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Bounded orthogonal systems and the $\Lambda(p)$-set problem

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0. Introduction

Let G be a compact Abelian group and Γ the dual group of G. For $p>2$, a subset Λ of Γ is called a $\Lambda(p)$-set, provided $L^p_\Lambda(G)=L^2_\Lambda(G)$. Here $L^p_\Lambda\equiv L^p_\Lambda(G)$ denotes the closure in $L^p(G)$ of the characters belonging to Λ and considered as functions on G. The reader will find an introduction to the subject in W. Rudin's 1960 paper [Ru] and the book of Lòpez and Ross [L-R].

The main problem in this area is to construct $\Lambda(p)$-sets which are not $\Lambda(r)$ for some $r>p$. This has so far only been done for p an even integer. In this case, the L^p-norm may be expressed in an algebraic way and the solution is of an arithmetic or combinatorial nature. In this paper, we consider the range $2<p<\infty$. Our approach is the point of view of general uniformly bounded orthogonal systems and no further properties of characters are exploited. The main result is the following fact.

THEOREM 1. *Let $\Phi=(\varphi_1,\ldots,\varphi_n)$ be a sequence of n mutually othogonal functions, uniformly bounded by* 1 *(i.e., $\|\varphi_i\|_\infty\leqslant 1$, $i=1,\ldots,n$). Let $2<p<\infty$. There is a subset S of $\{1,\ldots,n\}$, $|S|>n^{2/p}$ satisfying*

$$\left\|\sum_{i\in S} a_i\varphi_i\right\|_p\leqslant C(p)\left(\sum_{i\in S}|a_i|^2\right)^{1/2} \tag{0.1}$$

for all scalar sequences (a_i). Here $C(p)$ is a constant only dependent on p. In fact, (0.1) *holds for a generic set S of size $[n^{2/p}]$.*

Observe that the size $n^{2/p}$ is optimal. Indeed, if one considers for instance a finite Cantor group $G=\{1,-1\}^k$ and let $\Phi=G^*$, the space $L^p_S(G)$ is a Hilbertian subspace of

$L^p(G)\simeq \ell^p_n$, $n=2^k$, as soon as (0.1) is fulfilled. According to the results of [B-D-G-J-N] (cf. [F-L-M]), the largest possible dimension for such subspaces is $n^{2/p}$ (up to a constant). Previous observation shows the relation of Theorem 1 above to Dvoretsky's theorem on Hilbertian sections of convex bodies (se again [F-L-M] for more details).

An immediate corollary of Theorem 1 is the following.

THEOREM 2. *For* $2<p<\infty$, *there is a* $\Lambda(p)$-subset of $\mathbf{Z}$ *which is not a* $\Lambda(r)$*-set for any* $r>p$.

Let us point out that the situation for $p<2$ is different in this aspect. It was proved by Bachelis and Ebenstein ([B-E], based on earlier results of Rosenthal [Ro]) that for every set $\Lambda\subset\mathbf{Z}$

$$\{p\in]1,2[;\ L^1_\Lambda = L^p_\Lambda\}$$

is an open interval.

To deduce Theorem 2 from Theorem 1, consider for each $k=1,2,\ldots$ a set $S_k\subset[2^k\leq n<2^{k+1}]$ satisfying

$$|S_k| = [4^{k/p}] \tag{0.2}$$

and

$$\left\|\sum_{j\in S_k} a_j e^{2\pi ijx}\right\|_{L^p(\mathbf{T})} \leq C\left(\sum_{j\in S_k} |a_j|^2\right)^{1/2} \tag{0.3}$$

just applying Theorem 1 to the system $\Phi=\{e^{2\pi inx}|2^k\leq n<2^{k+1}\}$. Put $\Lambda=\bigcup_{k=1}^\infty S_k$. It follows from the Littlewood-Paley theory (cf. [St], for instance)

$$\left\|\sum_{j\in\Lambda} a_j e^{2\pi ijx}\right\|_p \sim \left\|\left(\sum_{k=1}^\infty \left|\sum_{j\in S_k} a_j e^{2\pi ijx}\right|^2\right)^{1/2}\right\|_p \tag{0.4}$$

which for $p\geq 2$ is bounded by

$$\left(\sum_{k=1}^\infty \left\|\sum_{j\in S_k} a_j e^{2\pi ijx}\right\|_p^2\right)^{1/2} \sim \left(\sum |a_j|^2\right)^{1/2}$$

invoking (0.3).

On the other hand, since by (0.2)

$$\left\|\sum_{j\in S_k} e^{2\pi ijx}\right\|_r \geq \left(\int_{-2^{-k}/10}^{2^{-k}/10}\left|\sum_{j\in S_k} e^{2\pi ijx}\right|^r dx\right)^{1/r} > c2^{-k/r}|S_k| \sim 2^{k(1/p-1/r)}|S_k|^{1/2}$$

Λ is not a $\Lambda(r)$-set for any $r>p$.

Our approach will first cover the range $p\in]2,4[$. At some points the cases $2<p\leq 3$ and $3<p<4$ will be distinguished, because of different behavior of the function $|x|^{p-2}$. In the last section, we show how to proceed for $p\geq 4$. Clearly, Theorem 1 need only be proven in the real context.

The letter C will be used for different constants, possibly depending on p. The rest of the paper is devoted to proving Theorem 1 and is organized as follows:

Section 1: A probabilistic inequality
Section 2: An entropy estimate
Section 3: Decoupling inequalities
Section 4: End of the proof ($p<4$)
Section 5: End of the proof ($p\geq 4$)
Section 6: Further comments

The exposition is completely self-contained.

1. A probabilistic inequality

For $x\in\mathbf{R}^n$, denote $|x|=(\Sigma_{i=1}^n x_i^2)^{1/2}$. If $\mathscr{E}\subset\mathbf{R}^n$ and $t>0$, denote $N_2(\mathscr{E},t)$ the metrical entropy number with respect to the l^2-distance, i.e., the minimum number of l^2-balls of radius t needed to cover $\mathscr{E}$.

LEMMA 1. *Let $\mathscr{E}$ be a subset of $\mathbf{R}_+^n$ and $B=\sup_{x\in\mathscr{E}}|x|$. Let $0<\delta<1$ and $(\xi_i)_{i=1}^n$ independent 0,1-valued random variables (=selectors) of mean $\delta=\int\xi_i(\omega)d\omega$. Let $1\leq m\leq n$. Then*

$$\left\|\sup_{x\in\mathscr{E},|A|\leq m}\left[\sum_{i\in A}\xi_i(\omega)x_i\right]\right\|_{L^{q_0}(d\omega)} \leq C\left[\delta m+\frac{q_0}{\log 1/\delta}\right]^{1/2} B+\left(\log\frac{1}{\delta}\right)^{-1/2}\int_0^B[\log N_2(\mathscr{E},t)]^{1/2}dt. \tag{1.1}$$

In the proof of Lemma 1, we use the following.

LEMMA 2. *If the (ξ_i) are as above, then for $q\geq 1$*

$$\left\|\sum_{i=1}^{l}\xi_i(\omega)\right\|_{L^q(d\omega)} \leq C\delta l+C\frac{q}{\log(2+q/\delta l)}. \tag{1.2}$$

Proof. It is clearly no restriction to assume $q>2\delta l$. Write (assuming q an integer)

$$\left\|\sum_{i=q}^{l}\xi_i\right\|_q^q=\int\left[\sum_{i=1}^{l}\xi_i(\omega)\right]^q d\omega$$
$$=\sum_{k=0}^{l}\binom{l}{k}\delta^k(1-\delta)^{l-k}k^q\le C\sum_{k=1}^{l}\left(\frac{\delta l}{k}\right)^k k^q$$

which may be evaluated by

$$q^q\int_0^\infty\left(\frac{\delta l}{\alpha q}\right)^{\alpha q}\alpha^q d\alpha<\left[\frac{C\cdot q}{\log(q/\delta l)}\right]^q.$$

This implies (1.2).

Proof of Lemma 1. By considering appropriate nets in $\mathscr{E}$ (taking the entropy information into account), there is a representation of the elements of $x\in\mathscr{E}$ as sums

$$x=\sum_{\substack{k\in\mathbf{Z}\\ 2^k\le B}}2^k y(k) \tag{1.3}$$

where $y(k)$ are vectors taken in a set $\mathscr{F}_k$ of vectors y, $|y|\le 1$ and where

$$\log|\mathscr{F}_k|\le C\log N(\mathscr{E},2^{k-2}). \tag{1.4}$$

Hence, from (1.3)

$$(1.1)\le\sum_{2^k\le B}2^k\left\|\sup_{y\in\mathscr{F}_k,|A|\le m}\left[\sum_{i\in A}\xi_i(\omega)\,|y_i|\right]\right\|_{q_0}. \tag{1.5}$$

Evaluating the individual terms of (1.5), we show that for $\mathscr{F}\subset\mathbf{R}_+^n$

$$\left\|\sup_{y\in\mathscr{F},|A|\le m}\left(\sum_{i\in A}\xi_i(\omega)\,y_i\right)\right\|_{q_0}\le C\sqrt{\delta m}+C\left(\log\frac{1}{\delta}\right)^{-1/2}[q_0+\log|\mathscr{F}|]^{1/2}. \tag{1.6}$$

Substitution of (1.4) and summing over k, $2^k\le B$, easily implies (1.1).

Define

$$\varrho_1=\delta^{1/2}m^{-1/2}\quad\text{and}\quad\varrho_2=\left(\log\frac{1}{\delta}\right)^{1/2}q^{-1/2}\quad\text{where}\quad q=q_0+\log|\mathscr{F}|. \tag{1.7}$$

Writing, since $|y|\le 1$, for $|A|\le m$

$$\sum_{i\in A}\xi_i(\omega)\,y_i\le\sum_{y_i\ge\varrho_2}y_i+\sum_{i\in A,\,y_i\le\varrho_1}y_i+\sum_{\varrho_1<y_i<\varrho_2}\xi_i(\omega)\,y_i\le\varrho_2^{-1}+m\varrho_1$$
$$+\sum_{\varrho_1<y_i<\varrho_2}\xi_i(\omega)\,y_i\sup_{y\in\mathcal{F},|A|\le m}\Big(\sum_{i\in A}\xi_i(\omega)\,y_i\Big)$$
$$\le\varrho_2^{-1}+m\varrho_1+\sup_{y\in\mathcal{F}}\Big(\sum_{\varrho_1<y_i<\varrho_2}\xi_i(\omega)\,y_i\Big)$$

it follows that (1.6) is bounded by

$$\varrho_2^{-1}+m\varrho_1+\sup_{|y|\le1}\Big\|\sum_{\varrho_1<y_i<\varrho_2}\xi_i(\omega)\,y_i\Big\|_{L^q(d\omega)}. \tag{1.8}$$

(If $\varrho_1\ge\varrho_2$, drop the last term.)

By considering level-sets and inequality (1.2), the last term of (1.8) may be estimated

$$\sum_{\substack{l\text{ diadic}\\ \varrho_1^{-2}>l>\varrho_2^{-2}}}\frac{1}{\sqrt{l}}\Big\|\sum_{i=1}^{l}\xi_i(\omega)\Big\|_q\le C\delta\varrho_1^{-1}+Cq\sum_{\substack{l\text{ diadic}\\ l>\varrho_2^{-2}}}\Big[l^{1/2}\log\Big(2+\frac{q}{\delta l}\Big)\Big]^{-1}$$
$$\sim\delta\varrho_1^{-1}+q\varrho_2\int_1^\infty t^{-3/2}\Big[\log\Big(2+\frac{\log 1/\delta}{\delta t}\Big)\Big]^{-1}dt$$
$$\sim\delta\varrho_1^{-1}+q\varrho_2\Big(\log\frac{1}{\delta}\Big)^{-1}$$

using the definition of ϱ_2.

Hence

$$(1.6)\le m\varrho_1+C\delta\varrho_1^{-1}+q\varrho_2(\log 1/\delta)^{-1}+\varrho_2^{-1}<C\sqrt{\delta m}+C(\log 1/\delta)^{-1/2}(q_0+\log|\mathcal{F}|)^{1/2},$$

completing the proof.

2. An entropy estimate

In a later application of inequality (1.1), the entropy numbers $N_2(\mathscr{E},t)$ will be related to entropy numbers $N_q(\mathscr{P},t)$ for certain sets of functions $\mathscr{P}$, considered as a subset of the corresponding L^q-space. More precisely, we will make use of the following

LEMMA 3. *Let $\Phi=\{\varphi_i\}_{i=1}^n$ be an orthogonal system of functions uniformly bounded by* 1, *$m\le n$ and $2\le q<\infty$. Define*

$$\mathcal{P}_m=\left\{\sum_{i\in A} a_i\varphi_i \,\middle|\, |a|\leq 1 \text{ and } |A|\leq m\right\}. \tag{2.1}$$

Then

$$\begin{cases}\log N_q(\mathcal{P}_m, t)\leq Cm\left(\log\left(\frac{n}{m}+1\right)\right)t^{-\nu} & \text{if } t>\frac{1}{2} \quad (2.2)\\ \log N_q(\mathcal{P}_m, t)\leq Cm\left(\log\left(\frac{n}{m}+1\right)\right)\log\frac{1}{t} & \text{if } 0<t\leq\frac{1}{2} \quad (2.3)\end{cases}$$

where $C=C_q$ *and* $\nu=\nu(q)>2$.

Remarks. (1) It suffices to prove (2.2) replacing $t^{-\nu}$ by $t^{-2}\log t$. Indeed, let $q<r$, $1/q=(1-\theta)/2+\theta/r$. One has in particular for each pair of elements $f, g\in\mathcal{P}_m$, by Hölder's inequality

$$\|f-g\|_q\leq\|f-g\|_2^{1-\theta}\|f-g\|_r^{\theta}\leq 2\|f-g\|_r^{\theta}.$$

Hence, for $t>1$

$$\log N_q(\mathcal{P}_m, t)\leq\log N_r\left(\mathcal{P}_m, \left(\frac{t}{2}\right)^{1/\theta}\right)\leq Cm\left(\log\left(1+\frac{n}{m}\right)\right)t^{-2/\theta}\log t,$$

where $t^{-2/\theta}\log t<t^{-\nu}$ for some $\nu>2$.

(2) It follows from the results of [B-L-M] (section 4) that for $t>1$

$$\log N_q(\mathcal{P}_m, t)\leq\log N_q(\mathcal{P}_n, t)\leq cqt^{-2}n. \tag{2.4}$$

This estimate turns out to be too crude for our purpose.

(3) Once (2.2) is obtained, it follows for $t<\frac{1}{2}$

$$\begin{aligned}\log N_q(\mathcal{P}_m, t)&\leq\log\binom{n}{m}+\sup_{|A|\leq m}\log N_q\left(\left\{\sum_{i\in A}a_i\varphi_i \,\middle|\, |a|\leq 1\right\}, t\right)\\ &\leq Cm\log\left(1+\frac{n}{m}\right)+Cm\log\frac{1}{t}+\sup_{|A|\leq m}\log N_q\left(\left\{\sum_{i\in A}a_i\varphi_i \,\middle|\, |a|\leq 1\right\}, 1\right)\\ &\leq C\left[\log\left(1+\frac{n}{m}\right)+\log\frac{1}{t}\right]m\end{aligned}$$

implying (2.3).

Thus it remains to verify (2.2) with $\nu=2$.

LEMMA 3′. *With the notations of Lemma* 3, *for* $2<t<\sqrt{m}$

$$\log N_q(\mathscr{P}_m, t) \leqslant Cm \log\left(1+\frac{n}{m}\right) t^{-2} \log t. \tag{2.5}$$

Proof. Let $t \sim 2^{k/2}$. Fix a function $f = \Sigma_{i\in A} a_i \varphi_i$, $|a|=1$ and write

$$\begin{aligned}\sum_A a_i\varphi_i &= \sum_A a_i \varepsilon_i^1 \varphi_i + \sum_A a_i(1-\varepsilon_i^1)\varphi_i \\ &= \sum_A a_i\varepsilon_i^1\varphi_i + \sum_A a_i(1-\varepsilon_i^1)\varepsilon_i^2\varphi_i + \sum_A a_i(1-\varepsilon_i^1)(1-\varepsilon_i^2)\varphi_i \\ &\vdots \\ &= \sum_A a_i\varepsilon_i^1\varphi_i + \sum_A a_i(1-\varepsilon_i^1)\varepsilon_i^2\varphi_i + \ldots + \sum_A a_i(1-\varepsilon_i^1)\ldots(1-\varepsilon_i^{k-1})\varepsilon_i^k\varphi_i \qquad (2.6) \\ &\quad + \sum_A a_i(1-\varepsilon_i^1)\ldots(1-\varepsilon_i^k)\varphi_i \qquad (2.7)\end{aligned}$$

where $(\varepsilon_i^j)_{1\leqslant i\leqslant n,\, 1\leqslant j\leqslant k}$ are ± 1, signs to be specified.

Denoting (2.6) by $\Phi(\varepsilon, u)$, it follows from Khintchine's inequality

$$\begin{aligned}\int \|\Phi(\varepsilon,u)\|_{L^q(du)}\, d\varepsilon &\leqslant \sum_{l\leqslant k} \int \left\| \sum a_i(1-\varepsilon_i^1)\ldots(1-\varepsilon_i^{l-1})\,\varepsilon_i^l\varphi_i(u)\right\|_{L^q(du\otimes d\varepsilon^l)} d\varepsilon^1\ldots d\varepsilon^{l-1} \\ &\leqslant \sum_{l\leqslant k}\sqrt{q}\int\left[\sum a_i^2(1-\varepsilon_i^1)^2\ldots(1-\varepsilon_i^{l-1})^2\right]^{1/2} d\varepsilon^1\ldots d\varepsilon^{l-1} \\ &< \sqrt{q}\sum_{l\leqslant k} 2^{l/2} < ct. \qquad (2.8)\end{aligned}$$

Also, denoting

$$A_\varepsilon = \{i\in A \mid \varepsilon_i^1 = \ldots = \varepsilon_i^k = -1\}$$

$$|A_\varepsilon| = 2^{-k}\sum_{i\in A}(1-\varepsilon_i^1)\ldots(1-\varepsilon_i^k)$$

and hence

$$\int |A_\varepsilon|\, d\varepsilon = 2^{-k} m < \frac{m}{t^2}. \tag{2.9}$$

Moreover,

$$\int\left[\sum_A a_i^2(1-\varepsilon_i^1)^2\ldots(1-\varepsilon_i^k)^2\right]^{1/2} d\varepsilon \leqslant 2^{k/2} \sim t. \tag{2.10}$$

Inequalities (2.8), (2.9), (2.10) permit to find a choice of signs ε_i^j such that

$$\varphi = \sum_A a_i(1-\varepsilon_i^1)\dots(1-\varepsilon_i^k)\,\varphi_i$$

satisfies the conditions

$$\left\| \left(\sum_A a_i\varphi_i\right) - \varphi \right\|_q \le ct \tag{2.11}$$

$$\varphi \in ct\,\mathcal{P}_{[m/t^2]}. \tag{2.12}$$

It is now easily seen that

$$\log N_q(\mathcal{P}_m, ct) \le \log\binom{n}{[m/t^2]} + \sup_{|I|=[m/t^2]} N_q\left(\left\{\sum_{i\in I} a_i\varphi_i \,\Big|\, |a|\le 1\right\}, 1\right) \tag{2.13}$$

$$\le \frac{m}{t^2}\log\frac{nt}{m} + c_q\frac{m}{t^2}. \tag{2.14}$$

The evaluation of the second term in (2.13) may be done from the results of [B-L-M], section 4 (cf. Remark (2) above) or, alternatively, using the method of support-reduction described above and yielding (2.13).

This concludes the proof of Lemma 3′ and hence of Lemma 3.

Remark. When defining the t-entropy number of the set $\mathcal{P}$, we do not require a priori the centers of the covering balls of radius t to belong to $\mathcal{P}$. This can however always be achieved, doubling the radius of the balls to $2t$. Observe that in proving (2.13), the initial centers of the balls do not belong to $\mathcal{P}_m$ and have to be substituted (to make Remark (1) on the improvement of the exponent $t^{-2}\log t \to t^{-\nu}$ applicable).

3. Decoupling inequalities

The first step in our probabilistic approach is a decoupling procedure which will be performed in this section.

The next lemma is formulated for 3 factors but easily generalizes.

LEMMA 4. *Consider for $\alpha=1,2,3$ real valued functions ϕ_α, $\alpha=1,2,3$ on* **R**, *satisfying*

$$|\phi_\alpha(x)| \le C(1+|x|)^{p_\alpha} \tag{3.1}$$

$$|\phi_\alpha(x)-\phi_\alpha(y)| \le C(1+|x|+|y|)^{p_\alpha-\delta}|x-y|^\delta \tag{3.2}$$

where $p_\alpha>0$, $\delta>0$.

Let $x=(x_i)_{1\le i\le n}$, $y=(y_i)_{1\le i\le n}$, $z=(z_i)_{1\le i\le n}$ *be scalar sequences with* $|x|, |y|, |z|\le 1$ *and* $\{\eta_i\}_{i=1}^n$, $\{\zeta_i\}_{i=1}^n$ *independent* 0, 1*-valued random variables of respective mean*

$$\int \eta_i(t)\,dt=\frac{1}{3} \quad \text{and} \quad \int \zeta_i(t)\,dt=\frac{1}{2} \quad (1\le i\le n). \tag{3.3}$$

Define the disjoint sets

$$R_t^1=\{1\le i\le n \mid \eta_i(t)=1\},\ R_t^2=\{1\le i\le n \mid \eta_i=0,\ \zeta_i=1\},\ R_t^3=\{1\le i\le n \mid \eta_i=0,\ \zeta_i=0\}.$$

Then

$$\left|\int \phi_1\Big(\sum_{i\in R_t^1} x_i\Big)\phi_2\Big(\sum_{R_t^2} y_i\Big)\phi_3\Big(\sum_{R_t^3} z_i\Big)dt-\phi_1\Big(\frac{1}{3}\sum x_i\Big)\phi_2\Big(\frac{1}{3}\sum y_i\Big)\phi_3\Big(\frac{1}{3}\sum z_i\Big)\right|$$
$$\le C\Big(1+\Big|\sum x_i\Big|+\Big|\sum y_i\Big|+\Big|\sum z_i\Big|\Big)^{p-\delta} \tag{3.4}$$

where $p=p_1+p_2+p_3$.

Proof. The argument is straightforward. Write by (3.2)

$$\left|\phi_1\Big(\frac{1}{3}\sum x_i\Big)-\phi_1\Big(\sum_{i\in R_t^1} x_i\Big)\right|\le C\left[1+\Big|\sum x_i\Big|+\Big|\sum\Big(\eta_i-\frac{1}{3}\Big)x_i\Big|\right]^{p_1-\delta}\Big|\sum\Big(\eta_i-\frac{1}{3}\Big)x_i\Big|^{\delta}$$

and the analogues with ϕ_1 replaced by ϕ_2 (resp. ϕ_3), p_1 by p_2 (resp. p_3), x by y (resp. z) and $\eta_i\equiv\eta_i^1$ by $\eta_i^2\equiv(1-\eta_i)\zeta_i$ (resp. $\eta_i^3=(1-\eta_i)(1-\zeta_i)$). Observe that by construction

$$\int \eta_i^1(t)\,dt=\int \eta_i^2(t)\,dt=\int \eta_i^3(t)\,dt=\frac{1}{3}.$$

Hence, by (3.1), the left member of (3.4) is bounded by

$$C\left[1+\Big|\sum x_i\Big|+\Big|\sum y_i\Big|+\Big|\sum z_i\Big|+\Big|\sum\Big(\eta_i^1-\frac{1}{3}\Big)x_i\Big|\right.$$
$$\left.+\Big|\sum\Big(\eta_i^2-\frac{1}{3}\Big)y_i\Big|+\Big|\sum\Big(\eta_i^3-\frac{1}{3}\Big)z_i\Big|\right]^{p-\delta}$$
$$\times\left(\Big|\sum\Big(\eta_i^1-\frac{1}{3}\Big)x_i\Big|+\Big|\sum\Big(\eta_i^2-\frac{1}{3}\Big)y_i\Big|+\Big|\sum\Big(\eta_i^3-\frac{1}{3}\Big)z_i\Big|\right)^{\delta}.$$

Since $\|\sum(\eta_i^1-\frac{1}{3})x_i\|_{L^p(dt)}\le C|x|<c$ etc. ..., (3.4) easily follows.

Remark. In what follows, Lemma 4 will be applied for functions ϕ_α being one of the following

$$\begin{array}{lll}\phi(x)=x & & (p=1,\ \delta=1)\\ \phi(x)=|x|^\sigma \text{ or } \phi(x)=(1+|x|)^\sigma; & 0<\sigma<1 & (p=\sigma,\ \delta=\sigma)\\ \phi(x)=|x|^\sigma; & \sigma\geq 1 & (p=\sigma,\ \delta=1).\end{array}$$

We will use the following scalar inequalities.

LEMMA 5. *Let $x,y\in\mathbf{R}$. Then*

$$|x+y|^p\leq|x+y|^2|y|^{p-2}+(1+|x|)^p+2x(1+|x|)^{p-2}y+(1+|x|)^{p-2}y^2\quad(2<p\leq 3)\tag{3.5}$$

$$|x+y|^p\leq|x+y|^{p-2}x^2+C(|x|+|y|)^{p-3}|y|^3+2x|x|^{p-2}y+(2p-3)\,|x|^{p-2}y^2\quad(3<p).\tag{3.6}$$

Proof. For (3.5), write

$$|x+y|^p\leq(x+y)^2(|x|^{p-2}+|y|^{p-2})\leq(x+y)^2|y|^{p-2}+(x+y)^2(1+|x|)^{p-2}.$$

For (3.6), write

$$|x+y|^p=|x+y|^{p-2}x^2+|x+y|^{p-2}(2xy+y^2)\tag{3.7}$$

and use the inequality

$$\big|\,|x+y|^{p-2}-|x|^{p-2}-(p-2)\,|x|^{p-4}xy\big|\leq C(|x|+|y|)^{p-4}y^2$$

to replace the second term of (3.7).

Let $\Phi=\{\varphi_i\}_{i=1}^n$ be a 1-bounded orthogonal system of functions and define for $S\subset\{1,\dots,n\}$ the number

$$K_S=\sup_{|a|\leq 1}\left\|\sum_{i\in S}a_i\varphi_i\right\|_p.\tag{3.8}$$

Case $2<p\leq 3$.

Choose $0<\gamma<1$ satisfying

$$(1-\gamma^2)^{(p-2)/2}+\gamma^p<1.\tag{3.9}$$

Fix $\bar a=(a_i)_{i\in S}$, $|\bar a|=1$. Choose $\varkappa(\bar a)>0$ and subsets $I=I_{\bar a}$, $J=J_{\bar a}\subset\{1,\dots,n\}$ satisfying

$$\{1,\dots,n\}\setminus(I\cup J)\quad\text{is at most 1 point}\tag{$*$}$$

$$\min_{i\in I}|a_i|\geq\varkappa\geq\max_{i\in J}|a_i|\tag{3.10}$$

$$\sum_{i\in I} a_i^2<\gamma^2 \quad\text{and}\quad \sum_{i\in J} a_i^2<1-\gamma^2. \tag{3.11}$$

Apply then (3.5) pointwise, letting

$$x(u)=\sum_I a_i\varphi_i(u) \quad\text{and}\quad y(u)=\sum_J a_i\varphi_i(u).$$

Integration in u and using Hölder's inequality and the definition of K_S then yields

$$\begin{aligned}\int |x(u)+y(u)|^p du &\leq \|x+y\|_p^2\,\|y\|_p^{p-2}+\|1+|x|\,\|_p^p+2|\langle y, x(1+|x|)^{p-2}\rangle|+|\langle y, y(1+|x|)^{p-2}\rangle|\\ &\leq K_S^p\Big(\sum_J a_i^2\Big)^{(p-2)/2}+K_S^p\Big(\sum_I a_i^2\Big)^{p/2}+CK_S^{p-1}\\ &\quad +2|\langle y, x(1+|x|)^{p-2}\rangle|+|\langle y, y(1+|x|)^{p-2}\rangle|.\end{aligned}$$

Hence, by (3.11), (3.9), $(*)$

$$K_S^p\leq[(1-\gamma^2)^{(p-2)/2}+\gamma^p]\,K_S^p+CK_S^{p-1}+\sup[2|\langle y, x(1+|x|)^{p-2}\rangle|+|\langle y, y(1+|x|)^{p-2}\rangle|]$$

$$K_S^p\leq C\sup[|\langle y, x(1+|x|)^{p-2}\rangle|+|\langle y, y(1+|x|)^{p-2}\rangle|+CK_S^{p-1} \tag{3.12}$$

where the supremum is taken over all vectors $x=\Sigma_{i\in I\cap S}a_i\varphi_i$, $y=\Sigma_S b_i\varphi_i$ with $|a|, |b|\leq 1$ and $\max|b_i|\leq|I|^{-1/2}$.

Next use Lemma 4. Let R_t^1, R_t^2, R_t^3 be as in Lemma 4. We have pointwise

$$\left|y(u)x(u)(1+|x(u)|)^{p-2}-3^p\int\Big[\sum_{S\cap R_t^1} b_i\varphi_i(u)\Big]\Big[\sum_{I\cap S\cap R_t^2} a_i\varphi_i(u)\Big]\Big(1+\Big|\sum_{I\cap S\cap R_t^3} a_i\varphi_i(u)\Big|\Big)^{p-2}dt\right|$$
$$\leq C(1+|x(u)|+|y(u)|)^2$$

where we considered $\phi(x)=x$, $\phi_2(x)=x$, $\phi_3(x)=(1+|x|)^{p-2}$.

Hence, integrating in u

$$\sup|\langle y, x(1+|x|)^{p-2}\rangle|\leq C\int\sup\left|\Big\langle\sum_{S\cap R_t^1} b_i\varphi_i, \Big(\sum_{I\cap S\cap R_t^2} a_i\varphi_i\Big)\Big(1+\Big|\sum_{I\cap S\cap R_t^3} a_i\varphi_i\Big|\Big)^{p-2}\Big\rangle\right|dt+C. \tag{3.13}$$

Again, the supremum is taken over sets $I\subset\{1,\ldots,n\}$, $|\bar{a}|\leq 1$, $|\bar{b}|\leq 1$ and $\bar{b}$ satisfying $\max|b_i|\leq|I|^{-1/2}$.

Replace similarly $\langle y, y(1+|x|)^{p-2}\rangle$, letting again $\phi_1(x)=\phi_2(x)=x$, $\phi_3(x)=(1+|x|)^{p-2}$. One gets now

$$\sup|\langle y, y(1+|x|)^{p-2}\rangle| \leq C\int \sup\left|\left\langle \sum_{S\cap R_t^1} b_i\varphi_i, \left(\sum_{S\cap R_t^2} b_i\varphi_i\right)\left(1+\left|\sum_{I\cap S\cap R_t^3} a_i\varphi_i\right|\right)^{p-2}\right\rangle\right| dt+C. \tag{3.14}$$

Collecting estimates, it follows that

$$K_S^p \leq \text{first term (3.13)}+\text{first term (3.14)}+CK_S^{p-1}. \tag{3.15}$$

Case $3<p$.

Choose now $0<\gamma<1$ such that

$$(1-\gamma^2)+C\gamma^3<1 \tag{3.16}$$

where C relates to the constant in (3.6). Take I, J satisfying (3.10) and now

$$\sum_{i\in I} a_i^2<1-\gamma^2 \quad\text{and}\quad \sum_{i\in J} a_i^2<\gamma^2. \tag{3.17}$$

Let $x(u)$, $y(u)$ be defined as above. Integrating (3.6), one gets

$$\int |x(u)+y(u)|^p du \leq K_S^p\left(\sum_I a_i^2\right)+CK_S^p\left(\sum_J a_i^2\right)^{3/2}+2|\langle y, x|x|^{p-2}\rangle|+(2p-3)|\langle y, y|x|^{p-2}\rangle|$$

and, by (∗)

$$K_S^p \leq C\sup[|\langle y, x|x|^{p-2}\rangle|+|\langle y, y|x|^{p-2}\rangle|]+CK_S^{p-1} \tag{3.18}$$

where the supremum is taken the same way as in (3.12).

Applying Lemma 4 with functions $\phi(x)=x$, $\phi(x)=|x|^{p-2}$, we now get

$$K_S^p \leq CK_S^{p-1}$$

$$+C\int \sup\left|\left\langle \sum_{S\cap R_t^1} b_i\varphi_i, \left(\sum_{I\cap S\cap R_t^2} a_i\varphi_i\right)\left|\sum_{I\cap S\cap R_t^3} a_i\varphi_i\right|^{p-2}\right\rangle\right| dt \tag{3.19}$$

$$+C\int \sup\left|\left\langle \sum_{S\cap R_t^1} b_i\varphi_i, \left(\sum_{S\cap R_t^2} a_i\varphi_i\right)\left|\sum_{I\cap S\cap R_t^3} a_i\varphi_i\right|^{p-2}\right\rangle\right| dt \tag{3.20}$$

the supremum being taken over sets $I\subset\{1,\ldots,n\}$, $|\bar{a}|, |\bar{b}|\leq 1$ and $\bar{b}$ satisfying

$$\max |b_i| \leqslant |I|^{-1/2}.$$

Let $\{\xi_i\}_{i=1}^n$ be independent 0, 1-valued random variables (selectors) of mean $\delta=\int \xi_i(\omega)\,d\omega$ satisfying

$$\delta n = n^{2/p} \tag{3.21}$$

and consider the random set

$$S_\omega = \{i=1, \dots, n \mid \xi_i(\omega)=1\}$$

which has expected size $\sim n^{2/p}$. Denote

$$K(\omega) = K_{S_\omega}.$$

We only consider the case $2<p\leqslant 3$. The case $3<p$ is identical at this stage and left to the reader (we will point out however how the argument has to be modified to deal with $p\geqslant 4$).

From (3.13), (3.14), (3.15), it clearly follows that

$$\int K(\omega)^p d\omega \leqslant C \int K(\omega)^{p-1} d\omega$$

$$+\int \sup_{(*)} \left| \left\langle \sum \xi_i(\omega_1)\, a_i \varphi_i, \left(\sum \xi_i(\omega_2)\, b_i \varphi_i\right) \left(1+\left|\sum \xi_i(\omega_3)\, c_i \varphi_i\right|\right)^{p-2} \right\rangle \right| d\omega_1\, d\omega_2\, d\omega_3 \tag{3.22}$$

$$+\int \sup_{(**)} \left| \left\langle \sum \xi_i(\omega_1)\, a_i \varphi_i, \left(\sum \xi_i(\omega_2)\, b_i \varphi_i\right) \left(1+\left|\sum \xi_i(\omega_3)\, c_i \varphi_i\right|\right)^{p-2} \right\rangle \right| d\omega_1\, d\omega_2\, d\omega_3 \tag{3.23}$$

where

$\sup_{(*)}$ refers to vectors $\bar a, \bar b, \bar c$; $|\bar a|, |\bar b|, |\bar c| \leqslant 1$ and $\max_{1\leqslant i\leqslant n} |a_i| \leqslant (|\operatorname{supp} \bar b|+|\operatorname{supp} \bar c|)^{-1/2}$

$\sup_{(**)}$ refers to vectors $\bar a, \bar b, \bar c$; $|\bar a|, |\bar b|, |\bar c| \leqslant 1$ and $\max_{1\leqslant i\leqslant n}(|a_i|, |b_i|) \leqslant |\operatorname{supp} \bar c|^{-1/2}$.

We denote here $\operatorname{supp} \bar a=\{i=1, \dots, n \mid a_i\neq 0\}$ and always assume its size $|\operatorname{supp} \bar a|\leqslant n_0 \equiv n^{2/p}=\delta n$. In order to obtain (3.22), (3.23), we performed a decoupling on the variable ω to independent variables ω_1, ω_2, ω_3, using the disjointness of the sets R_t^1, R_t^2, R_t^3 appearing in the scalar products in (3.13), (3.14), for individual t.

Define $q_0=\log n$ and for $1\leq m\leq n_0$, let

$$\Pi_m=\{\bar a=(a_i)_{1\leq i\leq n} \mid |\bar a|\leq 1 \text{ and } |\operatorname{supp}\bar a|\leq m\}.$$

Hence, with this notation

$$\mathcal{P}_m=\left\{\sum a_i\varphi_i \,\middle|\, \bar a\in\Pi_m\right\}.$$

Estimate (3.22) by

$$\int\left\{\sup_{m_1<n_0}\left\|\sup_{\substack{|a|\leq 1,\max|a_i|\leq m_1^{-1/2}\\ \bar b,\bar c\in\Pi_{m_1}}}\left|\left\langle f_{\bar a,\omega_1},f_{\bar b,\omega_2}(1+|f_{\bar c,\omega_3}|)^{p-2}\right\rangle\right|\right\|_{L^{q_0}(d\omega_1)}\right\}d\omega_2\,d\omega_3$$

denoting

$$f_{\bar a,\omega}=\sum \xi_i(\omega)\,a_i\,\varphi_i.$$

Splitting $\bar a$ in level sets, one gets further by triangle inequality

$$\int d\omega_2\,d\omega_3\left\{\sup_{m_1<n_0}\sum_{\substack{n_0>m>m_1\\ m\text{ diadic}}}\left\|\sup_{|A|\leq m;\,\bar b,\bar c\in\Pi_{m_1}}\frac{1}{\sqrt m}\sum_{i\in A}\xi_i(\omega_1)\,|\langle\varphi_i,f_{\bar b,\omega_2}(1+|f_{\bar c,\omega_3}|)^{p-2}\rangle|\right\|_{L^{q_0}(d\omega_1)}\right\}. \tag{3.24}$$

Evaluating (3.23), we proceed less crudely and write a representation

$$\bar a=\sum_{m_3<2^l<n_0}\lambda_l\,\bar a(l),\quad \bar b=\sum_{m_3<2^l<n_0}\mu_l\,\bar b(l)\quad (m_3=|\operatorname{supp}\bar c|)$$

where

$$\sum\lambda_l^2\leq 1,\quad \sum\mu_l^2\leq 1 \tag{3.25}$$

$$|\operatorname{supp}\bar a(l)|\leq 2^l,\quad |\operatorname{supp}\bar b(l)|\leq 2^l \tag{3.26}$$

$$|a_i(l)|\leq 2^{-l/2},\quad |b_i(l)|\leq 2^{-l/2} \tag{3.27}$$

decomposing in level sets (the existence of such representations is easily seen by considering a decreasing rearrangement).

Coming back to (3.23), estimate the scalar product

$$|\langle f_{\bar a,\omega_1}, f_{\bar b,\omega_2}(1+|f_{\bar c,\omega_3}|)^{p-2}\rangle| \leq \sum_{m_3<2^l,\, 2^{l'}<n_0} \lambda_l \mu_{l'} |\langle f_{\bar a(l),\omega_1}, f_{\bar b(l'),\omega_2}(1+|f_{\bar c,\omega_3}|)^{p-2}\rangle| \quad \text{(by 3.25)}$$

$$\leq \sum_{d\geq 0} \sup_{(l,l')\in \mathscr{L}_{m_3,d}} |\langle f_{\bar a(l),\omega_1}, f_{\bar b(l'),\omega_2}(1+|f_{\bar c,\omega_3}|)^{p-2}\rangle|$$

denoting for convenience $\mathscr{L}_{m,d}=\{(l,l')\,|\, m<2^l,\ 2^{l'}<n_0 \text{ and } |l-l'|=d\}$.

The following estimate for (3.23) may be written from the preceding

$$\int\left\{\sup_{m_3<n_0}\sum_{d>0}\left[\sup_{m_1,m_2|n_0>m_1\geq 2^d m_2;\, m_2\geq m_3} \|\bar K_{m_1,m_2,m_3}(\omega_1,\omega_2,\omega_3)\|_{L^{q_0}(d\omega_1)}\right]\right\} d\omega_2\, d\omega_3 \tag{3.28}$$

where

$$\bar K_{m_1,m_2,m_3}(\omega_1,\omega_2,\omega_3)=\sup_{|A|\leq m_1}\sup_{\bar b\in \Pi_{m_2}}\sup_{\bar c\in\Pi_{m_3}} \frac{1}{\sqrt{m_1}}\sum_{i\in A}\xi_i(\omega_1)|\langle \varphi_i, f_{\bar b,\omega_2}(1+|f_{\bar c,\omega_3}|)^{p-2}\rangle|$$

and $f_{\bar b,\omega}$ is defined as above.

We will prove the following estimate in the next section

$$\|\bar K_{m_1,m_2,m_3}(\omega_1,\omega_2,\omega_3)\|_{L^{q_0}(d\omega_1)} \leq C\left\{\delta m_3^{p/2-1}+\frac{m_2+m_3}{m_1}\right\}^{1/2}(1+K(\omega_2)+K(\omega_3))^{p-\sigma} \tag{3.29}$$

for some $\sigma>0$.

Substitution of (3.29) in (3.24) gives the estimate, since $\delta=n_0/n$, $n_0=n^{2/p}$

$$\sup_{m_1<n_0}\left[\sum_{\substack{n_0>m>m_1\\ m \text{ diadic}}}\left(\delta m_1^{p/2-1}+\frac{m_1}{m}\right)^{1/2}\right]\cdot\|K(\omega)\|_p^{p-\sigma}\leq C[1+(\delta n_0^{p/2-1})^{1/2}]\,\|K(\omega)\|_p^{p-\sigma}< C\|K(\omega)\|_p^{p-\sigma}. \tag{3.30}$$

Substitution of (3.29) in (3.28) gives

$$\sup_{m_3<n_0}\sum_{0<d<\log(n_0/m_3)}[(\delta m_3^{p/2-1})^{1/2}+2^{-d/2}]\cdot\|K(\omega)\|_p^{p-\sigma}\leq C\|K(\omega)\|_p^{p-\sigma}. \tag{3.31}$$

Collecting estimates (3.22), (3.23), (3.24), (3.28), (3.30), (3.31) gives

$$\|K(\omega)\|_p^p\leq C\|K(\omega)\|_p^{p-1}+C\|K(\omega)\|_p^{p-\sigma}\ \Rightarrow\ \|K(\omega)\|_p\leq C.$$

From the definition of $K(\omega)$, this means that a random set $S_\omega \subset \{1,\dots,n\}$, $|S_\omega|=n^{2/p}$, satisfies generically (0.1).

4. End of the proof ($2<p\leq 3$ and, similarly, $3<p<4$)

Let again $2<p\leq 3$. It remains to show (3.29). The argument is based on the results of the first two sections of this paper. With ω_2, ω_3 fixed, denote briefly

$$g_{\bar b}=f_{\bar b,\omega_2};\quad h_{\bar c}=f_{\bar c,\omega_3}.$$

Evaluate (3.29) by Lemma 1, taking $m=m_1$ and

$$\mathscr{E}=\{(|\langle\varphi_i, g_{\bar b}(1+|h_{\bar c}|)^{p-2}\rangle|)_{i=1}^n|\ \bar b\in\Pi_{m_2},\ \bar c\in\Pi_{m_3}\}.$$

Since $q_0=\log n\sim\log 1/\delta$, (1.1) yields the bound

$$C[\delta^{1/2}+m_1^{-1/2}]B+cm_1^{-1/2}(\log n)^{-1/2}\int_0^B[\log N_2(\mathscr{E},t)]^{1/2}dt \tag{4.1}$$

where $B=\sup_{x\in\mathscr{E}}|x|$.

It follows from Bessel's inequality that

$$\left(\sum|\langle\varphi_i, g_{\bar b}(1+|h_{\bar c}|)^{p-2}\rangle|^2\right)^{1/2}\leq\|g_{\bar b}(1+|h_{\bar c}|)^{p-2}\|_2$$

which by Hölder's inequality is further bounded by

$$\|g_{\bar b}\|_p\|1+|h_{\bar c}|\,\|_{2p}^{p-2}\leq\|g_{\bar b}\|_p(1+\|h_{\bar c}\|_p)^{p/2-1}(1+\|h_{\bar c}\|_\infty)^{p/2-1}\leq K(\omega_2)\,K(\omega_3)^{p/2-1}m_3^{\frac{1}{2}(p/2-1)}. \tag{4.2}$$

Similarly, there is the distance computation

$$\left(\sum_{i=1}^n\big||\langle\varphi_i, g_b(1+|h_c|)^{p-2}\rangle|-|\langle\varphi_i, g_{b'}(1+|h_{c'}|)^{p-2}\rangle|\big|^2\right)^{1/2}\leq\|g_b(1+|h_c|)^{p-2}-g_{b'}(1+|h_{c'}|)^{p-2}\|_2. \tag{4.3}$$

Using the inequality ($p\leq 3$)

$$|(1+|x|)^{p-2}-(1+|y|)^{p-2}|\leq(p-2)\,|x-y|$$

it follows

$$|g_b(1+|h_c|)^{p-2}-g_{b'}(1+|h_{c'}|)^{p-2}|\leq|g_b-g_{b'}|(1+|h_c|)^{p-2}+|g_{b'}|\,|h_c-h_{c'}|$$

hence, for $q=2p/(4-p)$, $r=2p/(p-2)$

$$\begin{aligned}(4.3) &\leq \|g_b-g_{b'}\|_q(1+\|h_c\|_p)^{p-2}+\|g_{b'}\|_p\|h_c-h_{c'}\|_r\\ &\leq K(\omega_3)^{p-2}\|g_b-g_{b'}\|_q+K(\omega_2)\|h_c-h_{c'}\|_r.\end{aligned}$$

Therefore

$$\log N_2(\mathscr{E},t)\leq \log N_q\left(\mathscr{P}_{m_2},\frac{t}{2}K(\omega_3)^{2-p}\right)+\log N_r\left(\mathscr{P}_{m_3},\frac{t}{2}K(\omega_2)^{-1}\right). \tag{4.4}$$

Substitution of (4.2), (4.4) in (4.1) yields

$$\begin{aligned}&C[\delta^{1/2}+m_1^{-1/2}]\,m_3^{\frac{1}{2}(p/2-1)}K(\omega_2)\,K(\omega_3)^{p/2-1}\\ &\quad+cm_1^{-1/2}(\log n)^{-1/2}K(\omega_3)^{p-2}\int_0^\infty[\log N_q(\mathscr{P}_{m_2},t)]^{1/2}dt\\ &\quad+cm_1^{-1/2}(\log n)^{-1/2}K(\omega_2)\int_0^\infty[\log N_r(\mathscr{P}_{m_3},t)]^{1/2}dt.\end{aligned} \tag{4.5}$$

Since $q,r<\infty$ for $2<p<4$, application of the entropy estimates (2.2), (2.3) yields the following bound on (4.5),

$$C\left[\delta m_3^{p/2-1}+\frac{m_3}{m_1}\right]^{1/2}K(\omega_2)\,K(\omega_3)^{p/2-1}+C\left(\frac{m_2}{m_1}\right)^{1/2}K(\omega_3)^{p-2}+C\left(\frac{m_3}{m_1}\right)^{1/2}K(\omega_2).$$

This proves (3.29), with $\sigma=p/2$.

5. End of the proof ($p\geq 4$)

In this section we show how to handle the case $p\geq 4$. If $p\geq 4$, a use of Bessel's inequality in evaluating the distance (cf. (4.3)) is inappropriate, since the resulting exponent $2(p-2)\geq p$, in this case. One proceeds in a different way and generates the random set S, $|S|\sim n^{2/p}$ in several steps.

Assume $p/2<p_1<p$ and $n^{2/p}=\delta' n^{2/p_1}$. Assume also the statement of Theorem 1 is verified for the exponent p_1. One may generate a random set S of size $n^{2/p}$ by considering S as a subset of a random set $S_1\subset\{1,\dots,N\}$ of size $|S_1|=n_1\sim n^{2/p_1}$ and satisfying, by hypothesis, the inequality

$$\left\|\sum_{i\in S_1}a_i\varphi_i\right\|_{p_1}\leq C|\bar{a}|. \tag{5.1}$$

Thus one considers a 1-bounded sequence $\{\varphi_i | 1 \leq i \leq n_1\}$ fulfilling (5.1) and its random subsets S of size $[\delta' n_1]$. We have to establish the analogue of inequality (3.29)

$$\|\bar{K}_{m_1,m_2,m_3}(\omega_1,\omega_2,\omega_3)\|_{L^{q_0}(d\omega_1)} \leq C\left\{\delta' m_3^{p/p_1-1} + \frac{m_2+m_3}{m_1}\right\}^{1/2}(1+K(\omega_2)+K(\omega_3))^{p-\sigma}. \quad (5.2)$$

One may then repeat the calculation of section 3 leading to inequality (3.31), using the fact that $\delta' n_0^{p/p_1-1}=O(1)$, where $n_0=n^{2/p}$. To establish (5.2), one proceeds as in the previous section, except in evaluating B and the distance, where the use of Bessel's inequality is replaced by (5.1) and duality. Thus

$$\left(\sum |\langle \varphi_i, g_{\bar{b}}|h_{\bar{c}}|^{p-2}\rangle|^2\right)^{1/2} \leq C\|g_{\bar{b}}|h_{\bar{c}}|^{p-2}\|_{p_1'} \leq CK(\omega_2)\,K(\omega_3)^{p/p_1'-1}m_3^{\frac{1}{2}(p/p_1-1)}. \quad (5.3)$$

Similarly, from (5.1) and Hölder's inequality

$$\begin{aligned}
&\left(\sum_{i=1}^{n}\Big|\,|\langle \varphi_i, g_b|h_c|^{p-2}\rangle| - |\langle \varphi_i, g_{b'}|h_{c'}|^{p-2}\rangle|\,\Big|^2\right)^{1/2} \\
&\leq C\|g_b|h_c|^{p-2}-g_{b'}|h_{c'}|^{p-2}\|_{p_1'} \\
&\leq C\|\,|g_b-g_{b'}|(|h_c|^{p-2}+|h_{c'}|^{p-2})\|_{p_1'}+\|\,|h_c-h_{c'}|(|h_c|^{p-3}+|h_{c'}|^{p-3})|g_{b'}|\,\|_{p_1'} \qquad (5.4)\\
&\leq C\|g_b-g_{b'}\|_q(\|h_c\|_p+\|h_{c'}\|_p)^{p-2}+C\|h_c-h_{c'}\|_q(\|h_c\|_p+\|h_{c'}\|_p+\|g_{b'}\|_p)^{p-2} \\
&\leq C[K(\omega_2)+K(\omega_3)]^{p-2}(\|g_b-g_{b'}\|_q+\|h_c-h_{c'}\|_q)
\end{aligned}$$

where $q=pp_1'/(p-p_1'(p-2))$ (notice that $p_1'(p-2)<p$).

Hence, similarly as in the previous section, the entropy-integral will contribute for

$$C\left(\frac{m_2+m_3}{m_1}\right)^{1/2}\left(\frac{\log n}{\log 1/\delta}\right)^{1/2}[K(\omega_2)+K(\omega_3)]^{p-2}. \quad (5.5)$$

(5.3), (5.5) and the fact that $p/p_1-1<1$ (cf. (4.1)) imply (5.2) with $p-\sigma=\max(p/p_1', p-2)$.

6. Further comments

(1) The hypothesis of uniform boundedness of the system Φ in Theorem 1 is essential. Weakening this assumption, the following statement may be shown:

Let $2<p<q<\infty$ and $\varphi_1,\dots,\varphi_n$ an orthogonal (or 1-Hilbertian) system of n functions satisfying $\|\varphi_j\|_q\leq 1$ $(1\leq j\leq n)$. Then (0.1) holds for a random set $S\subset\{1,\dots,n\}$ of size $|S|\sim n^{\alpha}$, $\alpha=(1/p-1/q)/(1/2-1/q)$. The proof uses the same techniques as developed above in this paper. A special case of the result (p even, $q<2p-2$) was obtained in [A], where also its optimality (as an existence result) is observed.

(2) The probabilistic techniques used in this paper to solve the $\Lambda(p)$ problem have some applications to Garsia's conjecture (see [G]) on the rearrangement of finite orthogonal systems.

The following result is obtained in [B]:

Let $\{\varphi_1, \dots, \varphi_n\}$ be an orthogonal system satisfying $\|\varphi_j\|_\infty \leq 1$ $(1\leq j\leq n)$. Then, there is a rearrangement $\pi \in \mathrm{Sym}(n)$ satisfying

$$\left\| \max_{m\leq n} \left| \sum_1^m a_j \varphi_{\pi(j)} \right| \right\|_2 \leq c(\log\log n) \left(\sum_1^n |a_j|^2 \right)^{1/2}. \tag{1.7}$$

The permutation π is chosen at random in the symmetric group $\mathrm{Sym}(n)$. The estimate (7.1) is the best one may reach by a purely probabilistic approach.

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