ON THE SPECTRAL SYNTHESIS OF BOUNDED FUNCTIONS.

By

ARNE BEURLING

of UPPSALA.

1. Introduction.

In this paper L^1, L^2 and L^∞ will denote the linear metric spaces formed by the measurable functions over $-\infty < x < \infty$ which are respectively summable, of summable square or equivalent to a bounded function. The corresponding norms will be denoted by $\|\varphi\|_1$, $\|\varphi\|_2$ and $\|\varphi\|_\infty$.

To each $\varphi(x) \in L^\infty$ corresponds a closed set Λ_φ of real numbers, termed the spectral set of φ, which is formed, briefly, by those λ for which the pure oscillation $e^{i\lambda x}$ is contained in the manifold spanned by the set

$$\varphi(x+\tau) \qquad (-\infty < \tau < \infty) \tag{1.1}$$

in the weak topology of bounded functions, i. e. for every $G(x) \in L^1$ the condition

$$\int_{-\infty}^{\infty} \varphi(x+\tau)\, G(x)\, dx = 0 \qquad (-\infty < \tau < \infty)$$

implies [1]

$$\int_{-\infty}^{\infty} e^{i\lambda x}\, G(x)\, dx = 0.$$

The main problem of spectral Synthesis of L^∞ is to decide whether each $\varphi(x) \in L^\infty$ is contained in the weak closure of the manifold spanned by the oscillations

$$e^{i\lambda x} \qquad (\lambda \in \Lambda_\varphi). \tag{1.2}$$

[1] As is easily proved this definition leads to the same Λ_φ as that obtained by the stronger topology used by the author in a previous paper; *Un Théorème sur les fonctions bornées* . . ., Acta math. vol. 77, 1945.

29 — 48173. *Acta mathematica.* 81. Imprimé le 27 avril 1949.

This delicate problem is still unsolved but recent progress is due to L. Schwartz, who has shown by an exemple that for a bounded function in an Euclidean space of dimension $n \geq 3$ the spectral Synthesis is in general not possible in the weak topology.[1] Even if this fact does not settle the question in the one-dimensional case, it has inspired the author to publish some older results which can be extended to the case of any dimension and therefore may be of some interest.

Let us first mention that both in spectral Analysis and in spectral Synthesis the main question is to find the best topology, i. e. the strongest, in which the closures of the sets (1.1) and (1.2) have certain properties given a priori. As for the Analysis this topology is known and proves to be considerably stronger than the ordinary weak topology, and it now seems probable that a similar property, but of opposite effect, may occur in the problem of spectral Synthesis, i. e. the weak topology is probably too narrow, even in the one-dimensional case. This evidently disturbs the symmetry of the theory, but on the other hand it gives rise to an interesting problem which we pose as follows:

For what positive weight-functions $w(x) \in L^1$ is it true that to any $\varphi(x) \in L^\infty$ and any $\varepsilon > 0$, a trigonometric polynomial

$$\sum c_\nu e^{i\lambda_\nu x} \qquad (\lambda_\nu \in \Lambda_\varphi)$$

can always be found such that

$$\int_{-\infty}^{\infty} |\varphi(x) - \sum c_\nu e^{i\lambda_\nu x}|\, w(x)\, dx < \varepsilon? \tag{1.3}$$

In this paper we shall neither consider the case where $w(x)$ is subject to a restrictive condition implying that

$$\int_{|x|>a} w(x)\, dx$$

can not tend to zero arbitrarily slowly as $a \to \infty$, nor the case where Λ_φ is restricted by special conditions. We aim merely at the following result:

Theorem I. *If $w(x) = w(|x|)$ is a non-increasing function of $|x|$, then the approximation (1.3) is always possible.*

In the following we shall briefly call a weight-function $w(x)$ regular if the approximation (1.3) can be realised.

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[1] The result is to be published in C. R. de l'Academie des Sciences, vol. 227.

2. On Contractions of Functions.

By a contraction of a domain D in the complex z-plane we shall mean a transformation T of D such that for every pair of points $z_1, z_2 \in D$

$$|Tz_1 - Tz_2| \leq |z_1 - z_2|.$$

If $f(t)$ is a function with values in D, we shall call the function $g(t) = Tf(t)$ a contraction of $f(t)$. More generally, $f(t)$ and $g(t)$ being two functions defined for the same t, $g(t)$ shall be called a contraction of $f(t)$ if for any pair of arguments t_1, t_2,

$$|g(t_1) - g(t_2)| \leq |f(t_1) - f(t_2)|.$$

If C is a closed convex set of points we are especially interested in the contraction T defined as the projection of the entire z-plane on C, viz.

$$Tz = \begin{cases} z & \text{if} \quad z \in C \\ z' & \text{if} \quad z \notin C \end{cases}$$

z' being the point $\in C$ whose distance to z is a minimum. In particular, when C is the circle $|z| \leq \varrho$, $(\varrho > 0)$, we set

(2.1) $$T_\varrho z = \begin{cases} z & \text{if} \quad |z| \leq \varrho \\ \varrho \dfrac{z}{|z|} & \text{if} \quad |z| > \varrho \end{cases}$$

and call T_ϱ a circular projection.

Consider now the space A of functions

(2.2) $$f(t) = \int_{-\infty}^{\infty} e^{-itx} F(x)\, dx, \qquad (F \in L^1)$$

with the metric

(2.3) $$\|f\| = \|F\|_1,$$

and let A^* be the subset of A formed by the functions $f(t)$ for which

(2.4) $$F^*(x) = \sup_{|\xi| > |x|} |F(\xi)|$$

is also summable over $-\infty < x < \infty$. In A^* we shall use the metric

(2.5) $$\|f\|^* = \|F^*\|_1.$$

A function $f(t)$ shall be called contractible in the space A if each of its contractions $g(t)$, normalized by the condition

$$\lim_{t=\infty} g(t) = 0, \tag{2.6}$$

also belongs to A. Moreover, we shall say that $f(t)$ is uniformly contractible in the space A, if for any sequence of normalized contractions $g_n(t)$, converging to zero as $n \to \infty$,

$$\lim_{n=\infty} \| g_n \| = 0. \tag{2.7}$$

Theorem II. *A sufficient condition for $w(x)$ to be a regular weight-function is that every measurable $F(x)$ of the class*

$$| F(x) | \leq w(x), \tag{2.8}$$

has a Fourier transform $f(t)$ which is uniformly contractible in the space A.

Consider the linear metric space of measurable functions $\psi(x)$ with the norm

$$\int_{-\infty}^{\infty} | \psi(x) | \, w(x) \, dx.$$

According to a classical theorem due to F. Riesz and Banach the approximation (1.3) is possible if and only if, for every linear and bounded functional $F(\psi)$ in this space, the relations

$$F(e^{i\lambda x}) = 0 \qquad (\lambda \in \Lambda_\varphi)$$

imply

$$F(\varphi) = 0.$$

It is, however, well known that $F(\psi)$ can be expressed in the form

$$F(\psi) = \int_{-\infty}^{\infty} \psi(x) \, \overline{F(x)} \, dx,$$

where $F(x)$ is a certain measurable function such that almost everywhere

$$| F(x) | \leq C \, w(x) \qquad (C = \text{const.} < \infty).$$

Without loss of generality we may assume $C = 1$, and we then have to prove that for any $F(x)$ of the class (2.8), the hypothesis

$$f(t) = \int_{-\infty}^{\infty} e^{-itx} F(x) \, dx = 0, \qquad (t \in \Lambda_\varphi) \tag{2.9}$$

implies

$$\int_{-\infty}^{\infty} \varphi(x)\,\overline{F(x)}\,dx = 0, \tag{2.10}$$

provided that $f(t)$ is uniformly contractible. A very useful tool for the discussion of this integral is the function

$$U_\varphi(\sigma, t) = \int_{-\infty}^{\infty} e^{-itx-\sigma|x|}\,\varphi(x)\,dx, \qquad (\sigma > 0). \tag{2.11}$$

$U_\varphi(\sigma, t)$ is a complex-valued harmonic function in the half-plane $\sigma > 0$, and it belongs to L^2 in t for any $\sigma > 0$, since $U_\varphi(\sigma, t)$ is the Fourier transform of $e^{-\sigma|x|}\varphi(x)$. Let us now state the following lemma:

Over any closed interval, entirely contained in the complementary set of Λ_φ, $U_\varphi(\sigma, t)$ tends uniformly to zero as $\sigma \to +0$.

Let a be a point not belonging to Λ_φ. By the definition of this set, a summable function, say $\overline{G(x)}$, then exists such that

$$\int_{-\infty}^{\infty} \varphi(x+\tau)\,\overline{G(x)}\,dx = 0, \qquad (-\infty < \tau < \infty)$$

whereas $g(a) \neq 0$, $g(t)$ being the Fourier transform of $G(x)$. Since $g(t)$ is continuous, the property $g(t) \neq 0$ must hold over a certain closed interval containing a as inner point. From an argument published elsewhere[3] it follows that $U_\varphi(\sigma, t)$ converges uniformly to zero over this interval as $\sigma \to +0$. On applying the Borel-Lebesgue covering theorem, the lemma follows.

Let now ϱ_n be an infinite sequence of positive numbers tending to zero, and let T_{ϱ_n} be the contraction defined by (2.1). Put $f_n(t) = T_{\varrho_n} f(t)$. By assumption $f(t)$ is uniformly contractible, hence

$$f_n(t) = \int_{-\infty}^{-\infty} e^{-itx}\,F_n(x)\,dx, \qquad (F_n \in L^1) \tag{2.12}$$

$$\lim_{n=\infty} \| F_n \|_1 = 0, \tag{2.13}$$

and we may write

$$\int_{-\infty}^{\infty} \varphi(x)\,\overline{F(x)}\,dx = \int_{-\infty}^{\infty} \varphi(x)\,\overline{F_n(x)}\,dx + \int_{-\infty}^{\infty} \varphi(x)\,(\overline{F(x)} - \overline{F_n(x)})\,dx. \tag{2.14}$$

[3] See BEURLING: *Sur une classe de fonctions presque-périodiques*, C. R. de l'Acad. des Sciences, t. 225, pp. 326—7, 1947.

Since $f(t)$ is uniformly continuous and tends to zero for $t \to \pm \infty$, it follows from the definition of T_{ϱ_n} that $f(t) - f_n(t)$ must vanish outside of a certain finite set E_n of finite intervals entirely contained in the complement of Λ_φ. According to the lemma, $U_\varphi(\sigma, t)$ then converges uniformly to zero over each of these sets E_n when $\sigma \to +0$, and we obtain by the Parseval relation

$$\lim_{\sigma=+0} \int_{-\infty}^{\infty} e^{-\sigma|x|} \varphi(x) (\overline{F(x)} - \overline{F_n(x)})\, dx = \lim_{\sigma=+0} \frac{1}{2\pi} \int_{-\infty}^{\infty} U_\varphi(\sigma, t) (\overline{f(t)} - \overline{f_n(t)})\, dt = 0.$$

Thus, the second integral of the right hand of (2.14) must vanish for every n. Hence

$$\left| \int_{-\infty}^{\infty} \varphi(x) \overline{F(x)}\, dx \right| \leq \|\varphi\|_\infty \|F_n\|_1 \qquad (n = 1, 2, \ldots)$$

which by virtue of (2.13) implies (2.10) and thus completes the proof.

It should be noted that the concept of minimal extrapolation[4] leads to a condition which is both necessary and sufficient for $w(x)$ to be a regular weight-function; viz. that the Fourier transform $f(t)$ of every $F(x)$ of the class (2.8) have a minimal extrapolation with respect to the set $|f(t)| < \delta$, the spectral mass of which converges to zero for $\delta \to 0$.

3. Negative Definite Functions Applied to Contractions.

An important tool for the study of contractions in the space A are integrals of the form

$$\lambda(x) = \int_0^\infty \frac{\sin^2 x\alpha}{\alpha^2}\, d\mu(\alpha), \qquad (\mu(0) = 0) \tag{3.1}$$

where $\mu(\alpha)$ is a non-decreasing function such that the integral converges for every real x[5]. Remembering that $2 \sin^2 x = 1 - \cos 2x$, we immediately find that

$$\lambda(x) = \lim_{n=\infty} (\psi_n(0) - \psi_n(x))$$

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[4] See BEURLING: *Sur les intégrales de Fourier absolument convergentes* ..., C. R. du congrès des mathématiques à Helsingfors 1938.

[5] For the general properties of these functions and their interesting connection with certain metrical problems in Hilbert space, see v. NEUMANN and SCHOENBERG: *Fourier Integrals and Metric Geometry*, Transactions of the Amer. Math. Soc. V. 50, pp. 467—487, 1941.

where $\psi_n(x)$ is a sequence of real positive definite functions in the Bochner sense. On account of this fact we shall in this paper briefly call $\lambda(x)$ a negative definite function.

Theorem III. *A function $f(t) \in A$ is uniformly contractible in the space A if there exists a negative definite $\lambda(x)$ such that $|F(x)|^2 \lambda(x) + 1/\lambda(x) \in L^1$. If*

$$C_1 = \int_{-\infty}^{\infty} \frac{dx}{\lambda(x)}; \quad C_2 = \int_{-\infty}^{\infty} |F(x)|^2 \lambda(x)\, dx, \tag{3.2}$$

then any normalized contraction $g(t)$ of $f(t)$ satisfies

$$\|g\| \leq \sqrt{C_1 C_2}. \tag{3.3}$$

Let us first consider the case that $\mu(+0) = 0$. Since $f(t+\alpha) - f(t-\alpha)$ is the Fourier transform of

$$(e^{-i\alpha x} - e^{i\alpha x})\, F(x),$$

we obtain by the Parseval relation, whether the integrals are finite or not,

$$\frac{1}{8\pi} \int_{-\infty}^{\infty} |f(t+\alpha) - f(t-\alpha)|^2\, dt = \int_{-\infty}^{\infty} |F(x)|^2 \sin^2 x\alpha\, dx. \tag{3.4}$$

Multiplying both sides by $\alpha^{-2} d\mu(\alpha)$ and integrating over $(+0, \infty)$ we get by inverting the order of integration,

$$\frac{1}{8\pi} \int_{-\infty}^{\infty} \int_{+0}^{\infty} |f(t+\alpha) - f(t-\alpha)|^2 \alpha^{-2}\, d\mu(\alpha)\, dt = \int_{-\infty}^{\infty} |F(x)|^2 \lambda(x)|\, dx. \tag{3.5}$$

Since the elements of integration are non-negative and the right-hand side is finite these operations are legitimate. Another consequence is that the set E of numbers $\alpha \neq 0$ for which (3.4) is finite cannot be empty.

For a normalized contraction we have obviously $|g(t)| \leq |f(t)|$. Hence

$$\lim_{t=\pm\infty} g(t) = 0. \tag{3.6}$$

Defining

$$G_n(x) = \frac{1}{2\pi} \int_{-n}^{n} e^{itx} g(t)\, dt, \qquad (n = 1, 2, \ldots)$$

we may write

$$\frac{1}{2\pi}\int_{-n}^{n} e^{itx}(g(t+\alpha)-g(t-\alpha))\,dt = (e^{-i\alpha x}-e^{i\alpha x})\,G_n(x) + H_n(x),$$

where the sequence $H_n(x)$, according to the Parseval relation and (3.6) converges to 0 in the metric of the space L^2 for any finite α. Let α be a number of the set E. Then $g(t+\alpha)-g(t-\alpha)$ must belong to L^2, and the Plancherel theorem yields that the sequence

$$(3.7)\qquad (e^{-i\alpha x}-e^{i\alpha x})\,G_n(x), \qquad (n = 1, 2, \ldots)$$

converges in the metric of L^2 to a certain function which we can write in the form

$$(e^{-i\alpha x}-e^{i\alpha x})\,G(x).$$

This $G(x)$ is defined almost everywhere and obviously possesses the property that for any $\beta \in E$ the sequence corresponding to (3.7) converges to

$$(e^{-i\beta x}-e^{i\beta x})\,G(x).$$

It follows by the Parseval relation, for any α, that

$$(3.8)\qquad \frac{1}{8\pi}\int_{-\infty}^{\infty}|g(t+\alpha)-g(t-\alpha)|^2\,dt = \int_{-\infty}^{\infty}|G(x)|^2\sin^2 x\alpha\,dx,$$

both sides being simultaneously finite or infinite. This integral being dominated by (3.4), the operations which led to (3.5) now yield,

$$(3.9)\qquad \frac{1}{8\pi}\int_{-\infty}^{\infty}\int_{+0}^{\infty}|g(t+\alpha)-g(t-\alpha)|^2\alpha^{-2}\,d\mu(\alpha)\,dt = \int_{-\infty}^{\infty}|G(x)|^2\lambda(x)\,dx.$$

Denoting this functional by $W(g)$, we have evidently for any contraction $g(t)$ of $f(t)$

$$(3.10)\qquad W(g) \le W(f) = C_2.$$

Thus, by the Schwarz' inequality

$$(3.11)\qquad \left\{\int_{-\infty}^{\infty}|G(x)|\,dx\right\}^2 \le W(g)\int_{-\infty}^{\infty}\frac{dx}{\lambda(x)} \le C_1\,C_2$$

Hence $G(x) \in L^1$, and the function

$$g_1(t) = \int_{-\infty}^{\infty} e^{-itx} G(x)\, dx$$

belongs to the space A. Let now α be a fixed number of the set E. According to (3.8) the two functions $g(t+\alpha) - g(t-\alpha)$ and $g_1(t+\alpha) - g_1(t-\alpha)$ belong to L^2 and their Fourier-Plancherel transform must coincide almost everywhere. Thus $g(t) - g_1(t)$ is a continuous function of period 2α, tending to zero for $t \to \infty$. This implies $g(t) \equiv g_1(t)$ and the relation (3.3) is established.

It remains to prove that $f(t)$ is uniformly contractible, i. e. that for any sequence $g_n(t)$ of normalized contractions converging to zero,

$$\lim_{n=\infty} \| g_n \| = 0.$$

Observing that

$$\| g_n \|^2 \leq C_1 W(g_n),$$

$$\lim_{n=\infty} | g_n(t+\alpha) - g_n(t-\alpha) |^2 = 0,$$

$$| g_n(t+\alpha) - g_n(t-\alpha) |^2 \leq | f(t+\alpha) - f(t-\alpha) |^2,$$

the Lebesgue theorem of dominated convergence yields

$$\lim_{n=\infty} W(g_n) = 0,$$

which proves the uniform contractibility of $f(t)$.

In the case $\mu(+0) > 0$ we have

$$\lambda(x) = \mu(+0)\, x^2 + \int_{+0}^{\infty} \frac{\sin^2 x\alpha}{\alpha^2}\, d\mu(\alpha),$$

and it is readily seen that the previous proof remains valid if we add the term

$$\frac{\mu(+0)}{2\pi} \int_{-\infty}^{\infty} | f'(t) |^2\, dt$$

to the expression for $W(f)$, $f'(t)$ being the derivative of $f(t)$ which in this case must exist and belong to L^2.

4. On Negative Definite $\lambda(x)$ Connected with Certain Monotonic Functions.

The result in this section depends on some elementary properties of monotonic functions, which we shall divide upon the following three lemmas.

Lemma I. *Let* $0 < \alpha < 1 < \beta$, *and let* $F(x)$ *be non-increasing and summable over* $0 < x < \infty$. *Then* $F(x)$ *possesses a majorant* $H(x)$ *with the properties:* $x^{\alpha} H(x)$ *is non-increasing,* $x^{\beta} H(x)$ *is non-decreasing and*

$$\int_0^{\infty} H(x)\,dx \leq \frac{\beta}{(1-\alpha)(\beta-1)} \int_0^{\infty} F(x)\,dx. \tag{4.1}$$

Assume $F(x) = F(x-0)$, and let $G(x)$ be the least majorant of $F(x)$ such that $x^{\beta} G(x)$ is non-decreasing over $0 < x < \infty$. Since $x^{\beta} F(x)$ must tend to zero for $x \to 0$, the existence of $G(x)$ is assured. If $F(x) \not\equiv G(x)$ the set where $F(x) < G(x)$ is formed by a sequence of non overlapping intervals. Let x_0 and $x_1 \leq \infty$ be the endpoints of one of them, say ω, then

$$x_0^{\beta} F(x_0) = x_1^{\beta} F(x_1)$$

$$G(x) = \left(\frac{x_0}{x}\right)^{\beta} F(x_0), \qquad (x \in \omega)$$

where the first relation is valid only if $x_1 < \infty$. Consider now in a rectangular system of coordinates (x, y) the graph C of the function $y = F(x)$. To the interval ω assigne the rectangle R_{ω},

$$0 < x < x_0, \quad F(x_1) < y < F(x_0),$$

lying entirely between C and the x-axis and covering an area

$$R_{\omega} = (F(x_0) - F(x_1))\,x_0.$$

Comparing R_{ω} with

$$A_{\omega} = \int_{x_0}^{x_1} G(x)\,dx = \frac{F(x_0)\,x_0^{\beta}}{\beta - 1}\left(x_0^{1-\beta} - x_1^{1-\beta}\right)$$

we obtain

$$\frac{A_{\omega}}{R_{\omega}} = \frac{1}{\beta - 1}\,\frac{1 - \left(\frac{x_0}{x_1}\right)^{\beta-1}}{1 - \left(\frac{x_0}{x_1}\right)^{\beta}} \leq \frac{1}{\beta - 1}.$$

Since the rectangles corresponding to different intervals have no point in common, we get by summation over all ω,

$$\int_{F<G} G(x)\,dx \leq \frac{1}{\beta-1}\sum R_\omega \leq \frac{1}{\beta-1}\int_0^\infty F(x)\,dx.$$

Consequently,

$$\int_0^\infty G(x)\,dx = \int_{F<G} + \int_{F=G} G(x)\,dx \leq \frac{\beta}{\beta-1}\int_0^\infty F(x)\,dx. \tag{4.2}$$

The function $G(x)$ thus obtained is continuous for $x > 0$, and furthermore non-increasing and summable. Hence $x^\alpha G(x)$ must tend to zero for $x \to \infty$, and $G(x)$ possesses a least majorant $H(x)$ such that $x^\alpha H(x)$ is non-increasing. From the fact that $x^{\beta-\alpha}$ is increasing with x, it follows that $x^\beta H(x)$, like $x^\beta G(x)$, is a non-decreasing function, and it remains only to prove that (4.1) holds for $H(x)$. If the set where $G(x) < H(x)$ is not empty, it is formed by open finite intervals, and if (x_0, x_1) is one of these,

$$x_0^\alpha G(x_0) = x_1^\alpha G(x_1)$$

$$H(x) = \left(\frac{x_0}{x}\right)^\alpha G(x_0); \quad G(x) \geq G(x_1), \qquad (x_0 < x < x_1).$$

Thus,

$$\frac{\int_{x_0}^{x_1} H(x)\,dx}{\int_{x_0}^{x_1} G(x)\,dx} \leq \frac{1}{1-\alpha}\,\frac{1-\left(\frac{x_0}{x_1}\right)^{1-\alpha}}{1-\frac{x_0}{x_1}} \leq \frac{1}{1-\alpha},$$

and we obtain

$$\int_0^\infty H(x)\,dx \leq \frac{1}{1-\alpha}\int_0^\infty G(x)\,dx, \tag{4.3}$$

which proves the lemma. It is also readily seen that $H(x)$ is the least majorant of its kind.

Lemma II. *Let $\mu(\alpha)$ of formula (3.1) be non-decreasing over $0 \leq \alpha < \infty$, and such that for a certain p, $0 < p < 2$, $\alpha^{p-2}\mu(\alpha)$ is non-increasing for $\alpha > 0$. Then $\lambda(x)$ satisfies the inequalities*

$$\frac{2}{3} \leq \frac{\lambda(x)}{x^2\mu\left(\frac{1}{x}\right)} \leq \frac{2}{p}, \qquad (x > 0). \tag{4.4}$$

Observing that $\sin 1 > 5/6$, we get in $0 \leq \alpha \leq 1$,

$$\left(\frac{\sin \alpha}{\alpha}\right)^2 > \frac{2}{3}. \tag{4.5}$$

Hence, if we replace the integration variable α with α/x,

$$\frac{\lambda(x)}{x^2} = \int_0^\infty \left(\frac{\sin \alpha}{\alpha}\right)^2 d\mu\left(\frac{\alpha}{x}\right) \geq \frac{2}{3}\mu\left(\frac{1}{x}\right).$$

On the other hand we have by assumption

$$\mu\left(\frac{\alpha}{x}\right) \leq \alpha^{2-p}\mu\left(\frac{1}{x}\right), \qquad (\alpha \geq 1),$$

and since $h(\alpha) = \text{Min}\,(1, \alpha^{-2})$ is a non-increasing majorant of the kernel (4.5) we obtain by partial integration,

$$\frac{\lambda(x)}{x^2} \leq \int_0^\infty h(\alpha)\, d\mu\left(\frac{\alpha}{x}\right) = -\int_0^\infty \mu\left(\frac{\alpha}{x}\right) dh(\alpha) \leq$$

$$\leq 2\mu\left(\frac{1}{x}\right)\int_1^\infty \alpha^{-1-p}\, d\alpha = \frac{2}{p}\mu\left(\frac{1}{x}\right).$$

Lemma III. *Let $F(x)$ be non-increasing and summable over $0 < x < \infty$. Then a negative definite function $\lambda(x)$ exists such that*

$$F(x) \leq \frac{1}{\lambda(x)}, \qquad (x > 0) \tag{4.6}$$

$$\int_0^\infty \frac{dx}{\lambda(x)} \leq 24\int_0^\infty F(x)\, dx, \tag{4.7}$$

$$\sum_1^\infty \frac{1}{\lambda(n)} \leq 24\int_0^\infty F(x)\, dx. \tag{4.8}$$

Let $H(x)$ be the majorant of $F(x)$ which correspond to the special case $\alpha = 1/2$, $\beta = 2$, of Lemma I. Then, according to (4.1),

$$\int_0^\infty H(x)\, dx \leq 4\int_0^\infty F(x)\, dx.$$

By taking

$$\mu(0) = 0$$
$$\mu\left(\frac{1}{x}\right) = \frac{1}{4x^2 H(x)}, \qquad (0 < x < \infty)$$

the conditions of Lemma II are satisfied with $p = 1/2$, and we obtained by (4.4),

$$\tfrac{1}{6} \leq \lambda(x) H(x) \leq 1.$$

Then (4.7) and (4.8) follow, the latter inequality by the fact that $H(x)$ is non-increasing.

Now we can immediately establish the following property of the subset A^* of A.

Theorem IV. *Every $f(t) \in A^*$ is uniformly contractible in the space A, and for any normalized contraction $g(t)$ of $f(t)$ $(\not\equiv 0)$ we have*

$$\| g \| < 5 \|f\|^*. \tag{4.9}$$

Applying Lemma III to the non-increasing function $F^*(x)$ defined by (2.4) we obtain a negative definite $\lambda(x)$ such that for C_1 and C_2 of formula (3.2),

$$C_1 = \int_{-\infty}^{\infty} \frac{dx}{\lambda(x)} \leq 24 \int_{-\infty}^{\infty} F^*(x)\, dx = 24 \|f\|^*$$
$$C_2 = \int_{-\infty}^{\infty} | F(x) |^2 \lambda(x) \leq \int_{-\infty}^{\infty} | F(x) |\, dx = \|f\|.$$

It follows then from Theorem III that $f(t)$ is uniformly contractible, and by (3.3),

$$\| g \| \leq \sqrt{24 \|f\| \, \|f\|^*},$$

which is sharper than (4.9). Furthermore, Theorem I of the Introduction follows directly from Theorem II and Theorem IV.

Let us also apply the previous argument to the absolute convergence of contracted Fourier series.

Theorem V. *Let*

$$f(\theta) = \sum_{-\infty}^{\infty} a_n e^{in\theta}, \qquad (a_0 = 0)$$

be an absolutely convergent Fourier series such that $|a_{\pm n}| \leq a_n^*$, $n \geq 1$, *where* a_n^* *is a non-increasing sequence of numbers with a finite sum. Then if*

$$g(\theta) \sim \sum_{-\infty}^{\infty} b_n e^{in\theta}, \qquad (b_0 = 0)$$

is a contraction of $f(\theta)$, *the Fourier series of* $g(\theta)$ *converges also absolutely and, setting* $a_{-n}^* = a_n^*$, $a_0^* = 0$, *we have*

$$\sum_{-\infty}^{\infty} |b_n| < 5 \sum_{-\infty}^{\infty} a_n^*.$$

Lemma III applied to the function

$$F(x) = a_n^*, \qquad (n - 1 < |x| \leq n, \ n = 1, 2, \ldots)$$

yields a negative definite $\lambda(x)$ such that according to (4.6) and (4.8),

$$a_n^* = F(n) \leq \frac{1}{\lambda(n)}, \qquad (n = 1, 2, \ldots)$$

$$\sum_{-\infty}^{-1} + \sum_{1}^{\infty} \frac{1}{\lambda(n)} \leq 24 \int_{-\infty}^{\infty} F(x)\,dx = 24 \sum_{-\infty}^{\infty} a_n^*.$$

For the rest the proof runs parallel to the previous case and will not be repeated.

Finally, let us point out that the obtained results are also rather immediate consequences of a quite different method pertaining to the potential theory that the author has developed elsewhere.[6] On the other hand, the main theorems concerning the equilibrum potential of a closed set may be deduced by the contraction method used in this paper.

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[6] *Sur les spectres de fonctions*, Conférence faite au Congrès sur l'Analyse harmonique à Nancy 1947, (unpublished but multigraphed).