ANALYTIC FUNCTIONS WITH FINITE DIRICHLET INTEGRALS ON RIEMANN SURFACES

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Since around 1950 the general classification theory of Riemann surfaces has been studied. Although many fruitful results have been obtained, there are still unsolved fundamental problems in the theory concerning the spaces of analytic functions with finite Dirichlet integrals.

In this paper we shall be concerned with the following problems I and II (cf. [5, pp. 50–51]).

Problem I. Let $AD(R)$ be the complex linear space of analytic functions on a Riemann surface R with finite Dirichlet integrals. Does there exist a Riemann surface R satisfying $1 < \dim_{\mathbf{C}} AD(R) < \infty$?

Problem II. Let O_{AD} (resp. O_{ABD}) be the class of Riemann surfaces on which there are no nonconstant AD functions (resp. bounded AD functions). Does the strict inclusion relation $O_{AD} \subsetneqq O_{ABD}$ hold?

Let $HD(R)$ be the real linear space of harmonic functions on R with finite Dirichlet integrals. Then, it is known that for every natural number n there is a Riemann surface R satisfying $\dim_{\mathbf{R}} HD(R) = n$ (cf. [5, p. 197]). In contrast to this result, we show that $R \notin O_{AD}$ if and only if $\dim_{\mathbf{C}} AD(R) = \infty$.

Problem II has been open since the beginning of the study of the classification theory of Riemann surfaces. We show that the equality $O_{AD} = O_{ABD}$ holds. Moreover, we prove that the space $ABD(R)$, the space of bounded AD functions on a Riemann surface R, is dense in $AD(R)$ in the sense that for every $f \in AD(R)$ there is a sequence $\{f_n\} \subset ABD(R)$ such that $f_n(\zeta) = f(\zeta)$ for a fixed point $\zeta \in R$ and $\int_R |f_n' - f'|^2 dx dy \to 0$ $(n \to \infty)$.

This paper consists of three sections. The purpose of § 1 is to prove Proposition 1.9 concerning modifications of positive measures. Its proof is relatively long. This proposition is used only in the proof of Proposition 2.2. On first reading one could omit § 1 except for the definition of admissible domains and the statement of Proposition 1.9.

In § 2 we define the kernel function M of the Hilbert space $AD(R, \zeta)$, the space of AD functions f on R with $f(\zeta)=0$, and prove Theorem 2.3. In our theorem, we obtain the following inequality:

$$\sup_{z\in R}|M(z)|\leqslant\left(\int_R |M'|^2 dx\,dy/\pi\right)^{1/2},$$

which implies that M is bounded. The results concerning the above two problems immediately follow from this theorem.

A generalization of our theorem and a complete condition when the equality holds in the above inequality are given in § 3. As an application, we obtain the inequality on conformal invariants c_D and c_B.

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§ 1. Modifications of positive measures

Let W be an open set in the complex plane C. For every $\varrho\geqslant 0$ put $W_{-\varrho}=\{z\in W\,|\,d(z,\partial W)>\varrho\}$, where $d(z,\partial W)$ denotes the distance from z to the boundary ∂W of W. The smallest open set G which satisfies $W\subset G_{-\varrho}$ is denoted by $W_{+\varrho}$. If $\varrho>0$, it follows that $W_{+\varrho}=\{z\in \mathrm{C}\,|\,d(z, W)<\varrho\}=\bigcup_{c\in W}\Delta_\varrho(c)$, where $\Delta_\varrho(c)$ denotes the open disc with radius ϱ and center at c.

Let W be a plane domain. We shall call it an admissible domain if it satisfies the following conditions:

(i) W is bounded.
(ii) $m(\partial W)=0$, where m denotes the Lebesgue measure.
(iii) The boundaries γ_n of connected components O_n, $n=0, 1, 2, \ldots$, of W^e are rectifiable Jordan curves and satisfy $\sum l(\gamma_n)<\infty$. Here W^e denotes the exterior of W and $l(\gamma_n)$ denotes the length of γ_n.

For an admissible domain W, we denote by $\partial_e W$ the union of γ_n. For an admissible domain W and $\varrho>0$, we consider the domain surrounded by the outer boundary γ_0 of the closure $\overline{W}$ of W and the other boundaries γ_n of connected components O_n of W^e such that $l(\gamma_n)\geqslant 2\pi\varrho$. We denote it by $W(\varrho)$.

Using this notation we have the following five lemmas. We omit the proofs of some of them.

LEMMA 1.1. *Let W_1 and W_2 be admissible domains such that $W_1 \cap W_2 \neq \emptyset$. Then $W_1 \cup W_2$ is also an admissible domain and satisfies*

$$l(\partial_e(W_1 \cup W_2)) \leqslant l(\partial_e W_1) + l(\partial_e W_2).$$

LEMMA 1.2. *Let Δ_j, $j = 1, \ldots, n$, be open discs whose radii are not less than a positive number r. Then*

$$l\left(\partial_e\left(\bigcup_{j=1}^{n} \Delta_j\right)\right) \leqslant \frac{2}{r} m\left(\bigcup_{j=1}^{n} \Delta_j\right),$$

where $\partial_e(\bigcup_{j=1}^{n} \Delta_j)$ denotes the union of $\partial_e W_i$ of the connected components W_i of $\bigcup_{j=1}^{n} \Delta_j$.

Proof. We prove the lemma by mathematical induction on the number n of open discs. If $n = 1$, then our assertion is trivial. Assume that our assertion is true when the number of open discs is equal to $n-1$. Let Δ_j, $j = 1, \ldots, n$, be open discs and assume that Δ_k has the minimum radius r_k. Set $W = \bigcup_{j \neq k} \Delta_j$. Then

$$\begin{aligned} l(\partial_e(W \cap \Delta_k)) &\geqslant 2\pi \sqrt{\frac{m(W \cap \Delta_k)}{\pi}} \\ &\geqslant \frac{2}{r_k} m(W \cap \Delta_k). \end{aligned}$$

Hence, by the assumption, we have

$$\begin{aligned} l(\partial_e(\bigcup \Delta_j)) &= l(\partial_e(W \cup \Delta_k)) \\ &= l(\partial_e W) + l(\partial \Delta_k) - l(\partial_e(W \cap \Delta_k)) \\ &\leqslant \frac{2}{r_k}\{m(W) + m(\Delta_k) - m(W \cap \Delta_k)\} \\ &= \frac{2}{r_k} m(W \cup \Delta_k) \\ &= \frac{2}{r_k} m(\bigcup \Delta_j). \end{aligned}$$

This completes the proof.

LEMMA 1.3. *For every admissible domain W and every $\varrho \geqslant 0$, $W_{+\varrho}$ is an admissible domain and satisfies the following inequalities:*

(1) $l(\partial_e W_{+\varrho}) \leqslant l(\partial_e W) + 2\pi\varrho$.

(2) $m(W_{+\varrho} - W) \leqslant \varrho l(\partial_e W) + \pi\varrho^2$.

LEMMA 1.4. *For every admissible domain W and every $\varrho>0$, $W(\varrho)$ is an admissible domain and satisfies the following:*

(1) $\partial_e W(\varrho)=\partial W(\varrho)$.

(2) $W\subset W(\varrho)\subset(W_{+\varrho})_{-\varrho}$.

(3) $W(\varrho)_{+\varrho}=W_{+\varrho}$.

(4) $l(\partial W(\varrho))=l(\partial_e W)-\sum_{n\geqslant 1,\, l(\gamma_n)<2\pi\varrho} l(\gamma_n)$.

(5) $m(W(\varrho)-W)\leqslant\frac{\varrho}{2}\sum_{n\geqslant 1,\, l(\gamma_n)<2\pi\varrho} l(\gamma_n)$.

LEMMA 1.5. *Let W be an admissible domain and set $\Delta_\varrho=\Delta_\varrho(c)$. If*

$$m(W^e\cap\Delta_\varrho)\leqslant\tfrac{1}{50}m(\Delta_\varrho), \tag{1.1}$$

then there is a number r such that $\varrho/2\leqslant r\leqslant\varrho$ and

$$l(W^e\cap\partial\Delta_r)\leqslant l(\Delta_r\cap\partial_e W), \tag{1.2}$$

where $\Delta_r=\Delta_r(c)$.

Proof. For every r with $0\leqslant r\leqslant\varrho$, set $l(r)=l(\Delta_r\cap\partial_e W)$ and $l^*(r)=l(W^e\cap\partial\Delta_r)$. Assume that $l(r)<l^*(r)$ for every r with $\varrho/2\leqslant r\leqslant\varrho$. This implies $W^e\cap\partial\Delta_r\neq\varnothing$. If there is a number r such that $\varrho/2\leqslant r\leqslant(49/50)\varrho$ and $\partial\Delta_r\subset W^e$, then we have either $\overline{\Delta_r}\subset W^e$ or $\Delta_\varrho-\Delta_r\subset W^e$. Both of them contradict (1.1). Hence $\partial_e W\cap\partial\Delta_r\neq\varnothing$ for every r with $\varrho/2\leqslant r\leqslant(49/50)\varrho$. Therefore

$$l^*(r)>l(r)\geqslant r-\frac{\varrho}{2}\qquad\left(\frac{\varrho}{2}\leqslant r\leqslant\tfrac{49}{50}\varrho\right).$$

Integrating both sides of this inequality, we have

$$m(W^e\cap\Delta_\varrho)\geqslant\int_{\varrho/2}^{(49/50)\varrho} l^*(r)\,dr\geqslant\int_{\varrho/2}^{(49/50)\varrho}\left(r-\frac{\varrho}{2}\right)dr>\tfrac{1}{50}\pi\varrho^2.$$

This also contradicts (1.1). The proof is complete.

Remark. In Lemma 1.5 assume further $W\cap\Delta_{\varrho/2}\neq\varnothing$. Then $W\cup\Delta_r$ is an admissible domain by Lemma 1.1. The inequality (1.2) implies

$$l(\partial_e(W\cup\Delta_r))\leqslant l(\partial_e W).$$

Let W be a plane domain and denote by $HB(W)$ the Banach space of bounded harmonic functions h on W with norm $\|h\|_\infty=\sup_{z\in W}|h(z)|$. If $h\in HB(W)$ can be extended continuously onto ∂W, then we say that h belongs to $HBC(W)$ and also denote by h its continuous extension.

The following lemma is well known.

LEMMA 1.6. *Let W be a bounded domain such that each point of ∂W is regular with respect to the Dirichlet problem. Then the mapping $h \mapsto h|\partial W$ is an isometric isomorphism of $HBC(W)$ onto $C(\partial W)$, where $C(\partial W)$ denotes the Banach space of continuous functions on ∂W with norm $\|\cdot\|_\infty$.*

Let μ be a totally finite signed measure on the closure $\overline{W}$ of a domain W mentioned in Lemma 1.6. Then we can find a measure $\beta = \beta(\mu, W)$ on ∂W such that

$$\int_{\overline{W}} h\, d\mu = \int_{\partial W} h\, d\beta$$

for every $h \in HBC(W)$.

For positive measures we show the following two lemmas. The proof of the first lemma is omitted.

LEMMA 1.7. *Let μ be a totally finite positive measure on $\mathbf{C}$ and define $\lambda(z) = \lambda(z; \mu)$ by*

$$\lambda(z) = \sup \{r \geq 0 \,|\, \mu(\overline{\Delta_r(z)}) \geq N\pi r^2\},$$

where N denotes a fixed positive number and $\overline{\Delta_r(z)} = \{z\}$ if $r = 0$. Then

(1) *λ is a nonnegative upper semicontinuous function on $\mathbf{C}$.*

(2) $\mu(\overline{\Delta_{\lambda(z)}(z)}) = N\pi(\lambda(z))^2$.

LEMMA 1.8. *Let μ be a positive measure on $\mathbf{C}$ and $\varrho > 0$. Suppose $\operatorname{supp} \mu \subset \overline{\Delta_{\varrho/8}(c)}$ and $\mu(\overline{\Delta_{\varrho/8}(c)}) = 144\pi(\varrho/8)^2$. Then, for every number r with $\varrho/2 \leq r \leq \varrho$, there is a bounded measurable function $f(z) = f(z; \mu, \Delta_r(c))$ on $\mathbf{C}$ such that*

(1) *$f(z) \geq 1$ on $\Delta_r = \Delta_r(c)$ and $f(z) = 0$ on Δ_r^c, where Δ_r^c denotes the complement of Δ_r.*

(2) *$\int_{\Delta_r} h\, d\mu = \int_{\Delta_r} hf\, dm$, for every $h \in HL^1(\Delta_r)$, where $HL^1(\Delta_r)$ denotes the class of harmonic L^1 functions on Δ_r.*

Proof. For a totally finite signed measure ν with compact support and a number $\alpha > 0$, set $(M_\alpha \nu)(z) = \nu(\Delta_\alpha(z))/\pi\alpha^2$. Then $M_\alpha \nu$ is a bounded L^1 function on $\mathbf{C}$. If W is a domain such that $\operatorname{supp} \nu \subset \overline{W_{-\alpha}}$, then $\int h\, d\nu = \int_W h(M_\alpha \nu)\, dm$ for every $h \in HL^1(W)$.

Now we consider the function $M_{3\varrho/8}\mu$. It is nonnegative and satisfies $(M_{3\varrho/8}\mu)(z) = 16$ on $\Delta_{\varrho/4}(c)$ and $(M_{3\varrho/8}\mu)(z) = 0$ on $(\Delta_{\varrho/2}(c))^c$. Let s be the solution of the following equation:

$$(s-1)\pi\left(\frac{\varrho}{4}\right)^2 = \pi\left\{r^2 - \left(\frac{\varrho}{4}\right)^2\right\}.$$

Since $\varrho/2 \leq r \leq \varrho$, s satisfies $4 \leq s \leq 16$.

Set

$$f(z)=\begin{cases}(M_{3\varrho/8}\mu)(z)-s+1 & \text{on } \Delta_{\varrho/4}(c)\\ (M_{3\varrho/8}\mu)(z)+1 & \text{on } \Delta_r(c)-\Delta_{\varrho/4}(c)\\ 0 & \text{on } \Delta_r(c)^c.\end{cases}$$

Then f satisfies (1) and

$$\int_{\Delta_r} hf\,dm=\int_{\Delta} h(M_{3\varrho/8}\mu)\,dm=\int_{\Delta_r} h\,d\mu$$

for every $h\in HL^1(\Delta_r)$. This completes the proof.

Every totally finite signed measure μ on $\mathbf{C}$ can be decomposed into an absolutely continuous part μ_a and a singular part μ_s with respect to the Lebesgue measure m. We denote by f_μ the Radon-Nikodym derivative $d\mu_a/dm$.

We shall now prove the following proposition which plays an important role in the next section.

PROPOSITION 1.9. *Let W be an admissible domain and ν a totally finite positive measure on $\mathbf{C}$ such that* $\operatorname{supp}\nu\subset\overline{W}$ *and $f_\nu\geqslant\chi_W$ a.e. on $\mathbf{C}$, where χ_W denotes the characteristic function of W. Then, for every $\varepsilon>0$, there are a bounded open set W_ε and a bounded domain $\hat{W}_\varepsilon$ such that*

(1) $W\subset W_\varepsilon\subset\hat{W}_\varepsilon$ *and* $\overline{W}\subset\hat{W}_\varepsilon$.

(2) $m(\hat{W}_\varepsilon-W_\varepsilon)<\varepsilon$.

(3) $\int_{\overline{W}} h\,d\nu=\int_{W_\varepsilon} h\,dm$, *for every* $h\in HL^1(\hat{W}_\varepsilon)$.

Proof.(1) We may assume $\varepsilon\leqslant 1$. We first show that to prove the proposition it is sufficient to construct the following W_n, $\hat{W}_n$ and ν_n, $n=0, 1, 2, \ldots$:

(a) W_n is an open set and $\hat{W}_n$ is an admissible domain.

(b) $W_n\subset W_{n+1}$ and $\overline{\hat{W}_n}\subset\hat{W}_{n+1}$.

(c) $W\subset W_n\subset\hat{W}_n$.

(d) $l(\partial_e\hat{W}_n)\leqslant l_0\left\{1+\frac{AB}{K}\sum_{m=0}^{n-1}\left(\frac{k}{K}\right)^m+\frac{4\pi K}{A(l_0)^2}\sum_{m=0}^{n-1}(K)^m\right\}$ for $n\geqslant 1$.

(e) $m(\hat{W}_0-W_0)\leqslant\frac{\varepsilon}{2}$ and

$$m(\hat{W}_n-W_n)\leqslant\frac{\varepsilon}{2}+4\varepsilon\left\{\frac{K}{A}\sum_{m=0}^{n-1}(K)^m+\frac{B}{2}\sum_{m=0}^{n-1}(k)^m+\frac{\pi(K)^2}{(Al_0)^2}\sum_{m=0}^{n-1}(K)^{2m}\right\}\quad\text{for } n\geqslant 1.$$

(1) In this proof we sometimes put the indices on the letters as superscripts. That is, ϱ^i is not the ith power of ϱ, but i is an index. For powers of ϱ, we shall put parentheses around ϱ and write $(\varrho)^i$.

(f) ν_n is a totally finite positive measure such that $\operatorname{supp}\nu_n \subset \overline{\hat{W}_n}$.

(g) $f_{\nu_n} \geqslant \chi_{W_n}$ a.e. on $\mathbf{C}$.

(h) $\operatorname{supp}\mu_n \subset \overline{(\hat{W}_n)_{-\delta(K)^n}}$ for $n \geqslant 1$ and

$\mu_n(\overline{\hat{W}_n}) \leqslant (k)^n B\varepsilon$ for $n \geqslant 0$, where $d\mu_n = d\nu_n - \chi_{W_n} dm$.

(i) $\int_{\hat{W}} h\,d\nu = \int_{\overline{\hat{W}_n}} h\,d\nu_n$ for every $h \in HBC(\hat{W}_n)$.

Here $k = 1-2/10^5$, $K = 1-1/10^5$, $l_0 = l(\partial_e \hat{W}_0)$ and $\delta = \varepsilon/(Al_0)$. The numbers A and B are positive and satisfy the following two inequalities:

$$\frac{AB}{K}\,\frac{1}{1-\dfrac{k}{K}} + \frac{4\pi}{(Al_*)^2}\,\frac{K}{1-K} < 1$$

and

$$\frac{1}{A}\,\frac{K}{1-K} + \frac{B}{2}\,\frac{1}{1-k} + \frac{\pi}{(Al_*)^2}\,\frac{(K)^2}{1-(K)^2} < \tfrac{1}{8},$$

where l_* denotes the length of the boundary of the largest open disc contained in W.

By virtue of (b), we can define W_ε and $\hat{W}_\varepsilon$ as $\lim W_n$ and $\lim \hat{W}_n$, respectively. Then, (1) is satisfied and (e) implies (2). To see (3), let $h \in HL^1(\hat{W}_\varepsilon)$ and set $\|h\|_1 = \int_{\hat{W}_\varepsilon} |h|\,dm$. Then, by (h) and (i), we have

$$\int_{\hat{W}} h\,d\nu - \int_{W_n} h\,dm = \int_{\overline{(\hat{W}_n)_{-\delta(K)^n}}} h\,d\mu_n.$$

Since h is harmonic on $\overline{\Delta_{\delta(K)^n}(z)}$ for every $z \in \overline{(\hat{W}_n)_{-\delta(K)^n}}$, we have

$$|h(z)| = \left|\frac{1}{\pi(\delta(K)^n)^2}\int_{\Delta_{\delta(K)^n}(z)} h\,dm\right| \leqslant \frac{\|h\|_1}{\pi(\delta(K)^n)^2}$$

for every $z \in \overline{(\hat{W}_n)_{-\delta(K)^n}}$. Hence, by (h),

$$\left|\int_{\hat{W}} h\,d\nu - \int_{W_n} h\,dm\right| \leqslant \frac{B\varepsilon\|h\|_1}{\pi(\delta)^2}\left(\frac{k}{(K)^2}\right)^n.$$

Combining this with the fact that h is also an L^1 function on W_ε, we conclude that (3) is satisfied.

Next we construct W_n, $\hat{W}_n$ and ν_n, $n = 0, 1, 2, \ldots$, by mathematical induction. Assume that W_n, $\hat{W}_n$ and ν_n are constructed. Set $N = 144$, $\varkappa = 1-1/200N = 0.999965\ldots$, $l_n = l(\partial_e \hat{W}_n)$ and

$$\lambda_n = \min\left\{4\delta(K)^{n+1}, \frac{(k-\varkappa)(k)^n B\varepsilon}{3N\pi l_n}, \left(\frac{(k-\varkappa)(k)^n B\varepsilon}{2N\pi}\right)^{1/2}\right\}.$$

Let $\tilde{W}_n(\lambda_n)$ be the domain as defined before Lemma 1.1, and $\beta_n=\beta(\mu_n, \tilde{W}_n(\lambda_n))$ be the measure as defined after Lemma 1.6. Suppose that there is a point $p^1\in\partial\tilde{W}_n(\lambda_n)$ such that

$$\beta_n(\overline{\Delta_{\lambda_n}(p^1)})\geqslant N\pi(\lambda_n)^2$$

and

$$m(\tilde{W}_n(\lambda_n)^e\cap\Delta_{\varrho^1}(p^1))\leqslant\tfrac{1}{50}m(\Delta_{\varrho^1}(p^1)),$$

where $\varrho^1=8\lambda(p^1;\beta_n)\geqslant 8\lambda_n$. For the definition of $\lambda(p^1;\beta_n)$, see Lemma 1.7. Then, by the remark to Lemma 1.5, we can find r^1 such that $\varrho^1/2\leqslant r^1\leqslant\varrho^1$ and

$$l(\partial_e(\tilde{W}_n(\lambda_n)\cup\Delta_{r^1}(p^1)))\leqslant l(\partial\tilde{W}_n(\lambda_n)). \tag{1.3}$$

Set $\Delta^1=\Delta_{r^1}(p^1)$, $W_n^1=W_n\cup\Delta^1$ and $\tilde{W}_n^1=\tilde{W}_n(\lambda_n)\cup\Delta^1$. Then, (1.3) implies

$$l(\partial_e\tilde{W}_n^1)\leqslant l_n-L_n,$$

where $L_n=l_n-l(\partial\tilde{W}_n(\lambda_n))$. Let

$$f(z)=f(z;\beta_n|\overline{\Delta_{\lambda(p^1;\beta_n)}(p^1)},\Delta^1)\quad .$$

be the function as defined in Lemma 1.8. Set

$$d\mu_n^1=d\beta_n|\{\partial\tilde{W}_n(\lambda_n)-\overline{\Delta_{\lambda(p^1;\beta_n)}(p^1)}\}+(f-\chi_{\Delta^1-W_n})\,dm$$

and

$$d\nu_n^1=\chi_{W_n^1}dm+d\mu_n^1.$$

Then μ_n^1 is nonnegative and ν_n^1 satisfies

$$\int_{\overline{\tilde{W}_n}}h\,d\nu_n=\int_{\overline{\tilde{W}_n^1}}h\,d\nu_n^1$$

for every $h\in HBC(\tilde{W}_n^1)$.

Set $\beta^1=\beta(\mu_n^1, \tilde{W}_n^1(\lambda_n))$. Suppose that there is a point $p^2\in\partial\tilde{W}_n^1(\lambda_n)$ such that

$$\beta^1(\overline{\Delta_{\lambda_n}(p^2)})\geqslant N\pi(\lambda_n)^2$$

and

$$m(\tilde{W}_n^1(\lambda_n)^e\cap\Delta_{\varrho^2}(p^2))\leqslant\tfrac{1}{50}m(\Delta_{\varrho^2}(p^2)),$$

where $\varrho^2=8\lambda(p^2;\beta^1)\geqslant 8\lambda_n$. Set $\Delta^2=\Delta_{\varrho^2}(p^2)$. By using the same argument as above we can construct $W_n^2=W_n^1\cup\Delta^2$, $\tilde{W}_1^2=\tilde{W}_n^1(\lambda_n)\cup\Delta^2$ and ν_n^2 so that $l(\partial_e\tilde{W}_n^2)\leqslant l_n-L_n-L^1$, where $L^1=l(\partial_e\tilde{W}_n^1)-l(\partial\tilde{W}_n^1(\lambda_n))$, $d\mu_n^2=d\nu_n^2-\chi_{W_n^2}dm$ is nonnegative and ν_n^2 satisfies

$$\int_{\overline{\tilde{W}_n^1}}h\,d\nu_n^1=\int_{\overline{\tilde{W}_n^2}}h\,d\nu_n^2$$

for every $h\in HBC(\tilde{W}_n^2)$.

We continue this process as long as possible. Since $\varrho^m \geqslant 8\lambda_n$ and $l(\partial_e \hat{W}_n^m) \leqslant l_n - L_n - \sum_{k=1}^{m-1} L^k \leqslant l_n$ for every m, our process must stop after a finite number of times. Therefore there are W_n^i, $\hat{W}_n^i$ and ν_n^i such that if a point $p \in \partial \hat{W}_n^i(\lambda_n)$ satisfies

$$\beta^i(\overline{\Delta_{\lambda_n}(p)}) \geqslant N\pi(\lambda_n)^2,$$

then

$$m(\hat{W}_n^i(\lambda_n)^e \cap \Delta_\varrho(p)) > \tfrac{1}{50} m(\Delta_\varrho(p)),$$

where $\beta^i = \beta(\mu_n^i, \hat{W}_n^i(\lambda_n))$ and $\varrho = 8\lambda(p; \beta^i)$.

Set $\lambda(z) = \lambda(z; \beta^i)$ and $E_1 = \{p \in \partial \hat{W}_n^i(\lambda_n) \,|\, \beta^i(\overline{\Delta_{\lambda_n}(p)}) \geqslant N\pi(\lambda_n)^2\}$. This set E_1 is compact. If $E_1 \neq \emptyset$, then λ attains its maximum on E_1 at a point p_1 of E_1. Set $\varrho_1 = 8\lambda(p_1)$ and $E_2 = E_1 - \Delta_{2\varrho_1}(p_1)$. If $E_2 \neq \emptyset$, we can again find $p_2 \in E_2$ at which λ attains its maximum on E_2. We can continue this process as long as $E_m \neq \emptyset$. Since $\lambda(p_m) \geqslant \lambda_n$ for each m, there is a number j such that $E_j \neq \emptyset$ and $E_{j+1} = \emptyset$.

Set $\Delta_m = \Delta_{\varrho_m}(p_m)$, $m = 1, 2, \ldots, j$. Then $\{\Delta_m\}$ is a set of mutually disjoint open discs. Now we define W_{n+1}, $\hat{W}_{n+1}$ and ν_{n+1} as follows:

$$W_{n+1} = W_n^i \cup \bigcup_{m=1}^{j} \Delta_m,$$

$$U_{n+1} = \hat{W}_n^i(\lambda_n) \cup \bigcup_{\varrho_m \geqslant \delta(K)^{n+1}} \Delta_m, \quad V_{n+1} = \hat{W}_n^i(\lambda_n) \cup \bigcup_{m=1} \Delta_m,$$

$$\hat{W}_{n+1} = (U_{n+1})_{+2\delta(K)^{n+1}},$$

$$f_m(z) = f(z; \beta^i | \overline{\Delta_{\lambda(p_m)}(p_m)}, \Delta_m),$$

$$d\mu_{n+1} = d\beta^i \left| \left\{ \partial \hat{W}_n^i(\lambda_n) - \bigcup_{m=1}^{j} \overline{\Delta_{\lambda(p_m)}(p_m)} \right\} + \sum_{m=1} (f_m - \chi_{\Delta_m - W_n^i})\, dm, \right.$$

$$d\nu_{n+1} = \chi_{W_{n+1}} dm + d\mu_{n+1}.$$

It is clear that W_{n+1}, $\hat{W}_{n+1}$ and ν_{n+1} satisfy (a), (b), (c), (f) and (g). To prove (d) and (e), we apply Lemma 1.2 to $\{\Delta_m\}_{\varrho_m \geqslant \delta(K)^{n+1}}$. Then, by (h),

$$\begin{aligned} l(\partial_e(\bigcup_{\varrho_m \geqslant \delta(K)^{n+1}} \Delta_m)) &\leqslant \frac{2}{\delta(K)^{n+1}} \sum_{m=1}^{j} \pi(\varrho_m)^2 \\ &= \frac{2 \cdot 8^2}{N} \frac{1}{\delta(K)^{n+1}} \sum_{m=1}^{j} \beta^i(\overline{\Delta_{\lambda(p_m)}(p_m)}) \\ &\leqslant \frac{1}{\delta(K)^{n+1}} \beta^i(\partial \hat{W}_n^i(\lambda_n)) \\ &\leqslant \frac{1}{\delta(K)^{n+1}} \mu_n(\overline{\hat{W}_n}) \\ &\leqslant \frac{A l_0 B}{K} \left(\frac{k}{K}\right)^n. \end{aligned}$$

Hence, by Lemma 1.1,

$$l(\partial_e U_{n+1}) \leqslant l(\partial \hat{W}_n^i(\lambda_n)) + \frac{Al_0 B}{K}\left(\frac{k}{K}\right)^n.$$

Therefore, by Lemma 1.3,

$$\begin{aligned} l_{n+1} &\leqslant l_n + \frac{Al_0 B}{K}\left(\frac{k}{K}\right)^n + \frac{4\pi\varepsilon K}{Al_0}(K)^n \\ &\leqslant l_0\left\{1 + \frac{AB}{K}\sum_{m=0}^{n}\left(\frac{k}{K}\right)^m + \frac{4\pi K}{A(l_0)^2}\sum_{m=0}^{n}(K)^m\right\}. \end{aligned}$$

By Lemma 1.4, we obtain

$$\begin{aligned} m(\hat{W}_n^i(\lambda_n) - W_n^i) &= m(\hat{W}_n^i(\lambda_n) - \hat{W}_n^i) + m(\hat{W}_n^i - W_n^i) \\ &\leqslant \frac{\lambda_n}{2}L^i + m(\hat{W}_n^{i-1}(\lambda_n) \cup \Delta^i - W_n^{i-1} \cup \Delta^i) \\ &\leqslant \frac{\lambda_n}{2}L^i + m(\hat{W}_n^{i-1}(\lambda_n) - W_n^{i-1}) \\ &\leqslant \frac{\lambda_n}{2}(L^i + L^{i-1} + \ldots + L^1 + L_n) + m(\hat{W}_n - W_n). \end{aligned}$$

Hence by Lemma 1.3 we have

$$\begin{aligned} m(\hat{W}_{n+1} - W_{n+1}) &\leqslant m((U_{n+1})_{+2\delta(K)^{n+1}} - U_{n+1}) + m(\hat{W}_n^i(\lambda_n) - W_n^i) \\ &\leqslant 2\delta(K)^{n+1}\left\{l(\partial\hat{W}_n^i(\lambda_n)) + \frac{Al_0 B}{K}\left(\frac{k}{K}\right)^n\right\} \\ &\quad + \pi(2\delta(K)^{n+1})^2 + \frac{\lambda_n}{2}\left(L_n + \sum_{m=1}^{i} L^m\right) \\ &\quad + m(\hat{W}_n - W_n). \end{aligned}$$

Since $l_n \leqslant 2l_0$ and $l(\partial\hat{W}_n^i(\lambda_n)) \leqslant l_n - L_n - \sum_{m=1}^{i} L^m$, we have

$$m(\hat{W}_{n+1} - W_{n+1}) \leqslant (\hat{W}_n - W_n) + 4\varepsilon\left\{\frac{K}{A}(K)^n + \frac{B}{2}(k)^n + \frac{\pi(K)^2}{(Al_0)^2}(K)^{2n}\right\}.$$

This implies (e).

By the definition of μ_{n+1}, we have

$$\begin{aligned} \operatorname{supp}\mu_{n+1} &\subset \overline{V_{n+1}} \\ &\subset \overline{(U_{n+1})_{+\delta(K)^{n+1}}} \\ &\subset \overline{(((U_{n+1})_{+\delta(K)^{n+1}})_{+\delta(K)^{n+1}})_{-\delta(K)^{n+1}}} \\ &= \overline{(\hat{W}_{n+1})_{-\delta(K)^{n+1}}}. \end{aligned}$$

To estimate $\mu_{n+1}(\overline{\tilde{W}_{n+1}})=\mu_{n+1}(\overline{V_{n+1}})$, we set $\xi_1=\mu_{n+1}|\bigcup_{m=1}^{j}\Delta_{2\varrho_m}(p_m)$ and $\xi_2=\mu_{n+1}|\partial\tilde{W}_n^i(\lambda_n)-\bigcup_{m=1}^{j}\Delta_{2\varrho_m}(p_m)$. Since $\{\Delta_m\}$ is a set of mutually disjoint open discs and $\beta^i(\overline{\Delta_{2\varrho_m}(p_m)})<N\pi(2\varrho_m)^2$, $m=1, 2, \ldots, j$, we have

$$\begin{aligned}\xi_1(\overline{V_{n+1}}) &= \mu_{n+1}(\bigcup\Delta_{2\varrho_m}(p_m))\\ &=\beta^i(\bigcup\Delta_{2\varrho_m}(p_m))-\sum_{m=1}^{j}\int\chi_{\Delta_m-W_n^i}\,dm\\ &\leqslant\beta^i(\bigcup\Delta_{2\varrho_m}(p_m))-\sum_{m=1}^{j}\int\chi_{\Delta_m-\tilde{W}_n^i(\lambda_n)}\,dm\\ &\leqslant\beta^i(\bigcup\Delta_{2\varrho_m}(p_m))-\sum_{m=1}^{j}\tfrac{1}{50}\pi(\varrho_m)^2\\ &\leqslant\beta^i(\bigcup\Delta_{2\varrho_m}(p_m))-\sum_{m=1}^{j}\tfrac{1}{50}\pi(\varrho_m)^2\frac{\beta^i(\Delta_{2\varrho_m}(p_m))}{N\pi(2\varrho_m)^2}\\ &\leqslant\varkappa\beta^i(\bigcup\Delta_{2\varrho_m}(p_m))\\ &\leqslant\varkappa(k)^nB\varepsilon.\end{aligned}$$

Since every point $p\in\partial\tilde{W}_n^i(\lambda_n)-\bigcup\Delta_{2\varrho_m}(p_m)$ satisfies

$$\beta^i(\overline{\Delta_{\lambda_n}(p)})>N\pi(\lambda_n)^2,$$

we have

$$\xi_2(\overline{\Delta_{\lambda_n/2}(p)})<N\pi(\lambda_n)^2$$

for every $p\in\partial\tilde{W}_n^i(\lambda_n)$.

Each component γ of $\partial\tilde{W}_n^i(\lambda_n)$ can be covered by at most $[[l(\gamma)/\lambda_n]]$ closed discs with radii $\lambda_n/2$ and centers on γ, where $[[l(\gamma)/\lambda_n]]$ denotes the smallest natural number not less than $l(\gamma)/\lambda_n$. If $l(\gamma)\geqslant 2\pi\lambda_n$, then

$$[[l(\gamma)/\lambda_n]]\leqslant\frac{l(\gamma)}{\lambda_n}+1\leqslant\frac{l(\gamma)}{\lambda_n}\left(1+\frac{1}{2\pi}\right)\leqslant\frac{3l(\gamma)}{2\lambda_n}.$$

Therefore

$$\begin{aligned}\xi_2(\overline{V_{n+1}}) &= \xi_2(\partial\tilde{W}_n^i(\lambda_n))\\ &\leqslant\left\{\frac{3l(\partial\tilde{W}_n^i(\lambda_n))}{2\lambda_n}+1\right\}N\pi(\lambda_n)^2\\ &\leqslant\frac{3}{2}l_n N\pi\lambda_n+N\pi(\lambda_n)^2\\ &\leqslant(k-\varkappa)(k)^nB\varepsilon.\end{aligned}$$

Thus (h) holds for μ_{n+1}.

To prove (i), let $h\in HBC(\tilde{W}_{n+1})$. Then $h|V_{n+1}\in HBC(V_{n+1})$, so that

$$\int_{\overline{W}} h\,d\nu=\int_{\overline{\tilde{W}_n}} h\,d\nu_n=\int_{\overline{\tilde{W}_n^l}} h\,d\nu_n^l=\int_{\overline{\tilde{W}_{n+1}}} h\,d\nu_{n+1}.$$

Finally, we construct W_0, $\tilde{W}_0$ and ν_0. Set

$$\lambda^0=\min\left\{\frac{1}{2}\frac{\varepsilon}{l},\ \frac{(k-\varkappa)\mu(\overline{W})}{3N\pi l},\ \left(\frac{(k-\varkappa)\mu(\overline{W})}{2N\pi}\right)^{1/2}\right\},$$

where $l=l(\partial_e W)$ and $d\mu=d\nu-\chi_W dm$. In the above argument, replace W_n, $\tilde{W}_n$, ν_n and λ_n by W, W, ν and λ^0, respectively. Then we can construct W^1, V^1 and ν^1 which correspond to W_{n+1}, V_{n+1} and ν_{n+1}, respectively. We see that W^1 and V^1 are admissible domains and satisfy

$$m(V^1-W^1)\leqslant\frac{\lambda^0}{2}l\leqslant\frac{\varepsilon}{2}\left(\frac{1}{2}\right)$$

and

$$\mu^1(V^1)\leqslant k\mu(\overline{W}),$$

where $d\mu^1=d\nu^1-\chi_{W^1}dm$. The positive measure ν^1 satisfies supp $\nu^1\subset\overline{V^1}$, $f_{\nu^1}\geqslant\chi_{W^1}$ a.e. on $\mathbf{C}$ and

$$\int_{\overline{W}} h\,d\nu=\int_{\overline{V^1}} h\,d\nu^1$$

for every $h\in HBC(V^1)$.

For $n\geqslant 1$, set

$$\lambda^n=\min\left\{\frac{1}{2^{n+1}}\frac{\varepsilon}{l^n},\ \frac{(k-\varkappa)(k)^n\mu(\overline{W})}{3N\pi l^n},\ \left(\frac{(k-\varkappa)(k)^n\mu(\overline{W})}{2N\pi}\right)^{1/2}\right\},$$

where $l^n=l(\partial_e V^n)$. Then, by using again the same argument as above, from W^n, V^n, ν^n and λ^n we can construct admissible domains W^{n+1}, V^{n+1} and a positive measure ν^{n+1} such that $W^{n+1}\subset V^{n+1}$, $W^n\subset W^{n+1}$, $V^n\subset V^{n+1}$, $m(V^{n+1}-W^{n+1})\leqslant(\varepsilon/2)(\sum_{m=1}^{n+1}1/2^m)$, supp $\nu^{n+1}\subset\overline{V^{n+1}}$, $f_{\nu^{n+1}}\geqslant\chi_{W^{n+1}}$ a.e. on $\mathbf{C}$, $d\mu^{n+1}=d\nu^{n+1}-\chi_{W^{n+1}}dm$, $\mu^{n+1}(V^{n+1})\leqslant(k)^{n+1}\mu(\overline{W})$ and

$$\int_{\overline{W}} h\,d\nu=\int_{\overline{V^{n+1}}} h\,d\nu^{n+1}$$

for every $h\in HBC(V^{n+1})$.

Choose n so that $(k)^n\mu(\overline{W})\leqslant B\varepsilon$, and set $W_0=W^n$, $\tilde{W}_0=V^n$ and $\nu_0=\nu^n$. Then these satisfy (a), (c), (e), (f), (g), (h) and (i). The proof is complete.

§ 2. Analytic functions with finite Dirichlet integrals on Riemann surfaces

In this section we deal with the kernel function of the Hilbert space $AD(R, \zeta)$ and prove Theorem 2.3 below. Main results of this paper follow from this theorem. First we give notation and a preliminary lemma.

Let R be a Riemann surface and ζ a point on R. We denote by $AD(R, \zeta)$ the complex linear space of analytic functions f on R such that $f(\zeta)=0$ and the Dirichlet integrals

$$D_R[f]=\int_R |f'(z)|^2 dx\,dy \qquad (z=x+iy)$$

of f on R are finite. An inner product on $AD(R, \zeta)$ is defined by

$$(f, g)=\frac{1}{\pi}\int_R f'(z)\overline{g'(z)}\,dx\,dy$$

for every pair of f and g in $AD(R, \zeta)$. With this inner product $AD(R, \zeta)$ becomes a Hilbert space. Set $\|f\|=(f, f)^{\frac{1}{2}}=(D_R[f]/\pi)^{\frac{1}{2}}$.

Let t be a local coordinate defined in a neighborhood of ζ. Then the functional $f\mapsto (df/dt)(\zeta)$ is bounded on $AD(R, \zeta)$, and hence there is a unique function $M(z)= M(z; \zeta, t, R)$ such that

$$\frac{df}{dt}(\zeta)=(f, M) \tag{2.1}$$

for every $f\in AD(R, \zeta)$. We call it the kernel function of $AD(R, \zeta)$. The differential dM is called the exact Bergman kernel differential.

The kernel function $M(z)=M(z;\ \zeta,\ t,\ R)$ is identically equal to zero if and only if $(df/dt)(\zeta)=0$ for every $f\in AD(R, \zeta)$. If $M\not\equiv 0$, then $(dM/dt)(\zeta)=\|M\|^2>0$.

In the case of a domain in the complex z-plane, we always set $t=z$ and abbreviate $M(z;\ \zeta,\ t,\ R)$ by $M(z; \zeta)$. Then

$$M(z; \zeta)=\tfrac{1}{2}(P_0(z; \zeta)-P_1(z; \zeta)),$$

where $P_0(z; \zeta)$ (resp. $P_1(z; \zeta)$) is the extremal horizontal (resp. vertical) slit mapping of the plane domain (cf. [6, pp. 125–132]).

By this equality and our Proposition 1.9, we can prove Lemma 2.1 below. But, because the referee and J. Burbea [2] have given another short proof by using Schiffer's equality ([7]), we omit the proof.

LEMMA 2.1. *Let R be a plane domain. Then the kernel function $M(z)=M(z;\zeta)$ on R satisfies*

$$\|M\|_\infty \leq \|M\|,$$

where $\|M\|_\infty = \sup_{z\in R}|M(z)|$.

By using above lemma, we shall prove the following proposition:

PROPOSITION 2.2. *Let R be a Riemann surface. If the valence function ν_M of $w=M(z)=M(z;\zeta,t,R)$ satisfies $\nu_M(w)\geq n$ on $M(R)$, then*

$$\|M\|_\infty \leq \|M\|/\sqrt{n}.$$

Proof. Without loss of generality we may assume $M(z)\not\equiv 0$. Let $\{R_j\}$ be an exhaustion of R such that each ∂R_j consists of a finite number of mutually disjoint analytic Jordan curves on R. We may assume that $\zeta\in R_j$ for every j. Set $M_j(z)=M(z;\zeta,t,R_j)$, $\nu_j=\nu_{M_j}$, $W_j=M_j(R_j)$ and $U_j=\{w\in W_j|\nu_j(w)\geq n\}$. Then, for every compact subset K of $M(R)$, there is a number J such that $K\subset U_J$. For every $\varepsilon>0$, we choose K so that $\int_{M(R)-K}\nu_M dm<\varepsilon$ and J so that $K\subset U_J$, $\int_K(\nu_M-\nu_J)dm<\varepsilon$ and $\int\nu_J dm-\int\nu_M dm<\varepsilon$. It follows that

$$\begin{aligned} m(W_J-U_J) &\leq \int_{W_J-U_J}\nu_J dm \\ &\leq \int\nu_J dm-\int\nu_M dm+\int_{M(R)-K}\nu_M dm+\int_K(\nu_M-\nu_J)\,dm \\ &< 3\varepsilon. \end{aligned}$$

Since M_J can be extended analytically onto $\bar{R}_J$ (cf. [8, pp. 114–137]), W_J is admissible. Define an L^1 function ν on $\mathbf{C}$ by $\nu=\max\{\nu_J, n\chi_{W_J}\}$ and apply Proposition 1.9 replacing W and $d\nu$ by W_J and $(\nu/n)\,dm$, respectively. Then there are an open set W_ε and a domain $\hat{W}_\varepsilon$ such that $W_J\subset W_\varepsilon\subset\hat{W}_\varepsilon$, $m(\hat{W}_\varepsilon)<\infty$, $m(\hat{W}_\varepsilon-W_\varepsilon)<\varepsilon$ and $\int_{W_J}h\nu dm=n\int_{W_\varepsilon}h dm$ for every $h\in HL^1(\hat{W}_\varepsilon)$. Set $M_\varepsilon(w)=M(w;0,w,\hat{W}_\varepsilon)$. Since $HL^2(\hat{W}_\varepsilon)\subset HL^1(\hat{W}_\varepsilon)$, by (2.1), we have

$$n\int_{\hat{W}_\varepsilon}f'\left(\frac{1}{\int\nu_J dm}-\frac{\overline{M'_\varepsilon}}{n\pi}\right)dm=\frac{n}{\int\nu_J dm}\int_{\hat{W}_\varepsilon-W_\varepsilon}f'\,dm+\frac{1}{\int\nu_J dm}\int_{W_J}f'(\nu-\nu_J)\,dm$$

for every $f\in AD(\hat{W}_\varepsilon,0)$. Since $\nu-\nu_J=0$ on U_J and $0\leq\nu-\nu_J\leq n$ on W_J-U_J, we have

$$D_{\hat{W}_\varepsilon}\left[\frac{w}{\int\nu_J dm}-\frac{M_\varepsilon}{n\pi}\right]^{1/2}\leq\frac{1}{\int\nu_J dm}\left\{\sqrt{m(\hat{W}_\varepsilon-W_\varepsilon)}+\sqrt{m(W_J-U_J)}\right\}<\frac{1}{\int\nu_J dm}(\sqrt{\varepsilon}+\sqrt{3\varepsilon}).$$

This implies that $M_\varepsilon(w)$ converges to $(n\pi/\int\nu_M dm)w$ uniformly on every compact subset of $M(R)$ as $\varepsilon\to 0$. By Lemma 2.1

$$
\begin{aligned}
|w| &= \frac{\int \nu_M dm}{n\pi} \lim_{\varepsilon\to 0} |M_\varepsilon(w)| \\
&\leqslant \frac{\int \nu_M dm}{n\pi} \lim_{\varepsilon\to 0} \sqrt{M'_\varepsilon(0)} \\
&= \frac{1}{\sqrt{n}} \sqrt{\frac{\int \nu_M dm}{\pi}} = \frac{\|M\|}{\sqrt{n}}
\end{aligned}
$$

for every $w \in M(R)$. Hence $\|M\|_\infty \leqslant \|M\|/\sqrt{n}$.

Since $\nu(w) \geqslant 1$ on $M(R)$, the following theorem immediately follows from Proposition 2.2.

THEOREM 2.3. *For an arbitrary Riemann surface, the kernel function* $M(z) = M(z; \zeta, t, R)$ *is bounded and satisfies*

$$\|M\|_\infty \leqslant \|M\|. \tag{2.2}$$

Finally we deal with the complex linear space $AD(R)$, the space of analytic functions on a Riemann surface R with finite Dirichlet integrals. By Theorem 2.3 we obtain the following corollaries:

COROLLARY 2.4. *If there is a nonconstant AD function on a Riemann surface R, then there is a nonconstant bounded AD function on R, namely, $O_{AD} = O_{ABD}$, where O_{AD} (resp. O_{ABD}) denotes the class of Riemann surfaces without nonconstant AD functions (resp. nonconstant bounded AD functions).*

Proof. Let f be a nonconstant AD function on R. Choose a point ζ on R so that $(df/dt)(\zeta) \neq 0$ for some local coordinate t defined in a neighborhood of ζ. Then the kernel function $M(z; \zeta, t, R)$ is nonconstant. Theorem 2.3 further implies that M is bounded.

COROLLARY 2.5. *A Riemann surface R is not of class O_{AD} if and only if* $\dim_{\mathbf{C}} AD(R) = \infty$.

Proof. It is sufficient to show that if there is a nonconstant AD function on R, then $\dim_{\mathbf{C}} AD(R) = \infty$. Let f be a nonconstant bounded AD function on R. Let P be a polynomial with complex coefficients. Then $P(f) \equiv 0$ if and only if $P \equiv 0$. Hence bounded AD functions f^n, $n = 0, 1, 2, \ldots$, are linearly independent, and so $\dim_{\mathbf{C}} AD(R) = \infty$.

COROLLARY 2.6. *Let $ABD(R)$ be the complex linear space of bounded AD functions f on a Riemann surface R and set $|||f||| = \|f\|_\infty + \|f - f(\zeta)\|$, where ζ is a fixed point on R. Then $ABD(R)$ becomes a Banach algebra with the norm $|||\cdot|||$ and dense in $AD(R)$ in the* *sense that for every $f \in AD(R)$ there is a sequence $\{f_n\} \subset ABD(R)$ such that $f_n(\zeta)=f(\zeta)$ for every n and $\|f_n - f\| \to 0$ as $n \to \infty$.*

Proof. The first assertion is evident. To prove the second assertion it is sufficient to show that $ABD(R, \zeta)$ is dense in the Hilbert space $AD(R, \zeta)$, where $ABD(R, \zeta)$ denotes the linear space of functions $f \in ABD(R)$ such that $f(\zeta)=0$. Choose a sequence $\{\zeta_n\}_{n=1}^{\infty} \subset R$ so that $\lim \zeta_n = \zeta$ and let t be a fixed local coordinate defined in a neighborhood of $\zeta \cup \bigcup_{n=1}^{\infty} \zeta_n$. Then $\{M(z; \zeta_n, t, R) - M(\zeta; \zeta_n, t, R)\}_{n=1}^{\infty}$ is complete in $AD(R, \zeta)$. In fact, if $f \in AD(R, \zeta)$ satisfies $(f(z), M(z; \zeta_n, t, R) - M(\zeta; \zeta_n, t, R)) = 0$ for every n, then $(df/dt)(\zeta_n)=0$ for every n. Hence $f \equiv 0$. By Theorem 2.3 $M(z; \zeta_n, t, R) - M(\zeta; \zeta_n, t, R)$ is bounded, and so we have the corollary.

Remark 1. It is easy to show that $\{M(z; \zeta^n, t, R)\}_{n=1}^{\infty}$ is also complete in $AD(R, \zeta)$ (for the definition, see § 3).

Remark 2. From Corollary 2.6 we know that the maximal ideal space of $ABD(R)$ will be useful to study functions of class $AD(R)$.

§ 3. Representing measures of the functional $\varphi \to \varphi(0)$

The valence function ν_M of the kernel function $M(z; \zeta, t, R)$ satisfies

$$f'(0) = \frac{1}{\int \nu_M dm} \int_{M(R)} f' \nu_M dm$$

for every analytic functions f on $M(R)$ such that $\int_{M(R)} |f'|^2 \nu_M dm < \infty$, namely, $\nu_M dm$ is a representing measures of the functional $\varphi \to \varphi(0)$ defined on the space of analytic $L^2(\nu_M dm)$ functions φ on $M(R)$ with single-valued integrals. In this section, we generalize Proposition 2.2 and prove Proposition 3.2 below.

First we give notation. Let R be a Riemann surface and ζ a point on R. Let ν be a measurable function on R such that $\nu(z) \geqslant c$ a.e. on R for a positive number c. We denote by $AD_\nu(R, \zeta)$ the complex linear space of analytic functions f on R such that $f(\zeta)=0$ and $\int_R |f'(z)|^2 \nu(z) dx dy < \infty$, where $z = x + iy$.

An inner product on $AD_\nu(R, \zeta)$ is defined by

$$(f, g)_\nu = \frac{1}{\pi} \int_R f' \overline{g'} \nu \, dx \, dy$$

for every pair of f and g in $AD_\nu(R, \zeta)$. With this inner product $AD_\nu(R, \zeta)$ becomes a Hilbert space. Set $\|f\|_\nu = (f, f)_\nu^{1/2} = (\int_R |f'|^2 \nu dx dy / \pi)^{1/2}$.

Let t be a fixed local coordinate defined in a neighborhood of ζ. Since the functional $f \mapsto (df/dt)(\zeta)$ is bounded, there is a unique function $M_\nu(z) = M_\nu(z; \zeta, t, R)$ such that

$$\frac{df}{dt}(\zeta) = (f, M_\nu)_\nu$$

for every $f \in AD_\nu(R, \zeta)$. We call M_ν the kernel function of $AD_\nu(R, \zeta)$. The kernel function $M_\nu(z; \zeta, t, R)$ is identically equal to zero if and only if $(df/dt)(\zeta) = 0$ for every $f \in AD_\nu(R, \zeta)$.

The following proposition is a generalization of Theorem 2.3.

PROPOSITION 3.1. *The kernel function $M_\nu(z) = M_\nu(z; \zeta, t, R)$ is bounded and satisfies*

$$\|M_\nu\|_\infty \leqslant \|M_\nu\|_\nu / \sqrt{c}.$$

Proof. We may assume $M_\nu(z) \not\equiv 0$. Suppose that ν is lower semicontinuous on R, $\nu(z)$ is a natural number for every $z \in R$ and $\nu(z) \geqslant n$ on R for a fixed natural number n. Set $W = M_\nu(R)$ and $\mu(\omega) = \sum_{z \in M_\nu^{-1}(\omega)} \nu(z)$. Then $\int_R |M'_\nu|^2 \nu \, dx dy = \int \mu \, dm < \infty$, μ is lower semicontinuous on $\mathbf{C}$, $\mu(\omega)$ is a natural number not less than n for almost all $\omega \in W$ and $\mu(\omega) = 0$ on W^c.

Now we construct a Riemann surface S and $F \in AD(S, \eta)$ for some η on S such that

(1) The valence function ν_F of F is equal to μ a.e. on $\mathbf{C}$.

(2) $(dF/d\tau)(\eta) \neq 0$ for some (and hence every) local coordinate τ defined in a neighborhood of η.

(3) For every $g \in AD(S, \eta)$, there is a function $f \in AD_\mu(W, 0)$ satisfying $g = f \circ F$.

Set $U_j = \{\omega \in \mathbf{C} \mid \mu(\omega) \geqslant j\}$, $j = 1, 2, \ldots$. Then each U_j is open and satisfies $U_j \supset U_{j+1}$. For every $j \geqslant 2$, let $U_{j,k}$, $k = 1, 2, \ldots, k(j)$, $k(j) \leqslant \infty$, be connected components of U_j. For each $j \geqslant 2$ and k with $1 \leqslant k \leqslant k(j)$, take a point $p_{j,k}$ and a neighborhood $V_{j,k}$ of $p_{j,k}$ so that $V_{j,k} \subset U_{j,k}$, $0 \notin \bigcup_j \bigcup_k V_{j,k}$, and $V_{j,k} \cap V_{i,h} = \varnothing$ $((j,k) \neq (i,h))$. Let $s_{j,k,l}$, $l = 1, 2, \ldots$, be mutually disjoint closed slits in $V_{j,k}$ converging to $p_{j,k}$.

Let S_1 be a copy of $U_1 - \bigcup_k \bigcup_l s_{2,k,l} = W - \bigcup_k \bigcup_l s_{2,k,l}$ and let S_j, $j = 2, 3, \ldots$, be copies of $U_j - \bigcup_k \bigcup_l s_{j,k,l} - \bigcup_k \bigcup_l s_{j+1,k,l}$. We joint these copies along their common slits identifying the upper edges of the slits of S_j with the corresponding lower edges of the slits of S_{j+1}, $j = 1, 2, \ldots$, and vice versa. This gives a ramified covering surface S of W.

Let F be its projection mapping and let η be a point of $F^{-1}(0)$. Then these S and F satisfy (1) and (2). It is easy to show that for every bounded analytic function g on S there is a bounded analytic function f on W satisfying $g = f \circ F$ (cf. Myrberg's example, e.g. [5, pp. 53–54]). By Corollary 2.6, every $g \in AD(S, \eta)$ can be approximated by bounded analytic functions on S, and so (3) is satisfied.

By virtue of (2) we can choose F as a local coordinate defined in a neighborhood of η. Set $M(w) = M(w; \eta, F, S)$. Then, for every $f \in AD_\mu(W, 0)$, we obtain

$$f'(0)\frac{dM_\nu}{dt}(\zeta) = \frac{1}{\pi}\int_R (f\circ M_\nu)'\overline{M'_\nu}\nu\, dx\, dy$$
$$= \frac{1}{\pi}\int_W f'\mu\, dm$$
$$= \frac{1}{\pi}\int_S (f\circ F)'\overline{F'}\, du\, dv \qquad (w = u + iv).$$

In particular, by taking $f(\omega) = \omega$, we have $(dM_\nu/dt)(\zeta) = \|M_\nu\|_\nu^2 = \|F\|^2$. Hence

$$\frac{dg}{dF}(\eta) = f'(0) = \frac{1}{\pi}\int_S g'\frac{\overline{F'}}{\|F\|^2}\, du\, dv$$

for every $g = f\circ F \in AD(S, \eta)$, and so $M = F/\|F\|^2$. Therefore, by Proposition 2.2, we have

$$\|M_\nu\|_\infty = \|F\|_\infty \leqslant \|F\|/\sqrt{n} = \|M_\nu\|_\nu/\sqrt{n}.$$

Suppose next that ν is lower semicontinuous on R, that $\nu(z)/\varepsilon$ is a natural number for some fixed $\varepsilon > 0$ and for every $z \in R$ and that c/ε is a natural number. Set $\nu_\varepsilon(z) = \max\{\nu(z)/\varepsilon, c/\varepsilon\}$. Then $M_\nu(z; \zeta, t, R) = M_{\nu_\varepsilon}(z; \zeta, t, R)/\varepsilon$ and ν_ε satisfies the above assumption. Hence

$$\|M_\nu\|_\infty = \|M_{\nu_\varepsilon}/\varepsilon\|_\infty \leqslant \sqrt{\varepsilon/c}\|M_{\nu_\varepsilon}/\varepsilon\|_{\nu_\varepsilon} = \|M_\nu\|_\nu/\sqrt{c}.$$

Finally, we consider an arbitrary measurable function ν on R such that $\nu(z) \geqslant c$ a.e. on R. We can construct measurable functions ν_j on R, $j = 1, 2, \ldots$, such that

(1) $0 \leqslant \nu_j \leqslant \nu_{j+1}$ a.e. on R and $\lim \nu_j = \nu$.
(2) $\nu_j(z) \geqslant c$ a.e. on R.
(3) ν_j is lower semicontinuous on R.
(4) $(2^j/c)\nu_j(z)$ is a natural number for every $z \in R$.

It is easy to show that $M_{\nu_j}(z; \zeta, t, R)$ converges to $M_\nu(z; \zeta, t, R)$ uniformly on every compact subset of R and $\|M_{\nu_j}\|_{\nu_j} \to \|M_\nu\|_\nu$ as $j \to \infty$. Since $\|M_{\nu_j}\|_\infty \leqslant \|M_{\nu_j}\|_{\nu_j}/\sqrt{c}$ by the above argument, we have $\|M_\nu\|_\infty \leqslant \|M_\nu\|_\nu/\sqrt{c}$. The proof is complete.

Let E be a compact set in the plane $\mathbf{C}$ and U be a domain containing E. We say that E is removable with respect to AD functions if every $f \in AD(U - E)$ can be extended analytically onto U. We denote by N_D the class of compact sets which are removable with respect to AD functions. A set E is of class N_D if and only if E^c is a domain of class O_{AD}, namely, there are no nonconstant AD functions on E^c (see e.g. [5, p. 261]). By using this notation we have

Proposition 3.2. *Let W be a plane domain containing the origin 0. Let ν be an L^1 function on $\mathbf{C}$ such that $\nu(z)\geqslant c$ a.e. on W for a positive number c and $\nu(z)=0$ a.e. on W^c. If*

$$f'(0)=\frac{1}{\int \nu\, dm}\int_W f'\nu\, dm$$

for every $f\in AD_\nu(W,0)$, then $W\subset\Delta_r(0)$, where $r=\{\int \nu\, dm/(c\pi)\}^{\frac12}$. The equality

$$\sup_{z\in W}|z|=r$$

holds if and only if $\nu(z)=c$ a.e. on W and $W=\Delta_r(0)-E$, where E is a relatively closed subset of $\Delta_r(0)$ such that $E\cap K\in N_D$ for every compact subset K of $\Delta_r(0)$.

Proof. From the uniqueness of the kernel function $M_\nu(z;0,z,W)$, it follows that $M_\nu(z;0,z,W)=(\pi/\int \nu\, dm)z$. Since $\|M_\nu\|_\nu=(\pi/\int \nu\, dm)^{\frac12}$, by Proposition 3.1, we have $W\subset\Delta_r(0)$. Thus the first assertion has been proved.

To show the second assertion, we assume $\inf_{z\in K}\nu(z)>c$ for a compact subset K of W with $m(K)>0$. Let $d=d(K,\partial W)$ and define a bounded nonnegative L^1 function ν_1 on $\mathbf{C}$ by $\nu_1=M_{d/2}\mu$, where $d\mu=(\nu-c)\chi_K dm$ and $(M_{d/2}\mu)(z)=\mu(\Delta_{d/2}(z))/\{\pi(d/2)^2\}$. Then there are a disc Δ_1 and a number $\alpha_1>0$ such that $\overline{\Delta_1}\subset W$ and $\nu_1(z)\geqslant\alpha_1$ on Δ_1. Let Δ_j, $j=2,\dots,n$, be discs with centers at p_j such that $p_n=0$, $\overline{\Delta}_j\subset W$ and $p_j\in\Delta_{j-1}$ for every j. Assume that there are a bounded nonnegative L^1 function ν_{j-1} on $\mathbf{C}$ and a number $\alpha_{j-1}>0$ such that $\operatorname{supp}\nu_{j-1}\subset W$, $\nu_{j-1}(z)\geqslant\alpha_{j-1}$ on Δ_{j-1} and $\int h\nu_{j-1}dm=\int h\nu_1 dm$ for every harmonic function h on W. Let Δ be a disc with center at p_j such that $\Delta\subset\Delta_{j-1}$. Then

$$\begin{aligned}\int h\nu_{j-1}\,dm&=\int h(\nu_{j-1}-\alpha_{j-1}\chi_\Delta)\,dm+\alpha_{j-1}\int_\Delta h\,dm\\&=\int h(\nu_{j-1}-\alpha_{j-1}\chi_\Delta)\,dm+\alpha_{j-1}\frac{m(\Delta)}{m(\Delta_j)}\int_{\Delta_j}h\,dm\end{aligned}$$

for every harmonic function h on W. Set $\nu_j=\nu_{j-1}-\alpha_{j-1}\chi_\Delta+(\alpha_{j-1}m(\Delta)/m(\Delta_j))\chi_{\Delta_j}$ and $\alpha_j=\alpha_{j-1}m(\Delta)/m(\Delta_j)$. Then ν_j and α_j satisfy the above conditions for j. Thus, by induction, we can construct ν_n and $\alpha_n>0$ such that $\operatorname{supp}\nu_n\subset W$, $\nu_n(z)\geqslant\alpha_n$ on Δ_n and $\int h\nu_n dm=\int h\nu_1 dm$ for every harmonic function h on W.

Set $\delta=\alpha_n m(\Delta_n)/\int \nu dm$ and $\nu^*=\nu-(\nu-c)\chi_K+\nu_n-\alpha_n\chi_{\Delta_n}+\delta\nu$. Then $\nu^*(z)\geqslant(1+\delta)c$ a.e. on W, $\nu^*(z)=0$ a.e. on W^c, $AD_{\nu^*}(W,0)=AD_\nu(W,0)$ and

$$f'(0)=\frac{1}{\int \nu\, dm}\int_W f'\nu\, dm=\frac{1}{\int \nu^*\, dm}\int_W f'\nu^*\, dm$$

for every $f\in AD_{\nu^*}(W,0)$. Hence $\sup_{z\in W}|z|\leqslant r/\sqrt{1+\delta}<r$.

Therefore if $\sup_{z\in W}|z|=r$, then $\nu(z)=c$ a.e. on W and $m(W)=\int \nu dm/c=\pi r^2=m(\Delta_r(0))$. This implies that $\chi_W=\chi_{\Delta_r(0)}$ a.e. on $\mathbf{C}$ and $M(z;0,z,W)=z/r^2$. By Theorem 1 of the author's paper [4], we see that W is a domain mentioned above.

Conversely, if W is a domain mentioned above and $\nu(z)=c$ a.e. on W, then every $f\in AD_\nu(W,0)$ can be extended analytically onto $\Delta_r(0)$, and hence

$$f'(0)=\frac{1}{c\pi r^2}\int_{\Delta_r(0)} f'c\, dm=\frac{1}{\int \nu\, dm}\int_W f'\nu\, dm$$

for every $f\in AD_\nu(W,0)$. Thus we have proved the second assertion.

Corollary 3.3. *The equality sign in (2.2) of Theorem 2.3 holds if and only if either*

(1) $M(z;\zeta,t,R)\equiv 0$, *or*

(2) R *is conformally equivalent to* $\Delta_1(0)-E$, *where* E *is a relatively closed subset of* $\Delta_1(0)$ *such that* $E\cap K\in N_D$ *for every compact subset* K *of* $\Delta_1(0)$.

Remark. If R is of finite genus, then $M(z;\zeta,t,R)\equiv 0$ if and only if $R\in O_{AD}$, namely, there are no nonconstant AD functions on R (cf. [5, pp. 50–52]).

For a natural number n, set

$$AD(R,\zeta^n)=\left\{f\in AD(R,\zeta)\,\middle|\,\frac{df}{dt}(\zeta)=\ldots=\frac{d^{n-1}f}{dt^{n-1}}(\zeta)=0\right\}.$$

This is a closed subspace of the Hilbert space $AD(R,\zeta)$ and there is a unique function $M(z)=M(z;\zeta^n,t,R)\in AD(R,\zeta^n)$ such that

$$\frac{d^n f}{dt^n}(\zeta)=(f,M)$$

for every $f\in AD(R,\zeta^n)$. Next we show

Proposition 3.4. *The function* $M(z)=M(z;\zeta^n,t,R)$ *is bounded and satisfies*

$$\|M\|_\infty\leqslant\|M\|.$$

The equality holds if and only if either

(1) $M \equiv 0$, *or*

(2) $n=1$ *and* R *is a planar surface mentioned in* (2) *of Corollary 3.3.*

Proof. Assume $M \not\equiv 0$, set $W = M(R)$ and let ν be the valence function of M. Since $f \circ M \in AD(R, \zeta^n)$ for $f \in AD_\nu(W, 0)$, we have

$$f'(0)\frac{d^n M}{dt^n}(\zeta) = \frac{d^n(f\circ M)}{dt^n}(\zeta)$$
$$= \frac{1}{\pi}\int_R (f\circ M)'\,\overline{M'}\,dx\,dy$$
$$= \frac{1}{\pi}\int_W f'\nu\,dm$$

for every $f \in AD_\nu(W, 0)$. Hence

$$f'(0) = \frac{1}{\int \nu\,dm}\int_W f'\nu\,dm$$

for every $f \in AD_\nu(W, 0)$ and so the proposition follows from Proposition 3.2.

For a fixed local coordinate t defined in a neighborhood of a point ζ on R we define $c_D(\zeta^n)$ and $c_B(\zeta^n)$ by

$$c_D(\zeta^n) = \sup\left\{\left|\frac{d^n f}{dt^n}(\zeta)\right| \,\middle|\, f \in AD(R, \zeta^n), \|f\| \leqslant 1\right\},$$
$$c_B(\zeta^n) = \sup\left\{\left|\frac{d^n f}{dt^n}(\zeta)\right| \,\middle|\, f \in AB(R, \zeta^n), \|f\|_\infty \leqslant 1\right\},$$

where $AB(R, \zeta^n)$ denotes the complex linear space of bounded analytic functions f on R satisfying $f(\zeta) = (df/dt)(\zeta) = \dots = (d^{n-1}f/dt^{n-1})(\zeta) = 0$ (cf. [6, pp. 256–257]). We denote by N_B the class of compact sets which are removable with respect to bounded analytic functions.

Finally we show

COROLLARY 3.5. $c_D(\zeta^n)$ *and* $c_B(\zeta^n)$ *satisfy*

$$c_D(\zeta^n) \leqslant c_B(\zeta^n). \tag{3.1}$$

The equality holds if and only if either

(1) $c_B(\zeta^n) = 0$, *or*

(2) $n=1$ *and* R *is conformally equivalent to* $\Delta_1(0) - E$, *where* E *is a relatively closed subset of* $\Delta_1(0)$ *such that* $E \cap K \in N_B$ *for every compact subset* K *of* $\Delta_1(0)$.

Proof. Assume $c_D(\zeta^n)>0$, and let $F\in AD(R,\ \zeta^n)$ be the extremal function such that $c_D(\zeta^n)=(d^nF/dt^n)(\zeta)$ and $\|F\|=1$. Then $F=M/\|M\|$, where $M(z)=M(z;\zeta^n,t,R)$. Hence, by Proposition 3.4, we have $\|F\|_\infty\leq\|F\|=1$, and so $c_D(\zeta^n)=(d^nF/dt^n)(\zeta)\leq c_B(\zeta^n)$.

Obviously either (1) or (2) implies $c_D(\zeta^n)=c_B(\zeta^n)$. Assume $c_D(\zeta^n)=c_B(\zeta^n)>0$. Then $\|F\|_\infty=\|F\|$. By Proposition 3.4, we see that $n=1$ and M is univalent. Hence our assertion follows from [3].

Remark. For the case of a plane domain and $n=1$, (3.1) was obtained by Ahlfors and Beurling [1]. This is also obtained by the relation of the Szegö and the exact Bergman kernel functions.

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