

# AN ELEMENTARY METHOD IN THE STUDY OF NONNEGATIVE CURVATURE

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A standard technique in classical analysis for the study of continuous sub-solutions of the Dirichlet problem for second order operators may be illustrated as follows. Suppose it is to be shown that a continuous real function  $f(x)$  is convex (respectively, strictly convex) at  $x_0$ ; then it suffices to produce a  $C^2$  function  $g(x)$  such that  $g(x) \leq f(x)$  near  $x_0$  and  $g(x_0) = f(x_0)$ , and such that  $g''(x_0) \geq 0$  (respectively  $g''(x_0) \geq$  some fixed positive constant). The main point of this procedure is to sidestep arguments involving continuous functions by working with differentiable functions alone. Now in global differential geometry, the functions that naturally arise are often continuous but not differentiable. Since much of geometric analysis reduces to second order elliptic problems, this technique then recommends itself as a natural tool for overcoming this difficulty with the lack of differentiability. In a limited way, this technique has indeed appeared in several papers in complex geometry (e.g. Ahlfors [1], Takeuchi [20], Elencwajg [7] and Greene–Wu [11]; cf. also Suzuki [19]). The main purpose of this paper is to broaden and deepen the scope of this method by making it the central point of a general study of nonnegative sectional, Ricci or bisectional curvature. The following are the principal theorems; the relevant definitions can be found in Section 1.

Let  $M$  be a noncompact complete Riemannian manifold and let  $0 \in M$  be fixed. Let  $\{C_t\}_{t \in I}$  be a family of closed subsets of  $M$  indexed by a subset  $I$  of  $\mathbf{R}$ . Assume that  $e_t \equiv d(0, C_t) \rightarrow \infty$  as  $t \rightarrow \infty$ , where  $d(p, q)$  will always denote the distance between  $p, q \in M$  relative to the Riemannian metric. The family of functions  $\eta_t: M \rightarrow \mathbf{R}$  defined by  $\eta_t(p) =$

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$e_t - d(p, C_t)$  is Lipschitz continuous (with Lipschitz constant 1) and also satisfies  $|\eta_t(p)| \leq d(p, 0)$  (by the triangle inequality). It is thus an equi-continuous family uniformly bounded on compact sets. By Ascoli's theorem, a subsequence of  $\{\eta_t\}$ , to be denoted by  $\{\eta_n\}$ , converges to a continuous function  $\eta: M \rightarrow \mathbf{R}$ , the convergence being uniform on compact subsets of  $M$ . To fix the ideas, it may be helpful to keep in mind two special cases.

*Example  $\alpha$ .*  $I = \mathbf{R}$  and  $C_t = \{y: d(0, y) = t\}$  = the geodesic sphere of radius  $t$  around 0.

*Example  $\beta$ .* Let  $\gamma: [0, \infty) \rightarrow M$  be a ray emanating from 0, i.e.  $\gamma$  is an arclength-parametrized geodesic such that  $\gamma(0) = 0$  and each segment of  $\gamma$  is distance minimizing. For each  $t \in \mathbf{R}$ , let  $C_t = \{\gamma(t)\}$  (i.e., the one-element set consisting of  $\gamma(t)$ ); then every subsequence  $\{\eta_n\}$  converges to a unique  $\eta$  in this case. This  $\eta$  is called the *Busemann function of  $\gamma$*  (Cheeger-Gromoll [4] and Eberlein-O'Neill [6]). In order to state the theorems, two definitions are needed. A continuous function  $f: M \rightarrow \mathbf{R}$  is called *essentially strictly convex* (respectively, *essentially strictly subharmonic*) iff for every  $C^\infty$  function  $\chi: \mathbf{R} \rightarrow \mathbf{R}$  satisfying  $\chi > 0$ ,  $\chi' > 0$  and  $\chi'' > 0$ ,  $\chi \circ f$  is strictly convex (respectively, strictly subharmonic). Essentially strictly convex (subharmonic) functions are convex (subharmonic), but the converse does not hold in general.

**THEOREM A.** *Let  $M$  be a complete noncompact Riemannian manifold. Notation as above:*

(a) *If  $M$  has nonnegative sectional curvature, then each  $\eta$  is convex; furthermore,  $\eta$  is essentially strictly convex at the points where the sectional curvature is positive.*

(b) *If  $M$  has nonnegative Ricci curvature, then each  $\eta$  is essentially strictly subharmonic; furthermore,  $\eta$  is strictly subharmonic at the points where the Ricci curvature is positive.*

(c) *If  $M$  is a Kähler manifold with nonnegative bisectional curvature, then each  $\eta$  is plurisubharmonic; furthermore,  $\eta$  is strictly plurisubharmonic at the points where the bisectional curvature is positive.*

In the next theorem, it is important to note that a general  $\eta$  is never bounded from above (Lemma 6 of Section 1).

**THEOREM B.** *Let  $M$  be a noncompact complete Riemannian manifold and let  $K$  be a compact subset of  $M$ . Fix an  $\eta$  as above. Then there exists an  $a_0 \in \mathbf{R}$ ,  $a_0$  depending only on  $K$ , such that (a), (b) and (c) of Theorem A hold with  $M$  replaced by  $M - K$  in the hypothesis and with the conclusion asserted only for the open set  $\{\eta > a_0\}$ .*

**THEOREM C.** *Let  $M$  be a complete noncompact Riemannian manifold and let  $K$  be a compact subset of  $M$ .*

(a) *If the sectional curvature is nonnegative everywhere and is positive in  $M - K$ , then each  $\eta$  is essentially strictly convex.*

(b) *If the Ricci curvature is nonnegative everywhere and is positive in  $M - K$ , then each  $\eta$  is strictly subharmonic.*

(c) *If  $M$  is a Kähler manifold with everywhere nonnegative bisectional curvature which is positive in  $M - K$ , then  $e^\eta$  is strictly plurisubharmonic for each  $\eta$ .*

Theorem C together with Greene–Wu [13], [14] yield two immediate corollaries; the first is due to Cheeger–Gromoll [4] and Poor [17] while the second has its roots in Greene–Wu [12] and [14].

**COROLLARY 1.** *If  $M$  is a noncompact complete Riemannian manifold whose sectional curvature is nonnegative everywhere and is positive outside a compact set, then  $M$  is diffeomorphic to Euclidean space.*

**COROLLARY 2.** *If  $M$  is a noncompact complete Kähler manifold such that for some compact set  $K \subset M$ , the bisectional curvature of  $M \geq 0$  in  $K$  and  $> 0$  in  $M - K$ , and the sectional curvature of  $M \geq 0$  in  $M - K$ , then  $M$  is a Stein manifold.*

The emphasis on Corollary 1, as in Greene–Wu [13], is that while the Cheeger–Gromoll–Poor proof of this involves intricate geometric arguments, the present proof is function-theoretic and is conceptually transparent. It goes as follows: Let  $\eta$  be the Busemann function of a ray issuing from 0 (Example  $\beta$  above) and let  $\xi = \sup \eta$ , where  $\sup$  is taken over all rays issuing from 0. By Cheeger–Gromoll [4],  $\xi$  is an *exhaustion function* (i.e., for every  $c \in \mathbf{R}$ ,  $\{x \in M: \xi(x) \leq c\}$  is a compact set; this is a simple argument using part (a) of Theorem A). By (a) of Theorem C,  $\chi \circ \xi$  is a strictly convex function where  $\chi$  is any  $C^\infty$  function on  $\mathbf{R}$  obeying  $\chi > 0$ ,  $\chi' > 0$  and  $\chi'' > 0$ . By Theorem 3 of Greene–Wu [13], the existence of this strictly convex exhaustion function implies that  $M$  is diffeomorphic to Euclidean space. (In outline, the proof is as follows:  $\chi \circ \xi$  can be smoothed to a  $C^\infty$  strictly convex exhaustion function  $\zeta$ , say;  $\zeta$  being strictly convex implies all its critical points are nondegenerate;  $\zeta$  being an exhaustion function implies there can be only one nondegenerate critical point  $p_0$  for  $\zeta$ ; mapping the integral curves of  $\text{grad } \zeta$  in  $M - \{p_0\}$  in the obvious manner to the radial rays of  $\mathbf{R}^n$  gives a diffeomorphism between  $M - \{p_0\}$  and  $\mathbf{R}^n - \{0\}$ ; a minor technical adjustment then extends the diffeomorphism to one between  $M$  and  $\mathbf{R}^n$ .)

The proof of Corollary 2 is equally simple. Let  $\xi$  be as above; sectional curvature being nonnegative on  $M - K$  implies that  $\xi$  is an exhaustion function (Cheeger–Gromoll [4] and Greene–Wu [10, Proposition 3]; it is a simple argument using part (a) of Theorem B). Then (c) of Theorem C implies that  $e^\xi$  is strictly plurisubharmonic. Grauert’s solution of the Levi problem in the form given by Narasimhan [16] then concludes the proof. Note that this corollary slightly extends Theorem 3 of Greene–Wu [14] and at the same time implies *both* part (B) and part (C) of Theorem 1 in Greene–Wu [12]. In the above proof, the assumption of the nonnegativity of the sectional curvature in  $M - K$  was only needed to insure that  $\xi$  is an exhaustion function. However, for the validity of Corollary 2 itself, it is natural to conjecture that this assumption is superfluous.

When  $\eta$  is the Busemann function of a ray (Example  $\beta$  above), (a) of Theorem A is in Cheeger–Gromoll [2] and Greene–Wu [10], a weaker version of (b) (the subharmonicity of  $\eta$ ) is in Cheeger–Gromoll [4], the positive half of (c) is in Greene–Wu [14], and finally (a) of Theorem B is in Greene–Wu [13]. These are the basic facts governing the behavior of noncompact complete Riemannian manifolds. Note however that these papers had to devise an *ad hoc* method for each case, which seems to work only for that particular case and only for the Busemann function. The method of this paper is by comparison elementary, simple, and more powerful as it applies to a general  $\eta$  and to all cases all at once. Thus the study of nonnegative curvature on noncompact manifolds is beginning to submit to order. A pertinent remark is that the original motivation for considering such a general  $\eta$  was that, by allowing the arbitrary closed sets  $\{C_i\}$  to enter into the definition of  $\eta$  (rather than just a ray), the resulting  $\eta$  might turn out to be an exhaustion function. (Thus far, the only exhaustion functions which can be constructed this way are those by Cheeger–Gromoll [4] on manifolds of nonnegative sectional curvature using only the Busemann functions; see also the generalization by Greene–Wu [10] to the case of nonnegative curvature outside a compact set as well as the proofs of Corollaries 1 and 2 above.) The available evidence suggests that the picture is more complicated than meets the eye, but this extra generality may prove to be useful in the eventual solution of this problem (cf. the remark after Lemma 7 of Section 1 in this regard). Such a solution would have many applications.

The scope of the present method is by no means confined to nonnegative curvature. As an illustration, Section 3 gives some sample theorems indicating other possible applications such as the study of minimal hypersurfaces in compact manifolds of nonnegative Ricci curvature or the global study of the heat kernel on complete manifolds. In a related manuscript (Wu [21]), this method is also applied to yield a general criterion for the volume of a Riemannian manifold to be infinite. Moreover, this method is particularly

sensitive to the presence of *positive* curvature when everywhere nonnegative curvature is already assumed; the implication of these considerations in the theory of  $q$ -complete spaces of Andreotti–Grauert [2] will be taken up in a future publication.

The theorems of this paper were obtained in January 1976, contemporaneously with those of a related paper Greene–Wu [14]. Due to an unfortunate set of circumstances the appearance in print of both papers has been much delayed.

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### Section 1

Here is a collection of definitions. Unless stated otherwise,  $M$  is a Riemannian manifold and  $D$  its covariant differential operator. Given  $p \in M$  and a function  $f: M \rightarrow \mathbf{R}$  continuous near  $p$ , three numbers  $Cf(p)$ ,  $Sf(p)$ , and  $Pf(p)$  will now be introduced to measure the deviation of  $f$  from being convex, subh (abbreviation for subharmonic) and psh (abbreviation for plurisubharmonic) at  $p$ , respectively; here as later, the third case will always be understood to be in the context of a Kählerian  $M$ .

Let  $X \in M_p$ ,  $|X| = 1$  and let  $\gamma: (-a, a) \rightarrow M$  be a geodesic such that  $\dot{\gamma}(0) = X$ . The extended real number  $Cf(p; X)$  is defined to be:

$$Cf(p; X) \equiv \liminf_{r \rightarrow 0} \frac{1}{r^2} \{[(f \circ \gamma)(r) + (f \circ \gamma)(-r)] - 2(f \circ \gamma)(0)\},$$

and then

$$Cf(p) \equiv \inf_X Cf(p; X)$$

where the infimum is taken over all unit vectors  $X$  in  $M_p$ .

Next recall that the Green's function  $\sigma_x^r$  of the ball  $B(r)$  of radius  $r$  around  $p$  is the fundamental solution of the Laplace-Beltrami operator  $\Delta$  with singularity at  $x$  (i.e.  $\Delta \sigma_x^r = \delta_x$ ) which vanishes on  $\partial B(r)$ . Then:

$$Sf(p) \equiv \liminf_{r \rightarrow 0} \frac{2(\dim M)}{r^2} \left\{ \int_{\partial B(r)} f * d\sigma_p^r - f(p) \right\},$$

where  $*$  is the usual star operator in Hodge theory.

Now let  $M$  be in addition a Kähler manifold. With the same setting as above, let  $f|L$  be the restriction of  $f$  to a 1-dimensional complex submanifold  $L$  through  $p$ . Then  $S(f|L)(p)$  is defined via the induced Kähler metric on  $L$ , and by definition,

$$Pf(p) = \inf_L S(f|L)(p),$$

where the infimum is taken over all 1-dimensional complex submanifold  $L$  of  $M$  through  $p$ . (Cf. Takeuchi [20], Elenowajg [7], and Greene–Wu [14].)

In general,  $-\infty \leq \{Cf(p), Sf(p), Pf(p)\} \leq +\infty$ . Assume from now on that all functions are continuous. Recall that given a continuous function  $f: M \rightarrow \mathbf{R}$ , then by definition:  $f$  is convex iff its restriction to every geodesic is a convex function of one variable,  $f$  is subh iff for all sufficiently small geodesic ball  $B$ , any harmonic function  $h$  on  $B$  which agrees with  $f$  on  $\partial B$  satisfies  $h \geq f$  in  $B$ , and  $f$  is psh iff its restriction to every 1-dimensional complex submanifold of  $M$  is subh;  $f$  is strictly convex (respectively strictly subh, strictly psh) iff in some neighborhood of each point of  $M$ ,  $f$  is the sum of a convex (resp. subh, psh) function and a  $C^2$  strictly convex (resp. strictly subh, strictly psh) function. The latter is understood in the sense of  $D^2\alpha > 0$ ,  $\Delta\alpha > 0$ , and  $\partial\bar{\partial}\alpha > 0$  respectively. A trivial but crucial definition is the following: a real-valued function  $g$  is said to support  $f$  at  $p \in M$  iff  $g$  is continuous near  $p$ ,  $g \leq f$  and  $g(p) = f(p)$ . The following lemmas shed light on this string of definitions. Their proofs are either standard or straightforward and hence omitted.

Let a continuous function  $f: M \rightarrow \mathbf{R}$  be given. Then:

LEMMA 1. If  $f$  is  $C^2$  at  $p$ , then  $Cf(p; X) = (f \circ \gamma)''(0)$  where  $\gamma$  is a geodesic such that  $|\dot{\gamma}| = 1$  and  $\dot{\gamma}(0) = X$ ,  $Cf(p) = \min_X D^2f(X, X)$  where  $X$  runs through the unit vectors at  $p$ ;  $Sf(p) = \Delta f(p)$ ; and finally  $Pf(p) = 4 \min_X \partial\bar{\partial}f(X, X)$  where  $X$  runs through all unit tangent vectors of type  $(1, 0)$  at  $p$ . (Cf. Feller [8] for  $Sf(p)$  and Greene-Wu [9] for  $Pf(p)$ .)

LEMMA 2.  $f$  is convex, subh, psh, respectively iff  $Cf \geq 0$ ,  $Sf \geq 0$ ,  $Pf \geq 0$ , respectively.

LEMMA 3.  $f$  is strictly convex, strictly subh, strictly psh, resp. iff for some positive function  $k$  on  $M$ ,  $Cf \geq k$ ,  $Sf \geq k$ ,  $Pf \geq k$ , resp. (Cf. Greene-Wu [13] for  $Cf$  and Richberg [18] for  $Pf$ .)

LEMMA 4. If  $g$  supports  $f$  at  $p$ , then  $Cf(p) \geq Cg(p)$ ,  $Sf(p) \geq Sg(p)$ , and  $Pf(p) \geq Pg(p)$ . If  $f$  is supported at every point of  $M$  by a (strictly) convex, subh or psh function, then  $f$  is (strictly) convex, subh, or psh.

LEMMA 5. Suppose there exists a sequence of continuous functions  $\{f_n\}$  converging uniformly to  $f$  and suppose  $Cf_n \geq \varepsilon_n$ ,  $Sf_n \geq \varepsilon_n$ , or  $Pf_n \geq \varepsilon_n$ , where  $\varepsilon_n$  is a sequence of real numbers converging to  $\varepsilon$ , then  $Cf \geq \varepsilon$ ,  $Sf \geq \varepsilon$ ,  $Pf \geq \varepsilon$  respectively.

Among these five lemmas, Lemma 4 is the most trivial and at the same time the most important for the purpose of this paper. For suppose it is to be proved that the function  $f$  is subh, then Lemma 4 implies that it suffices to find at each  $p \in M$  a function  $g$  supporting  $f$  at  $p$  and to prove that  $g$  is subh at  $p$ . In the situations under consideration (see next section), it turns out that this  $g$  is  $C^\infty$  so that the proof of  $\Delta g \geq 0$  can be accomplished by standard differential geometric arguments. This is the key observation underlying this paper.

For the proof of Theorem C, the following elementary facts about the function  $\eta$  of the introduction will be needed. Let  $M$  be a complete noncompact Riemannian manifold and let  $0 \in M$  be a fixed point. Recall:  $\{C_t\}_{t \in I}$  is a family of closed subsets of  $M$  ( $I \subset \mathbb{R}$ ),  $e_t \equiv d(0, C_t) \rightarrow \infty$  as  $t \rightarrow \infty$  by assumption,  $\eta_t \equiv e_t - d(\cdot, C_t)$ , and a subsequence  $\{\eta_n\}$  of  $\{\eta_t\}$  converges uniformly on compact subsets of  $M$  to an  $\eta: M \rightarrow \mathbb{R}$ .

Now fix  $p \in M$ . Choose  $r_n \in C_n$  such that  $d(p, r_n) = d(p, C_n)$ , and let  $\gamma_n: [0, l_n] \rightarrow M$  be a minimal geodesic joining  $p$  to  $r_n$ .  $\gamma_n$  is assumed parametrized by arclength so that  $l_n = d(p, r_n)$ . The tangent vectors  $\{\dot{\gamma}_n(0)\}$  are unit vectors in  $M_p$  and hence (by passing to a subsequence if necessary) converge to a unique unit vector  $X \in M_p$ . Let  $\gamma: [0, \infty) \rightarrow M$  be the maximal geodesic issuing from  $p$  such that  $\dot{\gamma}(0) = X$ ;  $\gamma$  is a ray, i.e., every segment of  $\gamma$  is minimizing.

LEMMA 6.  $(\eta \circ \gamma)(t) = t + \eta(p)$  for all  $t \in [0, \infty)$ .

*Proof.* By the definition of  $\eta$ ,  $(\eta \circ \gamma)(t) - \eta(p) = \lim_{n \rightarrow \infty} d(p, C_n) - d(\gamma(t), C_n)$ . Fix  $t$  in the following discussion. By the definition of  $\gamma$ , there exists a sequence  $\{t_n\} \subset [0, \infty)$  such that  $\lim_{n \rightarrow \infty} \gamma(t_n) = \gamma(t)$ ; since the geodesics are parametrized by arclength, necessarily  $\lim_{n \rightarrow \infty} t_n = t$ . Then:  $d(p, C_n) - d(\gamma(t), C_n) = \text{Length } \gamma_n - d(\gamma(t), C_n) = \text{Length } \{\gamma_n|_{[0, t_n]}\} + \text{Length } \{\gamma_n|_{[t_n, l_n]}\} - d(\gamma(t), C_n) = t_n + d(\gamma_n(t_n), C_n) - d(\gamma(t), C_n)$ . Equivalently,  $d(p, C_n) - d(\gamma(t), C_n) - t_n = d(\gamma_n(t_n), C_n) - d(\gamma(t), C_n)$ . Hence

$$\begin{aligned} |(\eta \circ \gamma)(t) - \eta(p) - t| &= \lim |d(p, C_n) - d(\gamma(t), C_n) - t_n| \\ &= \lim |d(\gamma_n(t_n), C_n) - d(\gamma(t), C_n)| \\ &\leq \lim |d(\gamma_n(t_n), \gamma(t))| \\ &= 0. \end{aligned}$$

Q.E.D.

LEMMA 7. Let  $a \in \mathbb{R}$  and let  $D = \{p: \eta(p) < a\}$ . Then for all  $p \in D$ ,  $\eta(p) = a - d(p, \partial D)$ .

*Proof.* Fix  $p \in D$  and let  $\eta(p) = b$ ; then the lemma is equivalent to:  $a - b = d(p, \partial D)$ . First show  $a - b \leq d(p, \partial D)$ . To this end, choose  $q \in \partial D$  so that  $d(p, q) = d(p, \partial D)$ . By the definition of  $\eta_n$ , there exists  $s_n \in C_n$  such that  $d(q, C_n) = d(q, s_n)$ . Thus  $\eta_n(q) = e_n - d(q, s_n)$  and  $\eta_n(p) \geq e_n - d(p, s_n)$ . Hence,  $a - b = \lim_{n \rightarrow \infty} \{\eta_n(q) - \eta_n(p)\} \leq \lim \{d(p, s_n) - d(q, s_n)\} \leq \lim d(p, q) = d(p, \partial D)$ , as desired. To prove the reverse inequality, use the notation of Lemma 6. Let  $\gamma$  be the ray issuing from  $p$  as in that lemma, and let  $q_0 = \gamma(a - b)$ . Then Lemma 6 implies  $\eta(q_0) = a$ , and  $\gamma$  being a ray implies  $d(p, q_0) = a - b$ . The former implies  $q_0 \in \partial D$  and hence  $d(p, \partial D) \leq d(p, q_0) = a - b$ .

Q.E.D.

*Remark.* To assess the exact degree of generality of this paper, the following observation may be of value. For the fixed point  $0 \in M$ , the construction leading up to Lemma 6

associates with 0 a ray  $\xi: [0, \infty) \rightarrow M$ . Explicitly, if  $s_n \in C_n$  satisfies  $e_n = d(0, s_n) = d(0, C_n)$  for all  $n$ , let  $\xi_n$  be a minimizing geodesic joining 0 to  $s_n$  parametrized by arclength. Pass to a subsequence if necessary, let  $\xi_n(0)$  converge in  $M_0$ , where  $\xi_n(0) = 0$ . Then  $\xi$  is by definition the maximal geodesic issuing from 0 such that  $\xi(0) = \lim \xi_n(0)$ . Let  $\beta$  be the Busemann function associated with the ray  $\xi$ . Then in general  $\eta \geq \beta$ , and examples show that the inequality is strictly on nonempty open subsets of suitably chosen  $M$ . This shows in some sense that the construction of  $\eta$  generates more functions than just be Busemann function.

It remains to recall the definitions of the various curvatures in order to fix the signs. Let  $M$  be a Riemannian manifold. If  $R$  is the Riemannian curvature tensor and  $X, Y$  are orthonormal basis of a plane in some  $M_p$ , then the sectional curvature of this plane is  $+R(X, Y, X, Y)$ . The Ricci tensor  $\text{Ric}$  is defined as follows: let  $X, Y \in M_p$ , then  $\text{Ric}(X, Y) = \sum_i R(X, X_i, Y, X_i)$ , where  $\{X_i\}$  is an orthonormal basis of  $M_p$ . If  $X$  is a unit vector, the Ricci curvature in the direction  $X$  is then  $\text{Ric}(X, X)$ . Now suppose  $M$  is a Kähler manifold with structure tensor  $J$ . Let  $P_1$  and  $P_2$  be two planes in  $M_p$  each invariant under  $J$ , and let  $X_1$  and  $X_2$  be unit vectors in  $P_1$  and  $P_2$  respectively. The (holomorphic) *bisectional curvature* of  $P_1$  and  $P_2$  is by definition  $H(P_1, P_2) \equiv R(X_1, JX_1, X_2, JX_2)$ . By the Bianchi identity, also  $H(P_1, P_2) = R(X_1, X_2, X_1, X_2) + R(X_1, JX_2, X_1, JX_2)$ , thus a sum of two sectional curvatures.

## Section 2

This section gives the proofs of the three theorems. Let  $\{\eta_n\}$  and  $\eta$  be as in the introduction; thus  $\eta_n = e_n - d(\cdot, C_n)$  and  $\eta_n \rightarrow \eta$  uniformly on compact subsets of  $M$ .

*Proof of Theorem A.* The proof must of necessity treat the three parts (a), (b) and (c) one by one. However, the reader will note that, except for superficial differences arising from the different kinds of curvature under consideration, the three proofs are identical.

First assume  $M$  is complete, noncompact and Riemannian, and the sectional curvature of  $M$  is nonnegative. Given  $p \in M$ , the first objective is to prove the first part of part (a), i.e.,

$$(C1) \quad C\eta(p) \geq 0.$$

Let  $B$  be a small open ball containing  $p$  (not necessarily with  $p$  as center) and let  $\zeta: (-a, a) \rightarrow B \subset M$  be a geodesic parametrized by arclength, i.e.,  $|\dot{\zeta}| = 1$ . Since  $\eta_n \rightarrow \eta$  uniformly on  $B$ , Lemma 5 and the definition of  $C\eta$  imply that it suffices to prove:

$$(C2) \quad \text{There exists a sequence } \varepsilon_n, \varepsilon_n \text{ depending only on } B \text{ and } \varepsilon_n \rightarrow 0, \text{ such that } C(\eta \circ \zeta) \geq \varepsilon_n \text{ for all geodesics } \zeta \text{ in } B.$$



To this end, fix an  $n$  and pick  $\zeta: (-a, a) \rightarrow B$  so that  $\zeta(0) = p$ . Also choose  $q_n \in C_n$  so that  $d(p, q_n) = d(p, C_n) \equiv l_n$ . Introduce the function  $f: (-a, a) \rightarrow \mathbf{R}$ , where  $f(s) = e_n - d(\zeta(s), q_n)$  and  $e_n \equiv d(0, C_n)$  as in the introduction;  $f$  depends on  $n$ . Then  $f$  supports  $\eta_n \circ \zeta$  at 0 and by Lemma 4,  $Cf(0) \leq C(\eta_n \circ \zeta)(0)$ . In view of (C2), (C1) would follow from:

(C3) There exists a sequence  $\varepsilon_n, \varepsilon_n$  depending only on  $B$  and  $\varepsilon_n \rightarrow 0$ , such that  $Cf(0) \geq \varepsilon_n$  for  $f$  as above.

As mentioned in Section 1,  $Cf(0)$  will be computed by first finding a suitable  $C^\infty$  function  $g$  which supports  $f$  at 0 and then estimating  $g''(0)$  from below via differential geometric methods. The construction of such a  $g$  proceeds as follows. Let  $\gamma: [0, l_n] \rightarrow M$  be a minimizing geodesic from  $p$  to  $q_n$  such that  $|\dot{\gamma}| \equiv 1$ . Thus  $l_n = \text{length of } \gamma$ . Recall:  $\zeta: (-a, a) \rightarrow B$  is a geodesic such that  $\zeta(0) = p$ . Both  $\dot{\zeta}$  and  $\dot{\gamma}$  are therefore unit vector fields. Let  $V(t)$  be a vector field along  $\gamma$  defined by:  $V(t) = \text{the parallel translate of } \dot{\zeta}(0) \text{ to } \gamma(t) \text{ along } \gamma$ , where  $0 \leq t \leq l_n$ . Define  $W(t) = (1 - (t/l_n))V(t)$ ; then  $W(0) = \dot{\zeta}(0)$  and  $W(l_n) = 0$ . Let  $k$  be a variation of  $\gamma$  which induces  $W(t)$ , i.e., for a small positive  $\delta$ ,  $k: [0, l_n] \times (-\delta, \delta) \rightarrow M$  is a  $C^\infty$  map such that: (i)  $k(t, 0) = \gamma(t)$  for all  $t \in [0, l_n]$ , (ii)  $k(0, s) = \zeta(s)$  for all  $s \in (-\delta, \delta) \cap (-a, a)$ , (iii)  $k(l_n, s) = q_n$  for all  $s \in (-\delta, \delta)$ , and (iv) for each  $t \in [0, l_n]$ , the tangent vector of the curve  $s \mapsto k(t, s)$  at  $s = 0$  is  $W(t)$ . Such a  $k$  can be obtained, for instance, via the exponential map along  $\gamma$ . Now define  $g: (-\delta, \delta) \rightarrow \mathbf{R}$  by  $g(s) = e_n - [\text{length of the curve } t \mapsto k(t, s)]$ . It follows that  $g \leq f$  in  $(-\delta, \delta) \cap (-a, a)$ , and  $g(0) = f(0)$  because of (i). Thus  $g$  supports  $f$  at 0 and by Lemma 1 and Lemma 4,  $g''(0) = Cg(0) \leq Cf(0)$ . Thus to prove assertion (C1), it suffices in view of assertion (C3) above to prove:

(C4) There exists a sequence  $\{\varepsilon_n\}$  such that  $g''(0) \geq \varepsilon_n$ ,  $\varepsilon_n$  depends only on  $B$ , and  $\varepsilon_n \rightarrow 0$  as  $n \rightarrow \infty$ .

By the very definition of  $g$  and  $k$ ,  $\{-g''(0)\}$  is just the second variation of arclength of the family induced by  $W(t)$ . Thus:

$$(C5) \quad g''(0) = \int_0^{l_n} [R(\dot{\gamma}, W, \dot{\gamma}, W) - \langle \dot{W}, \dot{W} \rangle + \{\langle W, \dot{\gamma}' \rangle\}^2] dt$$

where  $\langle \cdot, \cdot \rangle$  denotes the Riemannian metric, the prime denotes differentiation with respect to  $t$  and  $\dot{W} \equiv D_{\dot{\gamma}} W$ . By assumption,  $R(\dot{\gamma}, W, \dot{\gamma}, W) \geq 0$ . Also  $\langle \dot{W}, \dot{W} \rangle = 1/l_n^2$ . Thus (C5) implies  $g''(0) \geq -1/l_n$ . By assumption,  $e_n \equiv d(0, C_n) \rightarrow \infty$  as  $n \rightarrow \infty$ . Thus also  $l_n = d(p, C_n) \rightarrow \infty$  as  $n \rightarrow \infty$ . As  $p$  varies through  $B$ ,  $l_n \geq d(B, C_n) \rightarrow \infty$  as  $n \rightarrow \infty$ . Consequently, assertion (C4) is valid with  $\varepsilon_n = -1/d(B, C_n)$ . The proof of assertion (C1) is complete.

Now assume the sectional curvature to be in addition positive at  $p$ . To conclude the

proof of part (a) it is necessary to show that, given  $\chi: \mathbf{R} \rightarrow \mathbf{R}$  satisfying  $\chi > 0$ ,  $\chi' > 0$  and  $\chi'' > 0$ , the following holds (see Lemma 3):

(C6)  $C(\chi \circ \eta) > \varepsilon$  for some positive constant  $\varepsilon$  in some neighborhood of  $p$ .

Some simple reductions as in (C1)–(C4) would render (C6) more tractable. Let  $B$  be a small open ball of radius  $b$  containing  $p$ ,  $p$  not being necessarily the center of  $B$ .  $b$  may be assumed so small that the sectional curvature has a positive lower bound  $\beta$  in the ball of radius  $2b$  concentric with  $B$ . Now argue exactly as in (C1)–(C4) with  $\eta$ ,  $\eta_n$ ,  $f$ ,  $g$  replaced by  $\chi \circ \eta$ ,  $\chi \circ \eta_n$ ,  $\chi \circ f$ ,  $\chi \circ g$  respectively. The conclusion is that to prove (C6), it suffices to prove:

(C7) There exists a positive number  $\varepsilon_0$ ,  $\varepsilon_0$  depending only on  $b$ ,  $\beta$  and  $\chi$ , such that for all sufficiently large  $n$ ,  $(\chi \circ g)''(0) \geq \varepsilon_0$ .

Now  $(\chi \circ g)'' = (\chi'' \circ g)(g')^2 + (\chi' \circ g)g''$ ; it is thus necessary to evaluate  $g'(0)$  and  $g''(0)$ . At  $p$ , the following decomposition takes place relative to  $\dot{\gamma}(0)$  and  $\dot{\gamma}(0)^\perp$ :

$$\zeta(0) = a_1 N + a_2 \dot{\gamma}(0),$$

where  $\langle N, \dot{\gamma}(0) \rangle = 0$ ,  $|N| = 1$ , and  $a_1^2 + a_2^2 = 1$ . Define the vector field  $N(t)$  along  $\gamma$  by:  $N(t) \equiv$  the parallel translate of  $N$  to  $\gamma(t)$  along  $\gamma$ . Then from the definitions:

$$W(t) = \left(1 - \frac{t}{l_n}\right) \{a_1 N(t) + a_2 \dot{\gamma}(t)\}.$$

Hence  $\langle W, \dot{\gamma} \rangle' = a_2/l_n$ . (C5) implies:

$$g''(0) = \left[ \int_0^{l_n} R(\dot{\gamma}, W, \dot{\gamma}, W) dt \right] - \frac{1}{l_n} + \frac{a_2^2}{l_n}.$$

Let  $K(\dot{\gamma}, W)$  be the sectional curvature of span  $\{\dot{\gamma}, W\}$  at each  $t \in [0, l_n]$ . Then

$$g''(0) = \frac{-a_1^2}{l_n} + a_1^2 \int_0^{l_n} \left(1 - \frac{t}{l_n}\right)^2 K(\dot{\gamma}, W) dt$$

Recall, in the ball of radius  $2b$  concentric with  $B$ ,  $K(\dot{\gamma}, W) > \beta$ . Thus replacing the integral by  $\int_0^{l_n} \beta dt$  leads to:

$$g''(0) \geq a_1^2 \left\{ b\beta \left(1 - \frac{b}{l_n}\right)^2 - \frac{1}{l_n} \right\}.$$

Since  $n$  may be assumed arbitrarily large, in which case  $l_n = d(p, C_n)$  is also arbitrarily large,  $\{b\beta(1 - (b/l_n))^2 - 1/l_n\}$  may be assumed to exceed  $\frac{1}{2}b\beta$ . Thus

$$(C8) \quad g''(0) \geq a_1^2 b\beta/2.$$

Observe also that  $g'(0)$  = (the first variation of arclength of the family induced by  $W(t) = -a_2$ ). Hence,

$$\begin{aligned} (\chi \circ g)''(0) &= \{(\chi'' \circ g)(g')^2 + (\chi' \circ g)g''\}(0) \\ &\geq \{(\chi'' \circ g)a_2^2 + (\chi' \circ g)a_1^2 b\beta/2\}(0). \end{aligned}$$

Since  $g(0) = e_n - d(p, C_n)$ , for  $n$  large it is near  $\eta(p)$  and is hence within a fixed compact neighborhood  $K$  of  $\eta(B)$  in  $\mathbf{R}$ . On  $K$ , let  $\chi'$  and  $\chi''$  be bounded below by a positive constant  $a_3$ . It follows that:

$$(C9) \quad (\chi \circ g)''(0) \geq a_3(a_2^2 + \frac{1}{2}a_1^2 b\beta).$$

Let  $a_4 = \min\{1, \frac{1}{2}b\beta\}$ . Then from (C9) and the identity  $a_1^2 + a_2^2 = 1$ , it follows that  $(\chi'' \circ g)(0) \geq a_3 a_4 > 0$ . This proves (C7) with  $\varepsilon_0 = a_3 a_4$  and consequently also proves (C6). The proof of part (a) of Theorem A is complete.

Next, part (b). First suppose that the Ricci curvature of  $M$  is everywhere nonnegative but is positive at  $p$ . Let  $B$  be a small open ball of radius  $b$  containing  $p$  and let  $\beta > 0$  be a lower bound of the Ricci curvature in the ball of radius  $2b$  concentric with  $B$ . To show  $\eta$  is strictly subh near  $p$ , it suffices to show:

(S1)  $S\eta \geq \varepsilon$  in  $B$  for a fixed constant  $\varepsilon > 0$ .

Since  $\eta_n \rightarrow \eta$  uniformly on  $B$ , Lemma 5 becomes applicable, as follows. Fix an  $n$  in the following discussion. Let  $g \in C_n$  be chosen so that  $d(p, q) = d(p, C_n) \equiv l_n$ , and let  $\gamma: [0, l_n] \rightarrow M$  be a minimal geodesic joining  $p$  to  $q$  such that  $|\dot{\gamma}| \equiv 1$ . Consider the function  $f: B \rightarrow \mathbf{R}$  defined by  $f = e_n - d(\cdot, q)$ . Then  $f \leq \eta_n (= e_n - d(\cdot, C_n))$  and  $f(p) = \eta_n(p)$ . Thus  $f$  supports  $\eta_n$  at  $p$ , and Lemmas 4 and 5 imply that assertion (S1) would follow from:

(S2) There exists a positive number  $\varepsilon_0, \varepsilon_0$  depending only on  $\beta$  and  $b$ , such that for all sufficiently large  $n$ ,  $Sf \geq \varepsilon_0$  in  $B$ .

To this end, let  $A$  be the ball of radius  $b$  in  $M_p$  and consider the  $C^\infty$  map  $k: [0, l_n] \times A \rightarrow M$  defined as follows. For any  $X \in A$ , let  $X(t)$  be the parallel translation of  $X$  to  $\gamma(t)$  along  $\gamma$ . Then  $k(t, X) \equiv \exp_{\gamma(t)}[(1 - t/l_n)X(t)]$ .  $k$  so defined has the following properties: (i)  $k(t, 0) = \gamma(t)$  for all  $t \in [0, l_n]$ , where the first 0 denotes the origin of  $M_p$ ; (ii)  $k(0, X) = \exp_p X$  for all  $X \in A$ ; (iii)  $k(l_n, X) = q$  for all  $X \in A$ ; (iv) the length of the tangent vector to the curve  $s \mapsto k(t, sX)$  at  $s=0$  is  $(1 - t/l_n)|X|$ ; (v) if  $\langle X, Y \rangle = 0$ , then the tangent vectors to the curves  $s \mapsto k(t, sX)$  and  $s \mapsto k(t, sY)$  at  $s=0$  are orthogonal at  $\gamma(t) (= k(t, 0))$  for each  $t$ . Thus  $k$  is a  $(\dim M)$ -parameter variation of  $\gamma$ . Now define a  $C^\infty$  function  $g: \exp_p A \rightarrow \mathbf{R}$  by  $g(\exp_p X) = e_n - (\text{length of the curve } t \mapsto k(t, X))$ . It is straightforward to verify that  $g$  supports  $f$  at  $p$ . Lemma 4 and Lemma 1 imply that to prove (S2) it suffices to prove:

(S3) There exists a positive number  $\varepsilon_0, \varepsilon_0$  depending only on  $\beta$  and  $b$ , such that  $\Delta g(p) \geq \varepsilon_0$ .

Let  $\{x_i\}$  be normal coordinates centered at  $p$  obtained from the exponential map  $\exp_p$ ; to be specific, let the hyperplane orthogonal to  $\dot{\gamma}(0)$  be defined by  $x_1=0$  and let  $\{(\partial/\partial x_i)(0)\}$  be an orthonormal basis of  $M_p$ . In particular,  $\gamma$  near  $p$  is just the  $x_1$ -coordinate curve issuing from  $p$ . The following is an elementary computation:

$$(S4) \quad \Delta g(p) = \sum_i \frac{\partial^2 g}{\partial x_i^2}(0).$$

Since  $g(t, 0, \dots, 0) = e_n - (l_n - t)$  for all small  $t$ ,  $(\partial^2 g / \partial x_1^2)(0) = 0$ . Consider from now on only the case  $i > 1$ . For convenience, let the  $x_1$ -coordinate curve issuing from  $p$  be denoted by  $\zeta$ ;  $\zeta$  is a geodesic parametrized by arclength and is orthogonal to  $\gamma$  at  $\gamma(0) = \zeta(0)$ . The restriction of  $k$  to  $[0, l_n] \times \{A \cap \text{span } \dot{\zeta}(0)\}$  is a variation of  $\gamma$ ; this variation induces a vector field  $W_i(t)$  along  $\gamma$ .  $W_i(t)$  has the properties:  $|W_i(0)| = 1$ ,  $\langle W_i, \dot{\gamma} \rangle \equiv 0$ , and  $W_i(l_n) = 0$ . Thus  $(\partial^2 g / \partial x_i^2)(0)$  is the negative of the second variation of arclength corresponding to  $W(t)$ . Hence

$$(S5) \quad \frac{\partial^2 g}{\partial x_i^2}(0) = - \int_0^{l_n} (R(\dot{\gamma}, W_i, \dot{\gamma}, W_i) - \langle \dot{W}_i, \dot{W}_i \rangle) dt,$$

where the dot indicates covariant differentiation along  $\gamma$ . By property (iv) of  $k$  above,  $\langle \dot{W}_i, \dot{W}_i \rangle = 1/l_n^2$ . Moreover, if  $K(\dot{\gamma}, W_i)$  denote the sectional curvature of the plane spanned by  $\{\dot{\gamma}, W_i\}$ , property (v) of  $k$  above implies:

$$\begin{aligned} \Delta g(p) &= \sum_i \frac{\partial^2 g}{\partial x_i^2}(0) \\ &= \int_0^{l_n} \left(1 - \frac{t}{l_n}\right)^2 \text{Ric}(\dot{\gamma}, \dot{\gamma}) dt - \frac{(\dim M) - 1}{l_n} \\ &\geq \int_0^b \left(1 - \frac{t}{l_n}\right)^2 \text{Ric}(\dot{\gamma}, \dot{\gamma}) dt - \frac{(\dim M) - 1}{l_n} \\ &> \left(1 - \frac{b}{l_n}\right)^2 b\beta - \frac{(\dim M) - 1}{l_n}. \end{aligned}$$

Now  $l_n = d(p, C_n) \rightarrow \infty$  as  $n \rightarrow \infty$ . Thus for all large  $n$ ,  $\Delta g(p) > b\beta/2$ . This proves (S3) with  $\varepsilon_0 = b\beta/2$ , and therewith also (S1).

To complete the proof of part (b) of Theorem A, let  $\text{Ric} \geq 0$  everywhere and let  $\chi: \mathbf{R} \rightarrow \mathbf{R}$  be a  $C^\infty$  function such that  $\chi > 0$ ,  $\chi' > 0$  and  $\chi'' > 0$ . Again let  $p \in M$  be given and let  $B$  be a small open ball of radius  $b$  containing  $p$ . It is to be proved that:

$$(S6) \quad \text{For a fixed constant } \varepsilon > 0, \mathcal{S}(\chi \circ \eta) > \varepsilon \text{ in } B.$$

Repeat the argument from (S1) to (S3) verbatim, with  $\eta$ ,  $\eta_n$ ,  $f$  and  $g$  replaced by  $\chi \circ \eta$ ,  $\chi \circ \eta_n$ ,  $\chi \circ f$  and  $\chi \circ g$ . The upshot is that to prove (S5), it suffices to prove:

(S7) There exists a positive constant  $\varepsilon_0$ ,  $\varepsilon_0$  depending only on  $B$  and  $\chi$ , such that for all sufficiently large  $n$ ,  $\Delta(\chi \circ g)(p) > \varepsilon_0$ .

Now let  $k$ ,  $\{x_i\}$ ,  $W_i(t)$  be as above. Then (S5) and the argument immediately following it imply:

$$\Delta g(p) = \int_0^{l_n} \left(1 - \frac{t}{l_n}\right)^2 \text{Ric}(\dot{\gamma}, \dot{\gamma}) dt - \frac{(\dim M) - 1}{l_n} \geq -\frac{(\dim M) - 1}{l_n}.$$

(Note: a similar inequality has essentially been proved in Calabi [3].)

In general,  $\Delta(\chi \circ g) = (\chi'' \circ g) |dg|^2 + (\chi' \circ g) \Delta g$ . The first variation of arclength formula shows  $|dg|(p) = 1$ . Thus:

$$(S8) \quad \Delta(\chi \circ g)(p) \geq \left[ (\chi'' \circ g) - (\chi' \circ g) \frac{(\dim M) - 1}{l_n} \right] (p).$$

Now let  $n$  be large; then  $l_n$  is also large since  $l_n \rightarrow \infty$  as  $n \rightarrow \infty$ . Moreover  $g(p) = \eta_n(p)$  which is therefore close to  $\eta(p)$  when  $n$  is large. As  $p$  varies over  $B$ ,  $g(p)$  will remain in a fixed compact neighborhood  $K$  of  $\eta(B)$  in  $\mathbf{R}$ . In  $K$ , let  $a_1 = \min \chi''$  and  $a_2 = \max \chi'$ , and let  $n$  be so large that  $l_n > \frac{1}{2} a_1 a_2 (\dim M - 1)$ . Then (S8) implies that  $\Delta(\chi \circ g)(p) \geq a_1/2$ . Consequently, (S7) is valid with  $\varepsilon_0 = a_1/2$ . The proof of part (b) is complete.

Finally, part (c). For the remainder of the proof.  $M$  is a Kähler manifold in addition to being complete and noncompact; its bisectional curvature is assumed everywhere non-negative. Given  $p \in M$ , the claim is:

$$(P1) \quad P\eta(p) \geq 0.$$

By Lemma 2, this is equivalent to  $\eta$  being psh. Let  $B$  be a small open ball of radius  $b$  containing  $p$ . Recall  $\eta_n \rightarrow \eta$  uniformly in  $B$ . Take  $p \in B$  as above and let  $q \in C_n$  be a point satisfying  $d(p, q) = d(p, C_n) \equiv l_n$ . Define  $f: B \rightarrow \mathbf{R}$  by  $f = e_n - d(\cdot, q)$ . Since  $\eta_n = e_n - d(\cdot, C_n)$  and since  $f(p) = \eta_n(p)$ ,  $f$  supports  $\eta_n$  at  $p$ . By Lemma 4 and Lemma 5, (P1) follows from:

(P2) There exists a sequence  $\varepsilon_n$ ,  $\varepsilon_n$  depending only on  $B$  and  $\varepsilon_n \rightarrow 0$  as  $n \rightarrow \infty$ , such that  $Pf(p) \geq \varepsilon_n$  for  $f$  as above.

The next move, as usual, is to find a  $C^\infty$  function  $g: B \rightarrow \mathbf{R}$  that supports  $f$  at  $p$ . Let  $\gamma: [0, l_n] \rightarrow M$  be a minimizing geodesic from  $p$  to  $q \in C_n$  such that  $|\dot{\gamma}| \equiv 1$ . Let  $A$  be the ball of radius  $b$  in  $M_p$ . Now define the following  $(\dim_{\mathbf{R}} M)$ -parameter variation of  $\gamma$ , namely,  $k: [0, l_n] \times A \rightarrow M$  such that, if  $X \in A$  and  $X(t)$  is the parallel translate of  $X$  along  $\gamma$  to

$\gamma(t)$ ,  $k(t, X) \equiv \exp_{\gamma(t)} [(1-t/l_n)X(t)]$ . Note that since parallel translation preserves the complex structure tensor  $J$ ,  $J[X(t)] = (JX)(t)$  for every  $X \in A$ . This plays a role in the following summary of the properties of  $k$ : (i)  $k$  is  $C^\infty$ . (ii)  $k(t, 0) = \gamma(t)$  for every  $t \in [0, l_n]$ , where 0 is the origin of  $M_p$ . (iii)  $k(0, X) = \exp_p X$  for every  $X \in A$ . (iv)  $k(l_n, X) = q$  for every  $X \in A$ . (v) Let  $X \in M_p$  and let  $sX \in A$  for every  $s \in (-a, a)$ . Then the variation of  $\gamma$  given by  $[0, l_n] \times (-a, a) \rightarrow M$  such that  $(t, s) \mapsto k(t, sX)$  is a variation that induces the vector field  $(1-t/l_n)X(t)$  along  $\gamma$ . (vi) The variations of  $\gamma$  given by  $(t, s) \mapsto k(t, sX)$  and  $(t, s) \mapsto k(t, sJX)$  induce vector fields along  $\gamma$  such that  $J$  applied to the former yields the latter.

Now define  $g: B \cap \exp_p A \rightarrow \mathbf{R}$  by  $g(\exp_p X) = e_n - [\text{length of the curve } t \mapsto k(t, X)]$ .  $g$  is  $C^\infty$  and supports  $f$  at  $p (= \exp_p 0)$ . Thus in view of Lemma 4 and Lemma 1, (P2) would follow from:

(P3) There exists a sequence  $\varepsilon_n, \varepsilon_n$  depending only on  $B$  and  $\varepsilon_n \rightarrow 0$  as  $n \rightarrow \infty$ , such that the minimum eigenvalue of  $\partial\bar{\partial}g$  at  $p$  exceeds  $\varepsilon_n$ .

Since  $M$  is Kähler and  $g$  is  $C^\infty$ , the eigenvalues of  $\partial\bar{\partial}g$  can be calculated from those of the Hessian  $D^2g$  in the following way: if  $X_0 = X + \sqrt{-1}JX$ , then:

$$(P4) \quad \partial\bar{\partial}g(X_0, X_0) = D^2g(X, X) + D^2g(JX, JX).$$

(Cf. e.g., Greene-Wu [9, p. 646] or [12, § 4].) To put this to use, let  $X$  be any unit vector in  $M_p$ ;  $JX$  is then also a unit vector in  $M_p$ . Let  $\zeta_1: (-a, a) \rightarrow B$  and  $\zeta_2: (-a, a) \rightarrow B$  be geodesics such that  $\zeta_1(s) = \exp_p(sX)$  and  $\zeta_2(s) = \exp_p(sJX)$ ; in particular,  $\zeta_1(0) = X$ ,  $\zeta_2(0) = JX$ . Then  $D^2g(X, X) = (f \circ \zeta_1)''(0)$  and  $D^2g(JX, JX) = (g \circ \zeta_2)''(0)$ . From the definition of  $g$ ,  $-(g \circ \zeta_1)''(0)$  is nothing but the second variation of arclength of the family of curves  $t \mapsto k(t, sX)$ ,  $s \in (-a, a)$ ; similarly for  $-(g \circ \zeta_2)''(0)$ . Thus if  $V(t)$  and  $JV(t)$  are the vector fields along  $\gamma$  given by  $V(t) = (1-t/l_n)X(t)$  and  $JV(t) = (1-t/l_n)JX(t)$ , then properties (v) and (vi) of  $k$  above and the second variation of arclength formula give:

$$(P5) \quad D^2g(X, X) = \int_0^{l_n} [R(\dot{\gamma}, V, \dot{\gamma}, V) - \langle \dot{V}, \dot{V} \rangle + \{\langle V, \dot{\gamma} \rangle'\}] dt$$

$$D^2g(JX, JX) = \int_0^{l_n} [R(\dot{\gamma}, JV, \dot{\gamma}, JV) - \langle J\dot{V}, J\dot{V} \rangle + \{\langle JV, \dot{\gamma} \rangle'\}] dt$$

where the prime denotes  $d/dt$ . Now let  $P_1 = \text{span}\{\dot{\gamma}, V\}$  and  $P_2 = \text{span}\{\dot{\gamma}, JV\}$  at each  $\gamma(t)$ . Since  $|V| = 1 - (t/l_n)$ ,  $R(\dot{\gamma}, V, \dot{\gamma}, V) + R(\dot{\gamma}, JV, \dot{\gamma}, JV) = (1-t/l_n)^2 H(P_1, P_2)$ , where  $H$  denotes bisectional curvature (see end of Section 1). Moreover,  $\langle \dot{V}, \dot{V} \rangle = \langle J\dot{V}, J\dot{V} \rangle = 1/l_n^2$ . Hence adding the equations of (P5) and substituting into (P4) lead to

$$(P6) \quad \partial\bar{\partial}g(X_0, X_0) \geq \frac{-2}{l_n} + \int_0^{l_n} \left(1 - \frac{t}{l_n}\right)^2 H(P_1, P_2) dt,$$

where  $X_0 = X + \sqrt{-1}JX$ . Now  $H \geq 0$  by assumption. So  $\partial\bar{\partial}g(X_0, X_0) \geq -2/l_n$ . As  $p$  varies over  $B$ ,  $l_n \geq d(B, C_n)$ , so that  $\partial\bar{\partial}g(X_0, X_0) \geq -2/d(B, C_n)$ . Since  $d(B, C_n) \rightarrow \infty$  as  $n \rightarrow \infty$ , (P3) is valid with  $\varepsilon_n = -2/d(B, C_n)$ . (P1) is therefore proved.

To finish the proof of part (c), and hence the proof of Theorem A, it remains to show that if  $H > 0$  at  $p$ ,  $\eta$  is strictly psh near  $p$ . Let  $B$  be any small open ball of radius  $b$  containing  $p$ , but not necessarily centered at  $p$ , and let the lower bound of the bisectonal curvature in the ball of radius  $2b$  and concentric with  $B$  be positive, say  $\beta > 0$ . By Lemma 3, it suffices to show:

(P7)  $P\eta \geq \varepsilon$  in  $B$  for a fixed positive constant  $\varepsilon$ .

Notation exactly as above, the arguments leading up to (P3) show that (P7) is implied by:

(P8) There exists a positive constant  $\varepsilon_0$ ,  $\varepsilon_0$  depending only on  $b$  and  $\beta$ , such that for all sufficiently large  $n$ , the minimum eigenvalue of  $\partial\bar{\partial}g$  at  $p$  exceeds  $\varepsilon_0$ .

Now,  $\int_0^{l_n} (1 - t/l_n)^2 H(P_1, P_2) dt \geq \int_0^{l_n} (1 - t/l_n)^2 \beta dt > (1 - b/l_n)^2 b\beta$ . By (P7),  $\partial\bar{\partial}g(X_0, X_0) > \{(1 - b/l_n)^2 b\beta - 2/l_n\}$ . When  $n$  is sufficiently large,  $(1 - b/l_n)^2 > \frac{1}{2}$  and  $2/l_n < b\beta/4$ . Thus  $\partial\bar{\partial}g(X_0, X_0) > b\beta/4$  when  $n$  is sufficiently large. This shows that (P8) holds with  $\varepsilon_0 = b\beta/4$ . The proof of (P7), and hence of Theorem A is thereby concluded.

*Remark.* Since Theorem A has been dealt with in sufficient detail, each of the following proofs will deal with only one case, and then only in outline. The remaining cases are similar.

*Proof of Theorem B.* Let  $M$  be a complete noncompact Riemannian manifold with nonnegative Ricci curvature outside a compact  $K$ . The claim is that  $\eta$  will be essentially strictly subh on  $\{\eta > a_0\}$ , where  $a_0$  depends only on  $K$ .

Pick  $a_1$  so large that  $K$  is contained in the ball  $B$  of radius  $a_1$  about the fixed point 0 of  $M$ . Let  $a_0 = 2a_1$ . Suppose  $\eta(p) > a_0$ , then it has to be proved that  $\eta$  is essentially strictly subharmonic near  $p$ . Let  $N$  be an integer so large that  $n \geq N$  implies  $\eta_n(p) > a_0$ . For such an  $n$ , let  $q \in C_n$  satisfy  $d(p, q) = d(p, C_n) \equiv l_n$ , and let  $\gamma: [0, l_n] \rightarrow M$  be a minimizing geodesic from  $p$  to  $q$  parametrized by arclength. Then for every  $t \in [0, l_n]$ ,

$$\begin{aligned} d(0, \gamma(t)) &\geq d(0, q) - d(q, \gamma(t)) \\ &\geq d(0, C_n) - d(q, \gamma(t)) \\ &= e_n - (l_n - t) \\ &= e_n - l_n = e_n - d(p, C_n) \\ &= \eta_n(p) > a_0. \end{aligned}$$

Thus  $\gamma$  lies outside the ball of radius  $2a_1$  around 0. It follows that the set  $A$  of points of distance  $\leq a_1$  from  $\gamma$  still lies outside  $B$ , and hence outside  $K$ . On  $A$ , the Ricci curvature is non-negative. The arguments from assertion (S6) to (S8) in the preceding proof are therefore valid in  $A$  and  $\eta$  is consequently essentially strictly subh near  $p$ . By the same token, suppose the Ricci curvature  $> 0$  at  $p$ . Then the arguments from (S1) to (S5) transplanted to this situation again show that  $\eta$  is strictly subh near  $p$ . Q.E.D.

*Proof of Theorem C.* Let  $M$  be a complete noncompact Kähler manifold such that the bisectional curvature  $\geq 0$  everywhere, and  $> 0$  outside a compact set  $K$ . The claim is that  $e^\eta$  is strictly psh everywhere.

By Theorem A,  $\eta$  is known to be everywhere psh and strictly psh outside  $K$ ; *a fortiori*, so is  $e^\eta$  since  $e^x$  is a strictly convex and strictly increasing positive function. It remains to show that the psh function  $e^\eta$  is in fact strictly psh in  $K$ .

Let  $K_0 = \{x \in M: d(x, K) \leq 10\}$ , and let  $a_0$  be any number exceeding the maximum of  $\eta$  on  $K_0$ .  $\eta$  is strictly psh in the neighborhood  $\{a_0 - 1 < \eta < a_0 + 1\}$  of  $N \equiv \eta^{-1}(a_0)$ , and consequently  $1/(a_0 - \eta)$  is strictly psh in  $U_0 \equiv \{a_0 - 1 < \eta < a_0\}$ . By a theorem of Richberg [18],  $1/(a_0 - \eta)$  can be uniformly approximated in  $U_0$  by a  $C^\infty$  strictly psh function  $\omega$ . For definiteness, let  $|\omega - 1/(a_0 - \eta)| < 1$ ; since  $1/(a_0 - \eta) \rightarrow \infty$  towards  $N$ , so does  $\omega$ . If  $a_i$  is a sequence of regular values of  $\omega$  such that  $a_i \uparrow \infty$ , let  $N_i \equiv \omega^{-1}(a_i)$ . Each  $N_i$  is therefore the full boundary of a (possibly unbounded)  $C^\infty$  strictly pseudoconvex domain in  $M$ ; moreover,  $d(N, N_i) \rightarrow 0$  in the sense that  $\sup_L d(n, n_i) \rightarrow 0$  as  $i \rightarrow \infty$ , where  $L$  is any compact subset of  $M$  and  $n \in N \cap L$ ,  $n_i \in N_i \cap L$ . Here Lemma 7 enters. Let  $p \in K$ , then  $\eta(p) < a_0 - 10 < a_0$ . Lemma 7 implies that on  $K$ ,  $\eta = a_0 - d(\cdot, N)$ . Since  $d(p, N_i) \rightarrow d(p, N)$  uniformly as  $p$  varies over  $K$ , Lemma 5 says that  $e^\eta$  would be strictly psh if the modulus of plurisubharmonicity of each  $\exp\{-d(\cdot, N_i)\}$  can be bounded below by a positive constant independent of  $i$  when  $i$  is large. Since  $\eta$  is Lipschitzian with Lipschitz constant 1, the fact that  $\eta|_K$  and  $\eta|_N$  differ by at least 10 implies that  $d(K, N) \geq 10$ . Hence the following assertion suffices to prove that  $e^\eta$  is strictly psh:

(P9) Let  $M$  be a complete Kähler manifold with nonnegative bisectional curvature. Let  $D$  be a  $C^\infty$  strictly pseudoconvex domain in  $M$ , possibly unbounded, and let  $\delta: D \rightarrow [0, \infty)$  be the boundary distance function, i.e.,  $\delta(x) = d(x, \partial D)$ . Suppose the bisectional curvature is positive in  $D_\lambda = \{x \in D: \delta(x) \leq \lambda\}$ , where  $\lambda \in (0, 1]$ . Then  $e^{-\delta}$  is strictly psh in  $D^* = \{x \in D: \delta(x) \geq 1\}$ . More precisely, let  $\delta(p) = l \geq 1$  and suppose  $q \in \partial D$  satisfies  $\delta(p) = d(p, q)$ . Let  $\beta > 0$  be a lower bound of the bisectional curvature in  $D_\lambda \cap \{\text{ball of radius } 2l \text{ around } q\}$ . Then

$$P e^{-\delta(p)} \geq \frac{\beta \lambda^3 e^l}{3l^2}.$$



The proof of (P9) is in outline similar to the proof of Theorem 1 in Greene–Wu [14]. Let  $b$  be a small positive number so that  $B \equiv \{x \in M: d(x, p) < b\}$  lies in  $D$ . Let  $B_0$  be the ball of radius  $b$  in  $M_p$ ; assume  $b$  is so small that  $\exp_p$  is a diffeomorphism of  $B_0$  onto  $B$ . The main step is the construction of a  $C^\infty$  function  $g$  on  $B$  which supports  $-\delta$  at  $p$ . First construct a  $C^\infty$  function  $k: [0, l] \times B_0 \rightarrow M$  as follows. Let  $q \in \partial D$  be such that  $\delta(p) = d(p, q)$  as above, and let  $\gamma$  be a minimizing geodesic satisfying  $|\dot{\gamma}| \equiv 1$  and joining  $p$  to  $q$ . Thus  $\gamma: [0, l] \rightarrow \bar{D}$ , such that  $\gamma(0) = p$  and  $\gamma(l) = q$ . It follows from the first variation of arclength formula that  $\dot{\gamma}(l)$  is orthogonal to  $\partial D$  at  $q$ , in symbols:  $\dot{\gamma}(l) \perp \partial D$ . The tangent space  $(\partial D)_q$  of  $\partial D$  contains a maximal complex vector subspace  $C(l)$ ;  $C(l)$  has real codimension 1 in  $(\partial D)_q$ . For every  $t \in [0, l]$ , let  $C(t)$  be the parallel translate of  $C(l)$  to  $\gamma(t)$  along  $\gamma$ ;  $C(t) \perp \dot{\gamma}(t)$  for every  $t$  and, since  $M$  is Kähler,  $C(t)$  is a complex vector subspace of  $M_{\gamma(t)}$  of complex codimension 1. The orthogonal projection of  $M_{\gamma(t)}$  onto  $C(t)$  induces an orthogonal decomposition of  $M_{\gamma(t)}$  for every  $t$ , and it has the following property. For every  $X \in M_p$ , this decomposition at  $t=0$  gives  $X = a_1 \dot{\gamma}(0) + a_2 J\dot{\gamma}(0) + X^h$ , where  $X^h \in C(0)$  and  $a_1, a_2 \in \mathbb{R}$ . Let  $X(t)$  (resp.  $X^h(t)$ ) be the parallel translation of  $X$  (resp.  $X^h$ ) to  $\gamma(t)$  along  $\gamma$ . Then the above-mentioned property is that  $X(t) = a_1 \dot{\gamma}(t) + a_2 J\dot{\gamma}(t) + X^h(t)$  is the orthogonal decomposition of  $X(t)$  in  $M_{\gamma(t)}$  relative to  $C(t)$ , with the same real number  $a_1, a_2$  as above and with  $X^h(t) \in C(t)$ . Now the desired  $(\dim_{\mathbb{R}} M)$ -parameter variation  $k: [0, l] \times B_0 \rightarrow M$  is required to satisfy all the following properties: (i)  $k(t, 0) = \gamma(t)$  for all  $t \in [0, l]$ , where the first 0 is the origin of  $M_p$ ; (ii)  $k(0, X) = \exp_p X$  for all  $X \in B_0$ ; (iii)  $k(l, B_0) \subset \partial D$ . Next, let  $X \in A$  and let  $sX \in B_0$  for every  $s \in (-a, a)$ . Let the variation of  $\gamma$  given by  $[0, l] \times (-a, a) \rightarrow M$ ,  $(t, s) \mapsto k(t, sX)$ , induce the vector field  $V(t)$  along  $\gamma$ . Then  $V(t)$  has these properties: (iv) If  $X = a_1 \dot{\gamma}(0) + a_2 J\dot{\gamma}(0) + X^h$  is the orthogonal decomposition of  $X$  in  $M_p$  as above, then  $V(t) = (1-t/l)[a_1 \dot{\gamma}(t) + a_2 J\dot{\gamma}(t)] + X^h(t)$ , in the preceding notation; (v) in particular, every vector field  $V(t)$  along  $\gamma$  induced by  $k$  has the property that  $V(t) \in C(t)$ ; (vi) if  $V(t)$  and  $W(t)$  are the vector fields along  $\gamma$  induced by the variations  $(t, s) \mapsto k(t, sX)$  and  $(t, s) \mapsto k(t, sJX)$  respectively, then  $JV(t) = W(t)$ .

Here is one way to construct such a  $k$ . At  $q$ , let  $\partial D$  be locally defined by the real function  $\varphi$ , i.e., locally near  $q$ ,  $\partial D = \{\varphi = 0\}$ . Let  $\psi$  be another  $C^\infty$  function and let  $\{z_i\}$  be complex coordinates near  $q$  ( $z_i = x_i + \sqrt{-1}y_i$  as usual) such that  $\{\varphi, \psi, x_2, y_2, x_3, y_3, \dots\}$  forms a local coordinate system near  $q$ . (Observe:  $C(l) = \text{span}\{(\partial/\partial x_i)(q), (\partial/\partial y_i)(q)\}_{i \geq 2}$ .) The coordinate vector fields  $\{\partial/\partial \varphi, \partial/\partial \psi, \partial/\partial x_i, \partial/\partial y_i\}$ , where  $i \geq 2$ , when decreed to be everywhere orthonormal, define a Riemannian metric  $G_1$  near  $q$ . Now let  $G$  be a new Riemannian metric on  $M$  such that near  $p$ ,  $G$  equals the original Kähler metric, and that near  $q$ ,  $G$  equals  $G_1$ . Let  $E_t$  denote the exponential map of  $G$  at  $\gamma(t)$ . Then, in the notation of the preceding paragraph,  $k$  may be defined by

$$k(t, X) = E_t[(1-t/l)(a_1\dot{\gamma}(t) + a_2J\dot{\gamma}(t)) + X^h(t)].$$

Define a  $C^\infty$  function  $g: B \rightarrow \mathbf{R}$  by:  $g(\exp_p X) = -[\text{length of the curve } t \mapsto k(t, X)]$  for every  $X \in B_0$ . Then  $g(p) = -\delta(p) = -\text{length of } \gamma$ , and  $g \leq -\delta$  in general on account of property (iii) of  $k$  above. Thus  $g$  supports  $-\delta$  at  $p$ , and by Lemma 4 as well as Lemma 1, the following holds:

$$(P10) \quad Pe^{-\delta(p)} \geq \{\text{minimum eigenvalue of } \partial\bar{\partial}e^g(p)\}.$$

The next step is to evaluate  $\partial\bar{\partial}g(p)$  with the help of (P4). In the notation of (P4), suppose  $X_0 = X + \sqrt{-1}JX$  is given where  $X$  is a unit vector, then the computation of  $\partial\bar{\partial}g(X_0, \bar{X}_0)$  is simplified by noting  $\partial\bar{\partial}g(X_0, \bar{X}_0) = \partial\bar{\partial}g(X_0^*, \bar{X}_0^*)$ , where  $X_0^* \equiv X^* + \sqrt{-1}JX^*$  and  $X^*$  is any unit vector in  $\text{span}\{X, JX\}$ . Thus choosing  $X^*$  from  $\text{span}\{X, JX\}$  to have zero component in the  $J\dot{\gamma}(0)$  direction implies that there is no loss of generality in assuming  $X_0 = X + \sqrt{-1}JX$  where  $X$  now takes the special form  $X = a_1\dot{\gamma}(0) + a_2X^h$ ,  $X^h \in C(0)$ ,  $|X^h| = 1$  and  $a_1, a_2 \in \mathbf{R}$ . Since  $X$  is by choice a unit vector, it follows that  $a_1^2 + a_2^2 = 1$ . With such an  $X$ , let  $\zeta_1: (-a, a) \rightarrow B$ ,  $\zeta_2: (-a, a) \rightarrow B$  be the geodesics  $\zeta_1(s) = \exp_p(sX)$  and  $\zeta_2(s) = \exp_p(sJX)$ . Then  $D^2g(X, X) = (g \circ \zeta_1)''(0)$  and  $D^2g(JX, JX) = (g \circ \zeta_2)''(0)$ . By properties (ii) and (v) of  $k$  above,  $(g \circ \zeta_1)''(0)$  and  $(g \circ \zeta_2)''(0)$  are the second variations of  $\gamma$  with variation vector fields  $V(t) = (1-t/l)(a_1\dot{\gamma}(t) + a_2X^h(t))$  and  $JV(t) = (1-t/l)(a_1J\dot{\gamma}(t) + a_2JX^h(t))$ . The second variation formula now gives:

$$(P11) \quad \begin{aligned} D^2g(X, X) &= -\langle D_{V(t)}V(t), \dot{\gamma}(t) \rangle \Big|_0^l \\ &\quad - \int_0^l [-R(\dot{\gamma}, V, \dot{\gamma}, V) + \langle \dot{V}, \dot{V} \rangle - \{\langle V, \dot{\gamma}' \rangle\}^2] dt, \\ D^2g(JX, JX) &= -\langle D_{JV(t)}JV(t), \dot{\gamma}(t) \rangle \Big|_0^l \\ &\quad - \int_0^l [-R(\dot{\gamma}, JV, \dot{\gamma}, JV) + \langle J\dot{V}, J\dot{V} \rangle - \{\langle JV, \dot{\gamma}' \rangle\}^2] dt. \end{aligned}$$

Adding these equations leads to many simplifications, as follows. First at  $t=0$ , the boundary terms vanish. At  $t=l$ , note that  $\dot{\gamma}(l)$  is the outer unit normal to  $\partial D$  at  $q = \gamma(t)$  and hence (letting  $\varphi$  be the local defining function of  $\partial D$  at  $q$  as before):

$$[\langle V, D_V\dot{\gamma} \rangle + \langle JV, D_{JV}\dot{\gamma} \rangle](l) = 4\partial\bar{\partial}\varphi(V_0, \bar{V}_0) > 0,$$

where  $V_0 = V(l) + \sqrt{-1}JV(l)$ . The equality in this formula involves a standard computation using the Kähler property of the metric, while the inequality follows from the strict pseudoconvexity assumption of  $\partial D$  plus property (v) of  $k$  above. Thus

$$(P12) \quad -\langle D_VV, \dot{\gamma} \rangle \Big|_0^l - \langle D_{JV}JV, \dot{\gamma} \rangle \Big|_0^l = [\langle V, D_V\dot{\gamma} \rangle + \langle JV, D_{JV}\dot{\gamma} \rangle](l) > 0.$$

Next let  $H(t)$  be the bisectonal curvature determined by  $\text{span}\{\dot{\gamma}, J\dot{\gamma}\}$  and  $\text{span}\{V, JV\}$  at  $\gamma(t)$ . Then

$$(P13) \quad [R(\dot{\gamma}, V, \dot{\gamma}, V) + R(\dot{\gamma}, JV, \dot{\gamma}, JV)](t) = \left[ \left(1 - \frac{t}{l}\right)^2 a_1^2 + a_2^2 \right] H(t).$$

Since  $\langle V, \dot{\gamma} \rangle' = -a_1/l$  and  $\langle JV, \dot{\gamma} \rangle' = 0$ , (P4) and (P11)–(P13) together yield:

$$(P14) \quad \partial\bar{\partial}g(X_0, \bar{X}_0) > -\frac{a_1^2}{l} + \int_0^l \left[ \left(1 - \frac{t}{l}\right)^2 a_1^2 + a_2^2 \right] H(t) dt.$$

Now by assumption,  $H(t) > \beta$  along  $\gamma|[l-\lambda, l]$ . Since

$$\int_0^l ( \quad ) H(t) dt > \int_{l-\lambda}^l ( \quad ) H(t) dt,$$

(P14) implies:

$$(P15) \quad \partial\bar{\partial}g(X_0, \bar{X}_0) > -\frac{a_1^2}{l} + \beta\lambda \left( \frac{\lambda^2}{3l^2} a_1^2 + a_2^2 \right) \geq -a_1^2 + \frac{\beta\lambda^3}{3l^2},$$

where the last inequality is due to  $a_1^2 + a_2^2 = 1$  and  $l > 1$ . Since  $\partial\bar{\partial}e^g = e^g(\partial g \wedge \bar{\partial}g + \partial\bar{\partial}g)$ ,

$$(\partial\bar{\partial}e^g)(X_0, \bar{X}_0) = e^l[(Xg)^2 + (JXg)^2 + \partial\bar{\partial}g(X_0, \bar{X}_0)].$$

By the first variation of arclength formula,  $Xg = (g \circ \zeta_1)'(0) = -a_1$  while  $JXg = (g \circ \zeta_2)'(0) = 0$ .

Hence,

$$(\partial\bar{\partial}e^g)(X_0, \bar{X}_0) > e^l \frac{\beta\lambda^3}{3l^2}.$$

This together with (P10) proves (P9). The proof of this part of Theorem C is complete.

For the proofs of the remaining cases, the analogues of the theorem of Richberg [18] used in the above for strictly convex and strictly subharmonic functions were announced in Greene–Wu [11]; their proofs are contained in Greene–Wu [13] and [15].

Q.E.D.

*Remark.* The preceding proof of part (c) of Theorem C shows that, more generally,  $\chi \circ \eta$  is strictly psh if  $\chi$  is the usual  $C^\infty$  function of one variable such that (in addition to  $\chi > 0$ ,  $\chi' > 0$  and  $\chi'' > 0$ ) the quotient  $(\chi'/\chi'')$  is bounded above in  $[a, \infty)$  for some  $a \in \mathbb{R}$ .

### Section 3

This section records several applications and extensions of the method developed in the preceding section. No proofs will be offered, nor would they be necessary since no new ideas are involved beyond those already exposed above.

(I) Implicit in the proof of Theorem C, particularly (P9) and its convex and subh analogues, is a wealth of information of the following type. Let  $D$  be a domain in  $M$  and let  $\delta: D \rightarrow [0, \infty)$  be the distance to the boundary  $\partial D$ . Then suitable assumptions on  $M$ ,  $D$  or  $\partial D$  would lead to function-theoretic conclusions about  $\delta$ . The following are two among the many possible variations; the second one slightly extends Theorem 1(B) of Greene–Wu [14] which assumes everywhere positive bisectional curvature.

**THEOREM 1.** *Let  $M$  be an  $n$ -dimensional complete Riemannian manifold with non-negative Ricci curvature. Let  $D$  be a domain in  $M$  and let  $\delta: D \rightarrow [0, \infty)$  be the distance to  $\partial D$ . Then  $\delta^{2-n}$  is essentially strictly subh in  $D$ , is strictly subh wherever the Ricci curvature  $> 0$ , and is everywhere strictly subh already if the Ricci curvature is positive near  $\delta D$ . If  $\delta D$  is further assumed to be a  $C^\infty$  hypersurface whose second fundamental form has nonnegative trace relative to the outer unit normal, then the same conclusions hold if  $\delta^{2-n}$  is replaced by  $-\delta$ . (In case  $n=2$ ,  $-\log \delta$  must be used in place of  $\delta^{2-n}$ .)*

**COROLLARY.** *Suppose  $N$  is a closed imbedded minimal hypersurface in a complete Riemannian manifold  $M$  and let  $\delta: M - N \rightarrow [0, \infty)$  be the function  $\delta(x) = d(x, N)$ . Then  $-\delta$  is essentially strictly subh in  $M - N$  if  $M$  has nonnegative Ricci curvature, and is strictly subh in  $M - N$  if  $M$  has in addition positive Ricci curvature near  $N$ .*

**THEOREM 2.** *Let  $D$  be a locally pseudoconvex domain in a complete Kähler manifold  $M$  which has nonnegative bisectional curvature everywhere and positive bisectional curvature near  $\partial D$ ; let  $\delta: D \rightarrow [0, \infty)$  be the distance to the boundary. Then  $-\log \delta$  is strictly psh in  $D$ .*

(II) It is known that on manifolds, a continuous convex, subh, or psh function in the sense of Section 1 is convex, subh, psh respectively in the sense of distributions, but the proof for the first two cases is non-elementary (see e.g. Greene–Wu [9] and [15]). For the function  $\eta$  of Theorems A–C, however, the proof is much easier. For instance, suppose one wants to show that the  $\eta$  in part (b) of Theorem A is subh in the sense of distributions. It suffices to show that  $(\Delta\chi(\eta))(\alpha) \geq 0$  for all positive increasing convex  $C^\infty$  functions  $\chi$  and for all  $C_0^\infty$  nonnegative functions  $\alpha$ . To this end, fix  $\chi$ . By (S7), for each  $p$  there is a ball  $B$  containing  $p$  such that for all  $n$ : there is a  $C^\infty$  function  $g_n$  on  $B$ ,  $\chi(g_n)$  supports  $\chi(\eta_n)$  at  $p$ , and  $\Delta\chi(g_n) > 0$  on  $B$ . Then if  $\alpha_0$  is any nonnegative  $C_0^\infty$  function vanishing outside  $B$ ,  $\Delta(\chi(\eta))(\alpha_0) = \lim_n \chi(\eta_n)(\Delta\alpha_0) \sim \lim \chi(g_n)(\Delta\alpha_0) = \lim \Delta(\chi(g_n))(\alpha_0) > 0$ . The general conclusion for  $\alpha$  follows by a partition of unity argument. In the same way, the strict convexity, etc., of  $\eta$  in the sense of distributions can be immediately proved.

(III) The method of this paper is most effective whenever the curvature has a good lower bound. The paper Wu [21] gives one example in this direction. The following gives a

few more. Let  $M$  be a complete Riemannian manifold and let  $r$  be the distance function relative to a fixed  $0 \in M$ . Suppose  $A$  is a constant and  $U$  is a given open set containing  $0$ .

THEOREM 3. *There exists a constant  $K$  which depends only on  $A$  and  $U$  such that:*

$$\text{sectional curvature of } M \geq A \Rightarrow Cr \leq K \text{ on } M - U$$

$$\text{Ricci curvature of } M \geq A \Rightarrow Sr \leq K \text{ on } M - U.$$

$$\text{bisectional curvature of } M \geq A \Rightarrow Pr \leq K \text{ on } M - U.$$

The proof merely repeats those of Section 2 with  $(-r)$  replacing  $\eta_n$ . As an immediate consequence: if the Ricci curvature of the complete Riemannian manifold  $M$  is bounded below, then the Laplacian of  $r$  in the sense of distributions is bounded above outside any open  $U$  containing  $0$  (cf. the remarks in (II)); this fact was already known independently to S. T. Yau ([22]) in his global study of the heat kernel. Furthermore since  $r$  is (globally) Lipschitzian, the smoothing theorems of Greene–Wu ([11], [15]) imply:

THEOREM 4. *On a complete Riemannian manifold whose Ricci curvature is bounded below, there is a  $C^\infty$  Lipschitzian exhaustion function whose Laplacian is bounded above.*

The convex and psh analogues can be similarly proved. Theorem 4 should have applications.

(IV) The results of this paper prompt two observations of a technical nature. In the first place, one of the principal motivations of the approximation theorem for convex functions of Greene–Wu [1] was to be able to handle the Buseman functions on a Kähler manifold of nonnegative sectional curvature. By part (c) of Theorem A, the Busemann functions are now known to be psh. Therefore the option is now available to invoke instead Richberg's approximation theorem [18] for the function theory of Kähler manifolds of nonnegative sectional curvature. Second, all the smoothing (approximation) theorems thus far (Greene–Wu [9] and [15], Richberg [18]) are fairly elaborate affairs since they apply to general convex, subharmonic and psh functions. However, if it is merely a question of approximating the functions  $\eta$  of this paper by  $C^\infty$  functions of the same kind, the proofs given in this paper raise the hope that those proofs would be simpler. The reason is that such an  $\eta$  has now been shown to be supported at each point of  $M$  by a  $C^\infty$  function possessing (almost) the same function-theoretic properties.

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