \bar{C}(x) = (y;\, x - y \in C = C(q, \bar{\xi})),$$

and if we define I_1^- as I_1 with the only difference that it refers to q and $-\bar{\xi}$, that

$$D(I_1^-, x) < \bar{C}_-(x) = (y;\, x - y \in -C = C(q, -\bar{\xi})).$$

If $f \in C^0$ then $I_1 f$ and $I_1^- f$ are both in $C(\infty)$. It follows from Theorem 4.2 that Ig and hence also Jg has a sense and is in $C(\infty, T + S)$ if g is. Hence we can form $J I_1 f(x) = I_1 f(x) - Iq(\partial/\partial x) I_1 f(x) = I_1 f(x) - If(x)$ so that we get

(5) $$I_1 f(x) = J I_1 f(x) + If(x), \quad (x \in T).$$

Similarly

(6) $$I_1^- f(x) = J I_1^- f(x) + If(x), \quad (x \in T).$$

We can now prove that

(7) $$D(J, x) = S D(I, x)$$

where the right side stands for the common part of S and $D(I, x)$. Let $y \in SL(J, x)$ and put with $|y| = \max_k |y_k|$,

$$N_r(y) = (\bar{y};\, |y - \bar{y}| < r).$$

Then we can choose $r > 0$ so small that x is not in $N_r(y)$ and that $Jf(x)$ vanishes if f vanishes outside $N_r(y)$. Consider $\bar{C}(\bar{y})(T + S)$. It consists of all y' of the form $\bar{y} - z$ where $z \in C$ and $(z, \bar{\xi}) \leq (\bar{y}, \bar{\xi})$. It follows from Lemma 2.13 that $|z|$ has a finite maximum c_1 when $z \in C$ and $(z, \bar{\xi}) = 1$. Then $|\bar{y} - y'| \leq \leq c_1(\bar{y}, \bar{\xi})$ when $y' \in \bar{C}(\bar{y})(T + S)$. Hence we can choose $r' > 0$ so small that $\bar{C}(\bar{y})(T + S)$ is contained in $N_r(y)(T + S)$ whenever $\bar{y}$ is in $N_{r'}(y)$. Now if a point $\bar{x} \in T + S$ is also in $D(I_1^-, y)$ then it is necessary that $\bar{x} \in \bar{C}(\bar{y})$. Hence if $f \in C^0$ and vanishes outside $N_{r'}(y)$ it follows that $I_1^- f$ vanishes in $T + S$ outside $N_r(y)$ and hence $J I_1^- f(x) = 0$. Moreover, because $x \bar{\in} N_r(y) > N_{r'}(y)$ it follows that $I_1^- f(x) = 0$. But then (6) shows that $If(x) = 0$ and hence $y \in L(I, x)$. Conversely, assume that $y \in SL(I, x)$ and choose $r > 0$ so that x is not in $N_r(y)$ and that $If(x)$ vanishes if $f \in C^0$ and vanishes outside $N_r(y)$. Then also $Jf(x) = f(x) - Iq(\partial/\partial x)f(x) = f(x) = 0$ under the same conditions and this proves that $y \in L(J, x)$. Hence we have proved (7).

Next we want to show that

$$L(I, x) = T^- + (T + S)\, L(I_1, x) \tag{8}$$

where $x \in T$ and $T^- = T(-\xi) = (y; (y, \xi) < 0)$. If y is in T there is a neighborhood N of y which also is in T, and if $f \in C^0$ and vanishes outside N we get $If(x) = I_1 f(x)$ for all x in T. This proves that $TL(I, x) = TL(I_1, x)$. It is obvious that $T^- < L(I, x)$.

Consider

$$I_1 f(x) = (2\pi)^{-n} \int q(\zeta)^{-1} F(\zeta)\, e^{(x, \zeta)}\, d\eta$$

where $F(\zeta)$ is given by (3), and change variables in the integral so that $x = x' \tilde{M}^{-1}$ and $\zeta = \zeta' M = (\xi' + i\eta') M$, where M is a real square matrix whose determinant has absolute value 1 and choose M as in the beginning of the proof of Theorem 4.1. Then

$$I_1 f(x) = (2\pi)^{-n} \int q'(\zeta')^{-1} F'(\zeta')\, e^{(x', \zeta')}\, d\eta'$$

where $q'(\zeta') = q(\zeta)$ is a reduced polynomial in $\zeta_1', \ldots, \zeta_l'$, $(1 \leq l \leq n)$, and with $f'(y') = f(y)$,

$$F(\zeta) = F'(\zeta') = \int f'(y')\, e^{-(y', \zeta')}\, dy'.$$

Put $h_\varepsilon(y) = h_\varepsilon'(y')$, let h_ε' be in $C(\infty)$ and let $h_\varepsilon'(y') = h_\varepsilon'(y_1')$ be 1 when $y_1' > 0$ and 0 when $y_1' < -\varepsilon < 0$ and monotone for the other values of y_1'. Consider

$$F_\varepsilon'(\zeta') = \int f'(y')\, h_\varepsilon'(y')\, e^{-(\zeta', y')}\, dy'.$$

Integrating by parts in this formula we get

$$\zeta_1' \zeta_k'^N F_\varepsilon'(\zeta') = \int e^{-(\zeta', y')} \frac{\partial}{\partial y_1'} \left(h_\varepsilon'(y_1') \frac{\partial^N}{\partial y_k'^N} f'(y') \right) dy' \tag{9}$$

where $k > 1$. Because $f \in C^0$, the function f' vanishes outside some set of the form

$$A = (y'; |y_1', \ldots, y_n'| \leq \tfrac{1}{2} a).$$

Let a_1 be the maximum of $e^{-(\xi', y')}$ and a_2 the maximum of the absolute values of the derivatives of f' of order $\leq N + 1$ in A. Then because h'_ε is monotone, the absolute value of the right side in (9) is not greater than

$$a_1 a_2 a^{n-1} \int\limits_{-\frac{1}{2}a}^{+\frac{1}{2}a} \left(1 + \frac{\partial}{\partial y'_1} h'_\varepsilon(y'_1)\right) d y'_1 \leq a_1 a_2 a^{n-1} (a + 1).$$

Hence $|\zeta'_1| |\zeta'_2, \ldots, \zeta'_n|^N F'_\varepsilon(\zeta')$ is bounded for all $N \geq 0$, uniformly in ε. Now it follows from Lemma 3.1 that $|q'(\zeta')| \geq B(1 + |\eta'_1|)$ where $B > 0$. Hence because $F'_\varepsilon(\zeta')$ tends to

$$(2\pi)^{-n} \int\limits_{T(\bar{\xi}')} f'(y') e^{(\zeta', y')} d y' = (2\pi)^{-n} \int\limits_{T} f(y) e^{(\zeta, y)} d y,$$

the right side of

$$I_1 h_\varepsilon(x) f(x) = (2\pi)^{-n} \int q'(\zeta')^{-1} F'_\varepsilon(\zeta') e^{(\zeta', x')} d\eta'$$

tends to $If(x)$ as ε tends to zero. Hence if $y \in SL(I_1, x)$ and the neighborhood N of y is chosen so that $I_1 f(x)$ vanishes when f is in C^0 and vanishes outside N, it follows that $I_1 h_\varepsilon(x) f(x)$ and hence also its limit $If(x)$ vanish under the same circumstances. This proves that $SL(I_1, x) < SL(I, x)$.

To prove the converse inclusion, assume that $y \in SL(I, x)$ and choose $r > 0$ so small that if f is in C^0 and vanishes outside $N_r(y)$ then both $If(x)$ and $Jf(x)$ vanish. This is possible because we have proved (7). Let $r' > 0$, let $\bar{y} \in N_{r'}(y)$, put

$$T_1 = (\bar{z};\ |(\bar{z}, \bar{\xi})| < s_1)$$

and consider $T_1 \bar{C}_-(\bar{y})$. It consists of all points $\bar{y} + z$ in T_1 such that z is in C. Hence $|(\bar{y} + z, \bar{\xi})| < s_1$ so that $(z, \xi) < |(\bar{y}, \bar{\xi})| + s_1$. Now because y is in S so that $(y, \bar{\xi}) = 0$, the maximum of $|(\bar{y}, \bar{\xi})|$ when $\bar{y}$ is in $N_{r'}(y)$ is of the form $c_2 r'$ where $c_2 > 0$ and hence we get that $(z, \bar{\xi}) < c_2 r' + s_1$. But then if c_1 is the maximum of $|z|$ when $z \in C$ and $(z, \bar{\xi}) \leq 1$ it follows that $|z| < c_1(c_2 r' + s_1)$ and then

$$|y - \bar{y} - z| \leq |y - \bar{y}| + |z| < r' + c_1(c_2 r' + s_1) < r,$$

if $r' > 0$ and $s_1 > 0$ are small enough. Hence with this choice of r' and s_1 we get that $T_1 \bar{C}_-(\bar{y}) < T_1 N_r(y)$ whenever $\bar{y} \in N_{r'}(y)$. Let f be in C^0 and vanish outside $N_{r'}(y)$. It is clear then that $I_1 f(\bar{x})$ vanishes unless there are points $\bar{y}$ in $N_{r'}(y)$ such that $\bar{x} - \bar{y} \in C$, i.e. such that $x \in \bar{C}_-(\bar{y})$. Hence because $T_1 \bar{C}_-(\bar{y}) <$ $< T_1 N_r(y)$ when $\bar{y} \in N_{r'}(y)$ it follows that $I_1 f(x)$ vanishes *in* $T_1 - T_1 N_r(y)$ so that if g is in $C(\infty)$ and vanishes outside T_1 and equals one on the set

$(\bar{z}; (\bar{z}, \bar{\xi}) < \frac{1}{2} s_1)$ then the derivatives of $I_1 f - g I_1 f$ vanish on S and $g I_1 f$ vanishes outside T_1 and in T_1 outside $T_1 N_r(y) < N_r(y)$. Consequently, we get that

$$0 = J g(x) I_1 f(x) = J I_1 f(x).$$

Because f vanishes outside $N_{r'}(y) < N_r(y)$ it also follows that $If(x) = 0$ and hence (5) shows that $I_1 f(x) = 0$ and consequently $y \in L(I_1, x)$. This proves that $SL(I, x) = SL(I_1, x)$. Now the formulas (1) to (4) show that $I_1 f(x) = J_1 f_1$ where $f_1(y)$ is defined as $f(x-y)$ for fixed x. But then it is obvious that

$$L(I_1, x) = (y; x - y \in L_1)$$

and the lemma follows from (7) and (8).

Lacunas. We know that the domain of dependence D_1 of the operator J_1 defined by (1) and (2) is contained in C and in general one has in fact $D_1 = C$, but there are exceptions. In example 1 let $n = 4$.[1] By a simple passage to the limit in the formula following (2) one gets

$$J_1 f = \frac{1}{4\pi} \int f(t, x_2, x_3, x_4)\, t^{-1}\, dx_2\, dx_3\, dx_4,$$

where t is the positive square root of $x_2^2 + x_3^2 + x_4^2$. Hence D_1 is in this case the boundary of C. Its dimension is 3 while that of C is 4. The fact that $D_1 \neq C$ in this case is sometimes referred to as Huygens' principle for the wave equation and accounts for the possibility of emitting sharp light signals in space-time.[2] A still more striking example of the same kind of. anomaly is offered by example 2.[3] Then we have the formula[4]

$$J_1 f = \frac{1}{\pi^{\bar{n}-1}} \int f(\tilde{a}^* a)\, |a|\, da$$

where a is the vector $(a_1, \ldots, a_{\bar{n}})$ with real a_1 and complex $a_k = a_k' + i a_k''$ when $k > 1$, $\tilde{a}^*$ its transpose conjugate, $|a|$ the positive square root of $a_1^2 + |a_2'|^2 + \cdots$ and $da = da_1\, da_2'\, da_2'' \ldots$ Hence $J_1 f$ is a mean value of f over such hermitian matrices $x = \tilde{a}^* a$ which have all its roots zero except one which is not negative and it follows that D_1 consists of all such matrices. In this case the dimensions of C and D_1 are $\bar{n}^2$ and $2\bar{n} - 1$ respectively.

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[1] See pp. 25, 29 and 34.
[2] Riesz [11] p. 83—88.
[3] See pp. 25, 29 and 34.
[4] Gårding [4], Theorem H 10.2 p. 822.

Petrowsky [8] has made an extensive study of lacunary sets. He takes the case when q is homogeneous and not degenerate. Then if B is a domain of analyticity of $Q(1, \cdot)$[1] and f vanishes outside a closed and bounded set B' in B it follows that

$$J_1 f = \int_{B'} Q(1, y) f(y)\, dy.$$

Hence if B' has a not empty interior, it is in the lacunary set L_1 of J_1 if and only if $Q(1, \cdot)$ vanishes on B'. But then $Q(1, \cdot)$ vanishes in B and it follows that B is contained in L_1. Such a domain of analyticity of $Q(1, \cdot)$ is called a lacuna. According to Petrowsky L_1 is a sum of lacunas and he also gave a necessary and sufficient condition of a topological nature that a given domain of analyticity be a lacuna. Example 2 above shows that when q is degenerate, things are more complicated. Practically nothing is known about the existence of lacunas in C in the not homogeneous case. As is shown by the wave equation, terms of lower order tend to destroy them.[2]

Bibliography.

[1] S. BOCHNER. *Group invariance of Cauchy's formula in several variables.* Ann. of Math. (2) 45 (1944) p. 686—707.

[2] L. GÅRDING. *Equations différentielles linéaires hyperboliques à coefficients constants.* C. R. (Paris) t. 226 (1948) p. 539—541.

[3] —— *Une propriété caractéristique des équations hyperboliques à coefficients constants.* l.c. t. 228 (1949) p. 731—732.

[4] —— *The solution of Cauchy's problem for two totally hyperbolic linear differential equations by means of Riesz integrals.* Ann. of Math. (2) 48 (1947) p. 785—826.

[5] J. HADAMARD. *Le problème de Cauchy,* Paris, 1932.

[6] G. HERGLOTZ. *Über die Integration linearer partieller Differentialgleichungen mit konstanten Koeffizienten I—III.* Berichte Sächs. Akad. d. Wiss. 78, 80 (1926, 1928).

[7] A. OSTROWSKI. *Recherches sur la méthode de Graeffe et les zéros des polynômes et des séries de Laurent.* Acta math. 72 (1940—41) p. 99—257.

[8] I. PETROWSKY. *On the diffusion of waves and the lacunas for hyperbolic equations.* Mat. Sbornik N. S. 17 (1945) p. 289—370.

[9] —— *Über das Cauchysche Problem für ein System linearer partieller Differentialgleichungen im Gebiete der nichtanalytischen Funktionen.* Bull. de l'univ. Moscou Vol. 1 fasc. 7 (1938).

[1] See the end of Chapter 3.

[2] RIESZ [11] p. 88—90. See also footnote (3) p. 87.

[10] I. PETROWSKY. *Über das Cauchysche Problem für Systeme von Partiellen Differentialgleichungen.* Mat. Sbornik N. S. 2 (1937) p. 815.

[11] M. RIESZ. *L'intégrale de Riemann-Liouville et le problème de Cauchy.* Acta math. 81 (1948) p. 1—223.

[12] B. L. VAN DER WAERDEN. *Moderne Algebra.* 2. Aufl. Berlin 1937.

[13] N. ZEILON. *Sur les équations aux dérivees partielles à quatre dimensions et le problème optique des milieux biréfringents.* Nova Acta Regiae Soc. Sc. Upsaliensis, Ser. IV, Vol. 5. N:o 3.