Characters, asymptotics and $\mathfrak{n}$-homology of Harish-Chandra modules

by

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§ 1. Introduction

Let π be an irreducible representation of a semisimple Lie group G. As has been known for some time, there exist connections between three types of invariants of π: the asymptotic behavior of its "matrix coefficients", the character of π, and the set of induced representations into which π can be embedded. Most of the analytic arguments in the representation theory of G exploit these connections in some way. Harish-Chandra's construction of the discrete series, for instance, is based on a detailed analysis of the interaction between the growth rates of the character and of the matrix coefficients. To give a second example, Langlands classifies the irreducible representations π by realizing them as subrepresentations of certain induced representations, which he describes in terms of the asymptotic behavior of the matrix coefficients of π. In this paper, we systematically explore the relationships between characters, asymptotics, and embeddings into induced representations.

Our main tool is a character identity that was conjectured by Osborne [33]. In order to explain the conjecture, we consider a parabolic subgroup $P=LN$, with unipotent radical N and Levi factor L. In the special case of a finite dimensional representation π, the group L operates naturally on the Lie algebra homology groups $H_p(\mathfrak{n}, V)$ of the representation space V, with respect to the complexified Lie algebra $\mathfrak{n}$ of N; this action is induced by the action of L on the standard complex of Lie algebra homology

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$V \otimes \Lambda\mathfrak{n}$, via π on V and conjugation on $\mathfrak{n}$. As a general rule, if U is a module for a group H, we let $\Theta_H(U)$ denote the character. Then, for purely formal reasons,

$$\Theta_G(V) \sum_p (-1)^p \Theta_L(\Lambda^p \mathfrak{n}) = \sum_p (-1)^p \Theta_L(H_p(\mathfrak{n}, V)) \tag{1.1}$$

is an equality on L.

Properly interpreted, both sides of (1.1) still make sense if π is an irreducible unitary representation or, more generally, an admissible([1]) representation on a Fréchet space, with a composition series of finite length. Let $K \subset G$ be a maximal compact subgroup. The space V of all K-finite vectors corresponding to such a representation π (i.e., the linear span of the finite dimensional, K-invariant subspaces) consists entirely of differentiable vectors, and the complexified Lie algebra $\mathfrak{g}$ of G acts on V by differentiation. The resulting algebraic representation reflects all important properties of π except, loosely speaking, the choice of a topology [13]. We refer to V as the underlying Harish-Chandra module of π. The global character of π, as defined by Harish-Chandra [15], is an invariant of the Harish-Chandra module V; this justifies the notation $\Theta_G(V)$. Although the reductive group L no longer operates on the homology groups $H_p(\mathfrak{n}, V)$, its complexified Lie algebra $\mathfrak{l}$ does act, and one can show that the groups $H_p(\mathfrak{n}, V)$ are Harish-Chandra modules of global representations of L. In particular, the homology groups have well-defined characters $\Theta_L(H_p(\mathfrak{n}, V))$. Thus all ingredients of the identity (1.1) retain their meaning.

The two sides of (1.1) have different symmetry properties unless V is finite dimensional. For this reason one cannot expect an equality on all of L. In the case of a minimal parabolic subgroup P, Osborne [33] conjectured that the identity (1.1) holds on a certain large subset of L. The conjecture was later refined and extended to arbitrary parabolic subgroups by Casselman, who also verified its p-adic analogue [6]. Our proof establishes the identity (1.1) on a subset $L^- \subset L$, which contains the domain of validity predicted by Casselman, and which is "large", in the following sense:

(a) the G-conjugates of L^- contain a dense open subset of L;
(b) the projection $L \mapsto \mathrm{Ad}\, L$ maps L^- onto $\mathrm{Ad}\, L$. (1.2)

Because of (a), (b), the Euler characteristic of the $\mathfrak{n}$-homology groups determines

([1]) A technical hypothesis, which is satisfied in all known cases [13, 29].

$\Theta_G(V)$ not only on L^-, but on all of L, and conversely $\Theta_G(V)$ determines this Euler characteristic as a virtual Harish-Chandra module for L.

There are two important reasons for studying the $\mathfrak{n}$-homology of Harish-Chandra modules. In principle, at least, the $\mathfrak{n}$-homology groups in dimension zero contain complete information about intertwining operators into induced representations: the Frobenius reciprocity theorem [7] provides a natural isomorphism

$$\mathrm{Hom}_G(V, I_P^G(U)) \simeq \mathrm{Hom}_L(H_0(\mathfrak{n}, V), U); \tag{1.3}$$

here $I_P^G(U)$ denotes the Harish-Chandra module induced from $P=LN$ to G by a Harish-Chandra module U for L, and $\mathrm{Hom}_G(\ldots,\ldots)$ refers to the homomorphisms in the category of Harish-Chandra modules.(2) The rate of growth of the (K-finite) matrix coefficients of a global representation π is an invariant of the underlying Harish-Chandra module V. According to a result of Miličić [31], this rate of growth is controlled by the action of the noncompact part of the center of L on $H_0(\mathfrak{n}, V)$, for any particular minimal parabolic subgroup $P=LN$. In both instances the homology groups in dimension zero turn out to be the objects of interest. Nevertheless, the higher homology groups play a role, since they measure the obstruction to the exactness of the functor $V \rightsquigarrow H_0(\mathfrak{n}, V)$. We should add that all of the homology groups $H_p(\mathfrak{n}, V)$ are related to Ext groups in an appropriate category; this follows from (1.3) by a derived functor argument [3].

If the alternating sum formula (1.1) is to give information about $H_0(\mathfrak{n}, V)$ —and hence about asymptotics and intertwining operators into induced representations—, it must be coupled with a vanishing theorem. To state the result, we write the Levi factor of P as a direct product $L=MA$, of a reductive group M with compact center and a central vector subgroup A. The homology groups $H_p(\mathfrak{n}, V)$ have finite composition series, which makes it possible to put the action of the complexified Lie algebra $\mathfrak{a}$ into Jordan canonical form,

$$H_p(\mathfrak{n}, V) = \bigoplus_{\nu \in \mathfrak{a}^*} H_p(\mathfrak{n}, V)_\nu, \tag{1.4}$$

with $H_p(\mathfrak{n}, V)_\nu$=generalized $(\nu+\varrho_P)$-eigenspace(3) of $\mathfrak{a}$. We call ν a homology exponent if $H_p(\mathfrak{n}, V)_\nu \neq 0$ for some p. A leading homology exponent is one that cannot be

(2) The notation is merely symbolic, since G itself does not act on the modules in question.

(3) The shift by ϱ_P, which denotes one-half of the trace of $\mathrm{ad}\,\mathfrak{a}$ on $\mathfrak{n}$, makes the labeling compatible with Harish-Chandra's parametrization of the characters of the center of the universal enveloping algebra.

expressed as a sum $\nu'+S$, of another homology exponent ν', plus a non-zero sum S of positive restricted roots. The vanishing theorem asserts:

$$H_p(\mathfrak{n}, V)_\nu = 0 \quad \text{if } \nu \text{ is a leading homology exponent and } p > 0. \tag{1.5}$$

In particular, there can be no cancellation in the identity (1.1) among terms corresponding to a leading homology exponent ν.

Now let $P=LN=MAN$ be a minimal parabolic subgroup and V the Harish-Chandra module attached to a global representation π. The K-finite matrix coefficients of π have asymptotic expansions on the various Weyl chambers in A—on the negative chamber $A^- \subset A$, for example. Since $G=K\overline{A^-}K$, these expansions on A^- bound the growth of the matrix coefficients in all directions. The result of Miličić which was alluded to before describes the leading terms in the asymptotic expansions on A^- as the exponentials $e^{\nu+\varrho_P}$, with ν ranging over the leading homology exponents. Because of the character identity (1.1) and the vanishing theorem (1.5), the leading homology exponents also measure the growth of $\Theta_G(V)$ on A^-: the character and the matrix coefficients have exactly the same asymptotic behavior on A^-; similar arguments, based on the identity (1.1) for other choices of P, show that the growth of the character on A^- dominates its growth on every Cartan subgroup. This very precise relationship between characters and asymptotics contains various estimates of Harish-Chandra, Trombi-Varadarajan and Miličić.

The passage from characters to intertwining operators into induced representations is a more delicate matter. The character $\Theta_G(V)$, which determines the composition factors of V rather than V itself, cannot possibly give information about all intertwining operators. One can even argue heuristically that if there is no simple, explicit relationship in general, one should not expect it in the irreducible case. On the other hand, the existence of intertwining operators corresponding to leading homology exponents is an immediate consequence of the identity (1.1), the reciprocity theorem (1.3) and the vanishing theorem (1.5). Such leading intertwining operators play an important role in classification problems and irreducibility criteria. To give a concrete example, we consider a module $V=I_P^G(U)$ of the unitary principal series. Because of restrictions imposed by the action of the center of the universal enveloping algebra, every homology exponent ν, relative to the minimal parabolic subgroup P, is a leading homology exponent. Bruhat's irreducibility theorem follows readily: because of (1.1) and (1.5), the multiplicity of the inducing module U in $H_0(\mathfrak{n}, V)$ can be extracted from the character formula for the principal series, and this multiplicity bounds the dimension of

$$\mathrm{Hom}_G(V, V) \simeq \mathrm{Hom}_{MA}(H_0(\mathfrak{n}, V), U).$$

Similar arguments prove Harish-Chandra's generalization of Bruhat's theorem, results of Harish-Chandra and Trombi about tempered representations, and Langlands' classification of irreducible Harish-Chandra modules.

This paper was conceived several years ago, when we first saw how to deduce Osborne's conjecture from Langlands' classification and Harish-Chandra's estimates of tempered characters. As we put our argument into writing, we realized that the conjecture provides a unified, conceptual explanation of several important results—Langlands' classification and Harish-Chandra's estimates among them. For this reason we have chosen to prove the conjecture ab initio, and to include proofs of a number of known results for the convenience of the reader. We do assume basic facts about semisimple Lie groups and their representations, such as Harish-Chandra's definition of the character, his regularity theorem and the existence of asymptotic expansions of matrix coefficients. The latter is again due to Harish-Chandra, but was never published by him in its entirety; a very readable account has just become available [8]. Self-contained, efficient expositions exist also for the remaining ingredients of our proof: two algebraic lemmas of Casselman-Osborne [9, 10], the Artin-Rees lemma for $U(\mathfrak{n})$ [32, 30], the formula for induced characters [25, 42]. Finally, we should mention the result that every irreducible Harish-Chandra module lifts to a representation of G [29, 11]; although our proof is logically independent of it, we use it in order to avoid convoluted statements and hypotheses.

As for the organization of this paper, section 2 begins with a general discussion of Lie algebra homology. We go on to show that the homology groups $H_p(\mathfrak{n}, V)$ are Harish-Chandra modules for the Levi factor, and we deduce the vanishing theorem (1.5) from a lemma of Casselman-Osborne and the Artin-Rees lemma for $U(\mathfrak{n})$. We state Osborne's conjecture in § 3. We then reduce it to a very special case: it suffices to equate the contributions of certain extreme exponential terms on the two sides of (1.1). The mechanism is the process of coherent continuation, which we develop in the form most suited to our purposes. Section 4 reviews the Frobenius reciprocity theorem, the existence of asymptotic expansions and Miličić's characterization of the leading exponents; we use these ideas to construct the Langlands embedding of an arbitrary irreducible Harish-Chandra module into a module induced from a tempered Harish-Chandra module, with a negative inducing parameter. The induced character formula, which plays an important role both in the proof and the applications of Osborne's conjecture, is discussed in § 5. All threads run together in § 6, where we prove the identity (1.1) in the case of a minimal parabolic subgroup. After the earlier preparations, the proof is essentially formal, except for two analytic tools: a lemma of Miličić, which asserts that certain induced modules have unique irreducible submodules, and an estimate for the global character in terms of the leading exponents. This estimate is based on the Langlands embedding constructed in § 4; its proof takes up most of § 6. For the sake of completeness, we verify Miličić's lemma in an appendix, at the end of the paper. Some modifications are necessary to adapt the arguments of § 6 to the case of a general parabolic subgroup, which is treated in § 7; instead of the lemma of Miličić, we appeal to a corollary of the special case of Osborne's conjecture proved in § 6. Section 7 also contains a reformulation of the identity (1.1) that relates the character to the homology groups with respect to certain maximal nilpotent subalgebras of $\mathfrak{g}$. Vogan [41] uses this version of our results in his work on the generalized Kazhdan-Lusztig conjectures. Various applications of Osborne's conjecture are described in § 8: the relationship between characters and asymptotics, Harish-Chandra's irreducibility theorem, the Langlands classification and basic properties of tempered Harish-Chandra modules.

Our methods have further applications; we intend to pursue them in a future publication.

§ 2. A vanishing theorem for $\mathfrak{n}$-homology

Although we are primarily interested in representations of connected, semisimple Lie groups, our arguments can be carried out most efficiently in a slightly wider context. Throughout this paper, G will denote a reductive Lie group, subject to the following conditions, first introduced by Harish-Chandra [21]:

(a) G has finitely many connected components;
(b) the derived group $[G, G]$ has finite center; (2.1)
(c) $\operatorname{Ad} g$, for $g \in G$, is an inner automorphism of the complexified Lie algebra $\mathfrak{g}$.

Once and for all, we fix a maximal compact subgroup $K \subset G$; it is unique up to conjugation and meets every connected component of G.

By a Harish-Chandra module(¹) for G, we shall mean a module V over the universal enveloping algebra $U(\mathfrak{g})$, equipped with an action of K, such that

(a) V is finitely generated as $U(\mathfrak{g})$-module;

(¹) This terminology is not completely standard. Dixmier [11], for example, uses it in a different sense.

(b) every $v \in V$ lies in a finite dimensional, $\mathfrak{k}$-invariant subspace;
(c) the actions of $\mathfrak{g}$ and K are compatible; and (2.2)
(d) each irreducible K-module occurs only finitely often in V

($\mathfrak{k}$=complexified Lie algebra of K). Explicitly, the compatibility condition (c) means that the $\mathfrak{k}$-action is the differential of the K-action, and that

$$k(Xv) = \operatorname{Ad} k(X)\,(kv),$$

for $k \in K$, $X \in \mathfrak{g}$, $v \in V$. The space of K-finite vectors in an irreducible, admissible representation of G on a Banach space is the prototypical example of a Harish-Chandra module [13]. Because of (2.2c, d),

$$\text{every } v \in V \text{ lies in a finite dimensional, } Z(\mathfrak{g})\text{-invariant subspace} \tag{2.3}$$

($Z(\mathfrak{g})$=center of $U(\mathfrak{g})$). Conversely, if the module V satisfies (2.3), in addition to (2.2a–c), it also satisfies (2.2d) and consequently is a Harish-Chandra module [13].

The $Z(\mathfrak{g})$-finiteness property (2.3) makes it possible to decompose V under the action of $Z(\mathfrak{g})$:

$$V = \bigoplus_{\chi} V_{\chi}, \tag{2.4}$$

with χ ranging over the set of characters of $Z(\mathfrak{g})$, and

$$V_{\chi} = \text{largest submodule on which } Z-\chi(Z) \text{ acts nilpotently, for every } Z \in Z(\mathfrak{g}). \tag{2.5}$$

In particular, if V happens to be irreducible, $Z(\mathfrak{g})$ acts according to a character. As a finitely generated $U(\mathfrak{g})$-module, V is Noetherian, and hence has a decreasing filtration with irreducible quotients. Up to isomorphism, there exist only finitely many irreducible Harish-Chandra modules on which $Z(\mathfrak{g})$ acts according to any given character [13]. Thus, in view of (2.2d), the filtration must break off after finitely many steps. This shows:

$$\text{Harish-Chandra modules have finite composition series.} \tag{2.6}$$

Every irreducible Harish-Chandra module V lifts(2) to an irreducible, admissible representation π of G on a Hilbert space V_π, in the sense that V is (isomorphic to) the

(2) Our arguments can be made logically independent of this fact, which is not so easy to prove.

$U(\mathfrak{g})$-module of K-finite vectors in the representation space V_π [29, 11]. The distribution character Θ of π, as defined by Harish-Chandra, depends only on the infinitesimal equivalence class of π [15]—in other words, Θ may be regarded as an invariant of the Harish-Chandra module V. More generally, (2.6) allows us to define the characters Θ of an arbitrary Harish-Chandra module V, as the sum of the characters of the composition factors of V. The characters corresponding to any finite set of irreducible, pairwise non-isomorphic Harish-Chandra modules are linearly independent [15]. In particular,

$$\text{the character } \Theta \text{ of a Harish-Chandra module } V \text{ completely determines the composition factors of } V. \tag{2.7}$$

Like every invariant, $Z(\mathfrak{g})$-finite distribution, the character Θ is a locally L^1-function, real analytic on the set of regular semisimple elements [18].

Let P be a parabolic subgroup of G, with Langlands factorization

$$P = MAN. \tag{2.8}$$

Thus N is the unipotent radical of P, A the largest connected, central, $\mathbf{R}$-split subgroup of the Levi factor MA, and M is reductive, with compact center. On the level of the complexified Lie algebras, (2.8) corresponds to

$$\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}. \tag{2.9}$$

Although M and MA need not be connected, they do inherit the properties (2.1) from G. Conjugating P, if necessary, one can arrange that

$$M \cap K \text{ is a maximal compact subgroup of } M, \tag{2.10}$$

and hence also of MA.

The main objects of interest of this section are the homology and cohomology groups of a Harish-Chandra module V, with respect to the complexified Lie algebra $\mathfrak{n}$ of N. We briefly recall their definitions and general properties. The homology groups

$$H_p(\mathfrak{n}, V) \tag{2.11}$$

arise as the derived functors of the covariant, right exact functor

$$V \rightsquigarrow V/\mathfrak{n}V = H_0(\mathfrak{n}, V) \tag{2.12}$$

on the category of $U(\mathfrak{n})$-modules. They can be calculated as the homology groups of the standard chain complex $C_*(\mathfrak{n}, V)$, with

$$C_p(\mathfrak{n}, V) = V \otimes \Lambda^p \mathfrak{n} \tag{2.13}$$

[5]. Dually, the cohomology groups

$$H^p(\mathfrak{n}, V) \tag{2.14}$$

are derived from the covariant, left exact functor

$$V \rightsquigarrow \text{space of } \mathfrak{n}\text{-invariants in } V = H^0(\mathfrak{n}, V), \tag{2.15}$$

and can be identified canonically with the cohomology groups of the standard cochain complex $C^*(\mathfrak{n}, V)$, whose terms are

$$C^p(\mathfrak{n}, V) = V \otimes \Lambda^p \mathfrak{n}^*. \tag{2.16}$$

For any vector space or $U(\mathfrak{g})$-module V, we let V^* denote the (algebraic) dual of V. Thus, if V is a Harish-Chandra module, V^* is also a $U(\mathfrak{g})$-module and a K-module in a consistent manner, but V^* does not satisfy (2.2 b), unless $\dim V < \infty$. As can be verified by direct calculation, the natural isomorphisms

$$\begin{aligned} C^p(\mathfrak{n}, V^*) &\simeq C_p(\mathfrak{n}, V)^*, \\ C_p(\mathfrak{n}, V) &\simeq C^{d-p}(\mathfrak{n}, V \otimes \Lambda^d \mathfrak{n}) \end{aligned} \tag{2.17}$$

($d = \dim \mathfrak{n}$) are compatible with the appropriate boundary and coboundary operators, and consequently they induce natural isomorphisms

$$\begin{aligned} H^p(\mathfrak{n}, V^*) &\simeq H_p(\mathfrak{n}, V)^*, \\ H_p(\mathfrak{n}, V) &\simeq H^{d-p}(\mathfrak{n}, V) \otimes \Lambda^d \mathfrak{n} \end{aligned} \tag{2.18}$$

[5] (note: $\mathfrak{n}$ operates trivially on $\Lambda^d \mathfrak{n}$).

Since both $\mathfrak{m} \oplus \mathfrak{a}$ and $K \cap M$ normalize $\mathfrak{n}$, they act on the complexes (2.13, 2.16), and thus on the homology and cohomology groups (2.11, 2.14). These actions are canonical; in particular, they preserve the isomorphisms (2.18). Shortly we shall prove that the $\mathfrak{n}$-homology and -cohomology groups of a Harish-Chandra module V are Harish-Chandra modules with respect to the group MA. For this purpose, we recall the definition of the Harish-Chandra homomorphism (without renormalization)

$$\gamma'_P : Z(\mathfrak{g}) \to Z(\mathfrak{m} \oplus \mathfrak{a}) \tag{2.19}$$

between the centers of the universal enveloping algebras of $\mathfrak{g}$ and $\mathfrak{m} \oplus \mathfrak{a}$: $\gamma'_P(Z)$, for $Z \in Z(\mathfrak{g})$, is the unique element of $U(\mathfrak{m} \oplus \mathfrak{a})$ —which necessarily lies in $Z(\mathfrak{m} \oplus \mathfrak{a})$ —satisfying the congruence

$$\gamma'_P(Z) \equiv Z \pmod{\mathfrak{n}\, U(\mathfrak{g})} \tag{2.20}$$

[18]. The dual mapping, between the sets of characters of $Z(\mathfrak{m}\oplus\mathfrak{a})$ and $Z(\mathfrak{g})$, has finite fibres [18]; cf. § 3 below. In particular, if $I\subset Z(\mathfrak{g})$ is a maximal ideal, $\gamma'_P(I)$ generates an ideal of finite codimension in $Z(\mathfrak{m}\oplus\mathfrak{a})$.

LEMMA 2.21 (Casselman-Osborne [9]).(3) *If an ideal $I\subset Z(\mathfrak{g})$ annihilates a $U(\mathfrak{g})$-module V, then $\gamma'_P(I)$ annihilates the $\mathfrak{n}$-homology groups of V.*

Combined with (2.18), the lemma implies the analogous statement about $\mathfrak{n}$-cohomology. We shall also use the following result of Casselman-Osborne: let $\mathfrak{n}_m$ denote the complexification of a maximal nilpotent subalgebra of $\mathfrak{g}_0$ (=real Lie algebra of G); then

LEMMA 2.22 (Casselman-Osborne [10]). *Every Harish-Chandra module V is finitely generated over $U(\mathfrak{n}_m)$.*

Up to conjugacy over K, $\mathfrak{p}$ contains $\mathfrak{n}_m$, which allows us to conclude:

COROLLARY 2.23. *Every Harish-Chandra module V is finitely generated over $U(\mathfrak{p})$.*

We now state and prove the first main result of this section:

PROPOSITION 2.24. *Let V be a Harish-Chandra module for G. With respect to the natural actions of $U(\mathfrak{m}\oplus\mathfrak{a})$ and $M\cap K$, the $\mathfrak{n}$-homology and -cohomology groups of V become Harish-Chandra modules for MA.*

Proof. In view of (2.18), it suffices to consider the $\mathfrak{n}$-homology groups of V. We resolve V by free $U(\mathfrak{p})$-modules of finite rank,

$$\ldots\to V_p\to V_{p-1}\to\ldots\to V_0\to V\to 0; \tag{2.25}$$

this is possible because of corollary 2.23. Since $U(\mathfrak{p})$ is free over $U(\mathfrak{n})$ by Poincaré-Birkhoff-Witt, the functorial definition of $\mathfrak{n}$-homology allows us to identify $H_p(\mathfrak{n}, V)$ with the p-th homology group of the complex

$$\ldots\to V_p/\mathfrak{n}V_p\to V_{p-1}/\mathfrak{n}V_{p-1}\to\ldots\to V_0/\mathfrak{n}V_0\to 0. \tag{2.26}$$

The Lie algebra $\mathfrak{m}\oplus\mathfrak{a}$ normalizes $\mathfrak{n}$, hence operates on the complex (2.26), as well as on its homology groups $H_p(\mathfrak{n}, V)$. We claim that the resulting $U(\mathfrak{m}\oplus\mathfrak{a})$-action agrees with

(3) The proof has been simplified by Vogan and others; see, for example, the appendix of [36].

the one which was previously introduced([4]), i.e. with the $U(\mathfrak{m}\oplus\mathfrak{a})$-action inherited from the complex (2.13). For $p=0$, one can check this directly: both actions arise as the quotient of the $U(\mathfrak{m}\oplus\mathfrak{a})$-action on V. To verify the claim inductively, we assume it for some $p\geq 0$, and for all $U(\mathfrak{p})$-modules V. We express V as the quotient of a free $U(\mathfrak{p})$-module F,

$$0\to V'\to F\to V\to 0,$$

which is then also free as a $U(\mathfrak{n})$-module, and thus has no higher $\mathfrak{n}$-homology groups. In the exact homology sequence

$$0\to H_{p+1}(\mathfrak{n}, V)\to H_p(\mathfrak{n}, V')\to \ldots,$$

the coboundary maps commute with both $U(\mathfrak{m}\oplus\mathfrak{a})$-actions, since these actions have functorial origins. The claim follows by induction. The $U(\mathfrak{p})$-modules V_p in (2.25) are free, of finite rank. Hence, again by Poincaré-Birkhoff-Witt, each term of the complex (2.26) is a finitely generated, free $U(\mathfrak{m}\oplus\mathfrak{a})$-module. Using the claim that was just established, we deduce: the $U(\mathfrak{m}\oplus\mathfrak{a})$-modules $H_p(\mathfrak{n}, V)$ are finitely generated, and thus satisfy the first condition (2.2a) in the definition of Harish-Chandra module (with respect to MA). The next two conditions (2.2b, c) hold even on the level of the complex (2.13). As for (2.2d), an inductive argument, proceeding by the length of the composition series of V, reduces the problem to the case of an irreducible module V. In particular, we may assume that some maximal ideal $I\subset Z(\mathfrak{g})$ annihilates V. Its image $\gamma'_P(I)$ generates an ideal of finite codimension in $Z(\mathfrak{m}\oplus\mathfrak{a})$, which according to lemma 2.21 annihilates $H_p(\mathfrak{n}, V)$. But then $H_p(\mathfrak{n}, V)$ satisfies the $Z(\mathfrak{m}\oplus\mathfrak{a})$-finiteness condition analogous to (2.3). To complete the proof, we recall that (2.3), together with (2.2a–c), implies (2.2d).

The nilpotent radical $\mathfrak{n}$ of $\mathfrak{p}$ corresponds to a system of positive restricted roots([5])

$$\begin{gathered}\Phi^+(\mathfrak{g},\mathfrak{a}) = \text{set of all linear functions } \alpha\in\mathfrak{a}^* \\ \text{according to which } \mathfrak{a} \text{ operates on } \mathfrak{n}.\end{gathered} \tag{2.27}$$

([4]) This observation is the point of departure of Casselman-Osborne's proof of lemma 2.21 in [9].

([5]) This terminology is not meant to suggest that the restricted root system $\Phi(\mathfrak{g},\mathfrak{a})$ satisfies the axioms of a root system—which need not be the case, unless P is a minimal parabolic subgroup.

We let ϱ_P denote the half-sum of the positive restricted roots; more precisely,

$$\varrho_P = \frac{1}{2} \sum_{\alpha \in \Phi^+(\mathfrak{g}, \mathfrak{a})} n_\alpha \alpha, \tag{2.28}$$

with n_α=dimension of the α-root space. As one consequence of proposition 2.24, the action of $Z(\mathfrak{m}\oplus\mathfrak{a})$ on the homology groups $H_p(\mathfrak{n}, V)$ determines a decomposition analogous to (2.4). Since $Z(\mathfrak{m}\oplus\mathfrak{a})$ contains $U(\mathfrak{a})$, the action of $\mathfrak{a}$ can also be put into Jordan canonical form:

$$H_p(\mathfrak{n}, V) = \bigoplus_{\nu \in \mathfrak{a}^*} H_p(\mathfrak{n}, V)_\nu \tag{2.29}$$

(finite direct sum), where

$$\begin{aligned} H_p(\mathfrak{n}, V)_\nu = {} & \text{largest subspace of } H_p(\mathfrak{n}, V) \text{ on which } X - \langle \nu + \varrho_P, X\rangle \\ & \text{acts nilpotently, for all } X \in \mathfrak{a}. \end{aligned} \tag{2.30}$$

The shift by ϱ_P has the effect of making our notation compatible with Harish-Chandra's labelling of the characters of $Z(\mathfrak{g})$; cf. § 3 below.

We use the system of positive restricted roots (2.27) to partially order the dual space $\mathfrak{a}^*$, by defining

$$\begin{aligned} \mu > \nu \Leftrightarrow \mu - \nu & \text{ is a non-zero linear combination, with} \\ & \text{positive integral coefficients, of roots } \alpha \in \Phi^+(\mathfrak{g}, \mathfrak{a}). \end{aligned} \tag{2.31}$$

The next result plays a crucial role in both the proof and the applications of Osborne's conjecture. In the special case of a minimal parabolic subalgebra, it is due to Casselman (unpublished), and may be regarded as a weak version of the vanishing theorem for $\mathfrak{n}$-homology which was established in [36]. The much more precise version of [36] does have an analogue in our situation of a general parabolic subalgebra, but we shall not pursue this matter here. Our proof consists of a string of lemmas; it closely parallels the argument of [36], where further details can be found.

PROPOSITION 2.32. *Let V be a Harish-Chandra module, μ a linear function on $\mathfrak{a}$, and $p>0$ an integer, such that $H_p(\mathfrak{n}, V)_\mu \neq 0$. Then there exists a $\nu \in \mathfrak{a}^*$, with $\nu < \mu$ and $H_0(\mathfrak{n}, V)_\nu \neq 0$.*

Following Casselman's ideas, as described in [36, § 4], we consider

$$V^{[\mathfrak{n}]} = \{v^* \in V^* \mid \mathfrak{n}^k v^* = 0 \quad \text{for some } k \in \mathbf{N}\}. \tag{2.33}$$

A simple argument shows that $V^{[\mathfrak{n}]}$ is a $U(\mathfrak{g})$-submodule of the algebraic dual V^*: Since $\mathfrak{p}$ normalizes $\mathfrak{n}$, we only need to worry about root spaces $\mathfrak{g}^{-\alpha}, \alpha \in \Phi^+(\mathfrak{g}, \mathfrak{a})$; for any such α, $[\mathfrak{g}^{-\alpha}, \mathfrak{n}^{k+l}] \subset U(\mathfrak{g})\,\mathfrak{n}^k$ if l is large, hence $\mathfrak{g}^{-\alpha} V^{[\mathfrak{n}]} \subset V^{[\mathfrak{n}]}$. As before, $\mathfrak{n}_m$ shall denote the complexification of a maximal nilpotent subalgebra of the real Lie algebra $\mathfrak{g}_0$. We may and shall assume

$$\mathfrak{n} \subset \mathfrak{n}_m \subset \mathfrak{p},$$

in which case

$$\mathfrak{n}_m = (\mathfrak{n}_m \cap \mathfrak{m}) \oplus \mathfrak{n} \quad \text{(semidirect product).} \tag{2.34}$$

The definition (2.33) makes sense for any $U(\mathfrak{n}_m)$-module V, and thus we may regard

$$V \rightsquigarrow V^{[\mathfrak{n}]} \tag{2.35}$$

as a contravariant functor from the category of $U(\mathfrak{n}_m)$-modules to itself.

LEMMA 2.36. *Restricted to the category of finitely generated $U(\mathfrak{n}_m)$-modules, the functor* (2.35) *is exact.*

This follows directly from the Artin-Rees lemma for the ideal $\mathfrak{n}U(\mathfrak{n}_m) \subset U(\mathfrak{n}_m)$: let V' be a submodule of a finitely generated $U(\mathfrak{n}_m)$-module V, and k a positive integer; then $(\mathfrak{n}^{k+l}V) \cap V' \subset \mathfrak{n}^k V'$, provided l is large enough [32].

LEMMA 2.37. *For every finitely generated $U(\mathfrak{n}_m)$-module V, the inclusion $V^{[\mathfrak{n}]} \hookrightarrow V^*$ induces isomorphisms $H^p(\mathfrak{n}, V^{[\mathfrak{n}]}) \simeq H^p(\mathfrak{n}, V^*)$.*

Proof. Since V can be represented as a quotient of a free module of finite rank, an induction argument on the degree p reduces the problem to the special case $V = U(\mathfrak{n}_m)$. According to the Poincaré-Birkhoff-Witt theorem, $U(\mathfrak{n}_m)$ is free as a module over $U(\mathfrak{n})$, and thus has trivial higher homology groups. In particular, the augmented complex

$$\ldots \to U(\mathfrak{n}_m) \otimes \Lambda^p \mathfrak{n} \to \ldots \to U(\mathfrak{n}_m) \to H_0(\mathfrak{n}, U(\mathfrak{n}_m)) \to 0 \tag{2.38}$$

is exact. If we let $U(\mathfrak{n}_m)$ operate trivially on $\Lambda\mathfrak{n}$, and by right multiplication on itself and on

$$H_0(\mathfrak{n}, U(\mathfrak{n}_m)) = U(\mathfrak{n}_m)/\mathfrak{n}U(\mathfrak{n}_m),$$

(2.38) becomes an exact sequence of finitely generated (right) $U(\mathfrak{n}_m)$-modules, which remains exact when we apply either the duality functor or the functor (2.35)—cf. lemma

2.36. The resulting exact sequences, except for the initial terms, coincide with the standard complexes of Lie algebra cohomology for $U(\mathfrak{n}_m)^*$ and $U(\mathfrak{n}_m)^{[\mathfrak{n}]}$, respectively. Since

$$(U(\mathfrak{n}_m)/\mathfrak{n}U(\mathfrak{n}_m))^{[\mathfrak{n}]} = (U(\mathfrak{n}_m)/\mathfrak{n}U(\mathfrak{n}_m))^*,$$

one finds

$$H^p(\mathfrak{n}, U(\mathfrak{n}_m)^*) = H^p(\mathfrak{n}, U(\mathfrak{n}_m)^{[\mathfrak{n}]}) = \begin{cases} (U(\mathfrak{n}_m)/\mathfrak{n}U(\mathfrak{n}_m))^* & \text{if } p = 0, \\ 0 & \text{if } p > 0, \end{cases}$$

and this establishes the lemma.

LEMMA 2.39. *Let V be a Harish-Chandra module. Then every $v^* \in V^{[\mathfrak{n}]}$ lies in a finite dimensional, $\mathfrak{a}$-invariant subspace.*

Proof. We begin with a simple observation: if

$$0 \to M' \to M \to M'' \to 0$$

is an exact sequence of $\mathfrak{a}$-modules in which both M' and M'' satisfy the $\mathfrak{a}$-finiteness condition, then M also satisfies this condition. The increasing, $\mathfrak{a}$-invariant filtration

$$0 \subset V_1^{[\mathfrak{n}]} \subset V_2^{[\mathfrak{n}]} \subset \ldots \subset V^{[\mathfrak{n}]}, \quad \text{with } V_k^{[\mathfrak{n}]} = \{v^* \in V^* | \mathfrak{n}^k v^* = 0\}.$$

exhausts $V^{[\mathfrak{n}]}$ and has successive quotients

$$V_k^{[\mathfrak{n}]}/V_{k-1}^{[\mathfrak{n}]} \simeq (\mathfrak{n}^{k-1}V/\mathfrak{n}^k V)^*. \tag{2.40}$$

It therefore suffices to prove the $\mathfrak{a}$-finiteness property for the modules (2.40). The action of $\mathfrak{n}$ on V determines natural surjections

$$(\otimes^{k-1}\mathfrak{n}) \otimes (V/\mathfrak{n}V) \to \mathfrak{n}^{k-1}V/\mathfrak{n}^k V \to 0$$

and, dually, injections

$$0 \to (\mathfrak{n}^{k-1}V/\mathfrak{n}^k V)^* \to (\otimes^{k-1}\mathfrak{n}^*) \otimes (V/\mathfrak{n}V)^*.$$

This reduces the problem to the case of the $\mathfrak{a}$-module $(V/\mathfrak{n}V)^*$. Because of proposition 2.24, $H_0(\mathfrak{n}, V) = V/\mathfrak{n}V$ has a composition series of finite length, and in particular a finite, $\mathfrak{a}$-invariant filtration, with successive quotients on which $\mathfrak{a}$ acts according to a linear function. The dual filtration of $(V/\mathfrak{n}V)^*$ has these same properties. Appealing to our original observation, we may deduce the required $\mathfrak{a}$-finiteness property for $(V/\mathfrak{n}V)^*$, and hence also for $V^{[\mathfrak{n}]}$.

The complex $C^*(\mathfrak{n}, V^{[\mathfrak{n}]})$ inherits the $\mathfrak{a}$-finiteness property in lemma 2.39 from $V^{[\mathfrak{n}]}$, and hence has a decomposition

$$C^*(\mathfrak{n}, V^{[\mathfrak{n}]}) = \bigoplus_{\nu\in\mathfrak{a}^*} C^*(\mathfrak{n}, V^{[\mathfrak{n}]})_\nu, \text{ with}$$

$$C^*(\mathfrak{n}, V^{[\mathfrak{n}]})_\nu = \text{generalized } (\nu-\varrho_P)\text{-eigenspace of } \mathfrak{a},$$

which is analogous to (2.29), although in general infinite. On the level of cohomology,

$$H^*(\mathfrak{n}, V^{[\mathfrak{n}]}) = \bigoplus_{\nu\in\mathfrak{a}^*} H^*(\mathfrak{n}, V^{[\mathfrak{n}]})_\nu, \tag{2.41}$$

where $H^*(\mathfrak{n}, V^{[\mathfrak{n}]})_\nu$ denotes the generalized $(\nu-\varrho_P)$-eigenspace of $\mathfrak{a}$, acting on $H^*(\mathfrak{n}, V^{[\mathfrak{n}]})$; equivalently,

$$H^*(\mathfrak{n}, V^{[\mathfrak{n}]})_\nu = \text{cohomology of } C^*(\mathfrak{n}, V^{[\mathfrak{n}]})_\nu.$$

Combining lemma 2.37 with (2.18), one obtains $\mathfrak{a}$-invariant isomorphisms

$$H^p(\mathfrak{n}, V^{[\mathfrak{n}]}) \simeq H_p(\mathfrak{n}, V)^*,$$

which exhibit the decomposition (2.41) as dual to (2.29):

$$H^p(\mathfrak{n}, V^{[\mathfrak{n}]})_\nu \simeq (H_p(\mathfrak{n}, V)_{-\nu})^*. \tag{2.42}$$

Thus, if $H_p(\mathfrak{n}, V)_\mu \neq 0$ for some $p>0$,

$$C^p(\mathfrak{n}, V^{[\mathfrak{n}]})_{-\mu} = (V^{[\mathfrak{n}]} \otimes \Lambda^p \mathfrak{n}^*)_{-\mu}$$

must also be non-zero. The weights by which $\mathfrak{n}$ operates on $\Lambda^p\mathfrak{n}^*$ can be expressed as sums of p negative restricted roots. We conclude: there exists a non-zero $v^*\in V^{[\mathfrak{n}]}$, transforming according to some $\lambda\in\mathfrak{a}^*$, with $\lambda > -\mu-\varrho_P$. By definition of $V^{[\mathfrak{n}]}$, the $U(\mathfrak{n})$-translates of v^* lie in a finite dimensional, $\mathfrak{a}$-invariant subspace. Let $-\nu-\varrho_P$ be maximal, relative to the ordering $>$, among all generalized eigenvalues that contribute to this subspace. Then $-\nu-\varrho_P \geq \lambda$, and the generalized $(-\nu-\varrho_P)$-eigenspace of on $V^{[\mathfrak{n}]}$ meets the space of $\mathfrak{n}$-invariants. Thus $H^0(\mathfrak{n}, V^{[\mathfrak{n}]})_{-\nu}$ is non-zero, or dually $H_0(\mathfrak{n}, V)_\nu \neq 0$. Since $\nu<\mu$, this completes the proof of proposition 2.32.

§ 3. Osborne's conjecture and coherent continuation

The purpose of this section is to reduce Osborne's conjecture to a seemingly very special case. The mechanism behind our argument is the process of "coherent continuation" [35, 37], which we recall to the extent that is needed here. We begin with a precise statement of Osborne's conjecture.

As in § 2, $P=MAN$ shall denote a parabolic subgroup of G, and V a Harish-Chandra module. To bring out the dependence on G and V, we refer to the character of V as $\Theta_G(V)$. It is a real-analytic function on

$$G' = \text{set of all regular, semisimple } g \in G. \tag{3.1}$$

According to proposition 2.24, the $\mathfrak{n}$-homology groups of V are Harish-Chandra modules with respect to the group MA, and thus have MA-characters

$$\Theta_{MA}(H_p(\mathfrak{n}, V)). \tag{3.2}$$

Every $g \in MA \cap G'$ is regular also in MA; this ensures the real-analyticity of the characters (3.2) on $MA \cap G'$. The adjoint representation turns the exterior powers $\Lambda^p \mathfrak{n}$ into MA-modules. Since these are finite dimensional, their Harish-Chandra characters $\Theta_{MA}(\Lambda^p \mathfrak{n})$ agree with the MA-characters in the usual sense. Standard arguments in linear algebra imply the identity

$$\sum_p (-1)^p \Theta_{MA}(\Lambda^p \mathfrak{n}) = \det(1 - \mathrm{Ad}|_{\mathfrak{n}}); \tag{3.3}$$

in particular, the alternating sum (3.3) vanishes nowhere on $MA \cap G'$. The system of positive restricted roots (2.27) cuts out a negative Weyl chamber

$$A^- = \{a \in A \mid e^{\alpha}(a) < 1 \quad \text{for all } \alpha \in \Phi^+(\mathfrak{g}, \mathfrak{a})\} \tag{3.4}$$

in A. In terms of A^- and the function (3.3), we define

$$(MA)^- = \text{interior, in } MA, \text{ of the set}$$
$$\left\{ g \in MA \;\middle|\; \sum_p (-1)^p \Theta_{MA}(\Lambda^p \mathfrak{n})(ga) \geq 0 \quad \text{for all } a \in A^- \right\}. \tag{3.5}$$

The main result of this paper can now be stated as follows:

THEOEREM 3.6. *For every Harish-Chandra module V,*

$$\Theta_G(V)|_{(MA)^- \cap G'} = \left. \frac{\sum_p (-1)^p \Theta_{MA}(H_p(\mathfrak{n}, V))}{\sum_p (-1)^p \Theta_{MA}(\Lambda^p \mathfrak{n})} \right|_{(MA)^- \cap G'}$$

Two properties of the set $(MA)^-$ will be important in various applications; since they play no role in the present section, we postpone their verification until § 5:

$$\text{every } g\in MA \text{ which is semisimple and regular in } G \text{ is conjugate, under } G\text{, to an element of } (MA)^-; \tag{3.7}$$

and

$$\text{the projection } MA\to M \text{ maps } (MA)^- \text{ onto } M. \tag{3.8}$$

The character of a Harish-Chandra module for MA can be expressed as a sum of M-characters, multiplied by one dimensional characters of A. Hence

$$\text{every } MA\text{-character is completely determined by its restriction to } (MA)^-\cap G', \tag{3.9}$$

as can be deduced from (3.8)

In order to simplify various formulas, we abbreviate the right hand side of the identity (3.6),

$$\Theta_{\mathfrak{n}}(V)=\frac{\sum_p(-1)^p\,\Theta_{MA}(H_p(\mathfrak{n},V))}{\sum_p(-1)^p\,\Theta_{MA}(\Lambda^p\mathfrak{n})}. \tag{3.10}$$

Applying the usual Euler characteristic arguments to the long exact homology sequence, one finds:

LEMMA 3.11. *If* $0\to V'\to V\to V''\to 0$ *is an exact sequence of Harish-Chandra modules*, $\Theta_{\mathfrak{n}}(V)=\Theta_{\mathfrak{n}}(V')+\Theta_{\mathfrak{n}}(V'')$.

This shows, in particular, that $\Theta_{\mathfrak{n}}(V)$ depends only on the composition factors of V, and not on how they are put together. In other words, we may regard $\Theta_{\mathfrak{n}}(V)$ as an invariant of the character $\Theta_G(V)$. By a virtual character, we shall mean a finite, integral linear combination of characters of irreducible Harish-Chandra modules. Since

$$\Theta_G(V)\mapsto\Theta_{\mathfrak{n}}(V)$$

is an additive function, it extends to a $\mathbf{Z}$-linear mapping

$$\Theta\mapsto\Theta_{\mathfrak{n}}, \tag{3.12}$$

from the **Z**-module of virtual G-characters to the **Z**-module of virtual MA-characters. In terms of this notation, theorem 3.6 translates into the identity

$$\Theta|_{(MA)^{-}\cap G'} = \Theta_{\mathfrak{n}}|_{(MA)^{-}\cap G'}, \tag{3.13}$$

for all virtual G-characters Θ.

We shall have to recall Harish-Chandra's enumeration of the characters of $Z(\mathfrak{g})$. For this purpose, we pick a Cartan subalgebra $\mathfrak{b}\subset\mathfrak{g}$ and a maximal nilpotent subalgebra $\mathfrak{r}$, such that

$$\mathfrak{b}\oplus\mathfrak{r} \quad \text{(semidirect product)} \tag{3.14}$$

is a Borel subalgebra of $\mathfrak{g}$. The choice of $\mathfrak{r}$ corresponds to the choice of a system of positive roots

$$\Phi^{+}(\mathfrak{g},\mathfrak{b})\subset\Phi(\mathfrak{g},\mathfrak{b}) \tag{3.15}$$

($\Phi(\mathfrak{g},\mathfrak{b})$=root system of the pair $(\mathfrak{g},\mathfrak{b})$), whose half-sum we denote by ϱ:

$$\varrho=\frac{1}{2}\sum_{\alpha\in\Phi^{+}(\mathfrak{g},\mathfrak{b})}\alpha. \tag{3.16}$$

As can be deduced from the Poincaré-Birkhoff-Witt theorem, for each $Z\in Z(\mathfrak{g})$, there exists a unique $\gamma'_{\mathfrak{g}}(Z)\in U(\mathfrak{b})$ satisfying the congruence

$$Z\equiv\gamma'_{\mathfrak{g}}(Z) \mod \mathfrak{r}U(\mathfrak{g}); \tag{3.17}$$

moreover,

$$\gamma'_{\mathfrak{g}}: Z(\mathfrak{g})\to U(\mathfrak{b}) \tag{3.18}$$

is an algebra homomorphism—the Harish-Chandra homomorphism without normalization. Since $\mathfrak{b}$ is Abelian, we may identify $U(\mathfrak{b})$ with the algebra of polynomial functions on the dual space $\mathfrak{b}^*$. In particular, evaluation at any given $\lambda\in\mathfrak{b}^*$ defines a character of $U(\mathfrak{b})$. The translation

$$X\mapsto X+X(\varrho),\quad X\in\mathfrak{b},$$

extends to an automorphism

$$\gamma''_{\mathfrak{g}}: U(\mathfrak{b})\to U(\mathfrak{b}). \tag{3.19}$$

According to a fundamental result of Harish-Chandra [16],

$$\gamma_{\mathfrak{g}} = \gamma''_{\mathfrak{g}} \cdot \gamma'_{\mathfrak{g}} \text{ maps } Z(\mathfrak{g}) \text{ isomorphically onto the algebra of } W(\mathfrak{g}, \mathfrak{b})\text{-invariants in } U(\mathfrak{b}) \tag{3.20}$$

($W(\mathfrak{g}, \mathfrak{b})$=Weyl group of $\mathfrak{b}$ in $\mathfrak{g}$). Dually, every character of $Z(\mathfrak{g})$ is of the form

$$\chi_{\mathfrak{g},\lambda}: Z(\mathfrak{g}) \to \mathbf{C}, \quad \text{with } \chi_{\mathfrak{g},\lambda}(Z) = \gamma_{\mathfrak{g}}(Z)(\lambda), \tag{3.21}$$

for some $\lambda \in \mathfrak{b}^*$, and any two of these coincide precisely when their parameters lie in the same $W(\mathfrak{g}, \mathfrak{b})$-orbit; this is Harish-Chandra's enumeration of the characters of $Z(\mathfrak{g})$. It should be noted that the definitions of $\gamma_{\mathfrak{g}}$ and $\chi_{\mathfrak{g},\lambda}$ do not depend on the choice of $\mathfrak{r}$, nor even on the choice of $\mathfrak{b}$: any other Cartan subalgebra is conjugate to $\mathfrak{b}$ under an inner automorphism of $\mathfrak{g}$, and because of (3.20), such an inner automorphism relates the homomorphisms $\gamma_{\mathfrak{g}}$ and the characters $\chi_{\mathfrak{g},\lambda}$ corresponding to the two Cartan subalgebras.

The preceeding discussion applies equally to $\mathfrak{m} \oplus \mathfrak{a}$. In order to describe the effect of the homomorphism (2.19), we assume, as we may,

$$\mathfrak{b} \subset \mathfrak{m} \oplus \mathfrak{a}, \quad \text{and} \quad \mathfrak{n} \subset \mathfrak{r} \subset \mathfrak{p}. \tag{3.22}$$

The positive root system (3.15) is then compatible with the system of positive restricted roots (2.27), in the sense that

$$\alpha \in \Phi^+(\mathfrak{g}, \mathfrak{b}),\ \alpha|_{\mathfrak{a}} \neq 0 \ \Rightarrow\ \alpha|_{\mathfrak{a}} \in \Phi^+(\mathfrak{g}, \mathfrak{a}). \tag{3.23}$$

By intersecting $\Phi^+(\mathfrak{g}, \mathfrak{b})$ with

$$\Phi(\mathfrak{m} \oplus \mathfrak{a}, \mathfrak{b}) = \{\alpha \in \Phi(\mathfrak{g}, \mathfrak{b}) \mid \alpha|_{\mathfrak{a}} = 0\} \tag{3.24}$$

(=root system of $(\mathfrak{m}\oplus\mathfrak{a}, \mathfrak{b})$), one obtains a system of positive roots

$$\Phi^+(\mathfrak{m} \oplus \mathfrak{a}, \mathfrak{b}) \subset \Phi(\mathfrak{m} \oplus \mathfrak{a}, \mathfrak{b}), \tag{3.25}$$

which corresponds to the Borel subalgebra $\mathfrak{b}\oplus((\mathfrak{m}\oplus\mathfrak{a}) \cap \mathfrak{r}) \subset \mathfrak{m}\oplus\mathfrak{a}$. We define

$$\varrho_{\mathfrak{m}} = \text{half-sum of all } \alpha \in \Phi^+(\mathfrak{m} \oplus \mathfrak{a}, \mathfrak{b}); \tag{3.26}$$

then

$$(\varrho - \varrho_{\mathfrak{m}})|_{\mathfrak{a}} = \varrho|_{\mathfrak{a}} = \varrho_P \tag{3.27}$$

(cf. (2.28)). If we let $\mathfrak{m}\oplus\mathfrak{a}$, $\mathfrak{r}\cap\mathfrak{m}$, $\varrho_{\mathfrak{m}}$ play the roles of, respectively, $\mathfrak{g}, \mathfrak{r}, \varrho$, the identities (3.17–21) describe homomorphisms

$$\begin{aligned} &\gamma'_{\mathfrak{m}\oplus\mathfrak{a}}: Z(\mathfrak{m}\oplus\mathfrak{a})\to U(\mathfrak{b}),\\ &\gamma''_{\mathfrak{m}\oplus\mathfrak{a}}: U(\mathfrak{b})\xrightarrow{\sim} U(\mathfrak{b}),\\ &\gamma_{\mathfrak{m}\oplus\mathfrak{a}}=\gamma''_{\mathfrak{m}\oplus\mathfrak{a}}\circ\gamma'_{\mathfrak{m}\oplus\mathfrak{a}}: Z(\mathfrak{m}\oplus\mathfrak{a})\to U(\mathfrak{b}) \end{aligned} \tag{3.28}$$

and characters

$$\chi_{\mathfrak{m}\oplus\mathfrak{a},\lambda}: Z(\mathfrak{m}\oplus\mathfrak{a})\to \mathbf{C} \quad (\lambda\in\mathfrak{b}^*). \tag{3.29}$$

The image of $\gamma_{\mathfrak{m}\oplus\mathfrak{a}}$ consists of all $W(\mathfrak{m}\oplus\mathfrak{a},\mathfrak{b})$-invariants in $U(\mathfrak{b})$, and

$$\lambda\mapsto\chi_{\mathfrak{m}\oplus\mathfrak{a},\lambda}$$

sets up a bijection between $\mathfrak{b}^*/W(\mathfrak{m}\oplus\mathfrak{a},\mathfrak{b})$ and the set of characters of $Z(\mathfrak{m}\oplus\mathfrak{a})$.

A comparison of (2.20) and (3.17) leads to the equality

$$\gamma'_{\mathfrak{m}\oplus\mathfrak{a}}\circ\gamma'_P=\gamma'_{\mathfrak{g}}. \tag{3.30}$$

This in turn implies

$$\chi_{\mathfrak{g},\lambda}=\chi_{\mathfrak{m}\oplus\mathfrak{a},\lambda+\varrho-\varrho_{\mathfrak{m}}}\circ\gamma'_P. \tag{3.31}$$

for $\lambda\in\mathfrak{b}^*$. In particular, the ideal which $\gamma'_P(\operatorname{Ker}\chi_{\mathfrak{g},\lambda})$ generates in $Z(\mathfrak{m}\oplus\mathfrak{a})$ lies in kernel of each of the characters $\chi_{\mathfrak{m}\oplus\mathfrak{a},\mu}$, with $\mu=w\lambda+\varrho-\varrho_{\mathfrak{m}}$ for some $w\in W(\mathfrak{g},\mathfrak{b})$. Because of (3.20), no other character vanishes on this ideal. We also note that the character $\chi_{\mathfrak{m}\oplus\mathfrak{a},\mu}$, restricted to $U(\mathfrak{a})\subset Z(\mathfrak{m}\oplus\mathfrak{a})$, equals evaluation at $\mu|_{\mathfrak{a}}$, and we recall (3.27). Hence Casselman-Osborne's lemma 2.21 has the following

COROLLARY 3.32. *Let V be a Harish-Chandra module for G on which $Z(\mathfrak{g})$ acts according to $\chi_{\mathfrak{g},\lambda}$. Then* $Z(\mathfrak{m}\oplus\mathfrak{a})$ *acts on any irreducible constituent of $H_*(\mathfrak{n},V)$ according to a character $\chi_{\mathfrak{m}\oplus\mathfrak{a},\mu}$ indexed by $\mu=w\lambda+\varrho-\varrho_{\mathfrak{m}}$, for some $w\in W(\mathfrak{g},\mathfrak{b})$. Moreover, the decomposition* (2.29) *satisfies*

$$H_p(\mathfrak{n},V)=\bigoplus_{\nu=w\lambda|_{\mathfrak{a}}} H_p(\mathfrak{n},V)_\nu,$$

with w running over $W(\mathfrak{g},\mathfrak{b})$.

We now turn to the notion of coherent continuation. The $\mathbf{Z}$-module of virtual characters of Harish-Chandra modules, which we denote by $\mathscr{C}$, breaks up as a direct sum

$$\mathscr{C} = \bigoplus_{\lambda \in \mathfrak{b}^*/W(\mathfrak{g},\mathfrak{b})} \mathscr{C}_\lambda, \quad \text{with } \mathscr{C}_\lambda = \{\Theta \in \mathscr{C} \mid Z(\mathfrak{g}) \text{ acts on } \Theta \text{ by } \chi_{\mathfrak{g},\lambda}\}. \tag{3.33}$$

For future reference, we note that the projections

$$p_\lambda: \mathscr{C} \to \mathscr{C}_\lambda \tag{3.34}$$

map true characters to true characters, or to zero. If V is a Harish-Chandra module and F a finite dimensional G-module, the tensor product $V \otimes F$ again has the defining properties (2.2) of a Harish-Chandra module, and its character equals the product of the characters of V and F. In this manner, $\mathscr{C}$ becomes a module over the ring of finite dimensional virtual characters [43]. To avoid technical complications arising from the possible disconnectedness of G, we restrict our attention to the subring

$$\mathscr{F} = \text{ring of finite dimensional virtual characters of the adjoint group of } \mathfrak{g}, \tag{3.35}$$

which we pull back to G via the adjoint homomorphism.(1) This will result in a slightly weaker, but entirely adequate notion of coherent continuation. Henceforth we assume that the Cartan subalgebra $\mathfrak{b}$ arises as the complexified Lie algebra of a Cartan subgroup B, but we drop the condition (3.22), since it no longer plays a role. Every element μ of the weight lattice for the adjoint group,

$$\Lambda \subset \mathfrak{b}^*, \tag{3.36}$$

lifts to a character e^μ of B. The restriction to B of a finite-dimensional virtual character $\varphi \in \mathscr{F}$ can be expressed as

$$\varphi|_B = \sum_{\mu \in \Lambda} n_\mu e^\mu, \tag{3.37}$$

with n_μ=multiplicity of μ.

We shall say that a family of virtual characters $\{\Theta_\lambda \mid \lambda \in \Lambda + \lambda_0\}$, indexed by the Λ-translates of some $\lambda_0 \in \mathfrak{b}^*$, depends coherently on the parameter λ if

$$\begin{aligned} &\text{(a) } \Theta_\lambda \in \mathscr{C}_\lambda, \text{ and} \\ &\text{(b) } \varphi\Theta_\lambda = \textstyle\sum_{\mu \in \Lambda} n_\mu \Theta_{\lambda+\mu}, \end{aligned} \tag{3.38}$$

(1) The adjoint homomorphism takes values in the complex adjoint group because of (2.1 c).

for every $\lambda \in \Lambda+\lambda_0$ and every $\varphi \in \mathscr{F}$, as in (3.37) [35]. The choice of B does not affect this definition: if B_1 is a second Cartan subgroup of G, with complexified Lie algebra $\mathfrak{b}_1$, any particular inner automorphism of $\mathfrak{g}$ which maps $\mathfrak{b}$ to $\mathfrak{b}_1$ can be used to transfer the parametrization from $\mathfrak{b}^*$ to $\mathfrak{b}_1^*$; the reparametrized family is then coherent whenever the original family is.

Characters can be put into coherent families. To make this precise, we fix $\lambda_0 \in \mathfrak{b}^*$ and a virtual character $\Theta_0 \in \mathscr{C}_{\lambda_0}$, and we define W_0 = isotropy subgroup of $W(\mathfrak{g}, \mathfrak{b})$ at λ_0, N_0=order of W_0. Then

LEMMA 3.39. *There exists a unique coherently parametrized family* $\{\Theta_\lambda \mid \lambda \in \Lambda+\lambda_0\}$, *such that*

(i) $\Theta_{\lambda_0} = N_0 \Theta_0$,

(ii) $\Theta_{w\lambda} = \Theta_\lambda$, *for* $\lambda \in \Lambda+\lambda_0$, $w \in W_0$.

The condition (ii) serves the purpose of making the family unique. Without it, the integers N_0 in (i) can be replaced by one [37]—this is a more subtle result, and will not be needed here. If λ_0 is non-singular, (ii) becomes vacuous and $N_0=1$. In particular, a coherent family is completely determined by any one of its members with a non-singular parameter λ.

To simplify matters, we shall prove lemma 3.39, simultaneously with a related statement, which describes the notion of coherent continuation in more concrete terms. For the moment, we consider a single virtual character $\Theta_\lambda \in \mathscr{C}_\lambda$. After replacing B by a finite covering, if necessary, we can introduce a Weyl denominator on B by the formula

$$\prod_{\alpha>0} (e^{\alpha/2}-e^{-\alpha/2}); \tag{3.40}$$

the product extends over the positive roots in $\Phi(\mathfrak{g}, \mathfrak{b})$. Near any $b \in B \cap G'$, the product of Θ_λ with the Weyl denominator can be expressed as an exponential sum,

$$\left\{\Theta_\lambda \prod_{\alpha>0} (e^{\alpha/2}-e^{-\alpha/2})\right\}(b \exp X) = \sum_{w \in W(\mathfrak{g}, \mathfrak{b})} c_\lambda(b, w)\, e^{\langle w\lambda, X\rangle} \tag{3.41}$$

(Harish-Chandra [16]); here X ranges over a neighborhood of 0 in the real Lie algebra $\mathfrak{b}_0$ of B, and the $c_\lambda(b, w)$ are complex constants.(2) We now let Θ_λ denote the general

(2) Harish-Chandra [16] derives the identity (3.41) for invariant eigendistributions. In this more general situation, the $c_\lambda(b, w)$ may be polynomial functions of X whenever λ is singular. A simple tensoring argument of Fomin-Shapovalov [12], which is also implicit in our proof of lemma 3.39, excludes that possibility for virtual characters.

member of a family of virtual characters, subject to the condition (3.38 a), and thus obtain identities (3.41) for each value of the parameter λ. It should be observed that the constants $c_\lambda(b, w)$ are uniquely determined by Θ_λ only for nonsingular parameters λ, i.e., when the isotropy group

$$W_\lambda = \{w \in W(\mathfrak{g}, \mathfrak{b}) | w\lambda = \lambda\} \tag{3.42}$$

reduces to the identity. In general, the average

$$\frac{1}{\# W_\lambda} \sum_{v \in W_\lambda} c_\lambda(b, wv) \tag{3.43}$$

has canonical meaning.

LEMMA 3.44. *If the family $\{\Theta_\lambda | \lambda \in \Lambda + \lambda_0\}$ is coherently parametrized, the constants $c_\lambda(b, w)$ can be chosen so as to satisfy*

$$c_{\lambda+\mu}(b, w) = e^{w\mu}(b)\, c_\lambda(b, w),$$

for all $\lambda \in \Lambda + \lambda_0$, $\mu \in \Lambda$, $w \in W(\mathfrak{g}, \mathfrak{b})$ and all $b \in B \cap G'$. Conversely, if these relations hold for every(3) *Cartan subgroup $B \subset G$, the family is coherently parametrized.*

Proof of lemmas 3.39 *and* 3.44. To estabish the second half of lemma 3.44, we multiply the identity (3.41) with the virtual character φ:

$$\begin{aligned}\left\{\varphi\Theta_\lambda \prod_{\alpha>0} (e^{\alpha/2} - e^{-\alpha/2})\right\}(b \exp X) &= \sum_{\mu \in \Lambda} n_\mu \sum_{w \in W(\mathfrak{g}, \mathfrak{b})} e^{\mu}(b)\, c_\lambda(b, w)\, e^{\langle \mu + w\lambda, X\rangle} \\ &= \sum_{\mu \in \Lambda} n_\mu \sum_{w \in W(\mathfrak{g}, \mathfrak{b})} e^{w\mu}(b)\, c_\lambda(b, w)\, e^{\langle w(\lambda+\mu), X\rangle}\end{aligned} \tag{3.45}$$

(cf. (3.37); φ is $W(\mathfrak{g}, \mathfrak{b})$-invariant!). Because of the hypothesis about the $c_\lambda(b, w)$, this equals

$$\sum_{\mu} n_\mu \sum_{w \in W(\mathfrak{g}, \mathfrak{b})} c_{\lambda+\mu}(b, w)\, e^{\langle w(\lambda+\mu), X\rangle} = \sum_{\mu} n_\mu \left\{\Theta_{\lambda+\mu} \prod_{\alpha>0} (e^{\alpha/2} - e^{-\alpha/2})\right\}(b \exp X). \tag{3.46}$$

The coherence of the family follows, since B and $b \in B \cap G'$ were arbitrary.

The construction of the family whose existence is asserted by lemma 3.39 begins

(3) To state the conditions for some other Cartan subgroup B_1, one must transfer the parametrization to $\mathfrak{b}_1^*$, as described above.

with $\Theta_{\lambda_0}=N_0\Theta_0$. We pick constants $c_{\lambda_0}(b, w)$ in accordance with (3.41); to make them unique, we impose the symmetry

$$c_{\lambda_0}(b, wv) = c_{\lambda_0}(b, w), \quad \text{if } v \in W_0. \tag{3.47}$$

Next, we choose a weight $\mu \in \Lambda$ which is "highly dominant" in comparison to λ_0, in the sense that

$$(\mu, \alpha) >> |\lambda_0, \alpha)| \quad \text{for } \alpha \in \Phi^+(\mathfrak{g}, \mathfrak{b}),$$

and we define

$$\lambda = \lambda_0 + \mu.$$

In this situation,

$$\begin{gathered}\lambda_0+v\mu, \text{ with } v \in W(\mathfrak{g}, \mathfrak{b}), \text{ is } W(\mathfrak{g}, \mathfrak{b})\text{-conjugate to } \lambda \\ \text{if, and only if, } v \in W_0.\end{gathered} \tag{3.48}$$

Indeed, $\lambda_0+v\mu$ and $w\lambda=w(\lambda_0+\mu)$ are close to, respectively, $v\mu$ and $w\mu$—close relative to the size of λ_0. Any two distinct conjugates of μ lie far away from each other: $w\lambda$ cannot equal $\lambda_0+v\mu$ unless $w=v\in W_0$; the converse is immediate. As a $W(\mathfrak{g}, \mathfrak{b})$-invariant sum of characters of B,

$$\psi = \sum_{v \in W(\mathfrak{g}, \mathfrak{b})} e^{v\mu}$$

extends to a virtual character on G. Using the properties of the family that is to be constructed, one finds

$$\begin{aligned} p_\lambda(\psi\Theta_0) &= p_\lambda\left(\frac{1}{N_0}\psi\Theta_{\lambda_0}\right) \\ &= p_\lambda\left(\frac{1}{N_0}\sum_{v\in W(\mathfrak{g},\mathfrak{b})}\Theta_{\lambda_0+v\mu}\right) \\ &= p_\lambda\left(\frac{1}{N_0}\sum_{v\in W_0}\Theta_{\lambda_0+v\mu}\right) = \Theta_\lambda; \end{aligned} \tag{3.49}$$

the penultimate step depends on (3.48), coupled with the fact that $p_\lambda(\mathscr{C}_{\lambda'})=0$ whenever λ and λ' are non-conjugate, and the last equality is a consequence of the symmetry condition $\Theta_{v\lambda}=\Theta_\lambda$ for $v\in W_0$. Thus we are forced to define

$$\Theta_\lambda = p_\lambda(\psi\Theta_0).$$

To calculate the constants $c_\lambda(b, w)$ corresponding to our choice of λ, we apply (3.45) with λ_0 in place of λ and ψ in place of φ:

$$\left\{\psi\Theta_0\prod_{\alpha>0}(e^{\alpha/2}-e^{-\alpha/2})\right\}(b\exp X)=\frac{1}{N_0}\sum_{v,w\in W(\mathfrak{g},\mathfrak{b})} e^{wv\mu}(b)\,c_{\lambda_0}(b,w)\,e^{\langle w(\lambda_0+v\mu),X\rangle}$$
$$=\frac{1}{N_0}\sum_{w\in W(\mathfrak{g},\mathfrak{b})}\sum_{v\in W_0} e^{wv\mu}(b)\,c_{\lambda_0}(b,w)\,e^{\langle wv\lambda,X\rangle} \tag{3.50}$$
$$+\frac{1}{N_0}\sum_{w\in W(\mathfrak{g},\mathfrak{b})}\sum_{v\notin W_0} e^{wv\mu}(b)\,c_{\lambda_0}(b,w)\,e^{\langle w(\lambda_0+v\mu),X\rangle}.$$

In view of (3.41) and (3.48), the terms in the first of these two expressions match up with the character $\chi_{\mathfrak{g},\lambda}$ of $Z(\mathfrak{g})$, whereas those in the second sum belong to other characters. The symmetry (3.47) allows us to omit the sum over $v\in W_0$, in exchange for the N_0 in the denominator. Hence

$$\left\{p_\lambda(\psi\Theta_0)\prod_{\alpha>0}(e^{\alpha/2}-e^{-\alpha/2})\right\}(b\exp X)=\sum_{w\in W(\mathfrak{g},\mathfrak{b})} e^{w\mu}(b)\,c_{\lambda_0}(b,w)\,e^{\langle w\lambda,X\rangle}, \tag{3.51}$$

or equivalently,

$$c_\lambda(b,w)=e^{w\mu}(b)\,c_{\lambda_0}(b,w)=e^{w(\lambda-\lambda_0)}c_{\lambda_0}(b,w).$$

We turn our attention to an arbitrary parameter $\lambda\in\Lambda+\lambda_0$, and we pick a weight $\mu\in\Lambda$ which is "highly dominant" in comparison to both λ and λ_0. In the preceeding discussion, $\lambda_1=\lambda+\mu$ and $\mu_1=\lambda_1-\lambda_0$ can play the roles of λ and μ. Thus Θ_{λ_1} and the constants

$$c_{\lambda_1}(b,w)=e^{w(\lambda_1-\lambda_0)}(b)\,c_{\lambda_0}(b,w) \tag{3.52}$$

are already known. We use these to define Θ_λ and to calculate the $c_\lambda(b,w)$. Since the arguments closely parallel those in the previous case, we shall not go into detail. In analogy to (3.48),

$$\lambda_1-v\mu \text{ is conjugate to } \lambda \text{ only if } v=1. \tag{3.53}$$

The function

$$\psi^*=\sum_{v\in W(\mathfrak{g},\mathfrak{b})} e^{-v\mu}$$

extends to a virtual character $\psi^* \in \mathscr{F}$. Calculations based on (3.53), and similar to (3.49–51), show that we must define

$$\Theta_\lambda = p_\lambda(\psi^* \Theta_{\lambda_1}), \tag{3.54}$$

and this allows us to conclude

$$c_\lambda(b, w) = e^{-w\mu}(b)\, c_{\lambda_1}(b, w) = e^{w(\lambda-\lambda_0)}(b)\, c_{\lambda_0}(b, w); \tag{3.55}$$

the second equality is a consequence of (3.52). The definition (3.54) seems to depend on the choice of λ_1, but (3.55) ensures that this is not the case: the $c_\lambda(b, w)$, for all possible B and $b \in B \cap G'$, determine Θ_λ independently of λ_1.

If $\mu \in \Lambda$ is an arbitrary weight, (3.55) and the analogous formula for $\lambda+\mu$ give

$$c_{\lambda+\mu}(b, w) = e^{w\mu}(b)\, c_\lambda(b, w). \tag{3.56}$$

According to the second half of (3.44)—which has already been verified—the Θ_λ constitute a coherent family. We substitute $v\lambda$, with $v \in W_0$, for λ in (3.55) and appeal to (3.47), to find

$$c_{v\lambda}(b, w) = c_\lambda(b, wv),$$

and this in turn implies the symmetry

$$\Theta_{v\lambda} = \Theta_\lambda, \quad \text{if } v \in W_0.$$

The uniqueness of the family $\{\Theta_\lambda | \lambda \in \Lambda + \lambda_0\}$ is implicit in its construction, since (3.54) was forced on us. Because of the uniqueness, any coherent family may be viewed as arising from our construction; thus we have established (3.56) in general. The proof of lemmas 3.39 and 3.44 is now complete.

Very roughly speaking, the identity (3.13) is compatible with coherent continuation. What keeps this from being true in the technical sense is the fact that the characters of $Z(\mathfrak{g})$ and $Z(\mathfrak{m} \oplus \mathfrak{a})$ have different parametrizations. To deal with this problem we consider the local formulas for Θ and $\Theta_\mathfrak{n}$ on a particular Cartan subgroup $B \subset MA$. According to lemma 3.39, Θ can be inserted into a coherently parametrized family

$$\{\Theta_\lambda | \lambda \in \Lambda + \lambda_0\}, \tag{3.57}$$

at least if we replace Θ by a suitable multiple—as we may, without affecting the identity (3.13). The family (3.57) has local representations (3.41), near any $b \in B \cap G'$, with coefficients $c_\lambda(b, w)$ that satisfy the identities in lemma 3.44. Since we shall have to relate the actions of $Z(\mathfrak{g})$ and $Z(\mathfrak{m}\oplus\mathfrak{a})$, we again insist on (3.22). Each irreducible MA-character has local expressions analogous to (3.41), with a different Weyl denominator, of course, and with exponential terms corresponding to a $W(\mathfrak{m}\oplus\mathfrak{a}, \mathfrak{b})$-orbit in $\mathfrak{b}^*$. Because of corollary 3.32, if the character in question contributes to $(\Theta_\lambda)_\mathfrak{n}$, the orbit of potential exponents lies in the $W(\mathfrak{g}, \mathfrak{b})$-orbit of λ, translated by $\varrho-\varrho_\mathfrak{m}$. The quotient of the two Weyl denominators equals

$$\frac{\prod_{\alpha\in\Phi^+(\mathfrak{g},\mathfrak{b})} (e^{\alpha/2}-e^{-\alpha/2})}{\prod_{\alpha\in\Phi^+(\mathfrak{m}\oplus\mathfrak{a},\mathfrak{b})} (e^{\alpha/2}-e^{-\alpha/2})} = \pm e^{\varrho-\varrho_\mathfrak{m}} \prod_{\alpha\in\Phi^+(\mathfrak{g},\mathfrak{b}),\ \alpha\notin\Phi(\mathfrak{m}\oplus\mathfrak{a},\mathfrak{b})} (1-e^\alpha),$$

which—except for the sign and the exponential factor—is the denominator in the definition (3.10) of $(\Theta_\lambda)_\mathfrak{n}$; cf. (3.3). Combining this with the preceeding observations, one obtains local formulas for the product of $(\Theta_\lambda)_\mathfrak{n}$ with the Weyl denominator for G, of exactly the same appearance as (3.41):

$$\left\{(\Theta_\lambda)_\mathfrak{n} \prod_{\alpha>0} (e^{\alpha/2}-e^{-\alpha/2})\right\}(b \exp X) = \sum_{w\in W(\mathfrak{g},\mathfrak{b})} \tilde{c}_\lambda(b, w)\, e^{\langle w\lambda, X\rangle}. \tag{3.58}$$

The definition of coherent continuation is not limited to families of characters. It makes sense much more generally, for families of functions defined on open subsets of a Cartan subgroup B, and indexed by translates of the lattice Λ, provided they can be expressed locally as exponential sums of the form (3.58). Both statements and proofs of lemmas 3.39 and 3.44 apply without change in this wider context.

LEMMA 3.59. *The homomorphism $\Theta \mapsto \Theta_\mathfrak{n}$ is linear over $\mathcal{F}$.*

Proof. We consider characters Θ, φ, of a Harish-Chandra module V and a finite dimenisional G-module F, respectively. The latter has a P-invariant filtration

$$0 \subset F_1 \subset F_2 \subset \ldots \subset F_N = F$$

with irreducible quotients F_k/F_{k-1}, on which $\mathfrak{n}$ necessarily acts trivially. Hence

$$H_p(\mathfrak{n}, V\otimes(F_k/F_{k-1})) \simeq H_p(\mathfrak{n}, V)\otimes(F_k/F_{k-1}),$$

as Harish-Chandra modules for MA. For the purpose of calculating Euler characteristics of the $\mathfrak{n}$-homology groups, one may treat the short exact sequences

$$0 \to V \otimes F_{k-1} \to V \otimes F_k \to V \otimes F_k/F_{k-1} \to 0$$

as if they split. Hence $(\varphi\Theta)_{\mathfrak{n}} = \varphi\Theta_{\mathfrak{n}}$, which was to be shown.

The lemma ensures the coherence of the family of functions (3.58). Thus we may appeal to lemma 3.44, to conclude:

$$\tilde{c}_{\lambda+\mu}(b, w) = e^{w\mu}(b)\,\tilde{c}_\lambda(b, w), \tag{3.60}$$

for all $\lambda \in \Lambda + \lambda_0$, $\mu \in \Lambda$, $w \in W(\mathfrak{g}, \mathfrak{b})$, and $b \in B \cap G'$. To prove Osborne's conjecture (3.13) for the virtual characters Θ_λ in our family amounts to proving the identities

$$c_\lambda(b, w) = \tilde{c}_\lambda(b, w), \quad \text{for } b \in (MA)^- \cap B \cap G', \tag{3.61}$$

corresponding to every choice of Cartan subgroup $B \subset MA$. Because of the relations (3.60) and their counterparts for the $c_\lambda(b, w)$, most of these identities are redundant. For example, if C is any particular positive constant, it suffices to check (3.61) whenever

$$\operatorname{Re}(w\lambda, \alpha) < -C, \quad \text{for all } \alpha \in \Phi^+(\mathfrak{g}, \mathfrak{b}). \tag{3.62}$$

We are also free to disregard values of λ near any finite number of hyperplanes, none of which contain Λ. Concretely, we may impose additional restrictions

$$|\operatorname{Re}(\lambda, \mu_i)| > C, \quad 1 \leq i \leq N,$$

with $\mu_1, \ldots, \mu_N \in \mathbf{R}$-linear span of Λ, $\mu_i \neq 0$. The family $\{\Theta_\lambda\}$ has now served its purpose, and we may think of Θ_λ as a single, arbitrary virtual character in $\mathscr{C}_\lambda$. Since $\mathscr{C}_\lambda$ depends only on the $W(\mathfrak{g}, \mathfrak{b})$-orbit of λ, we can arrange that $\operatorname{Re}(\lambda, \alpha) \leq 0$ if $\alpha \in \Phi^+(\mathfrak{g}, \mathfrak{b})$. This, in conjunction with (3.62), implies $\operatorname{Re}(\lambda, \alpha) < -C$ and $w = e$.

To summarize, it suffices to check the identities (3.61) in the following special situation:

(a) $\operatorname{Re}(\lambda, \alpha) < -C$ for $\alpha \in \Phi^+(\mathfrak{g}, \mathfrak{b})$;
(b) $|\operatorname{Re}(\lambda, \mu_i)| > C$, $1 \leq i \leq N$;
(c) Θ_λ is a virtual character on which $Z(\mathfrak{g})$ acts according to $\chi_{\mathfrak{g},\lambda}$; and
(d) $w = e$.
(3.63)

Here $\mu_1, \ldots, \mu_N$ are non-zero elements of the **R**-linear span of Λ, and C is a positive constant, all of which can be chosen at will. We recall that the positive root system $\Phi^+(\mathfrak{g}, \mathfrak{b})$ was assumed to be compatible with $\Phi^+(\mathfrak{g}, \mathfrak{a})$, in the sense of (3.23). Although we have stated the conditon (3.63) in terms of a particular Cartan subgroup $B \subset MA$, their meaning does not depend on B, since they take the same form when rephrased in terms of any other Cartan subgroup.

We conclude this section with some remarks on how the conditions (3.63) will be used. First supose $P_m = M_m A_m N_m$ is a minimal parabolic subgroup, contained in P, with

$$MA \supset M_m A_m, \quad A \subset A_m, \quad N \subset N_m, \tag{3.64}$$

and $B_m \subset M_m A_m$ a Cartan subgroup of G. Let $\Phi^+(\mathfrak{g}, \mathfrak{a}_m)$ be the system of positive restricted roots corresponding to P_m, and $\Phi^+(\mathfrak{g}, \mathfrak{b}_m)$ a positive root system compatible with $\Phi^+(\mathfrak{g}, \mathfrak{a}_m)$. We transfer λ to a linear function λ_m on $\mathfrak{b}_m$, via an inner automorphism of $\mathfrak{m} \oplus \mathfrak{a}$ which makes λ_m antidominant on $\mathfrak{b}_m \cap \mathfrak{m}$, in the sense that

$$\operatorname{Re}(\lambda_m, \alpha) \leq 0 \quad \text{if } \alpha \in \Phi(\mathfrak{m} \oplus \mathfrak{a}, \mathfrak{b}_m) \cap \Phi^+(\mathfrak{g}, \mathfrak{b}_m). \tag{3.65}$$

In this situation

$$\begin{aligned} &\text{(a) } \operatorname{Re}(\lambda_m, \alpha) < -C \text{ for } \alpha \in \Phi^+(\mathfrak{g}, \mathfrak{b}_m), \text{ and} \\ &\text{(b) } \operatorname{Re}(\lambda_m|_{\mathfrak{a}_m}, \alpha) < -C \text{ for } \alpha \in \Phi^+(\mathfrak{g}, \mathfrak{a}_m). \end{aligned} \tag{3.66}$$

The first inequality follows from (3.63a), once we know that $\operatorname{Re}(\lambda_m, \alpha) \leq 0$, for every $\alpha \in \Phi^+(\mathfrak{g}, \mathfrak{b}_m)$. This is true by construction if α vanishes on $\mathfrak{a}$. If not, (3.64) implies $\alpha|_{\mathfrak{a}} \in \Phi^+(\mathfrak{g}, \mathfrak{a})$. The inner automorphism of $\mathfrak{m} \oplus \mathfrak{a}$ that maps λ_m to λ sends α to a root $\beta \in \Phi(\mathfrak{g}, \mathfrak{b})$—necessarily a positive root: $\Phi^+(\mathfrak{g}, \mathfrak{b})$ is compatible with $\Phi^+(\mathfrak{g}, \mathfrak{a})$, and $\Phi^+(\mathfrak{g}, \mathfrak{a})$ contains $\beta|_{\mathfrak{a}} = \alpha|_{\mathfrak{a}}$. Hence

$$\operatorname{Re}(\lambda_m, \alpha) = \operatorname{Re}(\lambda, \beta) < -C \leq 0,$$

as required. To deduce the second inequality from the first, we extend α to a root $\beta \in \Phi(\mathfrak{g}, \mathfrak{b}_m)$. Both β and its complex conjugate $\bar{\beta}$ are positive, because they restrict to α on $\mathfrak{a}_m$, and $\beta + \bar{\beta}$ vanishes on $\mathfrak{b}_m \cap \mathfrak{m}$. We conclude:

$$\operatorname{Re}(\lambda_m|_{\mathfrak{a}_m}, \alpha) = \frac{1}{2} \operatorname{Re}(\lambda_m, \beta + \bar{\beta}) < -C.$$

The preceeding discussion applies in particular if P itself is a minimal parabolic subgroup, in which case we set $P_m=P$, $B_m=B$, $\lambda_m=\lambda$. Then, if μ denotes the restriction of λ to $\mathfrak{a}_m$,

$$\operatorname{Re}(\mu, \alpha) < -C \quad \text{for } \alpha \in \Phi^+(\mathfrak{g}, \mathfrak{a}_m). \tag{3.67}$$

Now let $\tilde{P}=\tilde{M}\tilde{A}\tilde{N}$ be an arbitrary parabolic subgroup, $\tilde{B}\subset\tilde{M}\tilde{A}$ a Cartan subgroup, and $\tilde{\lambda}\in\tilde{\mathfrak{b}}^*$ a conjugate λ. By enlarging the set $\{\mu_1, \ldots, \mu_N\}$, one can make

$$\nu = \text{restriction of } \tilde{\lambda} \text{ to } \tilde{\mathfrak{a}}$$

"very non-singular":

$$\operatorname{Re}(\nu, \alpha) \leq 0 \Rightarrow \operatorname{Re}(\nu, \alpha) < -C, \tag{3.68}$$

simultaneously for all choices of $\tilde{P}$ and $\alpha\in\Phi^+(\mathfrak{g}, \tilde{\mathfrak{a}})$. Since there are only finitely many parabolic subgroups and restricted roots, up to conjugacy, one can treat one $\tilde{P}$ and α at a time. When α is extended to a linear function on $\tilde{\mathfrak{b}}$, trivial on $\tilde{\mathfrak{b}}\cap\tilde{\mathfrak{m}}$, and transferred back to $\mathfrak{b}$, the resulting linear function $\mu_\alpha\in\mathfrak{b}^*$ lies in the $\mathbf{R}$-linear span of Λ and satisfies $(\nu, \alpha)=(\lambda, \mu_\alpha)$. Hence (3.68) follows from (3.63 b), as soon as μ_α is included among the μ_i.

§ 4. Asymptotic and $\mathfrak{n}$-homology

In this section, we shall review the Frobenius reciprocity theorem for Harish-Chandra modules and the asymptotic expansion of matrix coefficients—two circles of ideas which provide a link between $\mathfrak{n}$-homology and global characters.

To begin the discussion, we consider a parabolic subgroup $P=MAN$, whose Levi component MA is stable under the Cartan involution, and a Harish-Chandra module U for MA. As was remarked in § 2, U lifts to a global representation of MA: there exists a continuous representation σ on a Hilbert space U_σ, such that U can be identified, as an $\mathfrak{m}\oplus\mathfrak{a}$-module, with the space of $M\cap K$-finite vectors in U_σ. We continue σ to P by letting N act trivially. Every linear functional $\nu\in\mathfrak{a}^*$ exponentiates to a character of A, which then extends to a character

$$e^\nu : P \to \mathbf{C}^*, \tag{4.1}$$

with trivial action on M and N; this applies in particular to $\nu=\varrho_P$, as defined in (2.28).

By left translation, the group G operates on the space of C^∞ functions $f: G\to U_\sigma$ satisfying

$$f(gp) = e^{\varrho_P}(p^{-1})\,\sigma(p^{-1})\,f(g), \quad \text{for } g\in G,\ p\in P. \tag{4.2}$$

Since $KP=G$, each such f is completely determined by its restriction to K. We let $I_P^G(U_\sigma)$ denote the Hilbert space completion with respect to the inner product

$$(f_1, f_2) = \int_K (f_1(k), f_2(k))\, dk. \tag{4.3}$$

Equivalently,

$$\begin{gathered} I_P^G(U_\sigma) = \text{space of } L^2 \text{ functions } f\colon K \to U_\sigma, \\ \text{with } f(km) = \sigma^{-1}(m) f(k), \quad \text{for } k \in K,\ m \in K \cap M. \end{gathered} \tag{4.4}$$

The G-action extends to the completion, and hence defines a continuous representation $I_P^G(\sigma)$ of G on the Hilbert space (4.4). To make this completely explicit,

$$\begin{gathered} [I_P^G(\sigma)(g)f](k) = e^{\varrho_P}(p^{-1})\,\sigma(p^{-1}) f(k_1, \\ \text{whenever } g \in G,\ k \in K,\ k_1 \in K,\ p \in P \quad \text{and} \quad g^{-1}k = k_1 p. \end{gathered} \tag{4.5}$$

One shows, using global character theory, for example, that $I_P^G(\sigma)$ is an admissible representation of finite length, which makes

$$I_P^G(U) = \text{space of } K\text{-finite vectors in } I_P^G(U_\sigma) \tag{4.6}$$

a Harish-Chandra module for G—the Harish-Chandra module obtained by (normalized) induction of U from P to G. As the notation suggests, $I_P^G(U)$ depends only on U, not on the particular globalization σ. For future reference, we record:

$$\begin{gathered} \text{if } Z(\mathfrak{m} \oplus \mathfrak{a}) \text{ acts on } U \text{ via the character } \chi_{\mathfrak{m}\oplus\mathfrak{a},\lambda}, \\ \text{then } Z(\mathfrak{g}) \text{ acts on } I_P^G(U) \text{ via } \chi_{\mathfrak{g},\lambda}; \end{gathered} \tag{4.7}$$

this follows from (2.20) and (3.31), by differentiation of the identity (4.2).

If V_1, V_2 are Harish-Chandra modules for G, we let $\mathrm{Hom}_G(V_1, V_2)$ denote the space of linear maps commuting with the actions of $\mathfrak{g}$ and K. The subscript G—instead of $(\mathfrak{g}, K)$, for example—may appear unnatural; it is meant to convey the idea that Harish-Chandra modules are skeletons of global representations, and will also help us avoid cumbersome notation. For $\nu \in \mathfrak{a}^*$, we define

$$\begin{gathered} \mathbf{C}_\nu = \text{one-dimensional Harish-Chandra module corresponding} \\ \text{to the character } e^\nu \text{ of } MA. \end{gathered} \tag{4.8}$$

We can now state the Frobenius reciprocity theorem. Because of its importance for our purposes we shall also sketch the proof.

THEOREM 4.9 [7]. *Let U and V be Harish-Chandra modules for MA and G, respectively. Then* $\mathrm{Hom}_G(V, I_P^G(U)) \simeq \mathrm{Hom}_{MA}(H_0(\mathfrak{n}, V), U \otimes \mathbf{C}_{\varrho_P})$.

Proof. As a K-finite function, each $f \in I_P^G(U)$ is smooth and can be evaluated at the identity. Because of (4.2), the map $f \mapsto f(e) \otimes 1$ from $I_P^G(U)$ to $U \otimes \mathbf{C}_{\varrho_P}$ (1=generator of $\mathbf{C}_{\varrho_P}$) preserves the actions of $\mathfrak{m} \oplus \mathfrak{a}$, $K \cap M$ and $\mathfrak{n}$, hence descends to a homomorphism of Harish-Chandra modules

$$\varphi: H_0(\mathfrak{n}, I_P^G(U)) = I_P^G(U)/\mathfrak{n} I_P^G(U) \to U \otimes \mathbf{C}_{\varrho_P}.$$

Define $i: \mathrm{Hom}_G(V, I_P^G(U)) \to \mathrm{Hom}_{MA}(H_0(\mathfrak{n}, V), U \otimes \mathbf{C}_{\varrho_P})$ by $i(T) = \varphi \cdot T_*$, where $T_*: H_0(\mathfrak{n}, V) \to H_0(\mathfrak{n}, I_P^G(U))$ is the map induced by T. To invert i, we reconstruct T from $S = i(T)$ as follows:

$$(Tv)(k) = T(k^{-1}v)(e) = S(q(k^{-1}v)), \quad \text{for } v \in V, \tag{4.10}$$

with $q(k^{-1}v)$=image of $k^{-1}v$ in $H_0(\mathfrak{n}, V)$. The $M \cap K$-invariance of S ensures that Tv satisfies the transformation rule (4,4); no matter how $v \in V$ and $S \in \mathrm{Hom}_{MA}(H_0(\mathfrak{n}, V), U \otimes \mathbf{C}_{\varrho_P})$ are chosen, the formula (4.10) describes a vector $Tv \in I_P^G(U)$. It remains to be shown that $v \mapsto Tv$ commutes with the actions of $\mathfrak{g}$ and K. As for K, this follows directly from (4.10). The transformation rule (4.2) for $f = Tv$ on the one hand, and the $\mathfrak{m} \oplus \mathfrak{a}$-invariance of S on the other give $T(Xv)(e) = X(Tv)(e)$, if $X \in \mathfrak{p}$, and hence—with kv in place of v— $T(kXk^{-1}v)(k) = (kXk^{-1})(Tv)(k)$, for all $k \in K$. This completes the proof, since $\mathrm{Ad}\, k(\mathfrak{p})$ and $\mathfrak{k}$ generate $\mathfrak{g}$.

We shall usually induce modules of the form $W \otimes \mathbf{C}_\nu$, where W is a Harish-Chandra module for M and ν a linear function on $\mathfrak{a}$. In the decomposition (2.29) of $H_*(\mathfrak{n}, V)$, the normalizing shift by ϱ_P is built into the indexing. Hence, if V and W are Harish-Chandra modules for G and M,

$$\mathrm{Hom}_G(V, I_P^G(W \otimes \mathbf{C}_\nu)) \simeq \mathrm{Hom}_{MA}(H_0(\mathfrak{n}, V)_\nu, W \otimes \mathbf{C}_{\nu+\varrho_P}). \tag{4.11}$$

It was observed by Casselman [7] that the theory of asymptotic expansions gives information about $H_0(\mathfrak{n}, V)$, and therefore also about homomorphisms of V into induced representations. To see this, we lift V to a global representation π of G on a Hilbert space V_π. The contragredient representation π' on the dual Hilbert space V'_π is again continuous, admissible, and has finite length. We denote by V' the Harish-

Chandra module of K-finite vectors in V'_π. Equivalently, we may regard V' as a subspace of the algebraic dual V^* of V:

$$V' = \{v' \in V^* | v' \text{ is } K\text{-finite}\}; \tag{4.12}$$

in particular, V' depends only on V, and not on π. Since K-finite vectors are analytic [13], the "matrix coefficients"

$$f_{v,v'}(g) = \langle v', \pi(g)\, v\rangle \tag{4.13}$$

corresponding to vectors $v \in V$, $v' \in V'$ are real-analytic functions on G. Their Taylor series can be calculated in terms of the actions of $\mathfrak{g}$ and K—in other words, the functions $f_{v,v'}$ are completely determined by the Harish-Chandra module V.

Once and for all, we fix a minimal parabolic subgroup $P_m \subset G$, with Langlands factorization

$$P_m = M_m A_m N_m, \tag{4.14}$$

such that $M_m \subset K$ and $\mathfrak{a}_m \subset (-1)$-eigenspace of the Cartan involution ($\mathfrak{a}_m$ = complexified Lie algebra of A_m). We define $\Phi^+(\mathfrak{g}, \mathfrak{a}_m)$, A_m^- and the partial ordering $>$ of $\mathfrak{a}_m^*$ as in (2.27), (2.31) and (3.4); thus

$$\begin{gathered} A_m^- = \exp(\mathfrak{a}_{m,0})^-, \quad \text{where} \\ (\mathfrak{a}_{m,0})^- = \{X \in \mathfrak{a}_{m,0} | \langle \alpha, X\rangle < 0 \quad \text{for all } \alpha \in \Phi^+(\mathfrak{g}, \mathfrak{a}_m)\}. \end{gathered} \tag{4.15}$$

To simplify the notation, we shall write ϱ_m instead of ϱ_{P_m}. The growth or decay of matrix coefficients $f_{v,v'}$ is governed by their behavior on A_m^-: the left and right K-translates of $f_{v,v'}$ are functions of the same type, and KA_m^-K is dense in G. We now recall the main facts about the asymptotic expansions of matrix coefficients on A_m^-. They are due to Harish-Chandra; for a complete and unified exposition we refer the reader to [8].

THEOREM 4.16. *There exists a collection of polynomial functions $p^\nu_{v,v'}$ on $\mathfrak{a}_m$, indexed by $v \in V$, $v' \in V'$ and by ν, which ranges over a countable set $\mathscr{E}(V) \subset \mathfrak{a}_m^*$, such that*

$$f_{v,v'}(\exp X) = \sum_{\nu \in \mathscr{E}(V)} p^\nu_{v,v'}(X)\, e^{\langle \nu + \varrho_m, X\rangle}$$

for $X \in (\mathfrak{a}_{m,0})^-$. The product of this series with any character e^μ converges uniformly *and absolutely on any translate of* $(\mathfrak{a}_{m,0})^-$ *whose closure is wholly contained in* $(\mathfrak{a}_{m,0})^-$, *and can be differentiated term-by-term. The degrees of the coefficient polynomials* $p^{\nu}_{v,v'}$ *are bounded by an integer which depends only on* V. *A suitably chosen finite subset* $\mathscr{E}_l(V)\subset\mathscr{E}(V)$ *has the property that every* $\nu\in\mathscr{E}(V)$ *lies above some* $\nu_l\in\mathscr{E}_l(V)$, *relative to the ordering* $>$.

We may and shall assume that $\mathscr{E}(V)$ and $\mathscr{E}_l(V)$ do not contain any redundant elements, in which case both sets are uniquely determined. The elements of $\mathscr{E}(V)$ are the "exponents of V" along A_m, those of $\mathscr{E}_l(V)$ the "leading exponents". Collectively the leading exponents dominate all other exponents on $(\mathfrak{a}_{m,0})^-$. Even though the asymptotic expansions do not converge uniformly near points of the boundary, some additional arguments show that the matrix coefficients can be bounded from above and below in terms of the leading exponents and their polynomial coefficients, uniformly on A_m^-. In particular the leading exponents determine whether or not the matrix coefficients lie in $L^p(G)$, $1\leq p<\infty$.

LEMMA 4.17 [8]. *Suppose G has compact center, and $1\leq p<\infty$. The following conditions are equivalent:*

(a) $f_{v,v'}\in L^p(G)$, *for every choice of* $v\in V$, $v'\in V'$;

(b) $\operatorname{Re}\langle\nu,X\rangle<((2/p)-1+\varepsilon)\langle\varrho_m,X\rangle$, *for all* $\nu\in\mathscr{E}_l(V)$ *and* $X\in(\mathfrak{a}_{m,0})^-$, *and some* $\varepsilon>0$.

The L^2 criterion, which is of particular importance, takes the slightly simpler form

$$\operatorname{Re}\nu<0 \text{ on the closure of } (\mathfrak{a}_{m,0})^-, \text{ except at } 0, \quad \text{for all } \nu\in\mathscr{E}_l(V), \tag{4.18}$$

again under the assumption that the center of G is compact.

Let us consider a particular leading exponent ν of V. The mapping

$$v\otimes v'\to p^{\nu}_{v,v'}(0) \tag{4.19}$$

sets up a bilinear pairing $V\otimes V'\to\mathbf{C}$. It is non-zero and invariant with respect to the actions of M_m and $\mathfrak{a}_m$. If Y lies in the root space corresponding to a positive restricted root α, a formal calculation leads to the identity

$$e^{-\langle\alpha,X\rangle}f_{Yv,v'}(\exp X)=-f_{v,Yv'}(\exp X),\quad \text{for } X\in(\mathfrak{a}_{m,0})^-.$$

If ν were to contribute to the asymptotic expansion of $f_{Yv,v'}$, $\nu-\alpha$ would be an exponent, contradicting the definition of $\mathscr{E}_l(V)$. Since the root spaces of positive restricted roots span $\mathfrak{n}_m$, this proves $p^{\nu}_{v,v'}(0)=0$ if $v \in \mathfrak{n}_m V$. In other words, the pairing (4.19) factors through a pairing

$$H_0(\mathfrak{n}_m, V) \otimes V' \to \mathbf{C}. \tag{4.20}$$

Differentiating the asymptotic expansion of $f_{v,v'}$ along some $X \in \mathfrak{a}_m$, one finds

$$p^{\nu}_{Xv,v'} - \langle \nu + \varrho_m, X \rangle \, p^{\nu}_{v,v'} = D_X p^{\nu}_{v,v'} \tag{4.21}$$

(D_X = directional derivative in the direction of X). If k exceeds the maximal degree of the coefficient polynomials, (4.21) implies

$$p^{\nu}_{v,v'}(0) = 0 \quad \text{whenever } v \in (X - \langle \nu + \varrho_m, X \rangle)^k V + \mathfrak{n}_m V,$$

and consequently the generalized $(\nu+\varrho_m)$-eigenspace $H_0(\mathfrak{n}_m, V)_\nu$ is the only one that can contribute to the induced pairing (4.20). The roles of V and V' in the preceeding arguments may be reversed, provided one also replaces P_m by the opposite parabolic subgroup $\bar{P}_m$ and ν by $-\nu$. We have shown:

LEMMA 4.22 (Casselman [7]). *If ν is a leading exponent, the mapping* (4.19) *descends to a non-zero bilinear pairing*

$$H_0(\mathfrak{n}_m, V)_\nu \times H_0(\bar{\mathfrak{n}}_m, V')_{-\nu} \to \mathbf{C},$$

which is invariant with respect to M_m and $\mathfrak{a}_m$. In particular $H_0(\mathfrak{n}_m, V)_\nu \neq 0$.

Coupled with the reciprocity theorem, the lemma proves Casselman's strengthened form of the subquotient theorem of Harish-Chandra [14]:

COROLLARY 4.23 [7]. *Every irreducible Harish-Chandra module can be realized as a submodule of a Harish-Chandra module induced from a minimal parabolic subgroup.*

If V is an irreducible Harish-Chandra module, $Z(\mathfrak{g})$ acts according to a character $\chi_{\mathfrak{g},\lambda}$. We may think of the parameter λ as a linear functional on a Cartan subalgebra $\mathfrak{b}$ which contains $\mathfrak{a}_m$. As another consequence of lemma 4.22, we note that

$$\begin{gathered} \text{every leading exponent of } V \text{ is of the form } \nu = w\lambda|_{\mathfrak{a}_m}, \\ \text{for some } w \in W(\mathfrak{g}, \mathfrak{b}); \end{gathered} \tag{4.24}$$

cf. corollary 3.32.

According to lemma 4.22, every leading exponent ν of a Harish-Chandra module V is an exponent of $H_0(\mathfrak{n}_m, V)$, in the sense that $H_0(\mathfrak{n}_m, V)_\nu \neq 0$. The converse of this statement is due to Miličić:

THEOREM 4.25 [31]. *The set of leading exponents of V coincides with the set of exponents of $H_0(\mathfrak{n}_m, V)$ which are minimal with respect to $>$.*

Proof. Let ν be a minimal exponent of $H_0(\mathfrak{n}_m, V)$; it must be shown that ν is greater than or equal to a leading exponent of V. We choose a finite-dimensional, irreducible G-module F, whose lowest $\mathfrak{a}_m$-weight μ satisfies

$$(\mu+\nu+\varrho_m, \alpha)<0 \quad \text{for all } \alpha \in \Phi^+(\mathfrak{g}, \mathfrak{a}_m). \tag{4.26}$$

The $\mathfrak{a}_m$-weight spaces other than the lowest weight space span an $\mathfrak{n}_m$-invariant subspace $F_1 \subset F$. Since $\mathfrak{n}_m$ acts trivially on the quotient F/F_1, $\ldots \to H_0(\mathfrak{n}_m, V\otimes F_1) \to H_0(\mathfrak{n}_m, V\otimes F) \to H_0(\mathfrak{n}_m, V)\otimes(F/F_1) \to 0$ is an exact sequence of $\mathfrak{a}_m$-modules. In particular, $H_0(\mathfrak{n}_m, V\otimes F)_{\mu+\nu}$ is non-zero, and thus has an irreducible M_m-module W as quotient. We may appeal to (4.11) to conclude

$$\mathrm{Hom}_G(V\otimes F, I^G_{P_m}(W\otimes \mathbf{C}_{\mu+\nu})) \neq 0. \tag{4.27}$$

In the situation (4.26), the induced Harish-Chandra module $I^G_{P_m}(W\otimes \mathbf{C}_{\mu+\nu})$ has a unique irreducible submodule V_0, and $\mu+\nu$ occurs as a leading exponent of V_0; this result, which plays a crucial role in our proof of Osborne's conjecture, will be proved in the appendix. Because of (4.27) V_0 is a quotient of $V\otimes F$, and $\mu+\nu$, as an exponent of V_0, must also be an exponent of $V\otimes F$. For entirely formal reasons, every exponent of the tensor product can be expressed as a sum $\mu+\nu_1+\eta$, with $\nu_1 \in \mathscr{E}(V)$ and η equal to a (possibly empty) sum of positive restricted roots. In the case of the exponent $\mu+\nu$ this gives the relation $\nu \geq \nu_1$, which proves the theorem.

Harish-Chandra calls a representation tempered if its character extends to a suitably defined space of Schwartz functions on G. For our purposes it is simpler to use the asymptotic behavior of the matrix coefficients as a criterion for temperedness. The resulting definition is known to be equivalent to Harish-Chandra's, as will also become apparent later in this paper. We enumerate the simple restricted roots in $\Phi^+(\mathfrak{g}, \mathfrak{a}_m)$ as $\alpha_1, \alpha_2, \ldots, \alpha_r$, and we let $\mu_1, \ldots, \mu_r$ denote the dual collection of fundamental highest weights; in other words,

$$2\frac{(\mu_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta_{ij}, \quad 1 \leq i,j \leq r. \tag{4.28}$$

We say that a Harish-Chandra module V is tempered modulo the center of G if

$$\operatorname{Re}(\nu, \mu_i) \geq 0 \quad \text{for } 1 \leq i \leq r \text{ and every exponent } \nu \text{ of } H_0(\mathfrak{n}_m, V); \tag{4.29}$$

V is tempered—without qualification—if, in addition,

$$e^\nu \text{ restricts to a unitary character of } Z_G \cap A_m \tag{4.30}$$

for every such ν (Z_G = center of G). This latter condition becomes vacuous in case G has a compact center. According to theorem 4.25, both (4.29) and (4.30) amount to restrictions on the leading exponents: we may replace "for every exponent ν of $H_0(\mathfrak{n}_m, V)$" by "for every $\nu \in \mathscr{E}_l(V)$" without altering the definition.

The matrix coefficients of a tempered Harish-Chandra module are at least "almost square-integrable", modulo the center: as soon as the inequalities in (4.29) become strict, the criterion (4.18) ensures that the matrix coefficients are truly square-integrable on the semisimple part of G. If $Z(\mathfrak{g})$ acts on V via a character $\chi_{\mathfrak{g},\lambda}$—for example, if V is irreducible—, the exponents ν of $H_0(\mathfrak{n}_m, V)$ are restrictions to $\mathfrak{a}_m$ of Weyl translates of λ; here we assume, as we may, that the parameter λ lies in the dual $\mathfrak{b}^*$ of a Cartan subalgebra $\mathfrak{b}$ which contains $\mathfrak{a}_m$. Hence, under the hypothesis

$$\operatorname{Re}(\lambda, w\mu_i) \neq 0 \quad \text{for } 1 \leq i \leq r,\ w \in W(\mathfrak{g}, \mathfrak{b}) \tag{4.31}$$

on the character $\chi_{\mathfrak{g},\lambda}$, the inequalities (4.29) hold strictly or not at all. In this situation the matrix coefficients of a tempered Harish-Chandra module V are necessarily square-integrable, modulo the center of G.

We close the present section with a result that is part of Langlands' classification of the irreducible Harish-Chandra modules [28]. By a collection of Langlands data, we shall mean a triple, consisting of a parabolic subgroup P with Langlands decomposition $P=MAN$, of a tempered irreducible Harish-Chandra module W for M, and a linear functional $\nu \in \mathfrak{a}^*$, such that

$$\operatorname{Re}(\nu, \alpha) < 0 \quad \text{for all } \alpha \in \Phi^+(\mathfrak{g}, \mathfrak{a}). \tag{4.32}$$

We do not exclude $P=G$, in which case $MA=G$ and A = the split part of the center of G.

LEMMA 4.33 (Langlands [28]). *Every irreducible Harish-Chandra module V for G can be realized as a submodule of an induced module $I_P^G(W \otimes \mathbf{C}_\nu)$, corresponding to a collection of Langlands data $P=MAN$, W, ν.*

The proof depends on certain geometric considerations. As before, we let

$P_m = M_m A_m N_m$ denote a minimal parabolic subgroup. The closed negative Weyl chamber

$$\mathscr{C} = \{\mu \in (\mathfrak{a}_{m,0})^* \mid (\mu, \alpha) \leqslant 0 \quad \text{for } \alpha \in \Phi(\mathfrak{g}, \mathfrak{a}_m)\} \tag{4.34}$$

is a closed convex cone in $(\mathfrak{a}_{m,0})^*$. Hence,

$$\begin{gathered}\text{the least distance from any particular } \mu \in (\mathfrak{a}_{m,0})^* \text{ to } \mathscr{C} \text{ is attained} \\ \text{at a single point } \mu^0 \in \mathscr{C}.\end{gathered} \tag{4.35}$$

On $\mathfrak{a}_m \cap$ center of $\mathfrak{g}_0$, whose dual lies in $\mathscr{C}$, μ^0 agrees with μ, and on the semisimple part of $\mathfrak{a}_m$, μ^0—like any element of $\mathscr{C}$—is equal to a unique linear combination of some of the fundamental highest weights (4.28), with strictly negative coefficients:

$$\begin{aligned}&\text{(a) } \mu^0 = \mu \text{ on } \mathfrak{a}_m \cap \text{center of } \mathfrak{g}_0,\\ &\text{(b) } \mu^0 = \textstyle\sum_{i \in S} c_i \mu_i \text{ on } \mathfrak{a}_m \cap [\mathfrak{g}, \mathfrak{g}],\ c_i < 0.\end{aligned} \tag{4.36}$$

The index set

$$S = S(\mu) \subset \{1, 2, \ldots, r\} \tag{4.37}$$

depends on μ^0, and thus on μ. For $1 \leqslant i \leqslant r$ and $t \geqslant 0$, the point $\mu^0 - t\mu_i$ also lies in $\mathscr{C}$ and cannot be closer to μ than μ^0. In other words,

$$0 \leqslant \|\mu - (\mu^0 - t\mu_i)\|^2 - \|\mu - \mu^0\|^2 = 2t(\mu - \mu^0, \mu_i) + t^2 \|\mu_i\|^2,$$

which implies $(\mu - \mu^0, \mu_i) \geqslant 0$. If the index i belongs to S, one also obtains the opposite inequality, since $\mu^0 - t\mu_i \in \mathscr{C}$ even for small negative values of t. In view of (4.28) and (4.36a), we may conclude

$$\mu - \mu^0 = \sum_{j \notin S} d_j \alpha_j, \quad d_j \geqslant 0. \tag{4.38}$$

The two properties (4.36), (4.38) of μ^0 and S not only follow from the characterization (4.35) of μ^0, they also imply it. Indeed, if $\mu^1 \in \mathscr{C}$,

$$\begin{aligned}\|\mu - \mu^1\|^2 - \|\mu - \mu^0\|^2 &= \|(\mu - \mu^0) + (\mu^0 - \mu^1)\|^2 - \|\mu - \mu^0\|^2 \geqslant 2(\mu - \mu^0, \mu^0 - \mu^1)\\ &= 2 \sum_{j \notin S} d_j (\alpha_j, \mu^0 - \mu^1) = -2 \sum_{j \notin S} d_j (\alpha_j, \mu^1) \geqslant 0;\end{aligned}$$

i.e., μ^0 minimizes the distance from μ to $\mathscr{C}$. We sumarize:

$$\begin{gathered}\mu^0 \in \mathscr{C} \text{ and } S = S(\mu) \text{ are uniquely determined by (4.36), (4.38);} \\ \mu^0 \text{ is the point in } \mathscr{C} \text{ closest to } \mu\end{gathered} \tag{4.39}$$

(cf. the "geometric lemmas" of [28]).

To prove lemma 4.33 for a particular irreducible Harish-Chandra module V, we write each of the exponents μ of $H_0(\mathfrak{n}_m, V)$ as

$$\mu = \operatorname{Re}\mu + i\operatorname{Im}\mu, \quad \operatorname{Re}\mu,\ \operatorname{Im}\mu \in (\mathfrak{a}_{m,0})^*.$$

Among these, we select one—to be called μ from now on—which maximizes the length of $(\operatorname{Re}\mu)^0$. Let us recall the standard construction that associates a parabolic subgroup to the set

$$S = S(\operatorname{Re}\mu) \subset \{1, \dots, r\}. \tag{4.40}$$

The root spaces indexed by roots in

$$\{\alpha \in \Phi^+(\mathfrak{g}, \mathfrak{a}_m) \mid (\alpha, \mu_i) > 0 \quad \text{for some } i \in S\} \tag{4.41}$$

span an ideal $\mathfrak{n} \subset \mathfrak{n}_m$, the complexified Lie algebra of a normal subgroup $N \subset N_m$. Its normalizer in G is a parabolic subgroup P, containing P_m. We define

$$\begin{aligned} &A = \{a \in A_m \mid e^{\alpha_j}(a) = 1 \quad \text{for all } j \notin S\}, \\ &M = \text{anisotropic part}(^1) \text{ of the centralizer of } A \text{ in } G; \end{aligned} \tag{4.42}$$

then

$$P = MAN \tag{4.43}$$

is a Langlands decomposition of P, and the Cartan involution preserves the Levi factor of MA.

The exponent μ of $H_0(\mathfrak{n}_m, V)$ which was used to construct P, restricts to a linear function $\nu \in \mathfrak{a}^*$,

$$\nu = \mu|_{\mathfrak{a}}. \tag{4.44}$$

We must show that ν satisfies the negativity condition (4.32). For this purpose, we may

(1) i.e., the common kernel of all $\mathbf{R}^+$-valued characters.

as well assume $\mathfrak{g}$ is semisimple, in which case the μ_i, $i \in S$, span the orthogonal complement of the annihilator of $\mathfrak{a}$ in $\mathfrak{a}_m^*$, whereas the α_j, $j \notin S$, vanish on $\mathfrak{a}$. Hence, for any $\eta \in \mathfrak{a}_m^*$,

$$\begin{aligned} &(\mu_i|_{\mathfrak{a}}, \eta|_{\mathfrak{a}}) = (\mu_i, \eta), \quad \text{if } i \in S, \\ &(\alpha_j|_{\mathfrak{a}}, \eta|_{\mathfrak{a}}) = 0, \quad \text{if } j \notin S. \end{aligned} \tag{4.45}$$

According to (4.36) and (4.38), with $\mathfrak{g}=[\mathfrak{g}, \mathfrak{g}]$, $(\operatorname{Re}\mu)^0$ is a linear combination of the μ_i, $i \in S$, with strictly negative coefficients c_i, and differs from μ by a linear combination of α_j's, $j \notin S$. If η is one of the roots in the set (4.41), this allows us to conclude

$$\begin{aligned} \operatorname{Re}(\nu, \eta|_{\mathfrak{a}}) &= ((\operatorname{Re}\mu)|_{\mathfrak{a}}, \eta|_{\mathfrak{a}}) = ((\operatorname{Re}\mu)^0|_{\mathfrak{a}}, \eta|_{\mathfrak{a}}) \\ &= \sum_{i \in S} c_i(\mu_i|_{\mathfrak{a}}, \eta|_{\mathfrak{a}}) = \sum_{i \in S} c_i(\mu_i, \eta) < 0. \end{aligned} \tag{4.46}$$

Every $\alpha \in \Phi^+(\mathfrak{g}, \mathfrak{a})$ arises as the restriction to $\mathfrak{a}$ of some such η, so (4.32) follows.

The final ingredient of lemma 4.33 is an irreducible, tempered Harish-Chandra module W for M, such that

$$\operatorname{Hom}_G(V, I_P^G(W \otimes \mathbf{C}_\nu)) \neq 0.$$

In view of the reciprocity theorem (4.11), any irreducible quotient W of $H_0(\mathfrak{n}, V)_\nu$ will do, provided

$$H_0(\mathfrak{n}, V)_\nu \text{ is non-zero and tempered,} \tag{4.47}$$

as Harish-Chandra module for M: the quotient W then inherits the temperedness, since the induced mapping on homology in dimension zero is surjective.

To begin the verification of (4.47), we note that the isomorphism

$$\begin{aligned} H_0(\mathfrak{n}_m, V) = V/\mathfrak{n}_m V &\simeq (V/\mathfrak{n}V)/(\mathfrak{m} \cap \mathfrak{n}_m)(V/\mathfrak{n}V) \\ &= H_0(\mathfrak{m} \cap \mathfrak{n}_m, H_0(\mathfrak{n}, V)) \end{aligned}$$

commutes with the action of $\mathfrak{a}$. Hence, if $\tilde{\nu} \in \mathfrak{a}^*$ denotes the restriction to $\mathfrak{a}$ of some $\tilde{\mu} \in \mathfrak{a}_m^*$,

$$H_0(\mathfrak{n}_m, V)_{\tilde{\mu}} \simeq H_0(\mathfrak{m} \cap \mathfrak{n}_m, H_0(\mathfrak{n}, V)_{\tilde{\nu}})_{\tilde{\mu}+\varrho_P}; \tag{4.48}$$

the shift by the appropriate ϱ occurs because of (3.27). In the special case of $\tilde{\mu}=\mu$, this gives

$$H_0(\mathfrak{m} \cap \mathfrak{n}_m, H_0(\mathfrak{n}, V)_\nu)_{\mu+\varrho_P} \neq 0,$$

so $H_0(\mathfrak{n}, V)_\nu$ cannot vanish.

As for the temperedness of $H_0(\mathfrak{n}, V)_\nu$, some preliminary observations will be helpful. The minimal parabolic subgroup P_m intersects M in a minimal parabolic subgroup of M, with Langlands decomposition

$$M \cap P_m = M_m(M \cap A_m)(M \cap N_m).$$

By restriction from $\mathfrak{a}_m$ to $\mathfrak{m} \cap \mathfrak{a}_m$, we can make the identification

$$\Phi^+(\mathfrak{m}, \mathfrak{m} \cap \mathfrak{a}_m) \simeq \Phi^+(\mathfrak{m} \oplus \mathfrak{a}, \mathfrak{a}_m) = \{\alpha \in \Phi^+(\mathfrak{g}, \mathfrak{a}_m) | (\alpha, \mu_i) = 0 \quad \text{for } i \in S\}.$$

This latter positive root system is spanned by the α_j, $j \notin S$, which we may regard as the simple roots for $(\mathfrak{m}, \mathfrak{m} \cap \mathfrak{a}_m)$. Dually, the μ_j, $j \notin S$, become the fundamental highest weights. In our present situation, then, the temperedness criterion (4.29) reduces to

$$\begin{gathered} \operatorname{Re}(\tilde{\mu}, \mu_j) \geq 0, \quad \text{for every } j \notin S \text{ and every exponent } \tilde{\mu} \\ \text{of } H_0(\mathfrak{n}_m, V) \text{ which restricts to } \nu \text{ on } \mathfrak{a}; \end{gathered} \tag{4.49}$$

cf. (4.48). The second condition (4.30) is irrelevant, since M has a compact center.

Let us collect various pieces of information about the exponents $\tilde{\mu}$ of $H_0(\mathfrak{n}_m, V)$ whose restriction to $\mathfrak{a}$ equals ν. The original choice of μ was made so as to maximize $\|(\operatorname{Re}\mu)^0\|$:

$$\|(\operatorname{Re}\mu)^0\| \geq \|(\operatorname{Re}\tilde{\mu})^0\|,$$

for all the $\tilde{\mu}$. According to (4.36) and (4.36),

$$(\operatorname{Re}\tilde{\mu} - (\operatorname{Re}\tilde{\mu})^0, (\operatorname{Re}\tilde{\mu})^0) = 0;$$

also, $(\operatorname{Re}\mu)^0$ lies in $\mathscr{C}$ and cannot be closer to $\operatorname{Re}\tilde{\mu}$ than $(\operatorname{Re}\tilde{\mu})^0$, i.e.

$$\|\operatorname{Re}\tilde{\mu} - (\operatorname{Re}\tilde{\mu})^0\| \leq \|\operatorname{Re}\tilde{\mu} - (\operatorname{Re}\mu)^0\|,$$

with equality hoding only if $(\operatorname{Re}\tilde{\mu})^0 = (\operatorname{Re}\mu)^0$. Since μ and $\tilde{\mu}$ have the same restriction to $\mathfrak{a}$, their difference can be expressed as a linear combination of the α_j, $j \notin S$. The restriction to $\mathfrak{a}_m \cap [\mathfrak{g}, \mathfrak{g}]$ of $(\operatorname{Re}\mu)^0$, on the other hand, is a linear combination of the μ_i, $i \in S$, so

$$(\operatorname{Re}\tilde{\mu} - (\operatorname{Re}\mu)^0, (\operatorname{Re}\mu)^0) = (\operatorname{Re}\tilde{\mu} - \operatorname{Re}\mu, (\operatorname{Re}\mu)^0) + (\operatorname{Re}\mu - (\operatorname{Re}\mu)^0, (\operatorname{Re}\mu)^0) = 0.$$

The preceeding statements, read in sequence, justify the following chain of equalities and inequalities:

$$\begin{aligned} \|(\operatorname{Re}\mu)^0\|^2 \geq \|(\operatorname{Re}\tilde{\mu})^0\|^2 &= \|\operatorname{Re}\tilde{\mu}\|^2 - \|\operatorname{Re}\tilde{\mu} - (\operatorname{Re}\tilde{\mu})^0\|^2 \\ &\geq \|\operatorname{Re}\tilde{\mu}\|^2 - \|\operatorname{Re}\tilde{\mu} - (\operatorname{Re}\mu)^0\|^2 = \|(\operatorname{Re}\mu)^0\|^2. \end{aligned}$$

Since equality must hold at all steps, $(\mathrm{Re}\,\tilde{\mu})^0$ and $(\mathrm{Re}\,\mu)^0$ coincide, hence

$$S = S(\mathrm{Re}\,\mu) = S(\mathrm{Re}\,\tilde{\mu}).$$

At this point another application of (4.36) and (4.38), with $\mathrm{Re}\,\tilde{\mu}$ playing the role of μ, give the inequality (4.49). The proof of lemma 4.33 is now complete.

Except for the geometric description (4.39) of the inducing parameter—which was discovered independently by several others—the preceeding argument is a modification of that of Langlands [28]. In this form it was shown to us by Miličić during a 1977 conference in Oberwolfach and is also contained in the monograph [3] of Borel-Wallach.

A final observation: if V satisfies the condition (4.31), then W satisfies the analogous condition with respect to M; indeed, the fundamental highest weights for $(\mathfrak{m}\oplus\mathfrak{a}, \mathfrak{a}_m)$—namely the μ_j, $j \notin S$—occur among the fundamental highest weights for $(\mathfrak{g}, \mathfrak{a}_m)$, and the parameter λ remains the same. Since W is tempered, its matrix coefficients are square-integrable in this situation. For future reference, we state

Remark 4.50. Under the hypothesis (4.31) on V, the inducing module W has matrix coefficients which are square-integrable on M.

§ 5. The induced character formula

The process of induction, which associates a Harish-Chandra module for G to a Harish-Chandra module of a Levi factor of a parabolic subgroup of G, has a direct analogue on the level of global characters. In this section we review the formulas for induced characters established in [25] and [42] and state them in a form suitable for the applications we have in mind. We start with some remarks of general nature.

For $g \in G$, set $D_G(g)$ = coefficient of $t^{r(G)}$ in the polynomial

$$\det\{(\mathrm{Ad}\,g + t - 1) : \mathfrak{g} \to \mathfrak{g}\}. \tag{5.1}$$

Here $r(G)$ = rank of G denotes the dimension of any Cartan subgroup of G. The function $g \mapsto D_G(g)$ is real analytic on G, as well as conjugation invariant. It detects regular semisimple points: $g \in G'$ if and only if $D_G(g) \neq 0$. Restricted to any Cartan subgroup of G,

$$|D_G| = \left| \prod_{\alpha>0} (e^{\alpha/2} - e^{-\alpha/2}) \right|^2 \tag{5.2}$$

where $\Pi_{\alpha>0}(e^{\alpha/2}-e^{-\alpha/2})$ is the Weyl denominator defined in (3.40). Although the Weyl denominator may not make sense unless we pass to a finite covering of G, its absolute value is always well defined, as it is equal to $|D_G|^{1/2}$. We remark that the function

$$|D_G|^{-1/2}\prod_{\alpha>0}(e^{\alpha/2}-e^{-\alpha/2}) \tag{5.3}$$

is continuous on $B\cap G'$ and can admit only the values ± 1, $\pm\sqrt{-1}$. In fact, $[\Pi_{\alpha>0}(e^{\alpha/2}-e^{-\alpha/2})]^2=\pm\Pi_{\alpha>0}(1-e^{\alpha})(1-e^{-\alpha})$ is invariant under complex conjugation, hence real, and therefore the Weyl denominator takes on only real or purely imaginary values.

Let V be a Harish-Chandra module for G. Fix $b\in G'$. Then B, the centralizer of b in G, is a Cartan subgroup of G. Assume that $Z(\mathfrak{g})$ acts on V according to a character $\chi=\chi_{\mathfrak{g},\lambda}(\lambda\in\mathfrak{b}^*)$. Recall from section 3 that the product of the character $\Theta_G(V)$ with the Weyl denominator can be expressed locally as an exponential sum:

$$\left[\prod_{\alpha>0}(e^{\alpha/2}-e^{-\alpha/2})\,\Theta_G(V)\right](b\exp X)=\sum_{w\in W(\mathfrak{g},\mathfrak{b})} c_\lambda(b,w)\,e^{\langle w\lambda,X\rangle}, \tag{5.4}$$

for all X in a neighborhood of 0 in $\mathfrak{b}_0$. The coefficients $c_\lambda(b,w)$ are complex constants depending on b and w. Let us emphasize again that (5.4) may make sense only on a finite covering of B. This ambiguity—which did not present a problem in § 3—can be circumvented by defining

$$[|D_G|^{1/2}\Theta_G(V)](b\exp X)=\sum_{w\in W(\mathfrak{g},\mathfrak{b})} c'_\lambda(b,w)\,e^{w\langle\lambda,X\rangle}; \tag{5.5}$$

the new coefficients $c'_\lambda(b,w)$ are related to the $c_\lambda(b,w)$ by a multiplicative constant equal to ± 1 or $\pm\sqrt{-1}$. In general the formula (5.4) holds uniformly on larger subsets of $B\cap G'$ than the right hand side of (5.5). On the other hand (5.5), because of its global flavor, behaves nicely under the process of induction. To keep the distinction between these two formulas, we reserve the primed symbols for the expansion (5.5).

Let $P=MAN$ be a parabolic subgroup of G, and U a Harish-Chandra module for MA. It is necessary for us to have explicit formulas for the character $\Theta_G[I_P^G(U)]$ of the induced module $I_P^G(U)$ in terms of $\Theta_{MA}(U)$. According to [25, 42], if $B\subset G$ is a Cartan subgroup,

$$\begin{gathered}\Theta_G[I_P^G(U)] \text{ vanishes on } B\cap G', \text{ unless } B \text{ is conjugate under } G\\ \text{to a Cartan subgroup of } MA.\end{gathered} \tag{5.6}$$

Thus, in order to compute $\Theta_G[I_P^G(U)]$, it is enough to determine its restriction to $B \cap G'$, for each Cartan subgroup B contained in MA. We fix such a B, and select representatives $B=B_1, B_2, \ldots, B_k$ from the various MA-conjugacy classes of Cartan subgroups in MA, which are G-conjugate to B. The Weyl group of B_i in G,

$$W(G, B_i) = N_G(B_i)/B_i,$$

acts on $\mathfrak{b}_i$ and $\mathfrak{b}_i^*$, and can be identified naturally with a subgroup of the complex Weyl group $W(\mathfrak{g}, \mathfrak{b}_i)$. It may not act on B_i, since B_i need not be Abelian. In any case vg, with $v \in W(G, B_i)$ and $g \in B_i$, makes sense up to conjugation by elements of B_i, and we shall use this suggestive, but slightly inaccurate notation when vg appears as an argument of a function invariant under conjugation by B_i. Finally, choose $e=y_1, y_2, \ldots, y_k \in G$ so that $y_i B y_i^{-1} = B_i$.

THEOREM 5.7 (Hirai [25], Wolf [42](1)). *The restriction of* $|D_G|^{1/2}\Theta_G[I_P^G(U)]$ *to* $B \cap G'$ *equals* $\Sigma_{i=1}^k \varphi_i$, *where*

$$\varphi_i(g) = \frac{1}{c_i} \sum_{v \in W(G, B_i)} [|D_{MA}|^{1/2} \Theta_{MA}(U)]\,(v(y_i g y_i^{-1})),$$

for $g \in B \cap G'$, *and* $c_i = \# W(MA, B_i)$.

We note the following important consequence:

$$\text{the character of } I_P^G(U) \text{ depends only on } MA \text{ and } U, \text{ and not on a particular choice of } P. \tag{5.8}$$

Suppose that $Z(\mathfrak{m} \oplus \mathfrak{a})$ acts on U according to a character χ. Choose $\lambda = \lambda_1, \lambda_2, \ldots, \lambda_k$ in $\mathfrak{b}_1^*, \ldots, \mathfrak{b}_k^*$, respectively, so that $\chi = \chi_{\mathfrak{g}, \lambda_i}$ for each i. By (5.5), we can express the function $|D_{MA}|^{1/2}\Theta_{MA}(U)$, restricted to a neighborhood in B_i of an MA-regular point $h \in B_i$, in the form

$$\sum_{w \in W(\mathfrak{m} \oplus \mathfrak{a}, \mathfrak{b}_i)} d'_{\lambda_i}(h, w)\, e^{w\lambda_i}, \tag{5.9}$$

with suitable complex constants $d'_{\lambda_i}(h, w)$. It follows from theorem 5.7 that

$$\varphi_i(b \exp X) = \frac{1}{c_i} \sum_{v \in W(G, B_i)} \sum_{w \in W(\mathfrak{m} \oplus \mathfrak{a}, \mathfrak{b}_i)} d'_{\lambda_i}(v^{-1}(y_i b y_i^{-1}), w)\, e^{\langle vw\lambda_i,\, \mathrm{Ad}\, y_i(X)\rangle}, \tag{5.10}$$

---

(1) The possibility that $k>1$ is overlooked in [42].

for all $b \in B \cap G'$ and X in a neighborhood of 0 in $\mathfrak{b}_0$. We want to transfer the data attached to $\mathfrak{b}_i$ back to $\mathfrak{b}$. Let u_i be an inner automorphism of $\mathfrak{m}\oplus\mathfrak{a}$ wich maps $\mathfrak{b}$ to $\mathfrak{b}_i$ and, dually, λ to λ_i—for $i=1$, we pick u_1 = identity. Then $\operatorname{Ad} y_i^{-1} \circ u_i$ normalizes $\mathfrak{b}$ and operates as an element w_i of $W(\mathfrak{g}, \mathfrak{b})$; w_1 is the identity. We remark that $u_i=1$ on $\mathfrak{a}$, hence

$$w_i X = \operatorname{Ad} y_i^{-1}(X) \quad \text{for } X \in \mathfrak{a}.$$

The map $w \mapsto \operatorname{Ad} y_i^{-1} \circ w \circ \operatorname{Ad} y_i$ establishes an isomorphism of $W(G, B_i)$ with $W(G, B)$, and sends $W(\mathfrak{m}\oplus\mathfrak{a}, \mathfrak{b}_i)$ isomorphically onto the subgroup

$$\{w \in W(\mathfrak{g}, \mathfrak{b}) \,|\, w \circ \operatorname{Ad} y_i^{-1}(X) = \operatorname{Ad} y_i^{-1}(X) \quad \text{for all } X \in \mathfrak{a}\} \tag{5.11}$$

of $W(\mathfrak{g}, \mathfrak{b})$. We can now rephrase (5.10) as follows:

$$\varphi_i(b \exp X) = \frac{1}{c_i} \sum_{v,w} d'_{\lambda_i}(y_i(v^{-1}b)\, y_i^{-1}, \operatorname{Ad} y_i \circ w \circ \operatorname{Ad} y_i^{-1})\, e^{\langle v w w_i \lambda, X\rangle}, \tag{5.12}$$

where v and w vary over $W(G, B)$ and the group (5.11), respectively. The above formula amounts to an expression for the constants a'_λ in the expansion

$$\sum_{w \in W(\mathfrak{g}, \mathfrak{b})} a'_\lambda(b, w)\, e^{w\lambda} \tag{5.13}$$

of $|D_G|^{1/2}\Theta[I_P^G(U)]$ near b, in terms of the constants d'_{λ_i}.

Any two maximally split Cartan subgroup in MA are MA-conjugate. Hence, if $B \subset MA$ is maximally split, the induced characters formula becomes considerably simpler: in a neighborhood in B of b, $|D_G|^{1/2}\Theta[I_P^G(U)]$ equals

$$\frac{1}{c} \sum_{v \in W(G,B)} \sum_{w \in W(\mathfrak{m}\oplus\mathfrak{a}, \mathfrak{b})} d'_\lambda(v^{-1}b, w)\, e^{vw\lambda}, \tag{5.14}$$

where $c = \# W(MA, B)$.

Our next result is of a technical nature and will be needed later. Again let V denote a Harish-Chandra module for G, on which $Z(\mathfrak{g})$ acts via $\chi_{\mathfrak{g},\lambda}$. For $\mu \in \mathfrak{a}^*$, define $\Theta_G(V)_\mu$, the μ-component along P of the character $\Theta_G(V)$, as follows. Fix a Cartan subgroup B in MA, and $b \in B \cap G'$. Then, restricted to a neighborhood of b in $B \cap G'$,

$$\Theta_G(V)_\mu = \frac{\sum_w c'_\lambda(b, w)\, e^{w\lambda}}{|D_{MA}|^{1/2}}\, e^{\varrho_P} \tag{5.15}$$

(cf. (5.5)), where w ranges over the set of all $w\in W(\mathfrak{g},\mathfrak{b})$ such that $w\lambda|_{\mathfrak{a}}=\mu$. This formula describes a real analytic, MA-conjugation invariant function on $(MA)\cap G'$. Formally $\Theta_G(V)_\mu$ looks like a virtual character of MA, and in fact coincides with such a character on $(MA)^-$, as will follow from Osborne's conjecture. At this point, however, we attach no representation theoretic meaning to it.

Let $\Phi^+(\mathfrak{g},\mathfrak{b})$ be a positive root system, compatible with $\Phi^+(\mathfrak{g},\mathfrak{a})$, as in (3.23). We suppose that $Z(\mathfrak{m}\oplus\mathfrak{a})$ acts on U according to a character $\chi_{\mathfrak{g},\lambda}$, with

$$\operatorname{Re}(\lambda,\alpha)<0 \quad \text{for all } \alpha\in\Phi^+(\mathfrak{g},\mathfrak{b}). \tag{5.16}$$

Then U can be expressed as a tensor product

$$U=W\otimes\mathbf{C}_\mu,$$

of the one dimensional $\mathfrak{a}$-module $\mathbf{C}_\mu$, $\mu=\lambda|_{\mathfrak{a}}$, and a Harish-Chandra module W for M, on which $Z(\mathfrak{m})$ acts according to $\chi_{\mathfrak{m},\tau}$, $\tau=\lambda|_{\mathfrak{b}\cap\mathfrak{m}}$.

LEMMA 5.17. *Under the hypotheses which were just stated,*

$$\Theta_G[I_P^G(W\otimes\mathbf{C}_\mu)]_\mu=\Theta_{MA}(W\otimes\mathbf{C}_{\mu+\varrho_P}).$$

Proof. Although the anti-dominance condition (5.16) is phrased in terms of a particular Cartan subgroup $B\subset MA$, it has independent meaning: if $B_1\subset MA$ is any other Cartan subgroup, one can transfer λ to $\lambda_1\in\mathfrak{b}_1^*$ and $\Phi^+(\mathfrak{g},\mathfrak{b})$ to a positive root system $\Phi^+(\mathfrak{g},\mathfrak{b}_1)$, both via the same inner automorphism of $\mathfrak{m}\oplus\mathfrak{a}$; then $\Phi^+(\mathfrak{g},\mathfrak{b}_1)$ is also compatible with $\Phi^+(\mathfrak{g},\mathfrak{a})$ and makes λ_1 anti-dominant. Let us check the identity which is to be proved near some $b\in MA\cap G'$—we may as well assume that b lies in the Cartan subgroup B which was used to state (5.16). We claim that φ_i, for $i>1$, cannot contribute to the μ-component of $\Phi_G[I_P^G(W\otimes\mathbf{C}_\mu)]$. Because of (5.16), this is equivalent to

$$vww_i\lambda|_{\mathfrak{a}}\neq\mu=\lambda|_{\mathfrak{a}},$$

for $v\in W(G,B)$ and w in the group (5.11). The condition (5.16) implies that a $W(\mathfrak{g},\mathfrak{b})$-conjugate $u\lambda$ of λ cannot restrict to μ on $\mathfrak{a}$ unless $u\in W(\mathfrak{m}\oplus\mathfrak{a},\mathfrak{b})$. Thus we must show $vww_i|_{\mathfrak{a}}\neq 1$, with v,w as before and $i>1$. We recall that $w_i|_{\mathfrak{a}}=\operatorname{Ad}y_i^{-1}|_{\mathfrak{a}}$ and $w\circ\operatorname{Ad}y_i^{-1}|_{\mathfrak{a}}=\operatorname{Ad}y_i^{-1}|_{\mathfrak{a}}$. In particular,

$$vww_i|_{\mathfrak{a}}=v\circ\operatorname{Ad}y_i^{-1}|_{\mathfrak{a}}=1 \Leftrightarrow v=\operatorname{Ad}m\circ\operatorname{Ad}y_i, \quad \text{for some } m\in M.$$

If so, $\mathfrak{b}_0 = v\mathfrak{b}_0 = \operatorname{Ad} m \circ \operatorname{Ad} y_i \mathfrak{b}_0 = \operatorname{Ad} m\mathfrak{b}_{i,0}$ is MA-conjugate to $\mathfrak{b}_{i,0}$, hence $i=1$. Now let us examine the local expression

$$\varphi_1 = \frac{1}{c_1} \sum_{v,w} d'_\lambda(v^{-1}b, w)\, e^{vw\lambda}. \tag{5.18}$$

Arguing just as before, we find that $vw\lambda|_\mathfrak{a} \neq \mu$ unless $v = \operatorname{Ad} m$, for some $m \in M$—in other words, unless $v \in W(MA, B)$. Thus, as far as the μ-component is concerned, only summands with $v \in W(MA, B)$ matter in (5.18). The character of $U = W \otimes \mathbf{C}_\mu$ is invariant under conjugation by MA, hence

$$d'_\lambda(v^{-1}b, w)\, e^{vw\lambda} = d'_\lambda(b, w)\, e^{w\lambda}$$

for $v \in W(MA, B)$. We conclude

$$\frac{1}{c_1} \sum_{v \in W(MA,B)} \sum_{w \in W(\mathfrak{m} \oplus \mathfrak{a}, \mathfrak{b})} d'_\lambda(v^{-1}b, w)\, e^{vw\lambda} = \sum_{w \in W(\mathfrak{m} \oplus \mathfrak{a}, \mathfrak{b})} d'_\lambda(b, w)\, e^{w\lambda},$$

which is the local expression for $|D_{MA}|^{1/2}\Theta_{MA}(W \otimes \mathbf{C}_\mu)$. The lemma follows.

Recall the definition (3.10) of the "Osborne character" $\Theta_\mathfrak{n}(V)$ of a Harish-Chandra module V for G. The local expressions (3.58) near any $b \in B \cap G'$ can be re-written with $|D_G|^{1/2}$ in place of the Weyl denominator,

$$[|D_G|^{1/2}\Theta_\mathfrak{n}(V)](b \exp X) = \sum_{w \in W(\mathfrak{g}, \mathfrak{b})} \tilde{c}'_\lambda(b, w)\, e^{\langle w\lambda, X\rangle}; \tag{5.19}$$

the new constants $\tilde{c}'_\lambda(b, w)$ are related to the $\tilde{c}_\lambda(b, w)$ by the same factor which relates the $c'_\lambda(b, w)$ of (5.5) to the $c_\lambda(b, w)$ of (3.41). In analogy to (5.15), we introduce the μ-component $\Theta_\mathfrak{n}(V)_\mu$, which is given locally near $b \in B \cap G'$ by the expression

$$\Theta_\mathfrak{n}(V)_\mu = \frac{\sum_w \tilde{c}'_\lambda(b, w)\, e^{w\lambda}}{|D_{MA}|^{1/2}}\, e^{\varrho_P}, \tag{5.20}$$

with w ranging over the set $\{w \in W(\mathfrak{g}, \mathfrak{b}) | \, w\lambda|_\mathfrak{a} = \mu\}$. We note that

$$\begin{aligned} &\Theta(V)_\mu = \Theta_\mathfrak{n}(V)_\mu \text{ in a neighborhood of } b \in B \cap G' \iff \\ &c_\lambda(b, w) = \tilde{c}_\lambda(b, w), \quad \text{for all } w \in W(\mathfrak{g}, \mathfrak{b}) \text{ such that } w\lambda|_\mathfrak{a} = \mu. \end{aligned} \tag{5.21}$$

Since $c'_\lambda(b, w)/c_\lambda(b, w) = \tilde{c}'_\lambda(b, w)/\tilde{c}_\lambda(b, w)$, this is equivalent to the corresponding statement about the primed constants, which in turn follows immediately from the definitions (5.15, 5.20).

As before, let B denote a Cartan subgroup in MA, and $\Phi^+(\mathfrak{g}, \mathfrak{b})$ a positive root system. Then, on B,

$$\frac{|D_G|^{1/2}}{|D_{MA}|^{1/2}} e^{\varrho_P} = e^{\varrho_P} \left| \prod_{\alpha>0,\, \alpha \neq 0 \text{ on } \mathfrak{a}} (e^{\alpha/2} - e^{-\alpha/2}) \right|$$
$$= \left| \prod_{\alpha>0,\, \alpha \neq 0 \text{ on } \mathfrak{a}} (1-e^{\alpha}) \right| = \left| \sum_p (-1)^p \Theta_{MA}(\Lambda^p \mathfrak{n}) \right|$$

(cf. (3.3)). The function $\Sigma_p(-1)^p \Theta_{MA}(\Lambda^p \mathfrak{n})$ assumes only positive values on $(MA)^-$, as can be seen from the definition (3.5), hence

$$\sum_p (-1)^p \Theta_{MA}(\Lambda^p \mathfrak{n}) = \frac{|D_G|^{1/2}}{|D_{MA}|^{1/2}} e^{\varrho_P} \quad \text{on } (MA)^- \cap G'.$$

Since $H_p(\mathfrak{n}, V)_\mu$ is the generalized $(\mu+\varrho_P)$-eigenspace of $\mathfrak{a}$ in $H_p(\mathfrak{n}, V)$, the identities (3.10) and (5.20) allow us to conclude:

$$\Theta_{\mathfrak{n}}(V)_\mu = \sum_p (-1)^p \Theta_{MA}(H_p(\mathfrak{n}, V)_\mu) \quad \text{on } (MA)^- \cap G'. \tag{5.22}$$

Our next lemma is a counterpart to lemma 5.17 for the "Osborne character". We suppose again that $Z(\mathfrak{g})$ acts on the Harish-Chandra module V according to $\chi_{\mathfrak{g},\lambda}$, with $\lambda \in \mathfrak{b}^*$ subject to the negativity condition (5.16), and we set

$$\mu = \text{restriction of } \lambda \text{ to } \mathfrak{a}. \tag{5.23}$$

Then

LEMMA 5.24. *On* $(MA)^- \cap G'$,

$$\Theta_{\mathfrak{n}}(V)_\mu = \Theta_{MA}(H_0(\mathfrak{n}, V)_\mu).$$

Proof. Because of (5.16), any Weyl translate $w\lambda$ differs from λ by a linear combination, with non-negative coefficients, of roots in $\Phi^+(\mathfrak{g}, \mathfrak{b})$. Each of these restricts to a root $\alpha \in \Phi^+(\mathfrak{g}, \mathfrak{a})$ on $\mathfrak{a}$, or to zero. In particular μ is minimal among the homology exponents, relative to the partial ordering (2.31). The lemma now follows from (5.22) and the vanishing theorem 2.32.

We close this section with a proof of the two statements (3.7–8) about the set

$(MA)^-$, and with an explicit description of $(MA)^- \cap B \cap G'$, whenever B is a Cartan subgroup contained in MA. Enumerate the connected components of B as

$$B^0 = \text{identity component}, B^1, B^2, \dots, B^N,$$

and define

$$\Phi_{\mathbf{R}}(\mathfrak{g}, \mathfrak{b}) = \{\alpha \in \Phi(\mathfrak{g}, \mathfrak{b}) \mid e^\alpha(B) \subset \mathbf{R}\}, \quad \Phi_{\mathbf{R},i} = \{\alpha \in \Phi_{\mathbf{R}}(\mathfrak{g}, \mathfrak{b}) \mid e^\alpha > 0 \quad \text{on } B^i\}; \tag{5.25}$$

both are sub-root systems of $\Phi(\mathfrak{g}, \mathfrak{b})$. We note that $\Phi_{\mathbf{R}}(\mathfrak{g}, \mathfrak{b})$ can be described equivalently as follows:

$$\Phi_{\mathbf{R}}(\mathfrak{g}, \mathfrak{b}) = \{\alpha \in \Phi(\mathfrak{g}, \mathfrak{b}) \mid \langle \alpha, \mathfrak{b}_0 \rangle \subset \mathbf{R}\}. \tag{5.26}$$

Indeed, if α lies in this latter set, $\chi = e^\alpha$ and $\bar{\chi}$ = complex conjugate of χ are characters by which B acts on $\mathfrak{g}$; their differentials $\alpha, \bar{\alpha}$ agree, and hence so do χ and $\bar{\chi}$. Fix a system of positive roots $\Phi^+(\mathfrak{g}, \mathfrak{b})$ which is compatible with $\Phi^+(\mathfrak{g}, \mathfrak{a})$; then

$$\Phi^+_{\mathbf{R}}(\mathfrak{g}, \mathfrak{b}) = \Phi_{\mathbf{R}}(\mathfrak{g}, \mathfrak{b}) \cap \Phi^+(\mathfrak{g}, \mathfrak{b}), \quad \Phi^+_{\mathbf{R},i} = \Phi_{\mathbf{R},i} \cap \Phi^+(\mathfrak{g}, \mathfrak{b}) \tag{5.27}$$

are systems of positive roots in $\Phi_{\mathbf{R}}(\mathfrak{g}, \mathfrak{b})$ and $\Phi_{\mathbf{R},i}$.

LEMMA 5.28. *The set* $(MA)^- \cap B^i \cap G'$ *coincides with*

$$\{b \in B^i \cap G' \mid e^\alpha(b) < 1 \quad \textit{for all } \alpha \in \Phi^+_{\mathbf{R},i} \textit{ such that } \alpha|_{\mathfrak{a}} \neq 0\}.$$

Proof. The definition (3.5) describes $(MA)^-$ as the interior, in MA, of

$$\left\{ g \in MA \;\middle|\; \sum_p (-1)^p \Theta_{MA}(\Lambda^p \mathfrak{n})(ga) \geq 0 \quad \text{for all } a \in A^- \right\}. \tag{5.29}$$

Let us suppose $b \in B^i \cap G'$ lies in the set (5.29). Then, for $a \in A$,

$$\sum_p (-1)^p \Theta_{MA}(\Lambda^p)(ba) = \det(1 - \operatorname{Ad}(ba))\Big|_{\mathfrak{n}} = \prod_{\alpha \in \Phi^+(\mathfrak{g}, \mathfrak{b}),\, \alpha \neq 0 \text{ on } \mathfrak{a}} (1 - e^\alpha(ba)) \geq 0 \tag{5.30}$$

(cf. (3.3)). Non-real roots occur in pairs $\alpha, \bar{\alpha}$. If one of the two lies in $\Phi^+(\mathfrak{g}, \mathfrak{b})$ and restricts non-trivially to $\mathfrak{a}$, then so does the other. In this situation the factor

$$(1 - e^\alpha(ba))(1 - e^{\bar{\alpha}}(ba))$$

appears in the product on the right hand side of (5.30), is non-negative and vanishes—with fixed b and variable $a \in A$—on a proper subvariety of A. It can therefore be omitted from the product without affecting its non-negativity. If α belongs to $\Phi^+_{\mathbf{R}}(\mathfrak{g}, \mathfrak{b})$ but not to $\Phi^+_{\mathbf{R},i}$, the factor $(1-e^\alpha(ba))$ is strictly positive and can also be omitted:

$$\prod_{\alpha\in\Phi^+_{\mathbf{R},i},\ \alpha\neq 0 \text{ on } \mathfrak{a}} (1-e^\alpha(ba)) \geq 0 \quad \text{for } a\in A^-. \tag{5.31}$$

Each factors of this product is strictly positive if $a\in A^-$ lies far from the boundary of A^-. Let a_0 be such a point, and $a(t)$, $0\leq t\leq 1$, a continuous path from $a(0)=a_0$ to $a(1)=e$, wholly inside A^- except for the endpoint e. Moving the path slightly, we can prevent it from crossing two or more of the hyperplanes

$$\{a\in A \mid e^\alpha(a)=e^{-\alpha}(b)\}, \quad \alpha\in\Phi^+_{\mathbf{R},i},\ \alpha|_{\mathfrak{a}}\neq 0$$

at any one time, until it reaches the endpoint. Since the product (5.31) remains non-negative along the entire path, no factor can change sign, and hence all factors are still non-negative at $a=e$. In fact, they are strictly positive: $b\in B^i\cap G'$ is regular, so $e^\alpha(b)\neq 1$. The first part of the proof can be reversed, and we may conclude: the set (5.29) intersects $B^i\cap G'$ in

$$(B^i)^- = \{b\in B^i\cap G' \mid e^\alpha(b)<1 \quad \text{if } \alpha\in\Phi^+_{\mathbf{R},i},\ \alpha|_{\mathfrak{a}}\neq 0\}.$$

Since $(MA)^-$ is the interior of the set (5.29),

$$(MA)^-\cap B^i\cap G' \subset (B^i)^-. \tag{5.32}$$

The MA-conjugacy classes which meet $(B^i)^-$ lie entirely inside the set (5.29), because the latter is MA-conjugation invariant, and they constitute an open subset of MA, because $(B^i)^-$ is open in B and consists of elements which are regular in MA. This proves the containment opposite to (5.32). The lemma follows.

We can now verify the statements (3.7–8). Let $g\in MA$ be semisimple and regular in G. Then g lies in a Cartan subgroup $B\subset MA$. The Weyl group $W(G,B)$ contains all reflections about roots $\alpha\in\Phi_{\mathbf{R}}(\mathfrak{g},\mathfrak{b})$. As was remarked before, $W(G,B)$ may not act on B, but wg, with $w\in W(G,B)$, does makes sense up to conjugation by an element of B. In particular, an appropriate conjugate $\tilde{g}\in B$ of g satisfies the inequalities

$$e^\alpha(\tilde{g})<1 \quad \text{for } \alpha\in\Phi^+_{\mathbf{R}}(\mathfrak{g},\mathfrak{b}),$$

necessarily strictly, because g is regular. Lemma 5.28 ensures the containment $\tilde{g}\in(MA)^-$, which proves (3.7).

As for (3.8), let $m \in M$ be given. If $a_0 \in A^-$ lies far away from the boundary of A^-, the eigenvalues of $\operatorname{Ad} a_0$ on $\mathfrak{n}$ will be small in relation to the operator norm of $\operatorname{Ad} m$. Hence $0 < \det(1 - \operatorname{Ad}(ma_0))|_{\mathfrak{n}} = \Sigma_p (-1)^p \Theta_{MA}(\Lambda^p \mathfrak{n})(ma_0)$. This inequality persists if we replace a_0 by $a_1 a$, with $a_1 \in A^-$ lying close to a_0 and $a \in A^-$, and m by any nearby $m_1 \in M$. We conclude that ma_0 lies in the interior of

$$\left\{ g \in MA \,\middle|\, \sum_p (-1)^p \Theta_{MA}(\Lambda^p \mathfrak{n})(ga) \geq 0 \quad \text{for all } a \in A^- \right\},$$

i.e., in $(MA)^-$.

§ 6. Osborne's conjecture: the case of a minimal parabolic

We are now ready to prove theorem 3.6—or equivalently, the identity (3.13)—in the case of a minimal parabolic subgroup $P_m \subset G$. We fix a Langlands decomposition

$$P_m = M_m A_m N_m, \tag{6.1}$$

with $M_m \subset K$, and extend A_m to a Cartan subgroup $B_m \subset M_m A_m$. In $\Phi(\mathfrak{g}, \mathfrak{b}_m)$ we pick a positive root system $\Phi^+(\mathfrak{g}, \mathfrak{b}_m)$ compatible with $\Phi^+(\mathfrak{g}, \mathfrak{a}_m)$; cf. (3.23). According to the discussion in § 3, we may assume that $Z(\mathfrak{g})$ acts on the virtual character Θ via $\chi_{\mathfrak{g},\lambda}$, for some $\lambda \in \mathfrak{b}_m^*$ subject to the conditions (3.63 a, b); the μ_i and the constants C will be specified later. Since the identity (3.13) is linear in Θ, we are also free to assume

$$\Theta = \Theta(V) \text{ is the character of an irreducible Harish-Chandra module } V. \tag{6.2}$$

Recall the definitions (5.15, 5.20) of the μ-components $\Theta_G(V)_\mu, \Theta_{\mathfrak{n}_m}(V)_\mu$ corresponding to $P = P_m$ and $\mu \in \mathfrak{a}_m^*$. Because of (5.21) and the reductions which were made in § 3, the problem is to prove

$$\Theta_G(V)_\mu = \Theta_{\mathfrak{n}_m}(V)_\mu \quad \text{on } (M_m A_m)^- \cap G', \tag{6.3}$$

with

$$\mu = \text{restriction of } \lambda \text{ to } \mathfrak{a}_m. \tag{6.4}$$

We note that (3.63 a) implies the hypothesis of lemma 5.24, hence

$$\Theta_{\mathfrak{n}_m}(V)_\mu = \Theta_{M_m A_m}(H_0(\mathfrak{n}_m, V)_\mu) \tag{6.5}$$

on $(M_m A_m)^- \cap G'$.

LEMMA 6.6. *The group $M_m A_m$ acts semisimply on $H_0(\mathfrak{n}_m, V)$.*

Proof. Since M_m is compact and A_m connected, we only need to worry about the action of the Lie algebra $\mathfrak{a}_m$. In fact, it suffices to show that $\mathfrak{a}_m$ acts semisimply on the lowest weight spaces in the M_m-module $H_0(\mathfrak{n}_m, V)$, or equivalently, on

$$H_0(\mathfrak{n}_m, V)/(\mathfrak{r} \cap \mathfrak{m}_m) H_0(\mathfrak{n}_m, V) \simeq H_0(\mathfrak{r}, V), \tag{6.7}$$

with $\mathfrak{r}$ = span of all root spaces $\mathfrak{g}^\alpha$ indexed by roots $\alpha \in \Phi^+(\mathfrak{g}, \mathfrak{b}_m)$. The $\mathfrak{a}_m$-action on the module (6.7) extends naturally to an action of $\mathfrak{b}_m$. We identify $U(\mathfrak{b}_m)$ with the algebra of polynomial functions on $\mathfrak{b}_m^*$. Then

$$I(\mathfrak{b}_m) = \text{algebra of } W(\mathfrak{g}, \mathfrak{b}_m)\text{-invariants in } U(\mathfrak{b}_m)$$

operates on the $\mathfrak{b}_m$-module $H_0(\mathfrak{r}, V) \otimes \mathbf{C}_{-\varrho}$ according to the character

$$p \mapsto p(\lambda) \quad (p \in I(\mathfrak{b}_m)). \tag{6.8}$$

This follows from the Casselman-Osborne lemma 2.21, coupled with (3.31); the normalization in the definition of the Harish-Chandra homomorphism $\gamma_\mathfrak{g}$ accounts for the shift by the one-dimensional $\mathfrak{b}_m$-module $\mathbf{C}_{-\varrho}$. Because of (3.63 a), λ is not a fixed point of $W(\mathfrak{g}, \mathfrak{b}_m)$—in geometric language, the covering

$$\mathfrak{b}_m^* \xrightarrow{\pi} \mathfrak{b}_m^*/W(\mathfrak{g}, \mathfrak{b}_m)$$

does not ramify over $\pi(\lambda)$. Let E be one of the generalized eigenspaces of $\mathfrak{b}_m$, acting on $H_0(\mathfrak{r}, V) \otimes \mathbf{C}_{-\varrho}$, and $X \in \mathfrak{b}_m$ a linear function on $\mathfrak{b}_m^*$ which separates the various $W(\mathfrak{g}, \mathfrak{b}_m)$-translates of λ. The character (6.8) annihilates the product

$$p = \prod_{w \in W(\mathfrak{g}, \mathfrak{b}_m)} w(X - X(\lambda) \cdot 1) \in I(\mathfrak{b}_m),$$

but only one of the factors can fail to be invertible on E. We conclude: the annihilator of E in $U(\mathfrak{b}_m)$ contains one of the maximal ideals lying above $\pi(\lambda)$. Since the generalized eigenspace E is arbitrary, $\mathfrak{b}_m$ must act semisimply on $H_0(\mathfrak{r}, V)$, and hence so does $\mathfrak{a}_m$.

To simplify the notation, we let W denote $H_0(\mathfrak{n}_m, V)_\mu$, viewed as an M_m-module. Because of lemma 6.6, A_m acts via the character $e^{\mu + \varrho_m}$ (the shift by $\varrho_m = \varrho_{P_m}$ is built into the definition (2.30)!), hence

$$H_0(\mathfrak{n}_m, V)_\mu \simeq W \otimes \mathbf{C}_{\mu + \varrho_m}, \tag{6.9}$$

as $M_m A_m$-modules.

LEMMA 6.10. *Suppose $H_0(\mathfrak{n}_m, V)_\mu \neq 0$. Then M_m acts irreducibly on W. The induced Harish-Chandra module $I^G_{P_m}(W \otimes \mathbf{C}_\mu)$ contains V as a submodule. Among the composition factors of $I^G_{P_m}(W \otimes \mathbf{C}_\mu)$, only V satisfies $H_0(\mathfrak{n}_m, V)_\mu \neq 0$.*

Proof. Let $W = \oplus_{i=1}^k W_i$ be a decomposition of W into irreducibles. The Frobenius reciprocity theorem 4.11 provides inclusions

$$V \hookrightarrow I^G_{P_m}(W_i \otimes \mathbf{C}_\mu), \quad 1 \leq i \leq k \tag{6.11}$$

(V is irreducible!). If we choose the constant C large enough, (3.67) implies

$$(\mu + \varrho_m, \alpha) < 0 \quad \text{for all } \alpha \in \Phi^+(\mathfrak{g}, \mathfrak{a}_m). \tag{6.12}$$

In this situation, if W_i is an arbitrary irreducible M_m-module,

$$\begin{gathered} I^G_{P_m}(W_i \otimes \mathbf{C}_\mu) \text{ has a unique irreducible submodule; it is the one and only} \\ \text{composition factor of } I^G_{P_m}(W_i \otimes \mathbf{C}_\mu) \text{ which has } \mu \text{ as a leading exponent.} \end{gathered} \tag{6.13}$$

Moreover,

$$\begin{gathered} \text{the unique irreducible submodules of } I^G_{P_m}(W_i \otimes \mathbf{C}_\mu) \text{ and } I^G_{P_m}(W_j \otimes \mathbf{C}_\mu) \\ \text{are non-isomorphic unless } W_i \simeq W_j. \end{gathered} \tag{6.14}$$

Both (6.13) and (6.14) are formal consequences of Langlands' classification, in conjunction with an observation of Miličić; an elementary, self-contained proof will be given in the appendix. From (6.14) we immediately deduce that all the irreducible summands $W_1, \ldots, W_k$ of W are isomorphic. According to the reciprocity theorem 4.11,

$$\operatorname{Hom}_{M_m A_m}(H_0(\mathfrak{n}_m, V), W_1 \otimes \mathbf{C}_{\mu+\varrho_m}) \simeq \operatorname{Hom}_G(V, I^G_{P_m}(W_1 \otimes \mathbf{C}_\mu)).$$

On the one hand, this space has dimension k, since $H_0(\mathfrak{n}_m, V)_\mu$ is a direct sum of k copies of $W_1 \otimes \mathbf{C}_{\mu+\varrho_m}$; on the other, it has dimension one, since V is the only irreducible submodule of $I^G_{P_m}(W_1 \otimes \mathbf{C}_\mu)$. Thus $W = W_1$ is irreducible. Any composition factor U of $I^G_{P_m}(W \otimes \mathbf{C}_\mu)$ which satisfies $H_0(\mathfrak{n}_m, U)_\mu \neq 0$ has μ as a leading exponent: the condition (3.63 a) ensures that μ is minimal among the exponents of $H_0(\mathfrak{n}_m, V)$, and so we may appeal to theorem 4.25. At this point (6.13) implies $U = V$, as was to be shown.

Our next result links homology to the global character: the μ-component $\Theta_{\mathfrak{n}_m}(V)_\mu$ can be non-zero on $(M_m A_m)^-$ only if $H_0(\mathfrak{n}_m, V)_\mu \neq 0$. In the following section we shall use the analogous statement for arbitrary parabolic subgroups. To avoid duplication, we treat this more general case now. Let $P=MAN$ be a proper parabolic subgroup of G, and B a Cartan subgroup of G, lying in MA. As before, $\Theta=\Theta(V)$ denotes the character of an irreducible Harish-Chandra module V, on which $Z(\mathfrak{g})$ acts via $\chi_{\mathfrak{g},\lambda}$, with $\lambda \in \mathfrak{b}^*$ subject to the conditions (3.63 a, b)—the choices of the μ_i and of the constant C will be specified during the course of the proof. We set μ = restriction of λ to $\mathfrak{a}$.

PROPOSITION 6.15. *Suppose $H_0(\mathfrak{n}, V)_\mu=0$. Then $\Theta_G(V)_\mu$ vanishes on $(MA)^- \cap G'$.*

Assuming the proposition, or merely its specialization to $P=P_m$, we can complete the proof of theorem 3.6 for the minimal parabolic subgroup P_m. If $H_0(\mathfrak{n}, V)_\mu=0$, the identity (6.3) follows from (6.5) and proposition 6.15; both $\Theta_G(V)_\mu$ and $\Theta_{\mathfrak{n}_m}(V)_\mu$ vanish on $(M_m A_m)^- \cap G'$. On the other hand, if $H_0(\mathfrak{n}_m, V)_\mu \neq 0$, we apply lemma 6.10: according to what we just saw, V is the only composition factor of $I^G_{P_m}(W \otimes \mathbf{C}_\mu)$ such that $\Theta_G(V)_\mu$ restricts non-trivally to $(M_m A_m)^- \cap G'$, hence

$$\Theta_G(V)_\mu = \Theta_G(I^G_{P_m}(W \otimes \mathbf{C}_\mu))_\mu \quad \text{on } (M_m A_m)^- \cap G'.$$

In this situation (6.5), (6.9) and lemma 5.17 imply the identity (6.3). As was pointed out before, theorem 3.6 for $P=P_m$ is a consequence of (6.3).

Proposition 6.15 is similar to a result of Miličić (Theorem 1 in part III of [31]). Miličić bounds the global character in terms of Harish-Chandra's Ξ function, but this is not quite enough for our needs. At least for $P=P_m$ the proposition can be deduced from Langlands' classification and Harish-Chandra's bound on tempered characters [19]. Here we proceed differently, since we shall use Osborne's conjecture to establish these results.

As a first step in the proof of the proposition, we estimate the "diagonal matrix coefficients" of V. For this purpose we realize V as a submodule of an induced Harish-Chandra module, in the manner described by lemma 4.33:

$$V \subset I^G_{\tilde{P}}(W \otimes \mathbf{C}_\nu), \tag{6.16}$$

where $\tilde{P}=\tilde{M}\tilde{A}\tilde{N}$, ν, W is a set of Langlands data. Enlarging the constant C in (3.68) if necessary, we ensure that $\nu \in \tilde{\mathfrak{a}}^*$ satisfies not only (4.32), but even the stronger condition

$$\operatorname{Re}(\nu+\varrho_{\tilde{P}}, \alpha)<0 \quad \text{for } \alpha \in \Phi^+(\mathfrak{g}, \tilde{\mathfrak{a}}). \tag{6.17}$$

The infinitesimal character of W is of the form $\chi_{\tilde{\mathfrak{m}},\tilde{\lambda}}$, with $\tilde{\lambda}$ = restriction to $\tilde{\mathfrak{m}}$ of λ, transferred to a Cartan subalgebra of $\tilde{\mathfrak{m}}\oplus\tilde{\mathfrak{a}}$ by an inner automorphism of $\mathfrak{g}$. By making an appropriate choice of the μ_i in (3.63), we arrange that W satisfies the conditions analogous to (4.31), with $\tilde{M}$ taking the place of G; cf. remark 4.50. Then W is square-integrable—in particular,

$$W \text{ is a unitary}(^1) \text{ Harish-Chandra-module for } \tilde{M}. \tag{6.18}$$

The quantities μ_i, C which were just specified appear to depend on $\tilde{P}$, but we are able to make a uniform choice because G contains only finitely many conjugacy classes of parabolic subgroups.

For each $i\in\hat{K}$, we let d_i denote the degree of i, and V_i the i-isotypic subspace of V. The dual space V_i^* is naturally isomorphic to the i^*-isotypic subspace of the dual Harish-Chandra module V' (i^* = contragredient of i). To every pair of dual bases $\{v_\alpha\}\subset V_i$, $\{v_\alpha^*\}\subset V_i^*$ one can therefore associate "diagonal matrix coefficients" f_{v_α, v_α^*}, as in (4.13). Their sum

$$F_i = \sum_\alpha f_{v_\alpha, v_\alpha^*} \tag{6.19}$$

depends only on i, not on the particular choice of bases. The character $\Theta=\Theta(V)$ is the sum of all "diagonal matrix coefficients",

$$\Theta = \sum_{i\in\hat{K}} F_i, \tag{6.20}$$

in the sense of distributions. We shall measure the growth of the F_i in terms of a certain spherical function. In order to describe it, we replace $\tilde{P}$ by a conjugate if necessary, so that $\tilde{M}\tilde{A}\supset M_m A_m$, $\tilde{A}\subset A_m$, $\tilde{N}\subset N_m$. With a slight abuse of notation, we refer to the extensions of ν and $\tilde{\varrho}$ (shorthand for $\varrho_{\tilde{P}}$) from $\tilde{\mathfrak{a}}$ to $\mathfrak{a}_m$ by the same letters; thus

$$\nu|_{\tilde{\mathfrak{m}}\cap\mathfrak{a}_m} = \tilde{\varrho}|_{\tilde{\mathfrak{m}}\cap\mathfrak{a}_m} = 0. \tag{6.21}$$

Every $g\in G$ can be expressed uniquely as $g=k(g)\,a(g)\,n(g)$, with $k(g)\in K$, $a(g)\in A_m$, $n(g)\in N_m$. For $\xi\in\mathfrak{a}_m^*$, define

$$\varphi_\xi(g) = \int_K e^{-(\xi+\varrho_m)}(a(g^{-1}k))\,dk. \tag{6.22}$$

(1) It is known that temperedness already implies unitarity. We have avoided using this fact, which is proved in § 8.

These are Harish-Chandra's spherical functions [17].

LEMMA 6.23 (cf. [31, p. 83]). *Let $S \subset G$ be compact. There exists a constant C, depending only on S and ν, such that*

$$|F_i(gxg^{-1})| \leq C\, d_i^3\, \varphi_{\mathrm{Re}\,\nu+\tilde{\varrho}-\varrho_m}(x),$$

for all $i \in \hat{K}$, $g \in S$ and $x \in G$.

Proof. The Harish-Chandra module W has a unitary globalization W_τ, which in turn determines a globalization $I_{\tilde{P}}^G(W_\tau \otimes \mathbf{C}_\nu)$ of $I_{\tilde{P}}^G(W \otimes \mathbf{C}_\nu)$. By definition, every $f \in I_{\tilde{P}}^G(W_\tau \otimes \mathbf{C}_\nu)$ is a right $\tilde{P}$-invariant function from G to $W \otimes \mathbf{C}_{\nu+\tilde{\varrho}}$. For $g \in G$ and $k \in K$, we write $gk = l\tilde{m}\tilde{a}\tilde{n}$, with $l \in K$, $\tilde{m} \in \tilde{M}$, $\tilde{a} \in \tilde{A}$, $\tilde{n} \in \tilde{N}$. Since τ is unitary,

$$\|f(gk)\| = \|e^{-(\nu+\tilde{\varrho})}(\tilde{a})\, \tau(\tilde{m}^{-1})\, f(l)\| = e^{-(\mathrm{Re}\,\nu+\tilde{\varrho})}(\tilde{a})\, \|f(l)\|.$$

Factoring $\tilde{m}$ as $l'a'n'$, with $l' \in K \cap \tilde{M}$, $a' \in A_m \cap \tilde{M}$, $n' \in N_m \cap \tilde{M}$, we find

$$gk = ll'a'\tilde{a}(\tilde{a}^{-1}n'\tilde{a})\,\tilde{n},$$

hence $a(gk) = a'\tilde{a}$, $e^{-(\mathrm{Re}\,\nu+\tilde{\varrho})}(\tilde{a}) = e^{-(\mathrm{Re}\,\nu+\tilde{\varrho})}(a(gk))$ (cf. (6.21)), and finally

$$\|f(gk)\| = e^{-(\mathrm{Re}\,\nu+\tilde{\varrho})}(a(gk))\, \|f(ll')\|. \tag{6.24}$$

The first factor on the right remains bounded as g ranges over the compact set S and k over K. Assuming f is bounded on K,

$$\sup_{k\in K} \|f(gk)\| \leq C_1 \sup_{k \in K} \|f(k)\|, \quad \text{for } g \in S, \tag{6.25}$$

with C_1 depending on S and ν. Now let us suppose that f has length one, relative to the K-invariant inner product (4.3), and lies in a K-invariant, K-irreducible subspace of type i; we claim:

$$\sup_{k \in K} \|f(k)\| \leq d_i^{1/2}. \tag{6.26}$$

To see this, we extend $f = f_1$ to an orthonormal basis $\{f_1, \ldots, f_d\}$, $d = d_i$, of the K-invariant subspace in question. For $k, l \in K$,

$$f_s(l^{-1}k) = \sum_r a_{rs}(l)\, f_r(k), \quad \text{where } a_{rs}(l) = \int_K (f_s(l^{-1}k), f_r(k))\, dk.$$

The a_{rs} satisfy the Schur orthogonality relations, hence

$$\begin{aligned}1 &= \int_{l\in K} (f(l), f(l))\, dl = \int_{l\in K} (f(l^{-1}k), f(l^{-1}k))\, dl \\ &= \sum_{r,s} (f_r(k), f_s(k)) \int_{l\in K} a_{r1}(l)\, \bar{a}_{s1}(l)\, dl = \frac{1}{d}\sum_r \|f_r(k)\|^2,\end{aligned}$$

which implies (6.26). The G-invariant pairing

$$\langle f^*, f\rangle = \int_K \langle f^*(k), f(k)\rangle\, dk \tag{6.27}$$

exhibits $I_P^G(W_\tau'\otimes \mathbf{C}_{-\nu})$ as the Hilbert space dual to $I_P^G(W_\tau\otimes\mathbf{C}_\nu)$. Let $\{f_1, \ldots, f_n\}$ be an orthonormal basis of $I_P^G(W\otimes\mathbf{C}_\nu)_i$, whose first m members constitute a basis of $V_i\subset I_P^G(W\otimes\mathbf{C}_\nu)_i$. The dual basis $\{f_1^*, \ldots, f_n^*\}$ of $I_P^G(W'\otimes\mathbf{C}_\nu)_{i^*}$ is then also orthonormal, and the natural projection $I_P^G(W'\otimes\mathbf{C}_{-\nu})\to V_i^*$ sends $\{f_1^*, \ldots, f_m^*\}$ to the basis dual to $\{f_1, \ldots, f_m\}$. In particular,

$$\begin{aligned}F_i(gxg^{-1}) &= \sum_{r=1}^m \langle f_r^*, I_P^G(\tau\otimes e^\nu)(gxg^{-1}) f_r\rangle \\ &= \sum_{r=1}^m \langle \tilde{f}_r^*, I_P^G(\tau\otimes e^\nu)(x)\tilde{f}_r\rangle,\end{aligned} \tag{6.28}$$

where $\tilde{f}_r=I_P^G(\tau\otimes e^\nu)(g^{-1})f_r$, $\tilde{f}_r^*=I_P^G(\tau'\otimes e^{-\nu})(g^{-1})f_r^*$. We may assume that the K-translates of each of the f_r and f_r^* span a K-irreducible subspace. According to (6.25–26), if g lies in S,

$$\sup_{k\in K}\|\tilde{f}_r(k)\| = \sup_{k\in K}\|f_r(gk)\| \le C_1\, d_i^{1/2}. \tag{6.29}$$

This bound has a direct analogue for $\tilde{f}_r^*$: there exists a constant C_2, such that

$$\sup_{k\in K}\|\tilde{f}_r^*(k)\| = C_2\, d_i^{1/2}. \tag{6.30}$$

Since V is irreducible,

$$m = \dim V_i \le d_i^2. \tag{6.31}$$

The statement of the lemma now follows from (6.24) and (6.27–31):

$$\begin{aligned}|F_i(gxg^{-1})| &\leq \sum_{r=1}^{m} |\langle \tilde{f}_r^*, I_P^G(\tau\otimes e^{\nu})(x)\tilde{f}_r\rangle| \\ &= \sum_{r=1}^{m} \left| \int_K \langle \tilde{f}_r^*(k), \tilde{f}_r(x^{-1}k)\rangle\, dk \right| \\ &\leq \sum_{r=1}^{m} \sup_{k\in K} \|\tilde{f}_r^*(k)\| \sup_{k\in K} \|\tilde{f}_r(k)\| \int_K e^{-(\mathrm{Re}\,\nu+\tilde{\varrho})}(a(x^{-1}k))\, dk \\ &\leq C\, d_i^3\, \varphi_{\mathrm{Re}\,\nu+\tilde{\varrho}-\varrho_m}(x),\end{aligned}$$

where $C=C_1C_2$.

Our next objective is to translate the estimate of the F_i into a bound on Θ. The arguments are inspired by the proof of proposition 6.10 of [1], and incorporate ideas of Harish-Chandra [19]. As before, B shall denote a Cartan subgroup in MA. To any $\varepsilon>0$ we associate the set

$$B_\varepsilon = \{b\in B \mid |e^{\alpha}(b)-1|>\varepsilon \quad \text{for } \alpha\in\Phi(\mathfrak{g},\mathfrak{b})\}; \tag{6.32}$$

then B_ε is open in B, and

$$B\cap G' = \bigcup_{\varepsilon>0} B_\varepsilon. \tag{6.33}$$

Since B^0, the connected component of the identity, lies in the center of B, the assignment

$$\xi\colon (gB^0, b)\mapsto gbg^{-1} \tag{6.34}$$

describes a map of $G/B^0\times B_\varepsilon$ onto B_ε^G, the union of all conjugacy classes passing through B_ε. It is a normal covering, with group $N_G(B)/B^0$, which is finite because of (2.1). Once and for all we fix a function $\varphi\in C_0^\infty(G/B^0)$, such that $\int_{G/B^0}\varphi\, dg^*=1$, and we define

$$T\colon C_0^\infty(B_\varepsilon)\to C_0^\infty(B_\varepsilon^G) \tag{6.35}$$

by the rule $Tf(g)=$ average, over the fiber $\xi^{-1}(g)$, of the function

$$(gB^0, b)\mapsto \varphi(gB^0)f(b)\,|D_G(b)|^{-1/2} \tag{6.36}$$

(cf. (5.2)). According to a standard integration formula,

$$\int_{B^G} h\,dg = c\int_B \int_{G/B^0} |D_G(b)|\,h(gbg^{-1})\,dg^*\,db, \tag{6.37}$$

for every compactly supported, continuous function h on B^G; the constant c reflects the normalization of the invariant measures and can be made equal to one. Both Θ and D_G are conjugation invariant, hence restrict to $N_G(B)$-invariant functions on B. Applying the integration formula and averaging with respect to $N_G(B)/B^0$, we find

$$\begin{aligned}\Theta(Tf) &= \int_B \int_{G/B^0} \varphi(gB^0)\,\Theta(b)\,|D_G(b)|^{1/2} f(b)\,dg^*\,db \\ &= \int_B \Theta |D_G|^{1/2} f\,db,\end{aligned} \tag{6.38}$$

provided the measures are suitably normalized.

LEMMA 6.39. *The distribution $f \mapsto \Theta(Tf)$ on B_ε can be expressed as $\Sigma_{j=1}^m X_j h_j$, in terms of translation invariant linear differential operators $X_1, \dots, X_m$ on B and continuous functions $h_1, \dots, h_m$ on $B \cap G'$, such that*

$$|h_j(b)| \leq C(\varepsilon)\,|D_G(b)|^{1/2} \varphi_{\mathrm{Re}\,\nu+\hat{\varrho}-\varrho_m}(b), \quad \textit{for all } b \in B_\varepsilon.$$

The X_j and h_j can be chosen independently of ε.

Proof. By infinitesimal right translation, each $X \in \mathfrak{g}$ determines a left invariant vector field $r(X)$ on G. The map $X \mapsto r(X)$ extends to $r\colon U(\mathfrak{g}) \to$algebra of left invariant linear differential operators. For $i \in \hat{K}$, the Casimir operator $\Omega_K \in U(\mathfrak{k}) \subset U(\mathfrak{g})$ of K acts on F_i according to the same constant $\omega_i \geq 0$ by which Ω_K operates on V_i, so

$$\Theta = \sum_{i \in \hat{K}} F_i = r(1+\Omega_K)^n \sum_{i \in \hat{K}} (1+\omega_i)^{-n} F_i, \tag{6.40}$$

in the sense of distributions. If the integer $n \in N$ is large enough,

$$\sum_{i \in \hat{K}} (1+\omega_i)^{-n} d_i^3 < \infty. \tag{6.41}$$

Indeed, the unitary dual of K^0 is parametrized by the non-singular, dominant points in the weight lattice; in terms of this parametrization, ω_i equals the square length of the lattice point minus a constant, and d_i is a polynomial function. Since K^0 has finite index in K, (6.41) follows. Combining lemma 6.23 with (6.40–41), we find: there exist an integer n and a continuous function h on G, such that

(a) $\Theta = r(1+\Omega_K)^n h$, in the sense of distributions;
(b) if $S \subset G$ is compact, $|h(gxg^{-1})| \leq C(S)\, \varphi_{\mathrm{Re}\,\nu+\tilde{\varrho}-\varrho_m}(x)$, for all $g \in S$, $x \in G$. (6.42)

The covering ξ pulls back the operator $r(1+\Omega_K)$ to an operator on $G/B^0 \times B_\varepsilon$. On general grounds,

$$\xi^* r(1+\Omega_K)^n = \sum_{r,s} \psi_{rs} Z_r \otimes Y_s, \tag{6.43}$$

independently of ε, where $Z_1, \ldots, Z_k$ are linear differential operators on G/B^0, $Y_1, \ldots, Y_l$ translation invariant linear differential operators on B, and $\psi_{rs} \in C^\infty(G/B^0 \times (B \cap G'))$. Let $w_1, \ldots, w_N \in N_G(B)$ be a set of representatives for $N_G(B)/B^0$. The differential operator (6.43) commutes with the action of the covering group, hence

$$\begin{aligned}\{r(1+\Omega_K)^n Tf\}\,(gbg^{-1}) &= \{\xi^* r(1+\Omega_K)^n (\xi^* Tf\}\,(gB^0, b)\\ &= \frac{1}{N}\sum_{r,s,t} \psi_{rs}(gw_tB^0, w_tbw_t^{-1})\, Z_r\varphi(gw_tB^0)\, Y_s(f|D_G|^{-1/2})(w_tbw_t^{-1}),\end{aligned}$$

for $f \in C_0^\infty(B_\varepsilon)$, $g \in G$, $b \in B_\varepsilon$. Since Ω_K is self-adjoint, the integral of this function against the function h of (6.42) equals $\Theta(Tf)$. We use (6.37) to re-write the integral as an integral over $G/B^0 \times B$; we can then dispense with the averaging procedure because $\xi^* h$ is invariant under $N_G(B)/B^0$:

$$\Theta(Tf) = \sum_{r,s} \int_B \int_{G/B^0} \psi_{rs}(gB^0, b)\, Z_r\varphi(gB^0)\, h(gbg^{-1})\, |D_G(b)|^{1/2} (|D_G|^{1/2} Y_s |D_G|^{-1/2}) f(b)\, dg^*\, db.$$

The differential operators $|D_G|^{1/2} Y_s |D_G|^{-1/2}$ on $B \cap G'$ can be expressed as linear combinations of translation invariant operators $X_1, \ldots, X_m$, with C^∞ coefficients,

$$|D_G|^{1/2} Y_s |D_G|^{-1/2} = \sum_j a_{js} X_j.$$

The formal adjoints X_j^* of the X_j are also translation invariant. Define

$$h_j(b) = \sum_{r,s} a_{js}(b) |D_G(b)|^{1/2} \int_{G/B^0} \psi_{rs}(gB^0, b)\, Z_r\varphi(gB^0)\, h(gbg^{-1})\, dg^*;$$

then

$$\Theta \circ T = \sum_j X_j^* h_j,$$

in the sense of distributions. In view of (6.42), this proves the lemma once it is known that the functions a_{js} and ψ_{rs} are uniformly bounded on the sets B_ε and $\operatorname{supp}\varphi \times B_\varepsilon$, respectively, for all $\varepsilon > 0$.

The boundedness of the a_{js} follows from the identity

$$|D_G|^{1/2} = c \prod_{\alpha>0} (e^{\alpha/2} - e^{-\alpha/2}),$$

with c constantly equal to ± 1 or $\pm i$ on any particular connected component of $B \cap G'$, as was pointed out in the beginning of § 5. In order to bound the ψ_{rs}, let us examine the differential of the map ξ. Via left translation by g, we identify the complexified tangent space $T(G/B^0)_{gB^0}$ with $\mathfrak{b}^\perp$ = orthogonal complement of $\mathfrak{b}$, relative to the Killing form. Similarly $T(G)_g \simeq \mathfrak{g}$ and $T(B)_b \simeq \mathfrak{b}$. In terms of these identifications, the differential ξ_* at (gB^0, b) is given by

$$\xi_*(X, Y) = \operatorname{Ad} g(Y + (\operatorname{Ad} b^{-1} - 1)X). \tag{6.44}$$

To see this, we note that $\xi_*(X, Y)$ is the tangent vector at $t=0$ to the curve

$$\begin{aligned} t \mapsto \xi(g \exp tX, b \exp tY) &= g \exp tX\, b \exp tY \exp(-tX)\, g^{-1} \\ &= gbg^{-1} \exp\{t \operatorname{Ad} g(Y + (\operatorname{Ad} b^{-1} - 1)X) + O(t^2)\}. \end{aligned}$$

The linear transformations

$$(\operatorname{Ad} b^{-1} - 1)^{-1} : \mathfrak{b}^\perp \to \mathfrak{b}^\perp, \quad b \in B \cap G',$$

are diagonal with respect to the root space decomposition and have eigenvalues $(e^\alpha(b) - 1)^{-1}$, $\alpha \in \Phi(\mathfrak{g}, \mathfrak{b})$, which are bounded on the sets B_ε, as are their derivatives of all orders. We can express the pullback to $G/B^0 \times (B \cap G')$ of any left invariant vector field $r(X)$ on G in terms of vector fields $\{Z_r\}$ on G/B^0 and translation invariant vector fields $\{Y_s\}$ on B,

$$\xi^* r(X) = \sum a_{rs} Z_r \otimes Y_s.$$

Inverting (6.44) and using the preceeding remark, we find that the coefficient functions a_{rs} have the following property: if Y is a translation invariant differential operator on B

and Z a linear differential operator on G/B^0, both of arbitrary order, then $(Z\otimes Y)a_{rs}$ remains bounded on any set of the form $\mathcal{O}\times B_\varepsilon$, with $\varepsilon>0$ and with a compact first factor $\mathcal{O}$. The analogous statement about left invariant linear differential operators $r(X)$ on G follows by induction on the order of X. In particular, this establishes the boundedness of the functions ψ_{rs} on $\operatorname{supp}\varphi\times B_\varepsilon$. The proof of the lemma is now complete.

The statement of proposition 6.15 remains unchanged if we replace P by a conjugate. We may therefore put P into "standard position", i.e.

$$MA\supset M_mA_m,\quad A\subset A_m,\quad N\subset N_m. \tag{6.45}$$

At this point we bring the hypothesis $H_0(\mathfrak{n}, V)_\mu=0$ into play.

LEMMA 6.46. *There exist a root $\alpha\in\Phi^+(\mathfrak{g},\mathfrak{a})$, a constant $\delta>0$ and, for each compact set $S\subset B$, a constant $C(S)$, such that*

$$|D_G(ba)|^{1/2}\varphi_{\operatorname{Re}\nu+\tilde{\varrho}-\varrho_m}(ba)\leq C(S)\,e^{\operatorname{Re}\mu+\delta\alpha}(a)$$

whenever $b\in S$, $a\in A^-$.

Proof. As linear functions on $\mathfrak{a}_m$ (cf. (6.21)), ν and $\tilde{\varrho}$ restrict to linear functions on $\mathfrak{a}$. We claim:

$$\begin{aligned}&\text{(a)}\ \ |D_G(ba)|^{1/2}\leq C_1\,e^{-\varrho_P}(a),\\ &\text{(b)}\ \ \varphi_{\operatorname{Re}\nu+\tilde{\varrho}-\varrho_m}(ba)\leq C_2\,e^{\operatorname{Re}\nu+\tilde{\varrho}}(a),\end{aligned} \tag{6.47}$$

again for all $b\in S$, $a\in A^-$, with constants C_1, C_2 which depend on S. Indeed, as b ranges over a compact set and a over A^-, $e^\alpha(ba)$ stays bounded if $\alpha\in\Phi(\mathfrak{g},\mathfrak{b})$ either restricts trivially to $\mathfrak{a}$ or projects to a root in $\Phi^+(\mathfrak{g},\mathfrak{a})$, and $e^\alpha(b)$ is bounded in any case; hence

$$\begin{aligned}|D_G(ba)|^{1/2}&=\left|\prod_{\alpha\in\Phi(\mathfrak{g},\mathfrak{b})}(e^\alpha(ba)-1)\right|^{1/2}\\ &\leq C_1\left\{\prod_{\alpha\in\Phi^+(\mathfrak{g},\mathfrak{a})}e^{-\alpha}(a)\right\}^{1/2}=C_1\,e^{-\varrho_P}(a).\end{aligned}$$

As for the second inequality, $A\subset B^0$ lies in the center of B, so

$$\varphi_{\operatorname{Re}\nu+\tilde{\varrho}-\varrho_m}(ba)=\varphi_{\operatorname{Re}\nu+\tilde{\varrho}-\varrho_m}(ab)=\int_K e^{-(\operatorname{Re}\nu+\tilde{\varrho})}(a(b^{-1}a^{-1}k))\,dk. \tag{6.48}$$

We write $a^{-1}k=l'a'n'$ and $b^{-1}l'=l''a''n''$, with $l', l''\in K$, $a', a''\in A_m$, $n', n''\in N_m$. Then

$$b^{-1}a^{-1}k = l''a''a'(a'^{-1}n''a')\,n',$$

hence $a(b^{-1}a^{-1}k)=a'a''=a(a^{-1}k)\,a(b^{-1}l)$. Since $a(b^{-1}l)$ is bounded as $b^{-1}l$ varies over the compact set $S^{-1}K$, there exists $C_2=C_2(S)$ such that

$$e^{-(\mathrm{Re}\,\nu+\tilde{\varrho})}(a(b^{-1}a^{-1}k)) \leqslant C_2\, e^{-(\mathrm{Re}\,\nu+\tilde{\varrho})}(a(a^{-1}k)). \tag{6.49}$$

Since $\tilde{P}$ was assumed to be in standard position, the negativity condition (6.17) implies

$$(\mathrm{Re}\,\nu+\tilde{\varrho}, \alpha)\leqslant 0 \quad \text{for } \alpha\in\Phi^+(\mathfrak{g}, \mathfrak{a}_m),$$

which in turn gives

$$e^{-(\mathrm{Re}\,\nu+\tilde{\varrho})}(a(a^{-1}k)) \leqslant e^{\mathrm{Re}\,\nu+\tilde{\varrho}}(a), \tag{6.50}$$

for all $a\in$ closure of A_m^-, $k\in K$. This inequality is a standard result (cf. [2, § 14]) and can be deduced from simple manipulations with finite dimensional representations; we shall sketch a proof in the appendix. Since $A^-\subset$ closure of A_m^-, (6.48–50) imply (6.47 b).

The lemma follows from (6.47) once we produce $\delta>0$ and $\alpha\in\Phi^+(\mathfrak{g}, \mathfrak{a})$, with

$$e^{\mathrm{Re}\,\nu+\tilde{\varrho}}(a)\leqslant e^{\mathrm{Re}\,\mu+\varrho_P+\delta\alpha}(a), \quad \text{for } a\in A^-. \tag{6.51}$$

We ennumerate the simple roots in $\Phi^+(\mathfrak{g}, \mathfrak{a}_m)$ as $\alpha_1, \ldots, \alpha_r$, so that

$$\alpha_i\neq 0 \text{ on } \tilde{\mathfrak{a}} \iff 1\leqslant i\leqslant s, \tag{6.52}$$

and we let $\mu_1, \ldots, \mu_r$ denote the corresponding fundamental highest weights in $\mathfrak{a}_m^*$. We transfer $\lambda\in\mathfrak{b}^*$ to a linear function λ_m on $\mathfrak{b}_m$ (=complexified Lie algebra of a Cartan subgroup $B_m\subset M_mA_m$), via an inner automorphism of $\mathfrak{m}\oplus\mathfrak{a}$ which makes λ_m anti-dominant on $\mathfrak{b}_m\cap\mathfrak{m}$, in the sense of (3.65). Then λ and λ_m have the same restriction to $\mathfrak{a}$, namely μ, and

$$\mathrm{Re}\,(\lambda_m, \alpha)<-C \quad \text{if } \alpha\in\Phi^+(\mathfrak{g}, \mathfrak{b}_m) \tag{6.53}$$

(cf. (3.66)). Recall how the embedding (6.16) was constructed in § 4: V has a leading exponent, necessarily of the form $w\lambda_m|_{\mathfrak{a}_m}$, $w\in W(\mathfrak{g}, \mathfrak{b}_m)$, such that

$$\begin{aligned} &\text{(a)}\ \ w\lambda_m|_{\mathfrak{a}_m} = \nu+\textstyle\sum_{i=s+1}^r a_i\alpha_i,\ \mathrm{Re}\,a_i\geqslant 0, \text{ and}\\ &\text{(b)}\ \ \nu = -\textstyle\sum_{i=1}^s b_i\mu_i \text{ on } \mathfrak{a}_m\cap[\mathfrak{g}, \mathfrak{g}],\ \mathrm{Re}\,b_i>0; \end{aligned} \tag{6.54}$$

here ν is regarded as a linear function on $\mathfrak{a}_m$, as in (6.21). According to theorem 4.25, the same exponent shows up as an exponent of

$$H_0(\mathfrak{n}_m, V) \simeq H_0(\mathfrak{n}_m \cap \mathfrak{m}, H_0(\mathfrak{n}, V)).$$

Since $H_0(\mathfrak{n}, V)_\mu$ was assumed to vanish,

$$w\lambda_m|_{\mathfrak{a}} + \mu \neq \lambda_m|_{\mathfrak{a}}. \tag{6.55}$$

The anti-dominance condition (6.53) implies

$$w\lambda_m|_{\mathfrak{a}_m} = \lambda_m|_{\mathfrak{a}_m} + \sum_{i=1}^{r} c_i \alpha_i, \quad \operatorname{Re} c_i \geqslant 0. \tag{6.56}$$

As a conjugate of λ, λ_m has a regular real part; cf. (3.66). Hence, if $\operatorname{Re} c_i = 0$, any minimal expression of w in terms of reflections about simple roots cannot involve a reflection about a simple root whose restriction to $\mathfrak{a}_m$ equals α_i, in which case c_i itself vanishes. Taking into account (6.55), we find:

$$\text{there exists an integer } j,\ 1 \leqslant j \leqslant r, \text{ such that } \operatorname{Re} c_j > 0 \quad \text{and} \quad \alpha_j|_{\mathfrak{a}} \neq 0. \tag{6.57}$$

It will be convenient to view μ and ϱ_P as linear functions on all of $\mathfrak{a}_m$, trivial on $\mathfrak{a}_m \cap \mathfrak{m}$. Since $\nu - \mu$ and $\tilde{\varrho} - \varrho_P$ vanish on the split part of the center of $\mathfrak{g}, \nu - \mu + \tilde{\varrho} - \varrho_P$ can be expressed as a linear combination of the α_i,

$$\nu - \mu + \tilde{\varrho} - \varrho_P = \sum_{i=1}^{r} d_i \alpha_i. \tag{6.58}$$

To prove (6.51), it suffices to show

$$\begin{aligned} &\text{(a) } \operatorname{Re} d_j \geqslant 0 \text{ whenever } \alpha_j|_{\mathfrak{a}} \neq 0, \text{ and} \\ &\text{(b) this inequality holds strictly for at least one such } \alpha_j. \end{aligned} \tag{6.59}$$

Since α_j does not vanish on $\mathfrak{a}$, μ_j is perpendicular to those α_i that do vanish on $\mathfrak{a}$—in other words, to the simple roots in $\Phi^+(\mathfrak{m}, \mathfrak{a}_m \cap \mathfrak{m})$. Hence μ_j vanishes on $\mathfrak{a}_m \cap \mathfrak{m}$, which implies

$$(\mu, \mu_j) = (\lambda_m, \mu_j) \tag{6.60 a}$$

(cf. (6.55)). Similarly

$$(\varrho_P, \mu_j) = (\varrho_m, \mu_j), \tag{6.60 b}$$

because $\varrho_P|_{\mathfrak{a}}=\varrho_m|_{\mathfrak{a}}$ (P is in standard position!). If α_i is non-zero on $\check{\mathfrak{a}}$, we can argue analogously, to deduce

$$(\tilde{\varrho}, \mu_i) = (\varrho_m, \mu_i) \quad \text{for } 1 \leqslant i \leqslant s. \tag{6.61}$$

Let us suppose first that $1\leqslant j\leqslant s$. Then (6.59 a) follows from (6.54–56) and (6.60–61):

$$\operatorname{Re} d_j = 2\frac{\operatorname{Re}(\nu-\mu+\tilde{\varrho}-\varrho_P, \mu_j)}{(\alpha_j, \alpha_j)} = 2\frac{\operatorname{Re}(\nu-\lambda_m, \mu_j)}{(\alpha_j, \alpha_j)} = \operatorname{Re} c_j \geqslant 0. \tag{6.62}$$

Now suppose $s+1\leqslant j\leqslant r$. We claim:

$$\mu_j = \sum_{i=1}^{s} e_{ji}\mu_i + \sum_{i=s+1}^{r} f_{ji}\alpha_i, \quad \text{with } e_{ji}\geqslant 0, f_{ji}\geqslant 0, \tag{6.63}$$

and

$$\text{for each } j, \text{ at least one } f_{ji} \text{ is strictly positive.} \tag{6.64}$$

To see this, we regard $\mu_j|_{\mathfrak{a}_m\cap\tilde{\mathfrak{m}}}$ as a fundamental highest weight for $\Phi^+(\tilde{\mathfrak{m}}, \mathfrak{a}_m\cap\tilde{\mathfrak{m}})$. The roots span the weight lattice over $\mathbf{Q}$, and any two fundamental highest weights have a non-negative inner product; hence

$$\mu_j|_{\mathfrak{a}_m\cap\tilde{\mathfrak{m}}} = \sum_{i=s+1}^{r} f_{ji}\,\alpha_i|_{\mathfrak{a}_m\cap\tilde{\mathfrak{m}}}, \quad f_{ji}\geqslant 0.$$

Not all f_{ji} can be zero, because μ_j restricts non-trivally to $\mathfrak{a}_m\cap\tilde{\mathfrak{m}}$. Both μ_j and the α_i vanish on the split part of the center of $\mathfrak{g}$, and $\mu_1, \ldots, \mu_s$ span the annihilator of $\mathfrak{a}_m\cap\tilde{\mathfrak{m}}$ in the dual space of $\mathfrak{a}_m\cap[\mathfrak{g},\mathfrak{g}]$. It follows that μ_j can be expressed as in (6.64)—we only need to check that $e_{ji}\geqslant 0$. For $1\leqslant i\leqslant s$, α_i is perpendicular to μ_j and has a non-positive inner product with α_l, $s+1\leqslant l\leqslant r$. Thus

$$e_{ji} = 2\frac{(\mu_j-\Sigma f_{jl}\alpha_l, \alpha_i)}{(\alpha_i, \alpha_i)} = -2\sum_{l=s+1}^{r} f_{jl}\frac{(\alpha_l, \alpha_i)}{(\alpha_i, \alpha_i)} \geqslant 0,$$

as was to be shown.

We recall that $\tilde{\varrho}$ vanishes on $\mathfrak{a}_m\cap\tilde{\mathfrak{m}}$, which implies

$$(\tilde{\varrho}, \alpha_i) = 0, \quad s+1\leqslant i\leqslant r. \tag{6.65}$$

The following chain of equalities and inequalities is justified by (6.58), (6.60), (6.63), (6.61) and (6.65), (6.54 a) and (6.56), (6.54 b), in the given order:

$$
\begin{aligned}
\tfrac{1}{2}(\alpha_j, \alpha_j)\operatorname{Re} d_j &= \operatorname{Re}(\nu-\mu+\tilde{\varrho}-\varrho_P, \mu_j) \\
&= \operatorname{Re}(\nu-\lambda_m+\tilde{\varrho}-\varrho_m, \mu_j) \\
&= \operatorname{Re}\Big(\nu-\lambda_m+\tilde{\varrho}-\varrho_m, \sum_{i=1}^{s} e_{ji}\mu_i+\sum_{i=s+1}^{r} f_{ji}\alpha_i\Big) \\
&= \sum_{i=1}^{s} e_{ji}\operatorname{Re}(\nu-\lambda_m, \mu_i)+\sum_{i=s+1}^{r} f_{ji}\operatorname{Re}(\nu-\lambda_m-\varrho_m, \alpha_i) \\
&\geq \sum_{i=s+1}^{r} f_{ji}\operatorname{Re}(\nu-\lambda_m-\varrho_m, \alpha_i) = -\sum_{i=s+1}^{r} f_{ji}\operatorname{Re}(\lambda_m+\varrho_m, \alpha_i).
\end{aligned}
\tag{6.66}
$$

If the constant C in (3.63) is chosen large enough, the negativity condition (6.53) implies

$$\operatorname{Re}(\lambda_m+\varrho_m, \alpha_i)<0, \quad 1\leq i\leq r. \tag{6.67}$$

At this point (6.64) and (6.66–67) prove (6.59 a) in case $s+1\leq j\leq r$:

$$\operatorname{Re} d_j \geq -2\sum_{i=s+1}^{r} f_{ji}\frac{\operatorname{Re}(\lambda_m+\varrho_m, \alpha_i)}{(\alpha_j, \alpha_j)}>0.$$

The inequality also proves (6.59 b), provided at least one of the roots $\alpha_{s+1}, \ldots, \alpha_r$ restricts non-trivially to $\mathfrak{a}$. If not, (6.59 b) follows from (6.57) and (6.62). As was pointed out before, (6.59) completes the verification of the lemma.

We shall prove proposition 6.15 by contradiction. Thus we suppose that $\Theta_G(V)_\mu$ is non-zero near some $b\in(MA)^-\cap G'$. We may also suppose $b\in B$, since the choice of the Cartan subgroup $B\subset MA$ has been left open until now. In terms of the local expression (5.5) for $|D_G|^{1/2}\Theta_G(V)$, this means: there exists a $w\in W(\mathfrak{g}, \mathfrak{b})$, such that

$$c_\lambda^i(b, w)\neq 0 \quad \text{and} \quad w\lambda|_{\mathfrak{a}}=\mu. \tag{6.68}$$

Let B^i be the connected component of B which contains b. Moving b slightly, we can arrange

$$e^\alpha(b)\notin \mathbf{R} \text{ unless } e^\alpha \text{ assumes only real values on } B^i, \tag{6.69}$$

for $\alpha \in \Phi(\mathfrak{g}, \mathfrak{b})$, without destroying (6.68) or the containment $b \in B^i$. We claim: if U is a sufficiently small compact neighborhood of the identity in $(M \cap B)^0$,

$$bUA^- \subset (MA)^- \cap B_\varepsilon, \tag{6.70}$$

provided $\varepsilon>0$ is small enough (cf. (6.32)). To see this, we recall the description (5.28) of $(MA)^- \cap B^i \cap G'$. If $\alpha \in \Phi^+_{\mathbf{R},i}$ restricts non-trivially to $\mathfrak{a}$ we know that $e^\alpha(b)<1$, and hence

$$e^\alpha(bma) = e^\alpha(bm)\, e^\alpha(a) < e^\alpha(bm) < 1, \tag{6.71}$$

for all $m \in U$ and $a \in A^-$, as long as U is not too large. In particular $(MA)^-$ contains bUA^-. It remains to be shown that

$$e^\alpha \text{ stays bounded away from 1 on } bUA^-, \tag{6.72}$$

for every $\alpha \in \Phi(\mathfrak{g}, \mathfrak{b})$. We distinguish four cases. If $\pm\alpha \in \Phi^+_{\mathbf{R},i}$ and $\alpha|_{\mathfrak{a}} \neq 0$, (6.72) follows from (6.71). A similar argument applies if α restricts trivially to $\mathfrak{a}$. If α belongs to $\Phi_{\mathbf{R}}(\mathfrak{g}, \mathfrak{b})$ but not to $\Phi_{\mathbf{R},i}$, e^α assumes no real positive values at all on B^i, so (6.72) is automatic. Finally, if α does not belong to $\Phi_{\mathbf{R}}(\mathfrak{g}, \mathfrak{b})$, we use (6.69) to conclude that $e^\alpha(bma)\,|e^\alpha(bma)|^{-1} = e^\alpha(bm)\,|e^\alpha(bm)|^{-1}$ stays away from 1, again assuming U is not too large.

We enumerate the restrictions to $\mathfrak{a}$ of λ and its $W(\mathfrak{g}, \mathfrak{b})$-conjugates as $\mu=\mu_1, \mu_2, \ldots, \mu_n$, without repetition. Because of the negativity assumption (3.63 a) on λ,

$$|e^{\mu_i-\mu}(a)| < 1, \quad \text{for } 2 \leq i \leq n,\ a \in A^-. \tag{6.73}$$

In view of (6.70) the local expression (5.5) can be analytically continued to all of BUA^-. Hence there exist C^∞ functions $\varphi_1, \ldots, \varphi_n$, defined on a neighborhood of U in $(M \cap B)^0$, such that

$$|D_G|^{1/2}\Theta(bma) = \sum_{i=1}^{n} \varphi_i(m)\, e^{\mu_i}(a) \quad (m \in U,\ a \in A^-). \tag{6.74}$$

The assumption (6.68) implies $\varphi_1 \not\equiv 0$ on U. Thus we can choose a C^∞ function ψ on $(M \cap B)^0$, having support in U, with

$$\int_{(M\cap B)^0} \varphi_1\, \psi\, dm = 1.$$

For $a \in A^-$, we define

$$\theta(a) = e^{-\mu}(a) \int_{m \in (M \cap B)^0} (|D_G|^{1/2} \Theta)(bma)\, \psi(m)\, dm. \tag{6.75}$$

Because of (6.74), we can express θ as an exponential sum,

$$\theta = 1 + \sum_{i=2}^{n} c_i e^{\mu_i - \mu}. \tag{6.76}$$

On the other hand, if we integrate the estimates in lemmas 6.39 and 6.46 for the distribution (6.38), we find: there exist translation invariant linear differential operators $X_1, \ldots, X_m$ on A, continuous functions $h_1, \ldots, h_m$ on A^-, a linear function $\tau \in \mathfrak{a}_0^*$ and constants $C_1, \ldots, C_m$ such that

$$\begin{gathered} \theta = \sum_{j=1}^{m} X_j h_j, \text{ in the sense of distributions, } |h_j| \leq C_j e^{\tau} \text{ for } 1 \leq j \leq m, \\ \text{and } e^{\tau}(a) < 1 \text{ for } a \in A^-. \end{gathered} \tag{6.77}$$

Let $f \in C_0^\infty(A^-)$ be such that $\int_A f\,da = 1$, and let $f_1, f_2, \ldots$ be the translates of f by a sequence of points $a_1, a_2, \ldots$, whose inverses tend to ∞ along a ray in A^-. Since the exponentials $e^{\mu_i - \mu}$, $2 \leq i \leq n$, decay on A^-, (6.76) tells us that $\int_A f_k \theta\, da$ tends to $\int_A f\,da = 1$ as $k \to \infty$. Arguing similarly, we deduce

$$\int_A f_k \theta\, da = \sum_{j=1}^{m} \int_A (X_j^* f_k)\, h_j\, da \to 0$$

from (6.77). This is the contradiction which proves proposition 6.15.

§ 7. Osborne's conjecture: the general case

In this section we complete the proof of theorem 3.6 for a general parabolic subgroup. We also reformulate the character identity in theorem 3.6, in terms of the homology groups with respect to certain maximal nilpotent subalgebras $\mathfrak{r}$ contained in $\mathfrak{g}$.

The proof of Osborne's conjecture in the general case proceeds along the same lines as that for a minimal parabolic subgroup, with one major exception: the verification of lemma 6.6 depended crucially on the compactness of M_m, and does not carry over to the present context. We get around this problem by proving an analogue of lemma 6.10 directly, as a consequence of theorem 3.6 for $P = P_m$.

We begin by drawing some conclusions from the special case of theorem 3.6.

PROPOSITION 7.1. *Let B_m be a maximally split Cartan subgroup, and B_m^0 its connected component containing the identity. The character $\Theta_G(V)$ of a Harish-Chandra module V cannot vanish on any non-empty open subset of B_m^0 unless $V=0$.*

We remark that the same statement about virtual characters fails spectacularly: the difference of characters of any two contragredient discrete series representations of $G=SL(2,\mathbf{R})$, for example, is identically zero on the hyperbolic set.

Proof. We may as well suppose $B_m \subset M_m A_m$. Osborne's conjecture for $P=P_m$ asserts the identity

$$\begin{aligned}\Theta_G(V)\sum_p(-1)^p\Theta_{M_mA_m}(\Lambda^p\mathfrak{n}_m)&=\sum_p(-1)^p\Theta_{M_mA_m}(H_p(\mathfrak{n}_m,V))\\&=\sum_\mu\sum_p(-1)^p\Theta_{M_mA_m}(H_p(\mathfrak{n}_m,V)_\mu),\end{aligned}\tag{7.2}$$

on $(M_mA_m)^-\cap G'$. Let us assume that $\Theta_G(V)$ vanishes on an open subset of B_m^0; it then vanishes also on an open subset of $(M_mA_m)^-\cap B_m^0\cap G'$—this follows from (3.7)—, as do the contributions of the various exponents $\mu\in\mathfrak{a}_m^*$ to the right hand side of (7.2). Each of these contributions is a finite dimensional virtual character, hence real analytic. We conclude:

$$\sum_p(-1)^p\Theta_{M_mA_m}(H_p(\mathfrak{n}_m,V)_\mu)=0\tag{7.3}$$

on B_m^0, for all $\mu\in\mathfrak{a}_m^*$. According to the vanishing theorem 2.32, there can be no cancellation in (7.3) if μ is minimal among the exponents of $H_0(\mathfrak{n}_m,V)$, thus $H_0(\mathfrak{n}_m,V)=0$. Casselman's lemma 4.22 now implies the vanishing of V.

As an immediate consequence of the proposition, we find

COROLLARY 7.4. *If $0\to V'\to V\to V''\to 0$ is an exact sequence of Harish-Chandra modules, and $\Theta_G(V)=\Theta_G(V'')$ on a non-empty open subset of B_m^0, then $V\simeq V''$.*

Let V be a Harish-Chandra module for G, and $P=MAN$ a parabolic subgroup, in standard position with respect to P_m, i.e.,

$$MA\supset M_mA_m,\quad A\subset A_m,\quad N\subset N_m.\tag{7.5}$$

Theorem 3.6 for minimal parabolic subgroups allows us to deduce the general statement, at least on $(M_m A_m)^-$:

LEMMA 7.6. *Restricted to* $(M_m A_m)^- \cap G'$,

$$\Theta_G(V) = \Theta_{\mathfrak{n}}(V).$$

Proof. It is enough to show that

$$\Theta_{\mathfrak{n}}(V) = \Theta_{\mathfrak{n}_m}(V),$$

or more explicitly,

$$\sum_p (-1)^p \Theta_{MA}(H_p(\mathfrak{n}, V)) = \sum_p (-1)^p \Theta_{M_m A_m}(H_p(\mathfrak{n}, V)) \times \frac{\sum_p (-1)^p \Theta_{MA}(\Lambda^p \mathfrak{n})}{\sum_p (-1)^p \Theta_{M_m A_m}(\Lambda^p \mathfrak{n}_m)}$$

$$= \frac{\sum_p (-1)^p \Theta_{M_m A_m}(H_p(\mathfrak{n}, V))}{\sum_p (-1)^p \Theta_{M_m A_m}(\Lambda^p(\mathfrak{n}_m \cap \mathfrak{m}))}, \tag{7.7}$$

both on $(M_m A_m)^- \cap G'$. Let us apply theorem 3.6, with MA and $P_m \cap MA$ in place of G and P_m, to the first term in (7.7). We note that $P_m \cap MA$ is a minimal parabolic subgroup of MA, with Langlands factorization $M_m A_m (N_m \cap M)$. Also, the set of validity of theorem 3.6, specialized to our present situation, includes $(M_m A_m)^- \cap G'$—to see this, we appeal to lemma 5.28. Hence, on $(M_m A_m)^- \cap G'$,

$$\Theta_{MA}(H_p(\mathfrak{n}, V)) = \frac{\sum_q (-1)^q \Theta_{M_m A_m}(H_q(\mathfrak{n}_m \cap \mathfrak{m}, H_p(\mathfrak{n}, V))}{\sum_q (-1)^q \Theta_{M_m A_m}(\Lambda^q(\mathfrak{n}_m \cap \mathfrak{m}))}. \tag{7.8}$$

The Hochschild-Serre spectral sequence for $H_*(\mathfrak{n}_m, V)$ corresponding to the semidirect product $\mathfrak{n}_m = \mathfrak{n} \oplus (\mathfrak{n}_m \cap \mathfrak{m})$, has E^2-term

$$E^2_{p,q} = H_q(\mathfrak{n}_m \cap \mathfrak{m}, H_p(\mathfrak{n}, V)).$$

Since there are only finitely many non-zero terms, the usual Euler characteristic argument gives

$$\sum_{p,q}(-1)^{p+q}\Theta_{M_mA_m}(H_q(\mathfrak{n}_m\cap\mathfrak{m}, H_p(\mathfrak{n}, V)) = \sum_p (-1)^p\Theta_{M_mA_m}(H_p(\mathfrak{n}_m, V)).$$

This coupled with (7.8), proves (7.7), and hence the lemma.

We fix a Cartan subgroup $B\subset MA$, and a linear function $\lambda\in\mathfrak{b}^*$ which is antidominant, relative to a positive root system $\Phi^+(\mathfrak{g},\mathfrak{b})$ compatible with $\Phi^+(\mathfrak{g},\mathfrak{a})$. In other words,

$$\operatorname{Re}(\lambda,\alpha)<0 \quad \text{for } \alpha\in\Phi^+(\mathfrak{g},\mathfrak{b}). \tag{7.9}$$

We let μ and τ denote the restrictions of λ to, respectively, $\mathfrak{a}$ and $\mathfrak{b}\cap\mathfrak{m}$.

LEMMA 7.10. *If W is an irreducible Harish-Chandra module for M, such that $Z(\mathfrak{m})$ acts according to $\chi_{\mathfrak{m},\tau}$, then*

$$H_0(\mathfrak{n}, I_P^G(W\otimes\mathbf{C}_\mu))_\mu \simeq W\otimes\mathbf{C}_{\mu+\varrho_P};$$

moreover, on $(MA)^-\cap G'$,

$$\Theta_G[I_P^G(W\otimes\mathbf{C}_\mu)]_\mu = \Theta_\mathfrak{n}[I_P^G(W\otimes\mathbf{C}_\mu)]_\mu.$$

Proof. The hypotheses of lemmas 5.17 and 5.24 are satisfied. Hence, on $(MA)^-\cap G'$,

$$\begin{aligned}&\text{(a)}\ \Theta_G[I_P^G(W\otimes\mathbf{C}_\mu)]_\mu = \Theta_{MA}(W\otimes\mathbf{C}_{\mu+\varrho_P}),\\ &\text{(b)}\ \Theta_\mathfrak{n}[I_P^G(W\otimes\mathbf{C}_\mu)]_\mu = \Theta_{MA}[H_0(\mathfrak{n}, I_P^G(W\otimes\mathbf{C}_\mu))_\mu].\end{aligned} \tag{7.11}$$

Let B_m^0 be the identity component of a Cartan subgroup $B_m\subset M_mA_m$. Lemma 5.28, applied to both $(MA)^-$ and $(M_mA_m)^-$, gives the containment

$$(MA)^-\cap B_m^0\cap G' \supset (M_mA_m)^-\cap B_m^0\cap G';$$

both sets are open in B_m^0 and non-empty. According to lemma 7.6, the two quantities on the left in (7.11 a, b) agree on $(M_mA_m)^-\cap G'$. We conclude:

$$\Theta_{MA}[H_0(\mathfrak{n}, I_P^G(W\otimes\mathbf{C}_\mu))_\mu] = \Theta_{MA}(W\otimes\mathbf{C}_{\mu+\varrho_P}), \tag{7.12}$$

on $(M_m A_m)^- \cap B_m^0 \cap G'$. The Frobenius reciprocity theorem 4.11 produces a surjection

$$H_0(\mathfrak{n}, I_P^G(W \otimes \mathbf{C}_\mu))_\mu \to W \otimes \mathbf{C}_{\mu+\varrho_P} \to 0$$

(W is irreducible!). At this point (7.12) and corollary 7.4, with MA playing the role of G, imply the first statement of the lemma. The second statement follows because of (7.11).

We can now prove theorem 3.6 for a general parabolic subgroup $P=MAN$, which we may as well put into standard position; cf. (7.5). Again let B denote a Cartan subgroup of G, lying inside MA. We make the same reductions as in the beginning of § 6: V is an irreducible Harish-Chandra module for G, on which $Z(\mathfrak{g})$ acts via $\chi_{\mathfrak{g},\lambda}$, with $\lambda \in \mathfrak{b}^*$ subject to the restrictions (3.63 a, b). It then suffices to prove the identity

$$\Theta_G(V)_\mu = \Theta_\mathfrak{n}(V)_\mu \quad \text{on } (MA)^- \cap G', \tag{7.13}$$

where

$$\mu = \text{restriction of } \lambda \text{ to } \mathfrak{a}. \tag{7.14}$$

From lemma 5.24 and proposition 6.15 we deduce:

$$\begin{gathered}\text{if } H_0(\mathfrak{n}, V)_\mu \text{ vanishes, both } \Theta_G(V)_\mu \text{ and } \Theta_\mathfrak{n}(V)_\mu \\ \text{are identically zero on } (MA)^- \cap G',\end{gathered} \tag{7.15}$$

which implies (7.13) in this particular situation.

Thus we suppose $H_0(\mathfrak{n}, V)_\mu \neq 0$, and we let W denote an irreducible quotient of $H_0(\mathfrak{n}, V)_\mu$, viewed as Harish-Chandra module for M. Because of the reciprocity theorem 4.11, there exists a short exact sequence

$$0 \to V \to I_P^G(W \otimes \mathbf{C}_\mu) \to Q \to 0, \tag{7.16}$$

and a corresponding long exact sequence

$$\ldots \to H_1(\mathfrak{n}, Q)_\mu \to H_0(\mathfrak{n}, V)_\mu \to H_0(\mathfrak{n}, I_P^G(W \otimes \mathbf{C}_\mu))_\mu \to H_0(\mathfrak{n}, Q)_\mu \to 0. \tag{7.17}$$

Since λ satisfies the negativity condition (3.63 a), μ is a minimal element of the set

$$\{w\lambda|_\mathfrak{a} \mid w \in W(\mathfrak{g}, \mathfrak{b})\},$$

relative to the partial order (2.31). We now appeal to the vanishing theorem 2.32:

$$H_p(\mathfrak{n}, Q)_\mu = 0 \quad \text{for } p > 0. \tag{7.18}$$

The negativity of λ also ensures that a $W(\mathfrak{g}, \mathfrak{b})$-conjugate $w\lambda$ restricts to μ an $\mathfrak{a}$ only if $w \in W(\mathfrak{m}\oplus\mathfrak{a}, \mathfrak{b})$. This, in conjunction with the Casselman-Osborne lemma 2.21, makes it possible to identify the $Z(\mathfrak{m})$-action on W as $\chi_{\mathfrak{m},\tau}$, with τ=restriction of λ to $\mathfrak{b}\cap\mathfrak{m}$. In particular, lemma 7.10 applies in the present context. Since

$$H_0(\mathfrak{n}, I_P^G(W\otimes \mathbf{C}_\mu))_\mu \simeq W\otimes \mathbf{C}_{\mu+\varrho_P}$$

is irreducible and $H_0(\mathfrak{n}, V)_\mu \neq 0$, (7.17–18) imply $H_0(\mathfrak{n}, Q)_\mu = 0$. We claim:

$$H_0(\mathfrak{n}, V_1)_\mu = 0, \quad \text{for every composition factor } V_1 \text{ of } Q. \tag{7.19}$$

Indeed, let $0\to Q'\to Q\to Q''\to 0$ be a short exact sequence; then $H_1(\mathfrak{n}, Q'')_\mu = 0$, for the same reason as (7.18), hence

$$H_0(\mathfrak{n}, Q')_\mu = H_0(\mathfrak{n}, Q'')_\mu = 0,$$

which establishes (7.19) by induction on the length of Q. In view of (7.15), $\Theta_G(Q)_\mu$ and $\Theta_\mathfrak{n}(Q)_\mu$ vanish on $(MA)^-\cap G'$. Equivalently,

$$\Theta_G(V)_\mu = \Theta_\mathfrak{n}(I_P^G(W\otimes \mathbf{C}_\mu))_\mu,$$
$$\Theta_\mathfrak{n}(V)_\mu = \Theta_\mathfrak{n}(I_P^G(W\otimes \mathbf{C}_\mu))_\mu,$$

both on $(MA)^-\cap G'$. The identity (7.13) follows from another application of lemma 7.10. The proof of theorem 3.6 is now complete.

We close this section with an alternate version of Osborne's conjecture, in terms of the homology $H_*(\mathfrak{r}, V)$ with respect to certain maximal nilpotent subalgebras $\mathfrak{r}\subset\mathfrak{g}$. Let B denote a Cartan subgroup of G. We conjugate B, if necessary, to make it invariant under the Cartan involution. Then

$$B = (B\cap K)A \quad \text{(direct product)}, \tag{7.20}$$

with A=split part of B. We fix a system of positive roots $\Phi^+(\mathfrak{g}, \mathfrak{b})$ which is compatible, in the sense of (3.23), with some system of positive restricted roots $\Phi^+(\mathfrak{g}, \mathfrak{a})$. The choice of $\Phi^+(\mathfrak{g}, \mathfrak{b})$ determines a Borel subalgebra $\mathfrak{b}\oplus\mathfrak{r}\subset\mathfrak{g}$, whose nilpotent radical $\mathfrak{r}$ is the direct sum of all root spaces $\mathfrak{g}^\alpha$ indexed by roots $\alpha\in\Phi^+(\mathfrak{g}, \mathfrak{b})$. The group $B\cap K$ and the Lie algebra $\mathfrak{a}$ commute, they both normalize $\mathfrak{r}$ and act on any Harish-Chandra module V. These actions induce commuting actions of $B\cap K$ and $\mathfrak{a}$ on the homology groups $H_p(\mathfrak{r}, V)$. As we shall see shortly, the homology groups are finite dimensional. Once this is known, we can lift the $\mathfrak{a}$-action uniquely to an action of the vector group

A—in other words, the homology groups $H_p(\mathfrak{r}, V)$ become B-modules. As was remarked in § 5, every root in

$$\Phi_{\mathbf{R}}(\mathfrak{g}, \mathfrak{b}) = \{\alpha \in \Phi(\mathfrak{g}, \mathfrak{b}) \mid \langle \alpha, \mathfrak{b}_0 \rangle \subset \mathbf{R}\}$$

exponentiates to a real-valued character of B. We define

$$\begin{aligned} B'' &= \{b \in B \mid e^{\alpha}(b) \neq 1 \quad \text{for } \alpha \in \Phi_{\mathbf{R}}(\mathfrak{g}, \mathfrak{b})\}, \\ B^- &= \{b \in B \mid e^{\alpha}(b) < 1 \quad \text{for } \alpha \in \Phi^+_{\mathbf{R}}(\mathfrak{g}, \mathfrak{b})\}, \end{aligned} \tag{7.21}$$

where $\Phi^+_{\mathbf{R}}(\mathfrak{g}, \mathfrak{b}) = \Phi_{\mathbf{R}}(\mathfrak{g}, \mathfrak{b}) \cap \Phi^+(\mathfrak{g}, \mathfrak{b})$.

THEOREM 7.22. *The $\mathfrak{r}$-homology groups of a Harish-Chandra module V are finite dimensional. For $b \in B^- \cap G'$,*

$$\Big\{\Theta_G(V) \prod_{\alpha \in \Phi^+(\mathfrak{g}, \mathfrak{b})} (1 - e^{\alpha})\Big\}(b) = \sum_p (-1)^p \Theta_B(H_p(\mathfrak{r}, V))(b).$$

Let us mention a useful corollary before we turn to the proof. If one passes to a suitable finite covering,

$$\varrho = \text{half-sum of all } \alpha \in \Phi^+(\mathfrak{g}, \mathfrak{b})$$

lifts to a character of B, and this makes it possible to introduce a "Weyl denominator"

$$\prod_{\alpha \in \Phi^+(\mathfrak{g}, \mathfrak{b})} (e^{\alpha/2} - e^{-\alpha/2}) = \pm e^{-\varrho} \prod_{\alpha \in \Phi^+(\mathfrak{g}, \mathfrak{b})} (1 - e^{\alpha}). \tag{7.23}$$

Since $W(G, B)$ contains all reflections about real roots, it permutes the positive root systems in $\Phi_{\mathbf{R}}(\mathfrak{g}, \mathfrak{b})$ transitively. Equivalently, every connected component of B'' is conjugate, under the normalizer $N_G(B)$ of B, to a subset of B^-. Thus:

COROLLARY 7.24. *The "Weyl numerator"*

$$\Theta_G(V) \prod_{\alpha \in \Phi^+(\mathfrak{g}, \mathfrak{b})} (e^{\alpha/2} - e^{-\alpha/2})$$

on $B \cap G'$ extends to a real-analytic function on all of B''. Its restriction to any particular $N_G(B)$-conjugate of B^- is equal to a finite, integral linear combination of characters(¹) *of B.*

(¹) Not necessarily one dimensional characters, since B need not be Abelian.

The first of the two assertions is also part of Harish-Chandra's matching conditions [26]. With some effort, the second can be deduced from the matching conditions, the induced character formula in theorem 5.7 and Harish-Chandra's character formula for the discrete series [19].

Proof of theorem 7.22. The system of positive restricted roots $\Phi^+(\mathfrak{g},\mathfrak{a})$, with which $\Phi^+(\mathfrak{g},\mathfrak{b})$ is compatible, corresponds to a parabolic subgroup $P=MAN$, such that

$$\mathfrak{r}=(\mathfrak{r}\cap\mathfrak{m})\oplus\mathfrak{n} \quad \text{(semidirect product)}. \tag{7.25}$$

This decomposition of $\mathfrak{r}$ leads to a Hochschild-Serre spectral sequence

$$E^2_{p,q}=H_q(\mathfrak{r}\cap\mathfrak{m}, H_p(\mathfrak{n},V))\Rightarrow H_{p+q}(\mathfrak{r},V). \tag{7.26}$$

We claim: if W is a Harish-Chandra module for M,

$$\begin{aligned}&\text{(a)}\ \dim H_*(\mathfrak{r}\cap\mathfrak{m},W)<\infty, \quad \text{and}\\ &\text{(b)}\ \Theta_M(W)\textstyle\prod_{\alpha\in\Phi^+(\mathfrak{m},\mathfrak{b}\cap\mathfrak{m})}(1-e^\alpha)=\sum_p(-1)^q\Theta_B(H_p(\mathfrak{r}\cap\mathfrak{m},W)), \quad \text{on } B\cap M''\end{aligned} \tag{7.27}$$

(M'=set of all $m\in M$ which are semisimple and regular in M). Let us assume this for the moment. There are only finitely many non-zero terms in the spectral sequence (7.26), all of them finite dimensional. Hence $H_*(\mathfrak{r},V)$ is finite dimensional also, and

$$\sum_p(-1)^p\Theta_B(H_p(\mathfrak{r},V))=\sum_{p,q}(-1)^{p+q}\Theta_B[H_q(\mathfrak{r}\cap\mathfrak{m},H_p(\mathfrak{n},V))]. \tag{7.28}$$

From (7.27 b) we deduce the identity

$$\sum_q(-1)^q\Theta_B[H_q(\mathfrak{r}\cap\mathfrak{m},H_p(\mathfrak{n},V))]=\Theta_{MA}(H_p(\mathfrak{n},V))\prod_{\alpha\in\Phi^+(\mathfrak{m}\oplus\mathfrak{a},\mathfrak{b})}(1-e^\alpha), \tag{7.29}$$

which holds on all of $B\cap G'$, because A commutes with $\mathfrak{r}\cap\mathfrak{m}$ and $B\cap G'\subset B\cap(M'A)$. The description in lemma 5.28 of $(MA)^-\cap B\cap G'$ shows that the character formula in theorem 3.6,

$$\sum_p(-1)^p\Theta_{MA}(H_p(\mathfrak{n},V))=\Theta_G(V)\sum_p(-1)^p\Theta_{MA}(\Lambda^p\mathfrak{n}), \tag{7.30}$$

is valid on $B^-\cap G'$. Since

$$\sum_p(-1)^p\Theta_{MA}(\Lambda^p\mathfrak{n})\prod_{\alpha\in\Phi^+(\mathfrak{m}\oplus\mathfrak{a},\mathfrak{b})}(1-e^\alpha)=\prod_{\alpha\in\Phi^+(\mathfrak{g},\mathfrak{b})}(1-e^\alpha),$$

the identities (7.28–30) imply the theorem.

We must still verify (7.27), which is nothing more than a special case of theorem 7.22: the Cartan subgroup $B\cap M$ of M has no split part, hence is compact and lies in the maximal compact subgroup $K\cap M$ of M; cf. (7.20). In other words, if we change notation, with $G, B, \mathfrak{r}$ taking the places of $M, B\cap M, \mathfrak{r}\cap\mathfrak{m}$, we have to prove theorem 7.22, under the additional hypothesis that

$$B\subset K \text{ is a compact Cartan subgroup of } G. \tag{7.31}$$

The analogous result about $\mathfrak{r}$-cohomology is implicit in [34], though well-hidden. We briefly recall the argument, translated back into homology.

Let $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{s}$ be the Cartan decomposition. The standard complex of $\mathfrak{r}$-homology,

$$V\otimes\Lambda\mathfrak{r}\simeq V\otimes\Lambda(\mathfrak{s}\cap\mathfrak{r})\otimes\Lambda(\mathfrak{k}\cap\mathfrak{r}),$$

has a natural increasing filtration

$$F_p(V\otimes\Lambda\mathfrak{r})=\bigoplus_{j\leq p} V\otimes\Lambda^j(\mathfrak{s}\cap\mathfrak{r})\otimes\Lambda(\mathfrak{k}\cap\mathfrak{r}).$$

A straightforward calculation shows that the boundary operator of $\mathfrak{r}$-homology induces the boundary operator of $(\mathfrak{k}\cap\mathfrak{r})$-homology on the quotients

$$F_p/F_{p-1}\simeq V\otimes\Lambda^p(\mathfrak{s}\cap\mathfrak{r})\otimes\Lambda(\mathfrak{k}\cap\mathfrak{r}).$$

In other words, the filtration determines a spectral sequence

$$E^1_{p,q}=H_q(\mathfrak{k}\cap\mathfrak{r}, V\otimes\Lambda^p(\mathfrak{s}\cap\mathfrak{r}))\Rightarrow H_{p+q}(\mathfrak{r}, V). \tag{7.32}$$

Under the action of the torus B^0 (=identity component of B)(²), V breaks up into an algebraic direct sum of weight spaces, which are generally infinite dimensional. Since B normalizes $\mathfrak{r}$, the homology groups $H_p(\mathfrak{r}, V)$ inherit a weight space decomposition. For each weight μ, we let $\mathbf{C}_\mu$ denote the one dimensional B^0-module corresponding to the character e^μ. Then, because of the Casselman-Osborne lemma,

$$\mathrm{Hom}_{B^0}(H_p(\mathfrak{r}, V), \mathbf{C}_\mu)=0 \quad \text{for all but finitely many weights } \mu. \tag{7.33}$$

Since $\mathfrak{b}\oplus(\mathfrak{k}\cap\mathfrak{r})$ is solvable, $\Lambda(\mathfrak{s}\cap\mathfrak{r})$ has an increasing filtration, invariant under B and $\mathfrak{k}\cap\mathfrak{r}$, with successive quotients $\mathbf{C}_{\eta_1}, \ldots, \mathbf{C}_{\eta_n}$, indexed by the weights $\eta_1, \ldots, \eta_n$ of $\Lambda(\mathfrak{s}\cap\mathfrak{r})$.

(²) $B^0=B$ if G is connected.

This filtration leads to a second spectral sequence

$$E^1_{p,q} = H_{p+q}(\mathfrak{k} \cap \mathfrak{r}, V \otimes \mathbf{C}_{\eta_p}) \Rightarrow H_{p+q}(\mathfrak{k} \cap \mathfrak{r}, V \otimes \Lambda(\mathfrak{s} \cap \mathfrak{r})). \tag{7.34}$$

The decomposition of V into its K-isotypic components,

$$V = \bigoplus_{i \in \hat{K}} V_i, \tag{7.35}$$

also decomposes the E^1-term:

$$H_{p+q}(\mathfrak{k} \cap \mathfrak{r}, V \otimes \mathbf{C}_{\eta_p}) \simeq \bigoplus_{i \in \hat{K}} H_{p+q}(\mathfrak{k} \cap \mathfrak{r}, V_i) \otimes \mathbf{C}_{\eta_p}; \tag{7.36}$$

the factor $\mathbf{C}_{\eta_p}$ can be pulled out of the parentheses because it is trivial as a module for $\mathfrak{k} \cap \mathfrak{r}$. For any particular weight μ,

$$\mathrm{Hom}_{B^0}(H_p(\mathfrak{k} \cap \mathfrak{r}, V_i), \mathbf{C}_\mu) = 0, \quad \text{for all but finitely many } i \in \hat{K}, \tag{7.37}$$

as follows from the Casselman-Osborne lemma, this time applied to $\mathfrak{k}$. Since $\dim V_i < \infty$, (7.32–37) prove

$$\dim H_p(\mathfrak{r}, V) < \infty, \tag{7.38}$$

which is the first assertion of theorem 7.22.

In view of the Weyl character formula, the formal series

$$\prod_{\alpha \in \Phi^+(\mathfrak{k}, \mathfrak{b})} (1 - e^\alpha) \sum_{i \in \hat{K}} \Theta_K(V_i) \tag{7.39}$$

is an infinite linear combination of characters of B, with integral coefficients. By the usual Euler characteristic arguments, the two spectral sequences (7.32), (7.34) imply the equality

$$\prod_{\alpha \in \Phi^+(\mathfrak{g}, \mathfrak{b})} (1 - e^\alpha) \sum_i \Theta_K(V_i) = \sum_p (-1)^p \Theta_B(H_p(\mathfrak{r}, V)), \tag{7.40}$$

of formal linear combinations of characters of B. Let Φ_c^+ denote the intersection of $\Phi^+(\mathfrak{g}, \mathfrak{b})$ with $\Phi(\mathfrak{k}, \mathfrak{b})$, and Φ_n^+ the complement of Φ_c^+ in $\Phi^+(\mathfrak{g}, \mathfrak{b})$. In an appropriate finite covering of K, the Weyl denominators of K and G become well-defined functions on B. Both are anti-symmetric with respect to $W(K, B)$. Their quotient

$$\prod_{\alpha \in \Phi_n^+} (e^{\alpha/2} - e^{-\alpha/2})$$

is a $W(K, B)$-invariant finite linear combination of characters of B, hence the restriction to B of a virtual K-character $\Delta_{G/K}$. We can re-write (7.40) as follows:

$$\prod_{\alpha\in\Phi_c^+}(e^{\alpha/2}-e^{-\alpha/2})\left\{\Delta_{G/K}\sum_{i\in\hat{K}}\Theta_K(V_i)\right\}=\pm e^{-\varrho}\sum_p(-1)^p\Theta_B(H_p(\mathfrak{r}, V)). \tag{7.41}$$

The term in curly parentheses is a formal linear combination of irreducible characters of K, with integral coefficients. Because of Weyl's character formula, and because the right hand side of (7.41) is a finite linear combination of B-characters, we conclude:

$$\Delta_{G/K}\sum_{i\in\hat{K}}\Theta_K(V_i) \text{ is a finite linear combination of irreducible characters of } K. \tag{7.42}$$

According to Harish-Chandra [16], the series

$$\Theta_K(V)=\sum_{i\in\hat{K}}\Theta_K(V_i) \tag{7.43}$$

converges to a distribution on K; this K-character $\Theta_K(V)$ is real-analytic on $K\cap G'$, and

$$\Theta_K(V)=\Theta_G(V), \quad \text{as functions on } K\cap G'. \tag{7.44}$$

At this point, (7.40–44) prove

$$\prod_{\alpha\in\Phi^+(\mathfrak{g},\mathfrak{b})}(1-e^{\alpha})\,\Theta_G(V)=\sum_p(-1)^p\Theta_B(H_p(\mathfrak{r}, V)),$$

as an identity between functions on $B\cap G'$.

§ 8. Characters, asymptotics and induced representations

Osborne's conjecture and the circle of ideas around it provide a very natural explanation of several important results in the representation theory of semisimple Lie groups: the relationship between characters and asymptotics [31, 23], basic facts about tempered representations [20, 38], the generic irreducibility of induced representations [4, 22], and Langlands' classification [28]. In this section we intend to show the close connection between them, which is not so apparent from the existing proof.

Recall the notion of a leading homology exponent of a Harish-Chandra module V, along a parabolic subgroup $P=MAN$: $\mu\in\mathfrak{a}^*$ is a homology exponent if $H_*(\mathfrak{n}, V)_\mu\neq 0$, and a leading homology exponent if it is minimal, relative to the partial order (2.31), among all homology exponents. According to proposition 2.32, a leading homology exponent contributes only in degree zero. Thus we can characterize the leading homology exponents also as the minimal elements in the set

$$\{\mu \in \mathfrak{a}^* \mid H_0(\mathfrak{n}, V)_\mu \neq 0\}.$$

We shall call μ a character exponent of V along P if $\Theta_G(V)_\mu$, as defined in (5.15), does not vanish identically on $(MA)^- \cap G'$. This can be restated, in terms of the local expressions[1] (5.4) or (5.5), as follows: there exist a Cartan subgroup $B \subset MA$, an element $b \in (MA)^- \cap B \cap G'$, $\lambda \in \mathfrak{b}^*$ and $w \in W(\mathfrak{g}, \mathfrak{b})$, such that $w\lambda|_\mathfrak{a} = \mu$ and $c_\lambda(b, w) \neq 0$ (or equivalently, $c'_\lambda(b, w) \neq 0$). A leading character exponent is one that is minimal with respect to the order (2.31).

THEOREM 8.1. *The set of leading character exponents along P coincides with the set of leading homology exponents along P. If μ is one of these leading exponents, $\Theta_G(V)_\mu = \Theta_{MA}(H_0(\mathfrak{n}, V)_\mu)$ on $(MA)^- \cap G'$.*

Proof. According to (5.22) and theorem 3.6,

$$\Theta_G(V)_\mu = \sum_p (-1)^p \Theta_{MA}(H_p(\mathfrak{n}, V)_\mu) \quad \text{on } (MA)^- \cap G'. \tag{8.2}$$

In particular every character exponent occurs also as homology exponent. If μ is a leading homology exponent, only the 0th summand contributes to the right hand side of (8.2). Thus $\Theta_G(V)_\mu \not\equiv 0$ on $(MA)^- \cap G'$, and μ is a character exponent—necessarily a leading one, because of our first observation. The theorem follows.

In the special case of a minimal parabolic subgroup P_m, the theorem was proved by one of us (Theorem 1.3 of [23]). When combined with theorem 4.25, it implies:

COROLLARY 8.3. *Every leading exponent in the asymptotic expansion in theorem 4.16 is a leading character exponent along P_m, and conversely.*

Since the three notions of leading character exponent along P, leading homology exponent along P and—for $P = P_m$ —leading exponent in the asymptotic expansion

(1) Since we do not require $Z(\mathfrak{g})$ to act on V according to a single character χ_λ, (5.4) and (5.5) must be modified slightly: the sums should extend not only over $W(\mathfrak{g}, \mathfrak{b})$, modulo the stabilizer of λ, but also over a finite set of $\lambda \in \mathfrak{b}^*$, pairwise non-conjugate under $W(\mathfrak{g}, \mathfrak{b})$.

coincide, there is no need to distinguish them; from now on, we shall simply speak of a "leading exponent along P".

Let V be a Harish-Chandra module, with composition factors $V_1, \ldots, V_n$. The character exponents of V, and hence the leading exponents depend only on the composition series of V:

COROLLARY 8.4. *The module V and its "semisimplification" $V_1 \oplus \ldots \oplus V_n$ have the same leading exponents.*

For $P=P_m$, this is a result of Miličić [31]: the matrix coefficients of V grow no more rapidly than those of $V_1 \oplus \ldots \oplus V_n$.

Suppose $P=MAN$ is in standard position with respect to the minimal parabolic subgroup $P_m=M_m A_m N_m$:

$$MA \supset M_m A_m, \quad A \subset A_m, \quad N \subset N_m. \tag{8.5}$$

LEMMA 8.6. *Every leading exponent $\nu \in \mathfrak{a}^*$ of V along P arises as the restriction to $\mathfrak{a}$ of a leading exponent $\mu \in \mathfrak{a}_m^*$ of V along P_m. Conversely, every leading exponent $\mu \in \mathfrak{a}_m^*$ along P_m restricts to a homology exponent $\nu \in \mathfrak{a}^*$ along P.*

Proof. If $\nu \in \mathfrak{a}^*$ is a leading exponent along P, $H_0(\mathfrak{n}, V)_\nu$ is a non-zero Harish-Chandra module for MA, which has at least one leading exponent along the minimal parabolic subgroup

$$P_m \cap MA = M_m A_m (N_m \cap M)$$

of MA. Whenever $\mu_1 \in \mathfrak{a}_m^*$ restricts to $\nu_1 \in \mathfrak{a}^*$ on $\mathfrak{a}$,

$$H_0(\mathfrak{n}_m, V)_{\mu_1} \simeq H_0(\mathfrak{n}_m \cap \mathfrak{m}, H_0(\mathfrak{n}, V)_{\nu_1})_{\mu_1+\varrho_P}; \tag{8.7}$$

the shift by ϱ_P reflects the normalization of the indexing. In particular $H_0(\mathfrak{n}_m, V)$ has an exponent $\mu_1 \in \mathfrak{a}_m^*$, which restricts to ν on $\mathfrak{a}$. If $\mu \leq \mu_1$ is a leading exponent, its restriction ν' to $\mathfrak{a}$ satisfies $\nu' \leq \nu$, is an exponent of $H_0(\mathfrak{n}, V)$ by (8.7), and hence coincides with the leading exponent ν. The second assertion of the lemma also follows from (8.7).

We shall say that a character $\Theta_G(V)$ has order of growth (at most) t along a parabolic subgroup $P=MAN$, provided

$$\begin{gathered}\operatorname{Re}\langle \mu, X\rangle \leq -t\langle \varrho_P, X\rangle \text{ for every leading exponent } \mu \in \mathfrak{a}^* \text{ of } V \\ \text{along } P \text{ and every } X \in \mathfrak{a}_0^-;\end{gathered} \tag{8.8}$$

as usual, $\mathfrak{a}_0^- \subset \mathfrak{a}_0$ denotes the negative Weyl chamber, relative to the system of positive restricted roots which corresponds to P. The character $\Theta_G(V)$ has order of growth t, without qualification, if it has order of growth t along every parabolic subgroup.

LEMMA 8.9. *A character has order of growth t if and only if it has order of growth t along any particular minimal parabolic subgroup.*

Proof. The "only if" is immediate. Conversely, we suppose $\Theta_G(V)$ has order of growth t along P_m; it suffices to check that it has the same order of growth along every parabolic subgroup $P=MAN$ which is in standard position with respect to $P_m=M_mA_mN_m$. If $\nu \in \mathfrak{a}^*$ is a leading exponent along P, lemma 8.6 guarantees the existence of a leading exponent $\mu \in \mathfrak{a}_m^*$ along P_m, with $\mu|_{\mathfrak{a}}=\nu$. Because of our assumption on P, $\mathfrak{a}_0^-$ lies in the closure of $(\mathfrak{a}_{m,0})^-$, and ϱ_{P_m} restricts to ϱ_P on $\mathfrak{a}$. Hence the inequality (8.8) for μ on $(\mathfrak{a}_{m,0})^-$ implies the analogous inequality for ν on $\mathfrak{a}_0^-$.

Our definition of a tempered Harish-Chandra module in § 4 characterizes temperedness in terms of the inequality (8.8), with $t=0$, for all leading exponents μ along a minimal parabolic subgroup. The L^p criterion in lemma 4.17, a byproduct of the asymptotic expansion of matrix coefficients, involves the same kind of bound, with $t<1-2/p$. Lemma 8.6 and theorem 8.1, which is implicit in our identification of leading character exponents and leading homology exponents, make the L^p criterion and the definition of temperedness equivalent to conditions on the growth of the character. To simplify the statements, we suppose that G has compact center.

COROLLARY 8.10 [19, 39, 31]. *A Harish-Chandra module V is tempered, as defined in § 4, if and only if the character $\Theta_G(V)$ has order of growth zero. For $1\leq p<\infty$, all matrix coefficients $f_{v,v'}$, with $v\in V$, $v'\in V'$, lie in $L^p(G)$ if and only if $\Theta_G(V)$ has order of growth strictly less than $1-2/p$.*

Except for the different terminology, the tempered case and the case $p=2$ are implicit in Harish-Chandra's work on the discrete series [19]. The "only if" for $1\leq p<2$ was first proved by Trombi-Varadarajan [39], and the remaining implications are due to Miličić [31]. Harish-Chandra, Trombi-Varadarajan and Miličić state their results in terms of global bounds on the character. It is not difficult to make the transition, but we shall do so only for $p=2$ and the tempered case. The general statement, which compares $\Theta_G(V)$ to Harish-Chandra's Ξ-function, can be established in the same way. Recall the definition of the conjugation invariant function D_G, as the coefficient of $t^{r(G)}$ in the polynomial (5.1).

LEMMA 8.11. *The following conditions on a character Θ are equivalent:*

(a) *Θ has order of growth zero;*

(b) $\sup_{g\in G'}|D_G(g)|^{1/2}|\Theta(g)|<\infty$.

Similarly, (c) *and* (d) *below are equivalent:*

(c) *Θ has order of growth strictly less than zero;*

(d) *for every Cartan subgroup B and $\varepsilon>0$, there exists a compact subset $\Omega\subset B$, such that* $\sup_{g\in B\cap G', g\notin\Omega}|D_G(g)|^{1/2}|\Theta(g)|<\varepsilon$.

Some remarks are in order. Harish-Chandra calls a character Θ, or more generally an invariant eigendistribution Θ, tempered if it extends continuously to a suitably defined Schwartz space, and he proves that this is the case precisely when $|D_G|^{1/2}\Theta$ grows at most polynomially. Thus temperedness is a slightly weaker notion than condition (b), and coincides with it for characters: the "Weyl numerators" (5.5) of a character Θ behave purely exponentially. Again according to Harish-Chandra, a representation is tempered if its character is, in the sense that was just described. The equivalence of our definition of temperedness in § 4 to Harish-Chandra's follows from his results [19], and is also one of the consequences of corollary 8.10 and lemma 8.11. The condition (d) on the characters of square-integrable representations plays a crucial role in the geometric construction of the discrete series [1], but is implicit already in Harish-Chandra's construction.

Proof of lemma 8.11. We consider a Cartan subgroup B, invariant under the Cartan involution, so that $B=(B\cap K)A$, with A=split part of B. To every choice of a system of positive restricted roots $\Phi^+(\mathfrak{g},\mathfrak{a})$ corresponds a parabolic subgroup $P=MAN$ and a negative Weyl chamber $\mathfrak{a}_0^-\subset\mathfrak{a}_0$, whose image in A we denote by A^-. Although it is possible to manage without it, we now use a consequence of the Harish-Chandra matching conditions(2) [26]: there exist (non-zero) C^∞ functions $\varphi_1,\dots,\varphi_N$ on $B\cap K$ and $\nu_1,\dots,\nu_N\in\mathfrak{a}^*$, such that

$$\Theta\prod_{\alpha>0}(e^{\alpha/2}-e^{-\alpha/2})=\sum_i\varphi_i e^{\nu_i}\quad\text{on } (MA)^-\cap B\cap G';\tag{8.12}$$

this set of validity contains $(B\cap K)A^-$, as follows from lemma 5.28, for example. We may have to pass to a finite covering to make the Weyl denominator well-defined. Each

(2) A consequence which can also be deduced from Osborne's conjecture; cf. Corollary 7.24.

ν_i is a character exponent along P, hence $\nu_i \geqslant \nu_i'$, for some leading exponent ν_i', in which case $\operatorname{Re}\nu_i'$ dominates $\operatorname{Re}\nu_i$ on $\mathfrak{a}_0^-$. Since $|D_G|^{1/2}$ coincides with the absolute value of the Weyl denominator,

$$\sup_{g \in G' \cap (B \cap K)A^-} |D_G(g)|^{1/2}\,|\Theta(g)| < \infty,$$

provided the leading exponents satisfy the bound (8.8) with $t=0$. As we let the system of positive restricted roots $\Phi^+(\mathfrak{g}, \mathfrak{a})$ vary, the sets $(B \cap K)A^-$ cover a dense subset of $B \cap G'$; also, G' can be covered by the conjugate of finitely many Cartan subgroups B, which we may assume are invariant under the Cartan involution. Thus (a) implies (b). To establish the converse, we apply (8.12) to a maximally split Cartan subgroup B_m. Then $P=P_m=M_mA_mN_m$ is a minimal parabolic subgroup, and all Cartan subgroups of M_mA_m are conjugate to B_m. Consequently all character exponents along P_m, and all leading exponents in particular, show up in (8.12). They must satisfy (8.8) with $t=0$, if $|D_G|^{1/2}\Theta$ is to be bounded on $(B \cap K)A^-$. This already proves the implication (b) $\Rightarrow$ (a): Lemma 8.9 makes it unnecessary(3) to bound the order of growth along other parabolic subgroups. The same arguments, with only small modifications, prove the equivalence of (c) and (d).

It will be convenient to call a Harish-Chandra module V square-integrable if its matrix coefficients $f_{v,v'}$ lie in $L^2(G)$. When this is the case, V can be embedded into a finite number of copies of $L^2(G) \cap C^\infty(G)$, equivariantly with respect to $\mathfrak{g}$ and K on the right:

$$v \mapsto (f_{v,v_1'}, \ldots, f_{v,v_n'});$$

here $\{v_1', \ldots, v_n'\}$ is a set of $U(\mathfrak{g})$-generators of the dual module V'.

Observation 8.13. Every square-integrable Harish-Chandra module V has a unitary globalization. In particular, any such V is completely reducible.

Let $P=MAN$ be a parabolic subgroup. Under the normalized induction procedure described in § 4, unitary representations of MA induce unitary representations of G, hence

Observation 8.14. If W is a square-integrable Harish-Chandra module for M and ν a

(3) Unlike lemma 8.9, lemma 8.11 also applies to virtual characters; with a little more effort one can avoid the use of lemma 8.9.

linear function on $\mathfrak{a}_0$ with purely imaginary values, the induced module $I_P^G(W \otimes \mathbf{C}_\nu)$ has a unitary globalization and is completely reducible.

According to Corollary 8.10 and Lemma 8.11, a Harish-Chandra module V is tempered precisely when $\Theta_G(V)$ satisfies the boundedness condition in lemma 8.11 (b), even if the center of G fails to be compact; cf. (4.30). The induced character formula in Theorem 5.7 shows that this boundedness condition is hereditary under induction. Temperedness is also shared by the composition factors of a tempered module, as follows from the definition in terms of leading exponents:

Observation 8.15. An induced module $I_P^G(W \otimes \mathbf{C}_\nu)$ and all its composition factors are tempered, provided W is square-integrable. or more generally tempered, and ν purely imaginary.

We now state three important results about Harish-Chandra modules. Their proofs, which have several common features, will be given at the end of this section.

THEOREM 8.16 [20, 38, 40]. *Every irreducible, tempered Harish-Chandra module for G occurs as a summand of a module $I_P^G(W \otimes \mathbf{C}_\nu)$, induced from a parabolic subgroup $P = MAN$ by an irreducible, square-integrable Harish-Chandra module W for M and $\nu \in i\mathfrak{a}_0^*$. Two induced modules $I_P^G(W \otimes \mathbf{C}_\nu)$, $I_{P'}^G(W' \otimes \mathbf{C}_{\nu'})$ of this type have no summands in common, unless the triples (MA, W, ν), $(M'A', W', \nu')$ are conjugate under G, in which case the induced modules are isomorphic.*

The existence of an embedding was proved by Trombi [38], and appears implicitly also in Harish-Chandra's earlier paper [20]. Langlands [28] points out that [20] contains the ingredients of a proof of the disjointness statement, but this is not obvious; a completely algebraic proof was given by Vogan [40]. The trivial parabolic subgroup $P = MA = G$ is not excluded as a possibility in theorem 8.16; square-integrable Harish-Chandra modules cannot be realized as summands of induced modules $I_P^G(W \otimes \mathbf{C}_\nu)$ of the type described above, unless $P = G$. One immediate consequence of the theorem deserves particular attention:

COROLLARY 8.17 [20, 38]. *Tempered, irreducible Harish-Chandra modules can be lifted to global unitary representations.*

We conjugate $P = MAN$ if necessary, to make the Levi factor MA invariant under the Cartan involution. Then each element of

$$W(G, A) = \text{normalizer of } A \text{ in } G/MA \tag{8.18}$$

has a representative in K; this is a standard fact in the special case of a minimal parabolic subgroup, to which the general case can be reduced. Thus $W(G, A)$ operates not only on $\mathfrak{a}$ and $\mathfrak{a}^*$, but also on the set of isomorphism classes of Harish-Chandra modules for M. With a slight abuse of notation, we write

$$W \mapsto vW, \quad v \in W(G, A). \tag{8.19}$$

A celebrated result of Bruhat [4] (for minimal parabolic subgroups) and Harish-Chandra [22] (in general) asserts that the induced representations in the statement of theorem 8.16 are generically irreducible. More precisely,

THEOREM 8.20 [4, 22]. *The number of irreducible summands of an induced module $I_P^G(W \otimes \mathbf{C}_\nu)$, corresponding to a square-integrable, irreducible Harish-Chandra module W and $\nu \in i\mathfrak{a}_0^*$, does not exceed the order of the stabilizer of $W \otimes \mathbf{C}_\nu$ in $W(G, A)$. In particular, $I_P^G(W \otimes \mathbf{C}_\nu)$ is irreducible whenever ν is non-singular.*

As will be apparent during the course of our proof, the theorem is a special case of a more complicated irreducibility criterion, in terms of the character exponents of W; cf. observation 8.47 below. In the special case of an algebraic group G, Knapp and Zuckerman [27] have explicitly determined the number of irreducible summands of $I_P^G(W \otimes \mathbf{C}_\nu)$. Their arguments are considerably more complicated, however.

Recall the notion of a collection of Langlands data: a parabolic subgroup $P = MAN$ —possibly $P = G$—, a tempered, irreducible Harish-Chandra module W for M, and a linear function $\nu \in \mathfrak{a}^*$, such that

$$\operatorname{Re}(\nu, \alpha) < 0 \quad \text{for all } \alpha \in \Phi^+(\mathfrak{g}, \mathfrak{a}). \tag{8.21}$$

THEOREM 8.22 (Langlands [28]). *The induced module $I_P^G(W \otimes \mathbf{C}_\nu)$, attached to a collection of Langlands data $P = MAN$, W, ν, has a unique irreducible submodule, to be denoted by $J_P^G(W \otimes \mathbf{C}_\nu)$. Every irreducible Harish-Chandra modules for G is isomorphic to one of these Langlands submodules $J_P^G(W \otimes \mathbf{C}_\nu)$. The isomorphism class of $J_P^G(W \otimes \mathbf{C}_\nu)$ determines the Langlands data uniquely, up to simultaneous conjugation by elements of G.*

Langlands states the classification dually, in terms of quotients and positive exponents. He characterizes the distinguished quotient as the image of the standard intertwining operator. The observation that the Langlands quotient can be described more simply as the unique irreducible quotient was made by Miličić [31]. The proof of theorem 8.22 has a corollary, also due to Langlands, which is useful in certain inductive applications of the classification. Let $\|\mathrm{Re}\,\nu\|$ denote the length of the real part of $\nu \in \mathfrak{a}^*$, measured with respect to the Killing form. Then:

PROPOSITION 8.23. *If $J_{P'}^G(W'\otimes\mathbf{C}_{\nu'})$ is a composition factor of $I_P^G(W\otimes\mathbf{C}_\nu)$, distinct from $J_P^G(W\otimes\mathbf{C}_\nu)$, the parameters ν, ν' satisfy* $\|\mathrm{Re}\,\nu'\|<\|\mathrm{Re}\,\nu\|$.

We now turn to the proofs of the three theorems. We fix a minimal parabolic subgroup $P_m=M_mA_mN_m$, a parabolic subgroup $P=MAN$ which is in standard position with respect to P_m, an arbitrary Harish-Chandra module W for M, and an arbitrary linear function $\nu\in\mathfrak{a}^*$.

LEMMA 8.24. (a) *Every character exponent $\mu\in\mathfrak{a}_m^*$ of $I_P^G(W\otimes\mathbf{C}_\nu)$ along P_m can be expressed as $\mu=w\xi$, where*

(i) *ξ is a character exponent of $W\otimes\mathbf{C}_\nu$ along $MA\cap P_m$,*

(ii) *$w\in W(G,A_m)$ maps $\Phi^+(\mathfrak{m}\oplus\mathfrak{a},\mathfrak{a}_m)$ into $\Phi^+(\mathfrak{g},\mathfrak{a}_m)$.*

(b) *Suppose $w\xi|_\mathfrak{a}\leq\nu$ implies $w\mathfrak{a}=\mathfrak{a}$ and $w\nu=\nu$, for all w, ξ as in* (i), (ii) *above. Then ν is a leading exponent of $I_P^G(W\otimes\mathbf{C}_\nu)$ along P, and*

$$\Theta_{MA}[H_0(\mathfrak{n}, I_P^G(W\otimes\mathbf{C}_\nu))_\nu] = \sum_{v\in W(G,A),\, v\nu=\nu} \Theta_{MA}((vW)\otimes\mathbf{C}_{\nu+\varrho_P}).$$

Proof. We choose a Cartan subgroup $B_m\subset M_mA_m$ and assume, without loss of generality, that $Z(\mathfrak{m}\oplus\mathfrak{a})$ acts on $W\otimes\mathbf{C}_\nu$ via a character $\chi_{\mathfrak{m}\oplus\mathfrak{a},\lambda}$, $\lambda\in\mathfrak{b}_m^*$. If $\mu\in\mathfrak{a}_m^*$ is a character exponent of $I_P^G(W\otimes\mathbf{C}_\nu)$ along P_m, there exists $b\in(M_mA_m)^-\cap B_m\cap G'$, such that the local expression (5.5) for the induced character around b,

$$[|D_G|^{1/2}\Theta_G(I_P^G(W\otimes\mathbf{C}_\nu))](b\exp X) = \sum_{s\in W(\mathfrak{g},\mathfrak{b}_m)} c'_\lambda(b,s)\,e^{\langle s\lambda, X\rangle},$$

involves a coefficient

$$c'_\lambda(b,s)\neq 0, \quad \text{with } s\lambda|_{\mathfrak{a}_m}=\mu.$$

In view of the induced character formula in theorem 5.7, this can happen only if the local expression

$$[|D_{MA}|^{1/2}\Theta_{MA}(W\otimes\mathbf{C}_\nu)](vb\exp X) = \sum_{u\in W(\mathfrak{m}\oplus\mathfrak{a},\mathfrak{b}_m)} d'_\lambda(vb,u)\,e^{\langle u\lambda, X\rangle}, \tag{8.25}$$

around a conjugate vb, $v \in W(G, B_m)$, has a non-zero contribution corresponding to $vs\lambda$. In other words, we have found $s \in W(\mathfrak{g}, \mathfrak{b}_m)$, $v \in W(G, B_m)$, with

$$d'_\lambda(vb, vs) \neq 0, \quad vs\lambda|_{\mathfrak{a}_m} = v\mu. \tag{8.26}$$

The Weyl groups $W(G, B_m)$, $W(MA, B_m)$ preserve A_m, and hence act on $\Phi(\mathfrak{g}, \mathfrak{a}_m)$. The group $W(MA, B_m)$ acts as the restricted Weyl group $W(MA, A_m)$, which includes all reflections about roots in $\Phi(\mathfrak{m} \oplus \mathfrak{a}, \mathfrak{a}_m)$. Since the expression (8.25) is $W(MA, B_m)$-invariant, we can modify v by an element of $W(MA, B_m)$ on the left, to arrange

$$v^{-1}\Phi^+(\mathfrak{m} \oplus \mathfrak{a}, \mathfrak{a}_m) \subset \Phi^+(\mathfrak{g}, \mathfrak{a}_m), \tag{8.27}$$

without destroying (8.26). Real roots of $(\mathfrak{g}, \mathfrak{b}_m)$ restrict non-trivially to $\mathfrak{a}_m$, so

$$v^{-1}\Phi^+_{\mathbf{R}}(\mathfrak{m} \oplus \mathfrak{a}, \mathfrak{b}_m) \subset \Phi^+_{\mathbf{R}}(\mathfrak{g}, \mathfrak{b}_m).$$

In conjunction with lemma 5.28, this shows that v maps $(M_m A_m)^- \cap B_m \cap G'$ into the subset of MA which plays the analogous role when $(M_m A_m)^-$ is defined with reference to MA, rather than G. At this point (8.26–27) prove (a):

$$\xi = \text{restriction of } vs\lambda \text{ to } \mathfrak{a}_m$$

is a character exponent of $W \otimes \mathbf{C}_\nu$ along $MA \cap P_m$,

$$w = \text{restriction of } v^{-1} \text{ to } \mathfrak{a}_m$$

satisfies the positivity condition (ii), and $w\xi = \mu$.

The reciprocity theorem 4.11, applied to the identity on $I^G_P(W \otimes \mathbf{C}_\nu)$, produces a surjection

$$H_0(\mathfrak{n}, I^G_P(W \otimes \mathbf{C}_\nu))_\nu \to W \otimes \mathbf{C}_{\nu + \varrho_P} \to 0. \tag{8.28}$$

Thus ν is a homology exponent of $I^G_P(W \otimes \mathbf{C}_\nu)$ along P, and there exists a leading exponent $\nu_l \leq \nu$. According to lemma 8.6, ν_l is the restriction to $\mathfrak{a}$ of a leading exponent along P_m. Hence there exist w, ξ as in (a), with $w\xi|_{\mathfrak{a}} = \nu_l \leq \nu$. As a character exponent of $W \otimes \mathbf{C}_\nu$, ξ restricts to ν on $\mathfrak{a}$. If the hypothesis of (b) is satisfied, ν must coincide with the leading exponent ν_l.

To prove the identity in (b), we initially drop the hypothesis on ν, and regard $\nu \in \mathfrak{a}^*$ as variable. Then λ in (8.25) becomes a function of ν, with

$$\lambda|_{\mathfrak{a}} = \nu, \quad \lambda|_{\mathfrak{a}_m \cap \mathfrak{m}} = \text{constant}. \tag{8.29}$$

We enumerate the restrictions to $\mathfrak{a}$ of the various translates $s\lambda$, $s \in W(\mathfrak{g}, \mathfrak{b}_m)$, as

$$\nu = \nu_1, \nu_2, \ldots, \nu_n, \tag{8.30}$$

without repeating those ν_i which coincide for all values of ν. The order relation $\nu_i \leqslant \nu$ holds only if $\nu - \nu_i$ lies in a discrete subset of $\mathfrak{a}^*$; cf. (2.28). Since $\nu \mapsto \nu - \nu_i$, $i \neq 1$, is non-constant and affine-linear, there exists an open, dense subset $U \subset \mathfrak{a}^*$, such that

$$\begin{aligned} &\nu \in U,\ \nu_i \leqslant \nu \Rightarrow \nu_i = \nu; \\ &\nu \in U,\ i \neq j \Rightarrow \nu_i \neq \nu_j. \end{aligned} \tag{8.31}$$

We claim:

$$\Theta_G(I_P^G(W \otimes \mathbf{C}_\nu))_\nu = \Theta_{MA}(W \otimes \mathbf{C}_{\nu+\varrho_P}) \quad \text{on } (MA)^- \cap G', \tag{8.32}$$

for all $\nu \in U$; here $\Theta_G(\ldots)_\nu$ refers to the ν-component of $\Theta_G(\ldots)$ along P, as defined in § 5. To see this, we first observe that ν is a leading exponent whenever $\nu \in U$: it is a homology exponent because of (8.28), and all potential homology exponents occur among the ν_i. According to theorem 8.1 and (8.28), the difference of the two terms in (8.32) is then an MA-character, restricted to $(MA)^- \cap G'$. We now appeal to proposition 7.1: it suffices to check the identity (8.32) on $(MA)^- \cap B_m \cap G'$. Since B_m is maximally split, the induced character formula in theorem 5.7 involves the single summand φ_1, with $y_1 = e$. When ν is confined to U, a $W(\mathfrak{g}, \mathfrak{b}_m)$-conjugate $s\lambda$ restricts to ν on only if $s \in W(\mathfrak{m} \oplus \mathfrak{a}, \mathfrak{b}_m)$. Hence, in (5.7), the term corresponding to a particular $v \in W(G, B_m)$ contributes to the ν-component of the induced character if and only if $v \in W(MA, B_m)$. As far as the ν-component is concerned, we can omit the sum over v altogether, if we also drop the factor $1/c_1$, because $W(MA, B_m)$ preserves the inducing character. This proves the identity (8.32) on $B_m \cap G'$, and therefore on $(MA)^- \cap G'$.

The $W(G, A)$-conjugates of ν play essentially symmetric roles. Thus, if we replace U by a suitable smaller open, dense subset of $\mathfrak{a}^*$,

$$\Theta_G(I_P^G((vW) \otimes \mathbf{C}_{v\nu}))_{v\nu} = \Theta_{MA}((vW) \otimes \mathbf{C}_{v\nu+\varrho_P}) \tag{8.33}$$

on $(MA)^- \cap G'$, for all $v \in W(G, A)$ and all ν in the newly re-defined set U. Conjugate inducing data yield isomorphic induced representations, and the induced character depends on MA, W, ν, but not on the choice of P; cf. (5.8). Hence

$$\Theta_G(I_P^G(W \otimes \mathbf{C}_\nu)) = \Theta_G(I_{vP}^G((vW) \otimes \mathbf{C}_{v\nu}) = \Theta_G(I_P^G((vW) \otimes \mathbf{C}_{v\nu}). \tag{8.34}$$

Combining (8.33–34), we find

$$\Theta_G(I_P^G(W\otimes \mathbf{C}_\nu))_{v\nu} = \Theta_{MA}((vW)\otimes \mathbf{C}_{v\nu+\varrho_P}) \quad \text{on } (MA)^- \cap G', \tag{8.35}$$

for every $\nu \in U$ and $v \in W(G,A)$.

As ν varies over U, the ν_i-components of $\Theta_G(I_P^G(W\otimes\mathbf{C}_\nu))$ depend smoothly on ν: the induced character formula, and (5.12) in particular show that the coefficients in their local expressions near any $b\in MA\cap G'$ are constant, whereas the exponents vary with ν in an affine-linear fashion. When ν approaches an arbitrary $\nu_0 \in \mathfrak{a}^*$ from inside U, the various ν_i-components have definite limits, and

$$\Theta_G(I_P^G(W\otimes \mathbf{C}_{\nu_0}))_{\nu_0} = \sum_{\nu_i(\nu_0)=\nu_0} \lim_{\nu\to\nu_0} \Theta_G(I_P^G(W\otimes\mathbf{C}_\nu))_{\nu_i}; \tag{8.36}$$

of course, we only need to sum over those ν_i which are actually character exponents for a generic choice of ν. Any such ν_i is dominated from below by a ν_j, which is generically a leading exponent. Because of Lemma 8.6 and 8.24(a) we can find a character exponent ξ of $W\otimes\mathbf{C}_\nu$ along P and $w\in W(G,A_m)$, subject to the positivity condition (ii), such that

$$w\xi|_{\mathfrak{a}} = \nu_j \leqslant \nu_i. \tag{8.37}$$

From now on we suppose that ν_0 satisfies the hypothesis in lemma 8.24(b). The relation (8.37) persists under the specialization $\nu\to\nu_0$, which also sends ν_i to ν_0. According to the hypothesis, w normalizes $\mathfrak{a}$ and fixes ν_0. Since ξ restricts to ν on $\mathfrak{a}$, ν_j is identically equal to $v\nu$, with v=restriction of w to $\mathfrak{a}$, which lies in the normalizer of ν_0 in $W(G,A)$. The condition $\nu_j\leqslant\nu_i$ is discrete, but ν_i and $\nu_j=v\nu$ both tend to ν_0, so $\nu_i=v\nu$. Thus only exponents $\nu_i=v\nu$ contribute to the right hand side of (8.36); their contributions can be read off from (8.35). As we saw before, ν_0 is a leading exponent, hence

$$\Theta_{MA}\big[H_0(\mathfrak{n},(I_P^G(W\otimes\mathbf{C}_{\nu_0}))_{\nu_0}\big] = \Theta_G(I_P^G(W\otimes\mathbf{C}_{\nu_0}))_{\nu_0} = \sum_{v\in W(G,A),\, v\nu_0=\nu_0} \Theta_{MA}((vW)\otimes\mathbf{C}_{\nu_0+\varrho_P})$$

on $(MA)^-\cap G'$, and because of (3.8) even on all of MA. This completes the proof of (b), with $\nu=\nu_0$.

LEMMA 8.38. *The hypothesis of Lemma* 8.24(b) *is satisfied in either of the following situations:*

(a) *W is tempered and* $\operatorname{Re}(\nu,\alpha)<0$ *for* $\alpha\in\Phi^+(\mathfrak{g},\mathfrak{a})$;

(b) *W is square-integrable and* $\nu\in i\mathfrak{a}_0^*$.

Proof. Let $\alpha_1, \ldots, \alpha_r$ be the simple roots in $\Phi^+(\mathfrak{g}, \mathfrak{a}_m)$, enumerated so that

$$\mathfrak{a} = \{X \in \mathfrak{a}_m | \langle \alpha_j, X \rangle = 0 \quad \text{for } s+1 \leqslant j \leqslant r\}, \tag{8.39}$$

and $\mu_1, \ldots, \mu_r$ the corresponding fundamental weights. Then $\alpha_{s+1}, \ldots, \alpha_r$ can be identified with the simple roots in $\Phi^+(\mathfrak{m} \oplus \mathfrak{a}, \mathfrak{a}_m)$, whereas $\mu_1, \ldots, \mu_s$ constitute a basis for the dual space of $\mathfrak{a} \cap [\mathfrak{g}, \mathfrak{g}]$. If ξ is a character exponent of $W \otimes \mathbf{C}_\nu$ along $MA \cap P_m$,

$$\xi = \sum_{i=1}^{s} a_i \mu_i + \sum_{j=s+1}^{r} b_j \alpha_j \quad \text{on } \mathfrak{a}_m \cap [\mathfrak{g}, \mathfrak{g}], \tag{8.40}$$

with suitable $a_i, b_j \in \mathbf{C}$. The first of the two sums represents ν, the second a character exponent of W. Character exponents are dominated from below by leading exponents, hence the assumptions (a), (b) imply, respectively,

$$\begin{aligned} &\text{(a) } \operatorname{Re} a_i < 0 \text{ for } 1 \leqslant i \leqslant s, \ \operatorname{Re} b_j \geqslant 0 \text{ for } s+1 \leqslant j \leqslant r, \\ &\text{(b) } \operatorname{Re} a_i = 0 \text{ for } 1 \leqslant i \leqslant s, \ \operatorname{Re} b_j \geqslant 0 \text{ for } s+1 \leqslant j \leqslant r. \end{aligned} \tag{8.41}$$

We suppose $w \in W(G, A_m)$ sends $\Phi^+(\mathfrak{m} \oplus \mathfrak{a}, \mathfrak{a}_m)$ into $\Phi^+(\mathfrak{g}, \mathfrak{a}_m)$. Then

$$w\alpha_j|_{\mathfrak{a}} \geqslant 0, \quad s+1 \leqslant j \leqslant r, \tag{8.42}$$

and equality in all cases means that w preserves the linear span of $\alpha_{s+1}, \ldots, \alpha_r$; in other words,

$$w\alpha_j|_{\mathfrak{a}} = 0 \quad \text{for } s+1 \leqslant j \leqslant r \Rightarrow w\mathfrak{a} = \mathfrak{a}. \tag{8.43}$$

Since μ_i is dominant, $\mu_i - w\mu_i$ is a non-negative, integral linear combination of $\alpha_1, \ldots, \alpha_r$. The coefficient of α_i,

$$2\frac{(\mu_i - w\mu_i, \mu_i)}{(\alpha_i, \alpha_i)} = \frac{2\|\mu_i\|^2 - 2(w\mu_i, \mu_i)}{(\alpha_i, \alpha_i)} = \frac{\|\mu_i - w\mu_i\|^2}{\|\alpha_i\|^2},$$

is strictly positive unless $w\mu_i = \mu_i$:

$$w\mu_i \neq \mu_i \Rightarrow (\mu_i - w\mu_i)|_{\mathfrak{a}} > 0, \quad 1 \leqslant i \leqslant s. \tag{8.44}$$

If w fixes $\mu_1, \ldots, \mu_s$, it belongs to the group generated by the reflections about $\alpha_{s+1}, \ldots, \alpha_r$. In view of the positivity condition on w, this happens only if $w = 1$:

$$w\mu_i = \mu_i \quad \text{for } 1 \leqslant i \leqslant s \Rightarrow w = 1. \tag{8.45}$$

According to (8.40–45),

$$\nu - w\xi|_{\mathfrak{a}} = \sum_{i=1}^{s} a_i(\mu_i - w\mu_i)|_{\mathfrak{a}} - \sum_{j=s+1}^{r} b_j w\alpha_j|_{\mathfrak{a}}$$
$$= \sum_{i=1}^{s} c_i \alpha_i|_{\mathfrak{a}}, \quad \text{with } \operatorname{Re} c_j \leq 0,$$

which cannot be greater than zero in the order $>$, and equals zero only if $w\mathfrak{a}=\mathfrak{a}$. Since ξ restricts to ν on $\mathfrak{a}$, w must then also fix ν. Hence the lemma.

LEMMA 8.46. *Suppose that W is irreducible, and that ν is a leading exponent of $I_P^G(W\otimes\mathbf{C}_\nu)$ along P. Let $V_1, \ldots, V_n$ be the composition factors of $I_P^G(W\otimes\mathbf{C}_\nu)$, repeated with appropriate multiplicities.*

(a) *If ν is a homology exponent of V_i, it is a leading exponent.*

(b) $\Theta_{MA}[H_0(\mathfrak{n}, I_P^G(W\otimes\mathbf{C}_\nu))_\nu] = \Sigma_i \Phi_{MA}(H_0(\mathfrak{n}, V_i)_\nu)$.

(c) *If $V_1 \oplus \ldots \oplus V_m$, $m \leq n$, can be realized as a completely reducible submodule of $I_P^G(W\otimes\mathbf{C}_\nu)$, then $W\otimes\mathbf{C}_{\nu+\varrho_P}$ occurs as a composition factor of $H_0(\mathfrak{n}, I_P^G(W\otimes\mathbf{C}_\nu))_\nu$ with multiplicity at least m.*

Proof. The assertion (a) follows from corollary 8.4. Together with theorem 8.1, (a) implies the identity (b), at least on $(MA)^- \cap G'$, but MA-characters which agree on this set agree everywhere on MA; cf. (3.8). According to the reciprocity theorem 4.11, $W_{\nu+\varrho_P}$ is a composition factor of $H_0(\mathfrak{n}, V_i)_\nu$ whenever V_i can be embedded into the induced representation. Thus (b) implies (c).

The irreducibility theorem 8.20 is an immediate consequence of the preceeding three lemmas: lemmas 8.24 and 8.38 guarantee that ν is a leading exponent of $I_P^G(W\otimes\mathbf{C}_\nu)$; they also identify the multiplicity of $W\otimes\mathbf{C}_{\nu+\varrho_P}$ in the composition series of $H_0(\mathfrak{n}, I_P^G(W\otimes\mathbf{C}_\nu))_\nu$, as the order of the stabilizer of $W\otimes\mathbf{C}_\nu$ in $W(G,A)$. Because of lemma 8.46, this integer bounds the number of irreducible summands of $I_P^G(W\otimes\mathbf{C}_\nu)$. More generally, lemmas 8.24 and 8.46 imply:

Observation 8.47. The conclusion of theorem 8.20 remains valid if W is an irreducible Harish-Chandra module with a unitary globalization, such that the character exponents of $W\otimes\mathbf{C}_\nu$ satisfy the hypothesis of lemma 8.24(b).

If $P=MAN$, W, ν is a collection of Langlands data, we argue similarly: ν is a leading exponent of $I_P^G(W\otimes\mathbf{C}_\nu)$ along P, and

$$H_0(\mathfrak{n}, I_P^G(W\otimes\mathbf{C}_\nu))_\nu \simeq W\otimes\mathbf{C}_{\nu+\varrho_P}$$

is irreducible, because only the identity in $W(G,A)$ fixes ν. In view of (4.11) and lemma 8.46, $I_P^G(W\otimes\mathbf{C}_\nu)$ has a distinguished composition factor $J_P^G(W\otimes\mathbf{C}_\nu)$, with the following properties:

$$\begin{aligned}&\text{(a) } J_P^G(W\otimes\mathbf{C}_\nu) \text{ is the unique irreducible submodule of } I_P^G(W\otimes\mathbf{C}_\nu);\\ &\text{(b) } H_0(\mathfrak{n}, J_P^G(W\otimes\mathbf{C}_\nu))_\nu \simeq H_0(\mathfrak{n}, I_P^G(W\otimes\mathbf{C}_\nu))_\nu \simeq W\otimes\mathbf{C}_{\nu+\varrho_P};\\ &\text{(c) } J_P^G(W\otimes\mathbf{C}_\nu) \text{ is the one and only composition factor of } I_P^G(W\otimes\mathbf{C}_\nu)\\ &\quad\text{which has } \nu \text{ as leading exponent along } P.\end{aligned} \tag{8.48}$$

The second statement in theorem 8.22 has already been verified; cf. lemma 4.33. The proof of the uniqueness of the Langlands data and of proposition 8.23 depends on certain geometric considerations. For simplicity, we suppose G has compact center, so that

$$\mathfrak{a}_m \subset [\mathfrak{g},\mathfrak{g}]. \tag{8.49}$$

The general case can easily be reduced to this. As in (4.34), we define

$$\mathscr{C} = \{\mu\in(\mathfrak{a}_{m,0})^* \mid (\mu,\alpha)\leq 0 \quad \text{for } \alpha\in\Phi^+(\mathfrak{g},\mathfrak{a}_m)\}. \tag{8.50}$$

Let $\alpha_1,\dots,\alpha_r$ be the simple roots in $\Phi^+(\mathfrak{g},\mathfrak{a}_m)$, and $\mu_1,\dots,\mu_r$ the corresponding fundamental weights. According to (4.39), for each $(\mathfrak{a}_{m,0})^*$, there exists a unique subset $S=S(\mu)\subset\{1,2,\dots,r\}$, such that

$$\mu = \sum_{i\in S} c_i\mu_i + \sum_{j\notin S} d_j\alpha_j, \quad \text{with } c_i<0,\ d_j\geq 0; \tag{8.51}$$

moreover,

$$\mu^0 = \sum_{i\in S} c_i\mu_i \text{ is the point in } \mathscr{C} \text{ closest to } \mu. \tag{8.52}$$

LEMMA 8.53 (cf. Langlands [28]). *If $t_1,\dots,t_r$ are non-negative,*

$$\left\|\left(\mu+\sum_{i=1}^{R} t_i\alpha_i\right)^0\right\| \leq \|\mu^0\|,$$

with equality if and only if $t_j=0$ for all $j\notin S(\mu)$.

Proof. It suffices to consider the case of a single non-zero $t=t_k$. As the d_j in (8.51) are increased, μ^0 remains unchanged. We may therefore suppose $k\in S(\mu)$. For continuity reasons, we only need to argue locally, i.e. $0 \leq t < \varepsilon$. As t tends to zero from above, the image $\mu(t)^0$ of

$$\mu(t) = \mu + t\alpha_k$$

approaches μ^0 on one of the faces of $\mathscr{C}$—this follows from geometric reasons, but also from the uniqueness of the representation (8.51). In other words, there exists a subset $S' \subset \{1, \dots, r\}$, such that $\mu(t) = \Sigma_{i \in S'} c_i(t)\mu_i + \Sigma_{j \notin S'} d_j(t)\alpha_j$, with $c_i(t) < 0$, $d_j(t) \geq 0$, for $0 < t < \varepsilon$. Some of the $c_i(t)$ may vanish at $t = 0$, and the remaining $i \in S'$ constitute $S(\mu)$, again because of the uniqueness of (8.51). In particular $c_k(0) < 0$. Let p denote the orthogonal projection onto the linear span of the μ_i, $i \in S'$. Then $p\alpha_j = 0$, for $j \notin S'$, hence $\mu(t)^0 = p\mu(t)$ on the interval $0 \leq t < \varepsilon$, and

$$\begin{aligned}
\|\mu(t)^0\|^2 &= \|p(\mu + t\alpha_k)\|^2 = \|\mu^0\|^2 + t^2\|p\alpha_k\|^2 + 2t(\mu^0, p\alpha_k) \\
&= \|\mu^0\|^2 + t^2\|p\alpha_k\|^2 + 2t \sum_{i \in S'} c_i(0)(\mu_i, \alpha_k) \\
&= \|\mu^0\|^2 + t^2\|p\alpha_k\| + tc_k(0)\,\|\alpha_k\|^2.
\end{aligned}$$

This is strictly decreasing near $t = 0$, as required.

Let $\mu \in \mathfrak{a}_m^*$ be a homology exponent along P_m, of any one of the composition factors of $I_P^G(W \otimes \mathbf{C}_\nu)$. Because of corollary 8.4, $I_P^G(W \otimes \mathbf{C}_\nu)$ has a leading exponent μ_l along P_m, such that $\mu_l \leq \mu$. Then $\mu_l = w\xi$, with w and ξ as in lemma 8.24. The identities (8.40–41 a) apply in the present context, and $\mathfrak{a}_m \subset [\mathfrak{g}, \mathfrak{g}]$. Hence

$$\begin{gathered}
\xi = \sum_{i=1}^{s} a_i\mu_i + \sum_{j=s+1}^{r} b_j\alpha_j, \quad \text{with } \operatorname{Re} a_i < 0,\ \operatorname{Re} b_j \geq 0, \\
(\operatorname{Re}\xi)^0 = \sum_{i=1}^{s} \operatorname{Re} a_i\mu_i, \quad S(\operatorname{Re}\xi) = \{1, \dots, s\}.
\end{gathered} \tag{8.54}$$

The first of two sums restricts to ν on $\mathfrak{a}$. Since $\mu_1, \dots, \mu_s$ are perpendicular to $(\mathfrak{a}_{m,0} \cap \mathfrak{m})^*$, this implies

$$\|\operatorname{Re}\nu\| = \left\| \sum_{i=1}^{s} (\operatorname{Re} a_i)\mu_i \right\| = \|(\operatorname{Re}\xi)^0\|. \tag{8.55}$$

As was pointed out in the proof of lemma 8.38, the $w\alpha_j$, $j\geq s+1$, and $\mu_i-w\mu_i$, $1\leq i\leq r$, are non-negative linear combinations of $\alpha_1,\ldots,\alpha_r$. We conclude

$$\begin{aligned} w\xi &= \sum_{i=1}^{s} a_i\mu_i+\sum_{i=1}^{s} a_i(w\mu_i-\mu_i)+\sum_{j=s+1}^{r} b_j w\alpha_j \\ &= \sum_{i=1}^{s} a_i\mu_i+\sum_{i=1}^{r} c_i\alpha_i, \quad \text{with } \operatorname{Re} c_i\geq 0. \end{aligned}$$

If $\operatorname{Re} c_i=0$ for all $i<s$, w must be the identity, as follows from (8.44–45). The order relation $\mu\geq\mu_l=w\xi$ gives

$$\mu=\sum_{i=1}^{s} a_i\mu_i+\sum_{i=1}^{r}(c_i+n_i)\,\alpha_i, \quad n_i\geq 0.$$

We now appeal to lemma 8.53: $\|(\operatorname{Re}\mu)^0\|\leq\|\Sigma_{i=1}^{s}(\operatorname{Re} a_i)\,\alpha_i\|$, and equality forces $w=1$, $\mu_l=\xi$, $n_i=0$ for $i\leq s$, hence $\mu|_{\mathfrak{a}}=\nu$, $S(\operatorname{Re}\mu)=\{1,\ldots,s\}$; conversely these last two conditions imply equality. To summarize,

$$\begin{aligned} &\|(\operatorname{Re}\mu)^0\|\leq\|\operatorname{Re}\nu\|, \quad \text{with equality if and only if } S(\operatorname{Re}\mu)=\{1,\ldots,s\} \\ &\qquad\qquad \text{and } \nu=\text{restriction of } \mu \text{ to } \mathfrak{a}, \end{aligned} \tag{8.56}$$

for every homology exponent μ along P_m of any composition factor of $I_P^G(W\otimes\mathbf{C}_\nu)$.

We can now reconstruct the data $P=MAN$, W,ν from $I_P^G(W\otimes\mathbf{C}_\nu)$. Let μ be a homology exponent along P_m which maximizes $\|(\operatorname{Re}\mu)^0\|$. We claim that the maximum value equals $\|\operatorname{Re}\nu\|$. Indeed, any leading exponent ξ of $W\otimes\mathbf{C}_\nu$ along $MA\cap P_m$ is a homology exponent of $J_P^G(W\otimes\mathbf{C}_\nu)$, as follows from (8.48 b) and the isomorphism

$$H_0(\mathfrak{n}_m, J_P^G(W\otimes\mathbf{C}_\nu))_\xi\simeq H_0(\mathfrak{n}\cap\mathfrak{m}, H_0(\mathfrak{n}, J_P^G(W\otimes\mathbf{C}_\nu))_\nu)_{\xi+\varrho_P}.$$

Any such ξ satisfies (8.54), hence $\|(\operatorname{Re}\xi)^0\|=\|\operatorname{Re}\nu\|$ by (8.56), which establishes our claim. Applying (8.56) to our original choice of μ, and recalling (8.39), we find

$$\begin{aligned} &\text{(a) } S(\operatorname{Re}\mu)=\{i\mid 1\leq i\leq r,\ \alpha_i\notin\Phi(\mathfrak{m}\oplus\mathfrak{a},\mathfrak{a}_m)\}, \\ &\text{(b) } \nu=\text{restriction of } \mu \text{ to } \mathfrak{a}. \end{aligned} \tag{8.57}$$

The equality (a) determines M and A, hence also P, which was assumed to lie in a standard position with respect to P_m, and (b) determines ν. The inducing module W can be recovered from (8.48 b). Thus $P=MAN$, W,ν are unique, up to the choice of P_m, which itself is unique up to conjugation.

We prove proposition 8.23 by contradiction. As we just saw, if $\|\operatorname{Re}\nu\|=\|\operatorname{Re}\nu'\|$,

$J_{P'}^{G}(W'\otimes\mathbf{C}_{\nu'})$ has a homology exponent μ' along P_m such that $\|(\mathrm{Re}\,\mu')^0\|=\|\mathrm{Re}\,\nu'\|=\|\mathrm{Re}\,\nu\|$. According to (8.56), μ' restricts to ν on $\mathfrak{a}$. Let $\mu''\leqslant\mu'$ be a leading exponent along P_m. Then μ'' restricts to a homology exponent $\nu''\in\mathfrak{a}^*$ along P (cf. lemma 8.6), and

$$\nu''=\mu''|_{\mathfrak{a}}\leqslant\mu'|_{\mathfrak{a}}=\nu.$$

Since ν is a leading exponent of $I_P^G(W\otimes\mathbf{C}_\nu)$, hence also of its "semisimplification", this makes $\nu''=\nu$ a leading exponent of $J_{P'}^{G}(W'\otimes\mathbf{C}_{\nu'})$, contradicting (8.48 c).

Now the proof of theorem 8.16. Let V be a tempered, irreducible Harish-Chandra module for G, and μ a leading exponent of V along the minimal parabolic subgroup $P_m=M_mA_mN_m$. Then

$$\mu \text{ takes purely imaginary values on } \mathfrak{a}_{m,0}\cap\text{center of } \mathfrak{g}_0, \tag{8.58}$$

and

$$\mu=\sum_{i=1}^{r}a_i\alpha_i \quad \text{on } \mathfrak{a}_m\cap[\mathfrak{g},\mathfrak{g}], \text{ with } \mathrm{Re}\,a_i\geqslant 0; \tag{8.59}$$

as before $\alpha_1,\dots,\alpha_r$ are the simple roots in $\Phi^+(\mathfrak{g},\mathfrak{a}_m)$. We suppose that μ has been choosen among all the leading exponent so as to maximize the cardinality of the set

$$T=\{i\,|\,\mathrm{Re}\,a_i=0\}\subset\{1,\dots,r\}. \tag{8.60}$$

This set determines a parabolic subgroup $P=MAN$, in standard position with respect to P_m, such that

$$\mathfrak{a}=\{X\in\mathfrak{a}_m\,|\,\langle\alpha_i,X\rangle=0 \quad \text{for } i\notin T\}. \tag{8.61}$$

Let ν denote the restriction of μ to $\mathfrak{a}$. We claim:

$$\begin{gathered} H_0(\mathfrak{n},V)_\nu \text{ is non-zero and square-integrable,}\\ \text{as Harish-Chandra module for } M. \end{gathered} \tag{8.62}$$

Indeed, ν is a homology exponent of V along P by lemma 8.6, hence dominated by a leading exponent $\nu'\leqslant\nu$, which is the restriction to $\mathfrak{a}$ of another leading exponent μ' along P_m, again by lemma 8.6. More generally, we consider a homology exponent μ' along P_m, which restricts to ν'. Since V is tempered and $\mu'|_{\mathfrak{a}}\leqslant\mu|_{\mathfrak{a}}$,

$$\begin{aligned} &\mu'=\sum_{i=1}^{r}b_i\alpha_i \quad \text{on } \mathfrak{a}_m\cap[\mathfrak{g},\mathfrak{g}], \text{ with } \mathrm{Re}\,b_i\geqslant 0,\\ &b_i=a_i-n_i \quad \text{for } i\in T, \text{ with } n_i\geqslant 0; \end{aligned} \tag{8.63}$$

moreover $\mu-\mu'$ vanishes on $\mathfrak{a}_m\cap$center of $\mathfrak{g}$, which lies in $\mathfrak{a}$. Comparing (8.63) to (8.60), we find $a_i=b_i$ if $i\in T$. Hence ν coincides with the leading exponent ν', and

$H_0(\mathfrak{n}, V)_\nu \neq 0$. Any leading exponent μ' of $H_0(\mathfrak{n}, V)_\nu$ along $MA \cap P_m$ restricts to ν on $\mathfrak{a}$ and is a homology exponent of V along P_m, because

$$H_0(\mathfrak{n}_m, V) \simeq H_0(\mathfrak{n}_m \cap \mathfrak{m}, H_0(\mathfrak{n}, V)).$$

In particular (8.63) applies to μ'. Since $\operatorname{Re} b_i = \operatorname{Re} a_i = 0$ for all $i \in T$, the maximality of T implies $\operatorname{Re} b_i > 0$ for $i \notin T$. This completes the verification of (8.62): the α_i, $i \notin T$, can be identified with the simple roots in $\Phi^+(\mathfrak{m} \oplus \mathfrak{a}, \mathfrak{a}_m)$. The exponent ν assumes purely imaginary values on $\mathfrak{a}_0$, as follows from (8.58–61). If W is an irreducible quotient of $H_0(\mathfrak{n}, V)_\nu$, the reciprocity theorem 4.11 provides the required embedding $V \subset I_P^G(W \otimes \mathbf{C}_\nu)$. The preceeding argument is virtually identical to that of Borel-Wallach [3].

We now suppose that V occurs as a summand of both $I = I_P^G(W \otimes \mathbf{C}_\nu)$ and $I' = I_{P'}^G(W' \otimes \mathbf{C}_{\nu'})$. The two parabolic subgroups $P = MAN$, $P' = M'A'N'$, which we put into standard position with respect to P_m, correspond to subsets

$$T = \{i \mid \alpha_i|_{\mathfrak{a}} \neq 0\}, \quad T' = \{i \mid \alpha_i|_{\mathfrak{a}'} \neq 0\} \tag{8.64}$$

of $\{1, \dots, r\}$. Since I, I' play symmetric roles, we may assume

$$\operatorname{card} T \leqslant \operatorname{card} T'. \tag{8.65}$$

According to Lemma 8.24, Lemma 8.38 and (4.11), ν' is a leading exponent of I' along P', and a homology exponent of its composition factor V. Thus ν' is a leading exponent of V along P', and can be realized as the restriction to $\mathfrak{a}'$ of a leading exponent μ' of V along P_m; cf. lemma 8.6. Since $V \subset I$, there exists a leading exponent $\mu \leqslant \mu'$ of I along P_m. lemma 8.24 allows us to write $\mu = w\xi$, where $w \in W(G, A_m)$ satisfies

$$w\alpha_i \in \Phi^+(\mathfrak{g}, \mathfrak{a}_m) \quad \text{if } i \notin T, \tag{8.66}$$

and ξ is a character exponent of $W \otimes \mathbf{C}_\nu$ along $MA \cap P_m$. Since W is square-integrable and ν imaginary,

$$\xi = \sum_{i=1}^{r} a_i \alpha_i \quad \text{on } \mathfrak{a}_m \cap [\mathfrak{g}, \mathfrak{g}], \text{ with } \operatorname{Re} a_i > 0 \quad \text{if } i \notin T, \quad \operatorname{Re} a_i = 0 \quad \text{if } i \in T. \tag{8.67}$$

Combining (8.66–67), we find

$$\begin{gathered} \mu = w\xi = \sum_{i=1}^{r} b_i \alpha_i \quad \text{on } \mathfrak{a}_m \cap [\mathfrak{g}, \mathfrak{g}], \\ \text{with } \operatorname{Re} b_i > 0 \quad \text{if } i \notin T'', \quad \operatorname{Re} b_i = 0 \quad \text{if } i \in T''; \end{gathered} \tag{8.68}$$

here $T''\subset\{1,\ldots,r\}$ is the largest subset such that

$$\{w\alpha_i|i\notin T\}\subset\text{span of }\{\alpha_i|i\notin T''\}. \tag{8.69}$$

In particular card $T''\leqslant$ card T. The exponent μ', which is $\geqslant\mu$, restricts to the imaginary linear function ν' on $\mathfrak{a}_0'$. Because of (8.64–65) and (8.68–69), this can happen only if card T''=card T, $T''=T'$, $\mu|_{\mathfrak{a}'}=\mu'|_{\mathfrak{a}}=\nu'$, $w\mathfrak{a}=\mathfrak{a}'$, and hence $wM=M'$. As a character exponent of $W\otimes\mathbf{C}_\nu$, $\xi=w^{-1}\mu$ restricts to ν on $\mathfrak{a}$, so $w\nu=\nu'$. Because of (5.8) and observation 8.14, $I_P^G(W\otimes\mathbf{C}_\nu)$ depends only on the data (MA, W, ν), not on P. As we just saw, w conjugates (MA,ν) to $(M'A',\nu')$. At this point, we may as well assume $P=P'$, $MA=M'A'$, $\nu=\nu'$. Once more we appeal to lemmas 8.24 and 8.38:

$$\Theta_{MA}[H_0(\mathfrak{n}, I_P^G(W\otimes\mathbf{C}_\nu))_\nu]=\sum_{v\in W(G,A),\,v\nu=\nu}\Theta_{MA}((vW)\otimes\mathbf{C}_{\nu+\varrho_P}).$$

Since $I_P^G(W\otimes\mathbf{C}_\nu)$, $I_P^G(W'\otimes\mathbf{C}_\nu)$ have a summand in common,

$$\begin{aligned}0&\neq\operatorname{Hom}_G(I_P^G(W\otimes\mathbf{C}_\nu), I_P^G(W'\otimes\mathbf{C}_\nu))\\&\simeq\operatorname{Hom}_{MA}(H_0(\mathfrak{n}, (I_P^G(W\otimes\mathbf{C}_\nu))_\nu, W'\otimes\mathbf{C}_{\nu+\varrho_P}).\end{aligned}$$

The last assertion of theorem 8.16 follows: $W'=vW$, for some v in the stabilizer of ν in $W(G,A)$.

Appendix

For the convenience of the reader, we supply proofs of certain results of Miličić, which were quoted in the proof of theorem 4.25 and in (6.13–14). In passing we also verify the inequality (6.50).

We fix a minimal parabolic subgroup $P_m=M_mA_mN_m$ with $M_m\subset K$, an irreducible M_m-module W, and a linear function $\mu\in\mathfrak{a}_m^*$, subject to the condition

$$\operatorname{Re}(\mu+\varrho_m,\alpha)<0\quad\text{for all }\alpha\in\Phi^+(\mathfrak{g},\mathfrak{a}_m). \tag{A 1}$$

The pairing (6.27) identifies $I_{P_m}^G(W'\otimes\mathbf{C}_{-\mu})$ with the dual of $I_{P_m}^G(W\otimes\mathbf{C}_\mu)$. In particular,

$$c_{f,f^*}(g)=\int_K\langle f^*(k), f(g^{-1}k)\rangle\,dk\quad(g\in G) \tag{A 2}$$

is the matrix coefficient corresponding to $f\in I_{P_m}^G(W\otimes\mathbf{C}_\mu)$ and $f^*\in I_{P_m}^G(W'\otimes\mathbf{C}_{-\mu})$. Let

$\bar{P}_m = M_m A_m \bar{N}_m$ denote the parabolic subgroup opposite to P_m. As we shall see presently, the integral

$$Jf^*(g) = \int_{\bar{N}_m} f^*(g\bar{n})\, d\bar{n} \tag{A 3}$$

converges(¹) in the situation (A 1); for purely formal reasons, $f^* \mapsto Jf^*$ defines a homomorphism of Harish-Chandra modules

$$J: I^G_{P_m}(W' \otimes \mathbf{C}_{-\mu}) \to I^G_{\bar{P}_m}(W' \otimes \mathbf{C}_{-\mu}), \tag{A 4}$$

the so-called "standard intertwining operator". We consider a particular $X \in (\mathfrak{a}_{m,0})^-$ and set $a_t = \exp(tX)$, $t \in \mathbf{R}$; thus $a_t \in A_m^-$ if $t > 0$.

PROPOSITION A 5 (Harish-Chandra [17]). *If the invariant measure $d\bar{n}$ is normalized appropriately,*

$$\lim_{t \to +\infty} e^{-\mu - \varrho_m}(a_t)\, c_{f,f^*}(a_t) = \langle Jf^*(e), f(e) \rangle.$$

The proof requires some preparation. Each $g \in G$ factors as $g = k(g)\, a(g)\, n(g)$, according to the Iwasawa decomposition $KA_m N_m$ of G. Suppose $\lambda \in \mathfrak{a}^*_{m,0}$ satisfies $(\lambda, \alpha) \geq 0$ for all $\alpha \in \Phi^+(\mathfrak{g}, \mathfrak{a}_m)$. Then

LEMMA A 6. *$e^\lambda(a(a_{-t}k)) \leq e^\lambda(a_{-t})$ whenever $k \in K$ and $t \geq 0$.*

In effect, this is the inequality (6.50). We assume the lemma for the moment. Because of (A 1),

$$\|e^{-\mu - \varrho_m}(a_t)\, f(a_{-t}k)\| = \frac{e^{-\mathrm{Re}\,\mu - \varrho_m}(a(a_{-t}k))}{e^{-\mathrm{Re}\,\mu - \varrho_m}(a_{-t})} \|f(k(a_{-t}k))\|$$

is uniformly bounded for $k \in K$, $t \geq 0$. We can therefore apply the dominated convergence theorem:

$$\begin{aligned} \lim_{t \to +\infty} e^{-\mu - \varrho_m}(a_t)\, c_{f,f^*}(a_t) &= \lim_{t \to \infty} \int_K e^{-\mu - \varrho_m}(a_t) \langle f^*(k), f(a_{-t}k) \rangle\, dk \\ &= \int_K \lim_{t \to \infty} e^{-\mu - \varrho_m}(a_t) \langle f^*(k), f(a_{-t}k) \rangle\, dk, \end{aligned} \tag{A 7}$$

(¹) In fact, the integral converges and proposition A5 below applies even if $\mathrm{Re}(\mu, \alpha) < 0$ for all $\alpha \in \Phi^+(\mathfrak{g}, \mathfrak{a}_m)$ [17]. The more restrictive hypothesis (A1), which is entirely adequate for our purposes, simplifies the arguments slightly.

provided the limit under the integral sign exists almost everywhere. The $\bar{P}_m$-orbit of the identity coset in G/P_m is open and dense, and can be identified with $\bar{N}_m$. Since $G/P_m \simeq K/M_m$, the map

$$\bar{N}_m \to K/M_m, \quad \bar{n} \mapsto k(\bar{n})\, M_m \tag{A 8}$$

describes a diffeomorphism of $\bar{N}_m$ onto an open, dense subset of K/M_m. A straightforward calculation of the Jacobian shows that

$$\int_{K/M_m} \varphi(kM_m)\, dk^* = \int_{\bar{N}_m} \varphi(k(\bar{n})\, M_m)\, e^{-2\varrho_m}(a(\bar{n}))\, d\bar{n} \tag{A 9}$$

for $\varphi \in L^1(K/M_m)$; this is the normalization we use to define J. The integrand in (A 7) is M_m-invariant on the right. For $k=k(\bar{n})$,

$$\begin{aligned} &e^{-\mu-\varrho_m}(a_t)\, \langle f^*(k(\bar{n})), f(a_{-t}k(\bar{n}))\rangle \\ &\quad = e^{-\mu-\varrho_m}(a_t)\, e^{2\varrho_m}(a(\bar{n}))\, \langle e^{\mu-\varrho_m}(a(\bar{n}))\, f^*(k(\bar{n})), e^{-\mu-\varrho_m}(a(\bar{n}))\, f(a_{-t}k(\bar{n}))\rangle \\ &\quad = e^{-\mu-\varrho_m}(a_t)\, e^{2\varrho_m}(a(\bar{n}))\, \langle f^*(\bar{n}), f(a_{-t}\bar{n})\rangle \\ &\quad = e^{2\varrho_m}(a(\bar{n}))\, \langle f^*(\bar{n}), f(a_{-t}\bar{n}a_t)\rangle . \end{aligned}$$

If α is a positive restricted root, $e^{-\alpha}(a_{-t})$ tends to 0 as $t \to +\infty$. Hence $a_{-t}\bar{n}a_t \to e$, and we conclude:

$$\lim_{t\to+\infty} e^{-\mu-\varrho_m}(a_t)\, \langle f^*(k(\bar{n})), f(a_{-t}k(\bar{n}))\rangle = e^{2\varrho_m}(a(\bar{n}))\, \langle f^*(\bar{n}), f(e)\rangle . \tag{A 10}$$

Now (A 7) and (A 9–10) imply

$$\lim_{t\to+\infty} e^{-\mu-\varrho_m}(a_t)\, c_{f,f^*}(a_t) = \int_{\bar{N}_m} \langle f^*(\bar{n}), f(e)\rangle\, d\bar{n}. \tag{A 11}$$

In particular, the integral on the right converges. As f varies over $I^G_{P_m}(W\otimes\mathbf{C}_\mu)$, the values $f(e)$ span the finite dimensional vector space W. The integral (A 3) therefore converges also, at least for $g=e$. Since we did not use the K-finiteness of f^*, we can apply the same argument to any g-translate of f^*, and this proves the convergence of the integral (A 3) in all cases. Clearly (A 11) establishes the proposition.

We must still prove lemma A 6, which is a standard tool in reduction theory [2]. On the intersection of $\mathfrak{a}_m$ with $[\mathfrak{g}, \mathfrak{g}]$, λ coincides with a non-negative linear combination of fundamental highest weights. Thus we may as well suppose that λ occurs as the highest $\mathfrak{a}_m$-weight of an irreducible, finite dimensional G-module F, and that the λ-weight space F^λ has dimension one. We introduce a K-invariant inner produce on F which makes the $\mathfrak{a}_m$-weight space decomposition

$$F = \bigoplus_{\nu \in \mathfrak{a}^*_{m,0}} F^\nu$$

orthogonal. If $v_\lambda \in F^\lambda$ is a unit vector,

$$\|gv_\lambda\| = \|k(g)\, a(g)\, n(g)\, v_\lambda\| = e^\lambda(a(g))\, \|k(g)\, v_\lambda\| = e^\lambda(a(g)).$$

We apply this identity for $g = a_{-t}k$, $t \geq 0$, and we write kv_λ as a sum of weight vectors v_ν. All weights ν satisfy $\nu \leq \lambda$, hence

$$\begin{aligned} e^{2\lambda}(a(a_{-t}k)) &= \|a_{-t}kv_\lambda\|^2 = \sum_\nu e^{2\nu}(a_{-t})\, \|v_\nu\|^2 \\ &\leq e^{2\lambda}(a_{-t}) \sum_\nu \|v_\nu\|^2 = e^{2\lambda}(a_{-t})\, \|kv_\lambda\|^2 = e^{2\lambda}(a_{-t}) \end{aligned}$$

(recall: a_{-t} lies in the closure of the highest Weyl chamber). This is the assertion of the lemma.

It is well known that the operator J does not vanish identically. One can also deduce this fact from the proof of proposition A 5, as follows. If $f \in I^G_{P_m}(W \otimes \mathbf{C}_\mu)$ assumes a non-zero value $f(e)$, and if one drops the K-finiteness condition on f^*, one can use the map (A 8) to produce a smooth function $f^*: G \to W' \otimes \mathbf{C}_{-\mu}$, subject to the transformation rule which characterizes $I^G_{P_m}(W' \otimes \mathbf{C}_{-\mu})$, such that

$$\int_{\bar{N}_m} \langle f^*(\bar{n}), f(e) \rangle \, d\bar{n} \neq 0. \tag{A 12}$$

The derivation of the identity (A 11) gives the bound

$$\left| \int_{\bar{N}_m} \langle f^*(\bar{n}), f(e) \rangle \, d\bar{n} \right| \leq \sup_{k \in K} \|f^*(k)\| \sup_{k \in K} \|f(k)\|. \tag{A 13}$$

There exists a sequence $\{f^*_n\} \subset I^G_{P_m}(W' \otimes \mathbf{C}_{-\mu})$, approximating f^* uniformly on K. Because of (A 12–13), $Jf^*_n(e)$ cannot vanish for all n, hence $J \neq 0$.

Let L denote the kernel of J, and V the annihilator of L in $I^G_{P_m}(W\otimes\mathbf{C}_\mu)$; then

$$V' \simeq I^G_{P_m}(W'\otimes\mathbf{C}_{-\mu})/L. \tag{A 14}$$

The next result is due to Miličić [31]. As Miličić points out, it constitutes a refinement of a lemma of Langlands [28].

PROPOSITION A 15 [31]. *$I^G_{P_m}(W\otimes\mathbf{C}_\mu)$ has a unique irreducible submodule, namely V. If W_1 is another irreducible M_m-module, the unique irreducible submodules of $I^G_{P_m}(W\otimes\mathbf{C}_\mu)$, $I^G_{P_m}(W_1\otimes\mathbf{C}_\mu)$ are non-isomorphic unless $W_1\simeq W$. Among the composition factors of $I^G_{P_m}(W\otimes\mathbf{C}_\mu)$, V and only V has μ as a leading exponent along P_m.*

We note that (6.13–14) are immediate consequences. Part of the proposition is also used in the proof of theorem 4.25.

Proof. In order to identify V as the unique irreducible submodule of $I^G_{P_m}(W\otimes\mathbf{C}_\mu)$, it suffices to show that $L=\operatorname{Ker} J$ contains every proper submodule L_1 of $I^G_{P_m}(W'\otimes\mathbf{C}_{-\mu})$. If L_1 is such a proper submodule, there exists a non-zero vector f in its annihilator. For all $f^*\in L_1$ and all $k_1,k_2\in K$, the matrix coefficient corresponding to $k_1^{-1}f$ and $k_2^{-1}f^*$ vanishes identically. Hence, by proposition A 5,

$$\langle Jf^*(k_2), f(k_1)\rangle = \lim_{t\to+\infty} e^{-\mu-\varrho_m}(a_t)\, c_{k_1^{-1}f,\,k_2^{-1}f^*}(a_t) = 0.$$

The values $f(k_1)$ span the irreducible M_m-module W. Consequently $Jf^*=0$ on K, and $f^*\in L$, as required.

The pairing $f\otimes f^*\mapsto\langle Jf^*(e), f(e)\rangle$ induces a bilinear form $\langle\ ,\ \rangle$ on $V\times V'$, which is equivariant with respect to M_m and $\mathfrak{a}_m$. This bilinear form does not depend on how V is realized as a submodule of $I^G_{P_m}(W\otimes\mathbf{C}_\mu)$: for $v\in V$, $v^*\in V'$,

$$\langle v, v^*\rangle = \lim_{t\to+\infty} e^{-\mu-\varrho_m}(a_t)\, c_{v,v^*}(a_t) \tag{A 16}$$

can be calculated in terms of the matrix coefficient c_{v,v^*}. In particular,

$$V/\text{radical of } \langle\ ,\ \rangle \tag{A 17}$$

is an invariant of V and μ. The original description of $\langle\ ,\ \rangle$, on the other hand, sets up an M_m-homomorphism between W and the quotient (A 17). This proves the second part of the proposition.

The expression (A 16) has a limit as $t \to +\infty$, for all $v \in V$, $v^* \in V'$ and $X \in (\mathfrak{a}_{m,0})^-$ (recall: $a_t = \exp tX$). Since the limit does not vanish identically in v, v^*, the convergence properties of the asymtotic expansion (4.16) force μ to occur among the leading exponents of V. In view of proposition A 5 and the description (A 14) of V', the expression analogous to (A 16), corresponding to any matrix coefficient of $I^G_{P_m}(W \otimes \mathbf{C}_\mu)/V$ and any choice of $X \in (\mathfrak{a}_{m,0})^-$, tends to zero as $t \to +\infty$. This precludes the occurrence of μ as a leading exponent of a composition factor other than V. The proof of proposition A 15 is now complete.

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