A CONTRIBUTION TO THE THEORY OF DIVERGENT SEQUENCES.[1]

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In this paper we define and examine a new method of summation which assigns a general limit $\operatorname{Lim} x_n$ to certain bounded sequences $x = \{x_n\}$. This method is analogous to the mean values which are used in the theory of almost periodic functions, furthermore it is narrowly connected with the limits of S. BANACH.[2] The sequences which are summable by this method F we shall call almost convergent. In spite of the fact that our method contains certain classes of matrix methods (for bounded sequences) it is not strong (§ 3). Its most remarkable property is that most of the commonly used matrix methods contain the method F (§ 5). In spite of this F is equivalent to none of the matrix methods (§ 7). In § 6 we shall examine a certain class of matrix methods and compare them with the method F.[1]

§ 1. Different Definitions of the Method *F*.

Let M be the entity of all bounded sequences of real numbers $x = \{x_n\}$. M is a Banach space, if we there define the linear operations in a natural manner and the norm of an element $x = \{x_n\}$ by

$$\|x\| = \sup_n |x_n|.$$

Then evidently the set C of all convergent sequences is a linear subspace of M. S. Banach proved the existence of certain functions of the element $x = \{x_n\}$ in

[1] Some preliminary results have been published in Zapiski Univ. Leningrad, Math. Ser. 12, 30—41 (1941).

[2] Cf. BANACH, Théorie des opérations linéaires, Warszawa 1932, p. 33—34.

M — the Banach limits — which we shall designate by $L(x) = L(x_n)$. These functions have the following properties:

1. $L(a x_n + b y_n) = a L(x_n) + b L(y_n) \qquad (a, b \text{ real}),$
2. $L(x_n) \geqq 0$, if $x_n \geqq 0$, $n = 0, 1, 2, \ldots$
3. $L(x_{n+1}) = L(x_n),$
4. $L(1) = 1.$

One can immediately see that for every $x \in M$ we have $\inf x_n \leqq L(x_n) \leqq \sup x_n$. Taking into account that $L(x_n)$ is independent of the first terms of the sequence $\{x_n\}$ i. e. independent of x_k for $k \leqq k_0$, we obtain a little more precisely

$$\underline{\lim}\, x_n \leqq L(x_n) \leqq \overline{\lim}\, x_n. \tag{1}$$

We introduce in M the functions

$$q(x) = q(x_n) = \inf_{n_1, n_2, \ldots, n_p} \overline{\lim_{k \to \infty}} \frac{1}{p} \sum_{i=1}^{p} x_{n_i+k} \tag{2}$$

and

$$q'(x) = q'(x_n) = -q(-x_n) = \sup_{n_1, n_2, \ldots, n_p} \underline{\lim_{k \to \infty}} \frac{1}{p} \sum_{i=1}^{p} x_{n_i+k}. \tag{3}$$

Here the infimum and supremum are taken with respect to all possible natural numbers p; $n_1, n_2, \ldots, n_p$. It is easy to prove that[1]

$$\begin{cases} q(a x) = a\, q(x) \quad \text{for} \quad a \geqq 0, \quad |q(x)| \leqq \|x\|, \\ \qquad q(x + y) \leqq q(x) + q(y). \end{cases} \tag{4}$$

From (4) it follows for $y_n = -x_n$ that

$$q'(x_n) \leqq q(x_n).$$

If now $\{x_n\} \in M$, $\{y_n\} \in C$ then

$$\begin{aligned} p(x_n + y_n) &= \inf_{n_1, \ldots, n_p} \overline{\lim_{k \to \infty}} \frac{1}{p} \sum_{i=1}^{p} (x_{n_i+k} + y_{n_i+k}) \\ &= \inf \left\{ \lim y_n + \overline{\lim} \frac{1}{p} \sum_{1}^{p} x_{n_i+k} \right\} = \lim y_n + q(x). \end{aligned} \tag{5}$$

[1] Cf. Banach, op. cit. p. 32.

With the aid of $q(x_n)$, $q'(x_n)$ the evaluation (1) can be rendered more accurate. We have

$$\overline{\lim_{k\to\infty}} \frac{1}{p} \sum_{i=1}^{p} x_{n_i+k} \geqq L\left(\frac{1}{p} \sum_{1}^{p} x_{n_i+k}\right) = L(x_n),$$

and similarly for $\underline{\lim}$. Therefore we have

$$q'(x_n) \leqq L(x_n) \leqq q(x_n). \tag{6}$$

It is now natural to ask for which of the sequences $x = \{x_n\}$ all limits $L(x_n)$ coincide. This is the case for the convergent sequences (according to (1)). But also for $x = \{1, 0, 1, 0, \ldots\}$ the value $L(x_n) = 1/2$ is the same for every Banach limit, as easily follows from 3., 1., and 4. We are thus led to the following

Definition: *The bounded sequence $\{x_n\}$ is called almost convergent and the number $s = \mathrm{Lim}\, x_n$ is called its F-limit if $L(x_n) = s$ holds for every limit L.*

We shall now give an internal characteristic of an almost convergent sequence.

The sequence $x = \{x_n\}$ is then and only then almost convergent, when the condition

$$q'(x_n) = q(x_n) \tag{7}$$

is fulfilled. From (6) it follows at once that this condition is sufficient. Conversely let $q'(x_n) < q(x_n)$ for a certain $x = \{x_n\}$. We then construct L in the following manner:[1] Let $L(y_n) = \lim y_n$ for $y \in C$. For the continuation of L into the space of the elements $y + ax$, $y \in C$, a real, the value of $L(x_n)$ can be chosen arbitrarily[1] in the interval

$$\sup_{y \in C} \{-q(-x_n - y_n) - \lim y_n\} \leqq L(x_n) \leqq \inf_{y \in C} \{q(x_n + y_n) - \lim y_n\}.$$

But according to (5) this inequality is equivalent to

$$q'(x_n) \leqq L(x_n) \leqq q(x_n).$$

Thus our proposition follows, as L can then be continued into the whole space M.

We shall now prove a theorem which brings the property (7) into a simpler

[1] We here make use of the known construction which leads to the proof of the theorem about the continuation of a linear functional, cf. BANACH op. cit. p. 27.

form. We shall thus obtain an analogy to the usual definition of the mean value of an almost periodic function.

Theorem 1. *In order that the F limit* $\operatorname{Lim} x_n = s$ *exists for the sequence* $\{x_n\}$, *it is necessary and sufficient that*

$$\lim_{p\to\infty} \frac{x_n + x_{n+1} + \cdots + x_{n+p-1}}{p} = s \tag{8}$$

holds uniformly in n.

Proof: If (8) is fulfilled, there is for $\varepsilon > 0$ a P such that for $p \geqq P$ and all n we have

$$s - \varepsilon < \frac{1}{p}(x_n + x_{n+1} + \cdots + x_{n+p-1}) < s + \varepsilon.$$

Therefore according to (2) and (3)

$$s - \varepsilon \leqq q'(x_n) \leqq q(x_n) \leqq s + \varepsilon$$

and as $\varepsilon > 0$ was arbitrary, $q'(x_n) = q(x_n) = s$. Therefore $\operatorname{Lim} x_n$ exists and is equal to s.

Conversely let $\operatorname{Lim} x_n = s$. Then for every $\varepsilon > 0$ there are natural numbers p; $n_1 \leqq \cdots \leqq n_p$ such that

$$\overline{\lim_{k\to\infty}} \frac{1}{p} \sum_{i=1}^{p} x_{n_i+k} < s + \varepsilon.$$

For sufficiently large k say $k \geqq k_0$ it follows that

$$\frac{1}{p} \sum_{1}^{p} x_{n_i+k} < s + \varepsilon \tag{9}$$

and if we replace n_i by $n_i + k_0$ and k by $k - k_0$ then (9) is even fulfilled for all natural k.

We now have for all natural n

$$\sum_{k=1}^{K} \frac{1}{p} \sum_{i=1}^{p} x_{n_i+k+n} = \frac{1}{p} \sum_{i=1}^{p} \sum_{j=n_i+1}^{n_i+K} x_{j+n} = \sum_{j=n_p+1}^{K+n_1} x_{j+n} + O(1) = \sum_{j=0}^{K-1} x_{j+n} + O(1),$$

where the last term is uniformly bounded in n. Then from (9) follows for K sufficiently large and all n

$$\frac{1}{K} \sum_{j=0}^{K-1} x_{j+n} < s + 2\varepsilon.$$

Similarly we may show, that the sum is $> s - 2\varepsilon$ so that we have really proved (8).[1]

Let F^* be the subset of M consisting of all almost convergent sequences. We sall sum up a few elementary properties of F^*. F^* is linear, not separable (for F^* contains a set of sequences $\{x_n\}$ of the power of continuum having the distance 1 from each other. Such a set for instance is formed by the sequences $x_n = 0$ for $n \neq k^2$ and $x_n = 0$ or $= 1$ for $n = k^2$, $k = 0, 1, \ldots$). F^* is nowhere dense in M, dense in itself and closed — therefore perfect (for the functions $q(x), q'(x)$ are continuous in M, as follows from (4)).

§ 2. The Method F and General Mean Values.

According to (8) the method F seems to be related to the method C_1 of the arithmetic means. In fact the method C_1 can be replaced in this definition by any other regular matrix method A at least when A fulfills certain simple conditions. A regular matrix method A is defined by

$$\lim_{m \to \infty} \sum_{n=0}^{\infty} a_{mn} x_n = s \tag{10}$$

where $a_{mn} \to 0$ and $\sum_n a_{mn} \to 1$ when $m \to \infty$, and furthermore $\sum_n |a_{mn}|$ remains bounded.

For such a method A we shall call the bounded sequence $\{x_n\}$ F_A summable to the value s if

$$y_{mk} = \sum_{n=0}^{\infty} a_{mn} x_{n+k}$$

uniformly in $k = 0, 1, \ldots$ tends to s as $m \to \infty$. We then have

Theorem 2. *An F_A summable sequence $x = \{x_n\}$ is also F summable, if the method A is regular.*

Proof: With $x = \{x_n\}$ the elements $x^{(k)} = \{x_{n+k}\}_{n=0,1,\ldots}$ $(k = 0, 1, \ldots)$ also belong to the space M and evidently $\|x^{(k)}\| \leqq \|x\|$. Let $y_{mk} = s + \alpha_{mk}$, where s is the F_A limit of $\{x_n\}$. Then for every $\varepsilon > 0$ an m_0 can be found such that

[1] It is no use to try to generalize by aid of (8) the notion of almost convergence also for unbounded sequences. In fact it easily follows from (8) that the sequence $\{x_n\}$ is bounded.

$|\alpha_{mk}| < \varepsilon$ for $m \geqq m_0$, $k = 0, 1, \ldots$. We examine the series whose terms are elements of the space M:

$$y^{(m)} = \sum_{n=0}^{\infty} a_{mn}\, x^{(n)}.$$

This series converges according to the norm (as $\sum |a_{mn}| \| x^{(n)} \|$ convergences) and represents an element $y^{(m)}$ of M whose k-th coordinate is just y_{mk}. According to the above we therefore have

$$y^{(m)} = s\, e + \alpha^{(m)}; \quad e = \{1, 1, \ldots, 1, \ldots\},\ \| \alpha^{(m)} \| \leqq \varepsilon.$$

From both sides of the last equation we take a Banach limit L. As L is continuous and additive and as furthermore $L(x^{(n)}) = L(x)$, we have

$$L(x) = (s + \alpha) \Big/ \sum_{n=0}^{\infty} a_{mn}, \qquad |\alpha| \leqq \varepsilon.$$

Since $\varepsilon > 0$ and m are arbitrary, it follows that

$$L(x) = s$$

i. e. $\operatorname{Lim} x_n = s$.

For special methods even more is true:

Theorem 3. *If the regular matrix method A has the property* (16) *of § 5 then the methods F and F_A are equivalent.*

This theorem will follow at once from the considerations of § 5.

§ 3. Examples of Almost Convergent Sequences.

1. For a complex z on the periphery of the unit circle $\operatorname{Lim} z^n = 0$ holds everywhere except for $z = +1$.[1] For from

$$\frac{1}{p}(z^n + z^{n+1} + \cdots + z^{n+p-1}) = z^n \frac{1 - z^p}{p(1 - z)}$$

the proposition follows immediately.

It is shown just as easily that the geometrical series Σz^n for $|z| = 1$, $z \neq +1$ is almost convergent to $1/(1 - z)$ (i. e. for its partial sums s_n we have $s_n \to 1/(1-z)$). Hence it follows that the Taylor series of a function $f(z)$, which for $|z| < 1$ is

[1] For a complex sequence $z_n = x_n + i\, y_n$ we define $\operatorname{Lim} z_n$ by the aid of (8) or put $\operatorname{Lim} z_n = \operatorname{Lim} x_n + i \operatorname{Lim} y_n$.

regular and on $|z| = 1$ has only simple poles, is almost convergent at every regular point of the periphery $|z| = 1$ with the sum $f(z)$.

By separating real and imaginary parts in $\operatorname{Lim} z^n = 0$ we have

$$\operatorname*{Lim}_{n\to\infty} \cos n\varphi = 0, \quad \operatorname*{Lim}_{n\to\infty} \sin n\varphi = 0 \qquad (\varphi \not\equiv 0 \bmod 2\pi).$$

Furthermore

$$\tfrac{1}{2} + F - \sum_{1}^{\infty} \cos n\varphi = 0 \qquad (\varphi \not\equiv 0 \bmod 2\pi)$$

holds, as this series represents the real part of the geometrical series $\frac{1}{2} + \Sigma e^{in\varphi}$.

2. A periodic sequence $x = \{x_n\}$ for which numbers N and p (the period) exist such that $x_{n+p} = x_n$ holds for $n \geqq N$ is almost convergent to the value

$$\operatorname{Lim} x_n = \frac{1}{p}(x_N + x_{N+1} + \cdots + x_{N+p-1}).$$

3. In analogy to the notion of an almost periodic function we call a sequence $\{x_n\}$ almost periodic if for every $\varepsilon > 0$ there are two natural numbers N and l, such that in every interval $(k, k+l)$, $k \geqq 0$ at least one »ε-period» p exists. More precisely $|x_{n+p} - x_n| < \varepsilon$ for $n \geqq N$ must hold for this p.[1] We have then: *Every almost periodic sequence is almost convergent.*

The proof of this statement can be given in the same manner as that for the similar fact of the existence of the mean value

$$\lim_{T\to\infty} \frac{1}{T} \int_{c}^{c+T} f(t)\, dt \qquad \text{(uniformly for all } c) \tag{11}$$

of an almost periodic function $f(t)$.[2] This proof can therefore be omitted here.

But naturally there are almost convergent sequences which are not almost periodic. For there is only a denumerable set of almost periodic sequences, whose terms take the values 0 and 1 only (namely the periodic ones only) whereas the set of almost convergent sequences of this kind has the power of continuum (cf. the last section of § 1).

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[1] A similar definition, where x_n is defined for all $-\infty < n < +\infty$ is given by A. Walther, Fastperiodische Folgen und Potenzreihen mit fastperiodischen Koeffizienten, Hamburger Abh., *6* (1928), p. 217—234.

[2] Cf. for example H. Bohr, Fastperiodische Funktionen, Ergebn. der Math., Berlin 1932, p. 34—38.

4. For the method with the matrix

$$(12) \qquad A = \left\| \begin{matrix} a_0 & a_1 & a_2 & \ldots & a_n & \ldots \\ 0 & a_0 & a_1 & \ldots & a_{n-1} & \ldots \\ 0 & 0 & a_0 & \ldots & a_{n-2} & \ldots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{matrix} \right\|$$

where $\Sigma a_n = 1$ and $\Sigma |a_n| < +\infty$ is supposed, one immediately sees that the method F_A means the same as A itself. Therefore from theorem 2 follows:

Theorem 4. *Every bounded sequence which is summable to the value s by a method* (12) *is also almost convergent to s.*

With restriction to matrices with finite lines of the form (12) H. HUNTEMANN[1] has introduced a method H, according to which the sequence $x = \{x_n\}$ is summable to the number s by definition, if it is A summable to s for some A of the form (12). (The number s is independent of A). From the above it follows at once that H is contained in F for bounded sequences. There are, however, sequences, which are F but not H summable as shown by the example of the sequence: $x_n = 1$ for $n = k^2$, $x_n = 0$ for $n \neq k^2$ $(k = 0, 1, 2, \ldots)$.

§ 4. Tauberian Theorems of the Method *F*.

We shall now look for Tauberian theorems for our method F. As usual we shall start from a series Σa_n with the partial sums $x_n = \sum_0^n a_\nu$. What condition on the terms of an almost convergent series Σa_n ensures the ordinary convergence of the series? We call such a condition a Tauberian condition. Such a Tauberian condition for example would be $a_n \to 0$ or even

$$(13) \qquad a_n^+ \to 0, \qquad a_n^+ = \mathbf{Max}\,(a_n, 0)$$

or finally the gap condition

$$(14) \qquad a_n = 0 \qquad \text{for } n \neq n_\nu,\ \nu = 1, 2, \ldots,$$

where $\{n_\nu\}$ is a lacunary sequence, i. e. a monotonously increasing sequence of natural numbers with $n_{\nu+1} - n_\nu \to +\infty$. Instead of these conditions we consider the more general conditions $a_n = O(c_n)$ and $a_n \leqq O(c_n)$, where c_n may have all values $0 \leqq c_n \leqq +\infty$.

[1] Deutsche Mathematik, *3* (1938), 390—402.

The above propositions are special cases of the following more general theorems: [1]

Theorem 5. *$a_n = O(c_n)$ is then and then only a Tauberian condition for almost convergent series Σa_n if*

(A) $$\begin{cases} \text{For every } \varepsilon > 0 \text{ a lacunary sequence } \{n_\nu\} \text{ exists with} \\ c_n < \varepsilon \text{ for } n \neq n_\nu,\ \nu = 1, 2, \ldots \end{cases}$$

Proof: a) Let (A) be fulfilled and for an almost convergent series let $|a_n| \leqq M c_n$ with a constant M. For a given $\varepsilon > 0$ we choose p so large, that for all n

$$\left|\frac{1}{p}(x_n + x_{n+1} + \cdots + x_{n+p-1}) - s\right| < \varepsilon \tag{15}$$

holds, where the x_n designates the partial sum of the series Σa_n and $s = \operatorname{Lim} x_n$.

Let $\{n_\nu\}$ be the sequence which belongs to $\varepsilon_1 = 2\varepsilon/(p-1)$ according to (A). If ν is defined by $n_\nu \leqq n < n_{\nu+1}$ then for sufficiently large n either $n + p - 1 < n_{\nu+1}$ or $n - p + 1 > n_\nu$. In the first case we have $c_\varrho < \varepsilon_1$ for $n < \varrho \leqq n + p - 1$ and therefore

$$\begin{aligned} |x_n - s| &\leqq \left|x_n - \frac{1}{p}(x_n + \cdots + x_{n+p-1})\right| + \varepsilon \\ &< M\frac{\varepsilon_1}{p}(1 + 2 + \cdots + (p-1)) + \varepsilon = (M+1)\varepsilon. \end{aligned}$$

This inequality also holds in the second case. Therefore $x_n \to s$.

b) If the numbers c_n do not satisfy the condition (A) then an $\varepsilon > 0$ and a sequence $\{n_\nu\}$ of natural numbers increasing monotonously to infinity exist such that $c_{n_\nu} \geqq \varepsilon$, $n_{2\nu+1} - n_{2\nu} \to +\infty$ and that $n_{2\nu} - n_{2\nu-1}$ is bounded. If we now let

$$a_{n_{2\nu-1}} = \varepsilon,\ a_{n_{2\nu}} = -\varepsilon,\ \nu = 1, 2, \ldots,\ a_n = 0 \text{ for the remaining } n,$$

then the sequence of the partial sums x_n is divergent, while evidently $\operatorname{Lim} x_n = 0$.

The one-sided condition of convergence can be treated just as easily.

Theorem 6. *$a_n \leqq O(c_n)$ is then and then only a Tauberian condition for almost convergent series Σa_n if*

(B) $$c_n \to 0 \quad \textit{holds.}$$

[1] The author gave similar Tauberian theorems for the methods of Cesàro and Abel in a paper »Tauberian theorems and Tauberian conditions» which is to appear in the Transactions Americ. Math. Soc.

Proof: a) Let these conditions be fulfilled and for an $\varepsilon > 0$ a number p be chosen so large that (15) holds. For $n \geqq N$, say, we then have $a_n < \varepsilon_1 = 2\varepsilon/(p-1)$ and therefore for $n \geqq N + p$

$$x_n < \frac{1}{p}(x_{n-p+1} + \cdots + x_n) + \frac{\varepsilon_1}{p}(1 + 2 + \cdots + (p-1)) < s + 2\varepsilon.$$

The inequality $x_n > s - 2\varepsilon$ is proved in a similar manner.

b) If (B) is not fulfilled, a lacunary sequence $\{n_\nu\}$ with $c_{n_\nu} \geqq \varepsilon > 0$ exists. Then the series Σa_n with $a_{n_\nu} = \varepsilon$, $a_{n_\nu+1} = -\varepsilon$, $a_n = 0$ for the remaining n is almost convergent and at the same time divergent, although $a_n \leqq O(c_n)$ is fulfilled. This proves the theorem.

§ 5. Strongly Regular Methods.

In spite of the fact that the method F contains certain regular matrix methods (with restriction to sequences contained in M) it is fairly weak. We shall show that it is contained in every »reasonable» matrix method. Almost convergence is a generalisation of ordinary convergence. From this point of view the method F seems to be rather akin to the ordinary convergence than to commonly used matrix methods. We shall therefore designate methods which sum all almost convergent sequences as strongly regular.

Theorem 7: *In order that the regular matrix method $A = \| a_{mn} \|$ sums all almost convergent sequences, it is necessary and sufficient that*

$$\lim_{m\to\infty} \sum_{n=0}^{\infty} | a_{mn} - a_{m,n+1} | = 0. \tag{16}$$

If this condition is fulfilled, then A-lim $x_n = \operatorname{Lim} x_n$ for every almost convergent sequence $\{x_n\}$.[1]

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[1] We shall not treat the similar problem of the regularity of a method with respect to all *almost periodic* sequences. A regular method for functions $f(t)$ which has the form

$$\lim_{x\to\infty} \int_0^{+\infty} K(x,t) f(t)\, dt = s$$

sums every *almost periodic function* $f(t)$ of a real argument $-\infty < t < +\infty$ to its mean value (11) exactly if the condition of »asymptotic orthogonality»

Proof: We assume first of all that (16) is fulfilled. Let $\{x_n\}$ be almost convergent and $\operatorname{Lim} x_n = s$. For any arbitrary $\varepsilon > 0$ we can then find a natural number p such that

$$\frac{1}{p}(x_n + x_{n+1} + \cdots + x_{n+p-1}) = s + \alpha_n, \quad |\alpha_n| < \varepsilon \qquad (n = 0, 1, 2, \ldots).$$

Multiplying by a_{mn} and adding we have

$$\frac{1}{p}\sum_{n=0}^{\infty} a_{mn}(x_n + x_{n+1} + \cdots + x_{n+p-1}) = s A_m + \sum_{n=0}^{\infty} a_{mn}\alpha_n \tag{17}$$

where $A_m = \sum_n a_{mn} \to 1$. As a_{mn} tends to zero for $m \to \infty$ we have on the other hand:

$$\begin{aligned} &\frac{1}{p}\sum_{n=0}^{\infty} a_{mn}(x_n + x_{n+1} + \cdots + x_{n+p-1}) \\ &= o(1) + \sum_{n=p-1}^{\infty} x_n \frac{1}{p}(a_{m,n-p+1} + \cdots + a_{mn}) \\ &= y_m + \sum_{n=p-1}^{\infty} x_n \left\{\frac{1}{p}(a_{m,n-p+1} + \cdots + a_{mn}) - a_{mn}\right\} + o(1). \end{aligned} \tag{18}$$

Here y^m designates the A transformation (10) of the sequence $\{x_n\}$ and the last term is infinitely small for $m \to \infty$ and the chosen p. Now the absolute value of the sum on the right hand side of (18) is not larger than

$$\begin{aligned} &\frac{1}{p}\sum_{n=p-1}^{\infty} |(a_{m,n-p+1} + \cdots + a_{mn}) - p\,a_{mn}| \cdot \|x\| \\ &\leqq \frac{1}{p}\|x\| \sum_{\varrho=0}^{p-1} \sum_{n=p-1}^{\infty} |a_{m,n-\varrho} - a_{mn}| \end{aligned}$$

$$\lim_{x\to\infty} \int_0^{+\infty} K(x,t) \begin{matrix}\cos\\ \sin\end{matrix}\, \lambda t\, dt = 0 \qquad (\lambda \text{ real} \neq 0)$$

is fulfilled. This is certainly the case, if the kernel $K(x,t)$ is equally distributed in the sense that

$$\lim_{x\to\infty} \int_E K(x,t)\,dt = \delta(E)$$

holds for every measurable set $E < (0, +\infty)$, for which the density in the interval $(0, +\infty)$, viz. $\delta(E) = \lim_{n\to\infty} \frac{1}{n} \operatorname{meas}\{E \cdot (0, n)\}$ has a sense.

$$\leqq \frac{1}{p}\|x\| \sum_{\varrho=0}^{p-1} \varrho \sum_{n=0}^{\infty} |a_{mn} - a_{m,n+1}|$$

$$= \frac{p-1}{2} \|x\| \sum_{n=0}^{\infty} |a_{mn} - a_{m,n+1}|.$$

From (17) and (18) we now have

$$y_m = s A_m + \sum_{n=0}^{\infty} a_{mn} \alpha_n + o(1).$$

Now

$$s A_m = s + o(1), \quad |\sum a_{mn} \alpha_n| \leqq M \varepsilon \quad \text{with } M = \sup \sum_n |a_{mn}|.$$

Thus for sufficiently large m we certainly have $|y_m - s| \leqq (M + 1)\varepsilon$. Therefore $\lim y_m = s$. This means that condition (16) is sufficient.

We now assume that (16) does not hold. We shall construct a sequence $\{x_n\}$ for which $\operatorname{Lim} x_n = 0$ but which is not summable by the matrix A. According to our hypotheses an $\varepsilon > 0$ exists, such that for an infinity of m

$$\sum_{n=0}^{\infty} |a_{mn} - a_{m,n+1}| > 8\varepsilon.$$

For every such m we either have

$$\sum_{l=0}^{\infty} |a_{m,2l} - a_{m,2l+1}| > 4\varepsilon$$

or

$$\sum_{l=0}^{\infty} |a_{m,2l+1} - a_{m,2l+2}| > 4\varepsilon.$$

By recurrence we now construct three increasing sequences of natural numbers $\{m_k\}$, $\{p_k\}$ and $\{q_k\}$ where $q_{-1} = 0 < p_1 < q_1 < p_2 < \cdots$ shall hold. We first choose m_1, p_1, q_1 such that

$$|a_{m_1,0}| < \frac{\varepsilon}{2},$$

$$\sum_{l=0}^{\frac{q_1-p_1-1}{2}} |a_{m_1,p_1+2l} - a_{m_1,p_1+2l+1}| > 2\varepsilon,$$

$$\sum_{n=q_1+1}^{\infty} |a_{m_1,n}| < \frac{\varepsilon}{2}.$$

If the numbers m_ν, p_ν, q_ν, $\nu = 1, 2, \ldots, k-1$ are already known, m_k, p_k, q_k (where $q_{k-1} < p_k < q_k$ and one of the numbers p_k, q_k even, the other, odd) are chosen such that

$$\sum_{n=0}^{q_{k-1}} |a_{m_k, n}| < \frac{\varepsilon}{2},$$

$$\sum_{l=0}^{\frac{q_k-p_k-1}{2}} |a_{m_k, p_k+2l} - a_{m_k, p_k+2l+1}| > 2\varepsilon,$$

$$\sum_{n=q_k+1}^{\infty} |a_{m_k, n}| < \frac{\varepsilon}{2}.$$

We now define the sequence $\{x_n\}$. Let

$$\left.\begin{aligned} x_{p_k+2l} &= (-1)^k \operatorname{sign}\left(a_{m_k, p_k+2l} - a_{m_k, p_k+2l+1}\right) \\ x_{p_k+2l+1} &= -x_{p_k+2l} \\ x_n &= 0 \quad \text{for} \quad q_{k-1} < n < p_k \end{aligned}\right\} \begin{aligned} &k = 1, 2, \ldots \\ &l = 0, 1, \ldots, \frac{q_k - p_k - 1}{2}. \end{aligned}$$

Under these conditions we have for our sequence

$$|y_{m_k}| = \left|\sum_{n=0}^{\infty} a_{m_k, n} x_n\right|$$

$$> \sum_{l=0}^{\frac{q_k-p_k-1}{2}} |a_{m_k, p_k+2l} - a_{m_k, p_k+2l-1}| - \frac{\varepsilon}{2} - \frac{\varepsilon}{2} > \varepsilon$$

and

$$\operatorname{sign} y_{m_k} = (-1)^k.$$

Hence it follows, that the sequence y_m diverges. It is further easy to see, that $\operatorname{Lim} x_n = 0$. This remark completes the proof. We shall investigate a few examples.

I. The Cesàro method C_α of the order $\alpha > 0$ is defined by

$$y_n = \frac{1}{A_n^\alpha} \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} x_\nu,$$

$$A_n^\alpha = \binom{n+\alpha}{n} = \frac{(\alpha+1)(\alpha+2)\ldots(\alpha+n)}{n!} \simeq \frac{n^\alpha}{\Gamma(\alpha+1)}.$$

If further we put $A_{-1}^{\alpha-1} = 0$, then the sum (16) is equal to

$$\frac{1}{A_n^\alpha}\sum_{\nu=0}^{n}|A_{n-\nu}^{\alpha-1}-A_{n-\nu-1}^{\alpha-1}|=\frac{1}{A_n^\alpha}(|1-A_n^{\alpha-1}|+1)\to 0,$$

as the numbers $A_\nu^{\alpha-1}$ are monotonous. Thus every almost convergent sequence is C_α summable for $\alpha>0$ to its F limit.

II. We examine the Euler method of summation $E_\alpha\,(\alpha>0)$ which is given by the transformation

$$y_n=\frac{1}{2^{\alpha n}}\sum_{\nu=0}^{n}\binom{n}{\nu}(2^\alpha-1)^{n-\nu}x_\nu=\sum_{\nu=0}^{n}\binom{n}{\nu}\left(\frac{1}{2^\alpha}\right)^\nu\left(1-\frac{1}{2^\alpha}\right)^{n-\nu}x_\nu.$$

We put $1/2^\alpha=t$ and use the notation

$$p_\nu(t)=p_{\nu,n}(t)=\binom{n}{\nu}t^\nu(1-t)^{n-\nu},\ \nu=0,1,\ldots,n;\ p_{n+1,n}(t)=0.$$

Then the sum (16) becomes

$$\sum_{\nu=0}^{n}|p_\nu(t)-p_{\nu+1}(t)|=\frac{1}{1-t}\sum_{\nu=0}^{n}\frac{n+1}{\nu+1}\left|\frac{\nu+1}{n+1}-t\right|p_\nu(t).$$

We split this sum into two parts, let Σ_1 be the sum for those ν, for which $\left|\frac{\nu}{n}-t\right|<n^{-\frac{1}{3}}$, and let Σ_2 be the remainder. For the evaluation of the sums we use the following known inequality, in which A signifies an absolute constant:

$$\sum_{\substack{0\leq\nu\leq n\\ \left|\frac{\nu}{n}-t\right|\geq n^{-\frac{1}{3}}}}p_\nu(t)<\frac{A}{n^2}.$$

With the aid of this inequality we obtain

$$|\Sigma_2|\leq\frac{A}{n^2}\operatorname*{Max}_{0\leq\nu\leq n}\left\{\frac{n+1}{\nu+1}\left|\frac{\nu+1}{n+1}-t\right|\right\}\leq\frac{A(n+1)}{n^2}.$$

For the terms of the sum Σ_1 we have

$$\left|\frac{\nu+1}{n+1}-t\right|<n^{-\frac{1}{3}}+\frac{1}{n}<2n^{-\frac{1}{3}},\ \frac{n+1}{\nu+1}<\frac{1}{t-2n^{-\frac{1}{3}}}<\frac{2}{t},$$

and therefore

$$|\Sigma_1|<\frac{2}{t}n^{-\frac{1}{3}}\sum_{\nu=0}^{n}p_\nu(t)=\frac{2}{t}n^{-\frac{1}{3}}.$$

Here also (16) is fulfilled, i. e. every almost convergent sequence $\{x_n\}$ is E_α summable to Lim x_n for $\alpha > 0$.

We now return to the proof of theorem 3 in § 2. For this purpose we replace x_n by x_{n+k} and y_m by $y_{mk} = \Sigma a_{mn} x_{n+k}$ in the first part of the above proof for theorem 7. Just as before one recognizes that from Lim $x_n = s$ under the supposition (16) the F_A summability of the sequence $\{x_n\}$ to the value s follows. That is, the method F is contained in F_A. This and theorem 2 imply theorem 3.

§ 6. The Class $\mathfrak{A}$ of Matrix Methods.

We have already seen that many matrix methods sum all almost convergent sequences. We shall now specify an even more comprehensive class of methods which have the property that they also sum certain bounded sequences, which are not almost convergent.

This class $\mathfrak{A}$ is the entity of regular methods $A = \| a_{mn} \|$ for which

$$\lim_{m \to \infty} \{\operatorname{Max}_n | a_{mn} |\} = 0 \tag{19}$$

is fulfilled.[1] We shall show thas the methods of this class can be characterized by the following »direct» theorem:

Theorem 8. *In order that a regular matrix method A sums every bounded sequence which has the property*

$$x_n = 0, \qquad n \neq n_1, n_2, \ldots \tag{20}$$

and for which n_ν increases sufficiently rapidly, it is necessary and sufficient that A belongs to $\mathfrak{A}$. Then A-lim $x_n = 0$.

»n_ν increases sufficiently rapidly» means more precisely: for all sequences $\{n_\nu\}$ for which $n_\nu \geqq N_\nu$ holds, where $\{N_\nu\}$ signifies a given sequence of real numbers.

Proof: a) If for the method A the condition (19) is not fulfilled there is an $\varepsilon_0 > 0$, a monotonously increasing sequence $m_\nu \to +\infty$ with

$$\operatorname{Max}_n | a_{m_\nu n} | \geqq \varepsilon_0$$

[1] It may be remarked here that the methods of class $\mathfrak{A}$ have been investigated by D. Menchoff, Bull. Acad. Sci. URSS, Moscou, ser. math., *1937*, 203—229. D. Menchoff proves an interesting theorem about the summability of orthogonal series by methods of class $\mathfrak{A}$. Cf. also R. P. Agnew, Bull. Americ. Math. Soc. *52*, 128—132 (1946), where a special case of our theorem 8 is proved.

and finally a sequence $\{n_\nu\}$ such that

$$|a_{m_\nu n_\nu}| \geqq \varepsilon_0. \tag{21}$$

As the sequence $\{n_\nu\}$ cannot be bounded, we can assume that n_ν tends to infinity and that n_ν increases as rapidly as we please.

We examine the matrix $A' = \| a'_{\mu\nu} \| = \| a_{m_\mu n_\nu} \|$. On account of (21) and according to a theorem of I. Schur[1] a bounded sequence $\{z_\nu\}$ exists which is not A' summable. Let

$$x_n = \begin{cases} z_\nu & \text{for } n = n_\nu,\ \nu = 1, 2, \ldots \\ 0 & \text{for all other } n. \end{cases}$$

This sequence $\{x_n\}$ is evidently not A summable, though n_ν can increase as rapidly as we please.

b). Suppose now the condition (19) to be fulfilled. Let

$$\varepsilon_m = \operatorname*{Max}_n |a_{mn}|.$$

It is possible to find two monotonously increasing sequences $\{p_m\}$, $\{q_m\}$ of natural numbers, such that the following conditions are fulfilled:

$$p_m \to \infty,\ p_m \leqq p_{m+1},\ p_m \varepsilon_m \to 0,$$
$$\sum_{n=q_m+1}^{\infty} |a_{mn}| = \eta_m \to 0,\ q_m < q_{m+1}.$$

Then a third sequence of natural numbers $\{N_\nu\}$ exists, such that for every m the inequality $N_n \leqq q_m$ is fulfilled for at most p_m of the numbers N_ν. To obtain such a sequence $\{N_\nu\}$ we choose $m_1 < m_2 < \cdots$, such that

$$1 \leqq p_{m_1} < p_{m_2} < \cdots \qquad \text{and therefore } p_{m_\nu} \geqq \nu$$

and put

$$N_1 = q_{m_1},\ N_2 = q_{m_2}, \ldots, N_\nu = q_{m_\nu}, \ldots.$$

The number l of the N_ν for which the inequality $N_\nu \leqq q_m$ is fulfilled, is determined by $m_l \leqq m < m_{l+1}$. For this l we therefore have

$$l \leqq p_{m_l} \leqq p_m$$

which prowes the existence of the sequence $\{N_\nu\}$.

[1] I. Schur, Journ. für reine und angew. Math. *151* (1921), 79—111 Theorem III. According to this theorem a regular method $A' = \| a'_{\mu\nu} \|$ with elements $a'_{\mu\nu}$ converging to zero for $\mu \to \infty$ sums all bounded sequences exactly when $\lim \sum_\nu |a'_{\mu\nu}| = 0$ holds.

Now this sequence has the property stated in the theorem. For suppose that for a sequence $\{n_\nu\}$ of natural numbers $n_\nu \geqq N_\nu$ for $\nu = 1, 2, \ldots$. Then the n_ν also have the property that $n_\nu \leqq q_m$ holds for at most p_m of the numbers n_ν. If $\{x_n\}$ is a bounded sequence with $|x_n| \leqq M$ and $x_n = 0$ for $n \neq n_\nu$ then

$$|y_m| = |\sum_{n=0}^{\infty} a_{mn} x_n| \leqq \sum_{n=0}^{q_m} |a_{mn} x_n| + \sum_{n=q_m+1}^{\infty}$$
$$\leqq \varepsilon_m p_m M + \eta_m M \to 0.$$

This proves the theorem.

We shall add some remarks to this theorem. For this purpose we introduce the notion of the density of a finite or infinite sequence $n_1 < n_2 < \cdots$ of natural numbers: That is a numerical monotonous not decreasing function $\omega(n)$ defined for all real $n \geqq 0$ such that for every n there are exactly $\omega(n)$ numbers n_ν satisfying the inequality $n_\nu \leqq n$. Evidently $\omega = \omega(n)$ is defined by

$$n_\omega \leqq n < n_{\omega+1}.$$

If for two sequences $\{n_\nu\}$ and $\{n'_\nu\}$ the inequality $n_\nu \leqq n'_\nu$ holds for every ν, then for their densities we have $\omega'(n) \leqq \omega(n)$. And conversely: from $\omega'(n) \leqq \omega(n)$ follows $n_\nu \leqq n'_\nu$. If $\omega(n)$ is the density of $\{n_\nu\}$ then $\}n_{2\nu}\}$ has a density $\leqq \omega(n)/2$.

We can now state theorem 8 in the following manner:

Theorem 8*. *The condition* (19) *is necessary as well as sufficient for the existence of a function $\Omega(n)$ (which has integral values only) increasing monotonously towards $+\infty$, such that every bounded sequence $x = \{x_n\}$ for which the indices n_ν with $x_{n_\nu} \neq 0$ have a density $\leqq \Omega(n)$ is certainly A summable (to zero).*

For if the sequence $\{N_\nu\}$ of theorem 8 exists, we designate its density by $\Omega(n)$. If then $x = \{x_n\}$ is a bounded sequence for which $x_n = 0$ for $n \neq n_\nu$, $\nu = 1, 2, \ldots$ and if $\{n_\nu\}$ has a density $\omega(n) \leqq \Omega(n)$ we have $n_\nu \geqq N_\nu$ and therefore according to theorem 8 certainly A-lim $x_n = 0$. Conversely, if a function $\Omega(n)$ of the kind required in theorem 8* exists, it may be assumed that $\Omega(n)$ only has jumps of the magnitude 1 for integral values of n. Then $\Omega(n)$ is the density of some sequence $\{N_\nu\}$ and for this sequence theorem 8 holds.

We shall call a function of the type treated in theorem 8* a summability function of the method A. Thus for example it can be seen at once that for the method C_1 of the arithmetic means all functions $\Omega(n) = o(n)$

and only these are summability functions. As the methods $C_\alpha (\alpha > 0)$ and the Abel method A are equivalent to C_1 for bounded sequences, they also have the same summability functions. The methods of class $\mathfrak{A}$ are characterized by the fact that they do *at all* possess summability functions. If a method A is stronger than a method B in the space M of the bounded sequences then every summability function of B is at the same time a summability function of A. The magnitude of $\Omega(n)$ is in a certain sense a measure for the strengh of the method in the space M.

We shall now describe ways for the determination of the summability functions of a given method. They are provided by the following two theorems.

Theorem 9. *A function $\Omega(n)$ with integral values only, increasing monotonously to $+\infty$ is a summability function for a regular matrix method A exactly when for every sequence $\{n_\nu\}$ whose density is $\leqq \Omega(n)$*

$$\lim_{m\to\infty} \sum_{\nu=1}^{\infty} |a_{m n_\nu}| = 0 \tag{22}$$

holds.

The proof follows immediately from the theorem of I. Schur mentioned in footnote p. 182 if one applies it to the methods with the matrices $\| a_{m n_\nu} \|$.

In order to obtain a more convenient form of the above criterion we introduce the numbers A_m^Ω where $\Omega = \Omega(n)$ is a function with integral values, increasing monotonously to $+\infty$. A_m^Ω is defined as the upper bound of all sums $\sum\limits_\nu |a_{m n_\nu}|$ in which $\{n_\nu\}$ goes through all sequences of a density $\leqq \Omega(n)$. (It is enough only to admit finite sequences $\{n_\nu\}$).

Theorem 10. *A function $\Omega(n)$ is a summability function of a matrix method A if and only if*

$$\lim_{m\to\infty} A_m^\Omega = 0. \tag{23}$$

Proof: (22) evidently follows from (23) so that it only remains to be shown that (23) follows from (22). Let (22) be satisfied but let (23) be wrong. Then an $\varepsilon > 0$ exists such that for an infinity of m the inequality $A_m^\Omega > 3\varepsilon$ holds. On account of (19) one can, by taking the sequence $\{n_{2\nu}\}$ or $\{n_{2\nu-1}\}$ instead of $\{n_\nu\}$ and only keeping a suitable segment of this sequence, deduce the following:

For every $N > 0$ there is a finite set of numbers n_ϱ:

$$N < n_1 < n_2 < \cdots < n_k$$

whose density is $\leqq \Omega(n)/2$ such that for an infinity of m

$$\sum_{\varrho=1}^{k} | a_{m n_\varrho} | \geqq \varepsilon. \tag{24}$$

From these m we now choose a partial sequence $\{m_\nu\}$ increasing monotonously to $+\infty$ such that, if $n_1^\nu, n_2^\nu, \ldots, n_{k_\nu}^\nu$ signify the respective n_ϱ, the inequalities $n_{k_{\nu-1}}^{\nu-1} < n_1^\nu$ and

$$\Omega(n_{k_1}^1) + \Omega(n_{k_2}^2) + \cdots + \Omega(n_{k_{\nu-1}}^{\nu-1}) \leqq \Omega(n_1^\nu)$$

hold. Then the sequence

$$n_1^1, n_2^1, \ldots, n_{k_1}^1, n_1^2, \ldots$$

for which we also write $\{n_\varrho\}$ has a density $\leqq \Omega(n)$. For the number of the n_ϱ with $n_\varrho \leqq n$ is

$$\begin{aligned} &\leqq \tfrac{1}{2}\,\Omega(n_{k_1}^1) + \tfrac{1}{2}\,\Omega(n_{k_2}^2) + \cdots + \tfrac{1}{2}\,\Omega(n_{k_{\nu-1}}^{\nu-1}) + \tfrac{1}{2}\,\Omega(n) \\ &\leqq \tfrac{1}{2}\,\Omega(n_1^\nu) + \tfrac{1}{2}\,\Omega(n) \leqq \Omega(n), \end{aligned}$$

ν being defined by $n_1^\nu \leqq n < n_1^{\nu+1}$.

For this sequence $\{n_\varrho\}$, however, (22) would not be fulfilled, as according to (24)

$$\sum_{\varrho=1}^{\infty} \| a_{m_\nu n_\varrho} \| \geqq \varepsilon.$$

This contradiction proves the assertion.

Usually the coefficients a_{mn} are $\gtreqq 0$ and monotonous. They increase up to $n = N_m$ and then decrease to zero for $n \to \infty$. In this case it is easy to evaluate the numbers A_m^Ω. Let N_m^* be chosen such that the sum $\Sigma\, a_{mn}$ for n varying in the interval $N_m^* - h_m < n \leqq N_m^*$ becomes as large as possible. (This will be the case in the neighbourhood of N_m). The set of these n has a density $\omega(n) \leqq \Omega(n)$.[1] Therefore according to the definition of A_m^Ω

[1] We assume that $\Omega(n)$ does not alter more than by 1 in every interval of the length 1, a natural condition as a density $\omega(n)$ always has this property.

$$A_m^\Omega \geqq \sum_{N_m^* - h_m < n \leqq N_m^*} a_{mn}. \tag{25}$$

One obtains the evaluation of A_m^Ω from the other side by choosing N_m^{**} so large, that $\sum\limits_{n > N_m^{**}} a_{mn} \leqq \varepsilon$. Then every sum $\sum\limits_{\nu} a_{mn_\nu}$ splits up into two parts of which one, with $n > N_m^{**}$, is $\leqq \varepsilon$, while the other contains at most $\Omega(N_m^{**}) = h'_m$ terms and is therefore smaller than Σa_{mn} for $N_m - h'_m < n < N_m + h'_m$. Hence

$$A_m^\Omega \leqq \varepsilon + \sum_{|N_m - n| < h'_m} a_{mn}. \tag{26}$$

As an example we consider the Euler method E_α,

$$y_n = \sum_{\nu=0}^{n} \binom{n}{\nu} t^\nu (1-t)^{n-\nu} x_\nu, \qquad 0 < t = \frac{1}{2^\alpha} < 1.$$

The Newton probability

$$p_\nu(t) = \binom{n}{\nu} t^\nu (1-t)^{n-\nu} = a_{n\nu}$$

becomes the larger as ν is the closer to nt, and approximately equally large on both sides of nt. We make use of the known fact that for every t there are positive numbers C_1, C_2, D_1, D_2, δ independent of n, ν such that

$$\frac{C_1}{\sqrt{n}} e^{-D_1 n \left(\frac{\nu}{n} - t\right)^2} \leqq a_{n\nu} \leqq \frac{C_2}{\sqrt{n}} e^{-D_2 n \left(\frac{\nu}{n} - t\right)^2} \qquad \text{for } \left|\frac{\nu}{n} - t\right| \leqq \delta. \tag{27}$$

Then according to (26) we have by putting $N_n^{**} = n$, $N_n = nt$ and $\varepsilon = 0$

$$A_n^\Omega \leqq C_2 \sum_{\left|\frac{\nu}{n} - t\right| \leqq \Omega(n)/n} e^{-D_2 n \left(\frac{\nu}{n} - t\right)^2} \frac{1}{\sqrt{n}}$$

$$= C_2 \sum_{|l_\nu| \leqq \Omega(n)/n} e^{-D_2 l_\nu^2} \Delta l_\nu,$$

where further $l_\nu = \sqrt{n}\left(\frac{\nu}{n} - t\right)$. For $\Omega(n) = o(\sqrt{n})$ the last sum tends to zero for $n \to \infty$: It is the Riemann sum of the integral

$$C_2 \int e^{-D_2 u^2} du$$

for an interval around the point o whose length tends to zero. Similarly if in (25) we put $N_n^* = nt$, we see that $\overline{\lim} A_n^{\Omega} > 0$ if $\Omega(n) \neq o(\sqrt{n})$.

Therefore *all functions* $\Omega(n) = o(\sqrt{n})$ *and only these functions are summability functions for the methods* $E_\alpha (\alpha > 0)$.

§ 7. Impossibility of the Representation of the Method *F* by Matrix Methods.

The method F cannot be expressed in the form of a matrix method. Every such method containing F, i. e. fulfilling (16) also sums certain bounded and not almost convergent sequences. As we stated above this even holds for all methods of the class $\mathfrak{A}$. But we want to prove a little more and therefore introduce the following definitions:

Let A_k, $k = 1, 2, \ldots$ be a sequence of regular matrix methods. By the product $A = \Pi A_k$ we understand the method A which is defined by the property that it sums a sequence $\{x_n\}$ to the value s exactly when this sequence is summed by *all* methods A_k to this value s. The sum $A = \Sigma A_k$ is defined as a method which sums a sequence $\{x_n\}$ then and then only to the value s, if it is summed to s by *at least one* of the methods A_k. (Here it must further be assumed that the methods A_k are consistent with each other.)

Now the method F is neither equivalent to a product of regular matrix methods nor to a sum. (For the matrix methods we only consider bounded sequences.) The first of these propositions follows from

Theorem 11. *For every sequence* $\{A_k\}$ *of methods of the class* $\mathfrak{A}$ *there is a bounded sequence* $x = \{x_n\}$ *which is not almost convergent but is summable to the value zero by every one of the methods* A_k.

Proof: According to theorem 8* a summability function $\Omega_k(n)$ exists for every method A_k. It is then possible to define a monotonous not decreasing function $\Omega(n)$ for which $\Omega(n) \to +\infty$ for $n \to \infty$ and which satisfies the inequality $\Omega(n) \leqq \Omega_k(n)$ for every k from a certain n onwards. Clearly $\Omega(n)$ is a summability function for all methods A_k. Now it is easy to specify a sequence $\{x_n\}, x_n = 0$ or $= 1$, which contains segments of arbitrary length consisting of 0's only or of 1's only and for which the sequence of the n with $x_n = 1$ has a density $\leqq \Omega(n)$. This sequence is not almost convergent and according to theorem 8* we have A_k-lim $x_n = 0$ for $k = 1, 2, \ldots$, which proves the theorem.

Theorem 11 states that almost convergence cannot be attained »from above» by products of matrix methods. We shall now show that it is also impossible to get almost convergence from below by means of sums.

Theorem 12. *If for every regular matrix method A_k $(k = 1, 2, \ldots)$ an almost convergent sequence exists which is not A_k summable, then an almost convergent sequence exists which is not summable by any method A_k.*

Let A_k^* signify the entity of the bounded and A_k summable sequences. We have to prove that from $F^* - A_k^* \neq 0$, $k = 1, 2, \ldots$ it follows that $F^* - \Sigma A_k^* \neq 0$ (cf. § 1, last section). Now F^* is a linear closed subspace of M and therefore of the second category in itself. $F^* A_k^*$ is a linear closed subset of F^* different from F^* and therefore nowhere dense in F^*. Therefore $F^* \Sigma A_k^*$ is of the first category F^* and cannot coincide with it.[1]

§ 8. Strongly Regular Hausdorff Methods.

A Hausdorff method H has the form

$$y_n = \sum_{\nu=0}^{n} a_{n\nu} x_n = \sum_{\nu=0}^{n} \binom{n}{\nu} \Delta^{n-\nu} \mu_\nu x_\nu, \tag{28}$$

where $\{\mu_\nu\}$ is a fixed real sequence and $\Delta^k \mu_\nu$ signifies the difference

$$\Delta^k \mu_\nu = \mu_\nu - \binom{k}{1} \mu_{\nu+1} + \cdots + (-1)^k \mu_{\nu+k}.$$

According to Hausdorff[2] the method (28) is exactly then regular when there is a function $g(t)$ of bounded variation in $[0, 1]$ which solves the moment problem

$$\mu_n = \int_0^1 t^n \, d g(t), \qquad n = 0, 1, 2, \ldots \tag{29}$$

(we assume $g(t)$ to be normed by $g(0) = 0$), satisfies the condition $g(1) = 1$ and is continuous in $t = 0$.

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[1] Theorem 12 evidently follows also from the known theorem about the condensation of singularities. Cf. Kaczmarz und Steinhaus, Theorie der Orthogonalreihen, Warschau, 1935 p. 24. The proof given above is nothing but a »geometrical» proof of this theorem.

[2] F. Hausdorff, Summationsmethoden und Momentenfolgen, Math. Zeitschrift 9 (1921), 74—109.

The k-th difference of t^ν is $t^\nu(1-t)^k$. On account of (29) we obtain for the coefficients $a_{n\nu}$ of the transformation (28)

$$a_{n\nu} = \binom{n}{\nu} \Delta^{n-\nu} \mu_\nu = \int_0^1 \binom{n}{\nu} t^\nu (1-t)^{n-\nu}\, d\,g(t) = \int_0^1 p_\nu(t)\, d\,g(t)$$

using the notation of § 5 example II.

Theorem 13. *A Hausdorff method H is strongly regular if it belongs to class $\mathfrak{A}$. For this the necessary and sufficient condition is*

$$a_{nn} = \mu_n \to 0 \tag{30}$$

or the continuity of the function $g(t)$ for $t = 1$.

Proof: Let $V(t)$ be the total variation of $g(t)$ in the interval $[0, t]$. If $g(t)$ is continuous in $t = 1$ then also $V(t)$ is continuous. For an arbitrary $\varepsilon > 0$ we choose $\delta > 0$ so small, that $V(1) - V(1-\delta) < \varepsilon$. We then have

$$|\mu_n| \leqq \left| \int_0^{1-\delta} t^n\, d\,g(t) \right| + \left| \int_{1-\delta}^{1} \right| \leqq (1-\delta)^n\, V(1) + \varepsilon < 2\,\varepsilon$$

for n sufficiently large, i. e. μ_n converges to zero.

If on the other hand $g(t)$ has a jump $\sigma = g(1) - g(1-0) \neq 0$ in $t = 1$, then we designate by $h(t)$ the function $h(t) = 0$ for $t < 1$, $h(1) = \sigma$. Then $g_1(t) = g(t) - h(t)$ is continuous and

$$\mu_n = \int_0^1 t^n\, d\,h + \int_0^1 t^n\, d\,g_1 = \sigma + o(1) \to \sigma \neq 0.$$

Hence the condition (30) is equivalent to the continuity of $g(t)$ in $t = 1$.

From $H \in \mathfrak{A}$ (30) evidently follows. It therefore suffices to show that the condition

$$\sum_{\nu=0}^{n} |a_{n\nu} - a_{n,\nu+1}| = \sum_{\nu=0}^{n} \left| \int_0^1 (p_\nu(t) - p_{\nu+1}(t))\, d\,g(t) \right| \to 0 \tag{31}$$

for a regular Hausdorff method H follows from the continuity of $g(t)$ in $t = 1$. The sum (31) is obviously not greater than

$$\int_0^1 \sum_{\nu=0}^{n} |p_\nu(t) - p_{\nu+1}(t)|\, d\,V(t).$$

$V(t)$ is continuous in $t = 0$ and $t = 1$ and the sum under the integral sign is $\leqq 2$ for all t. It therefore suffices to show that for every $\delta > 0$

$$\int_{\delta}^{1-\delta} \sum_{\nu=0}^{n} |p_\nu(t) - p_{\nu+1}(t)| \, d\, V(t)$$

tends to zero for $n \to \infty$. But in the interval $(\delta, 1 - \delta)$ the sum $\Sigma |p_\nu - p_{\nu+1}|$ converges uniformly to zero as shown by the evaluations of § 5 example II. From this the proposition follows. Examples I and II are only special cases of this theorem.