Acta Math., 187 (2001), 59–137
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# The $L_\infty$-norm of the $L_2$-spline projector is bounded independently of the knot sequence: A proof of de Boor's conjecture

by

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## Contents

0. Introduction . . . . . . . . . . 60
0.1. Preface . . . . . . . . . . 60
0.2. Formulation of Theorem I . . . . . . . . . . 61
0.3. Outline of the proof . . . . . . . . . . 62
1. Main ingredients of the proof . . . . . . . . . . 65
1.1. B-splines and their properties . . . . . . . . . . 65
1.2. $L_2$-projector and the inverse of the B-spline Gramian . . . . . . 66
1.3. Analytic version of de Boor's Lemma 1.2.4 . . . . . . . . . . 68
1.4. Main idea. Formulation of Theorem $\Phi$ . . . . . . . . . . 69
1.5. Proof of Theorem I and its corollaries . . . . . . . . . . 70
1.6. Proof of Theorem $\Phi$: proof of $(A_1)$ . . . . . . . . . . 71
1.7. An invariant . . . . . . . . . . 72
1.8. Proof of Theorem $\Phi$: proof of $(A_2)$ . . . . . . . . . . 74
1.9. Vectors $z_\nu$. Formulation of Theorem Z . . . . . . . . . . 75
1.10. Proof of Theorem $\Phi$: proof of $(A_3)$ . . . . . . . . . . 76
2. Proof of Theorem Z: intermediate estimates for $z_\nu$ . . . . . . . . . . 78
2.1. Notation and auxiliary statements . . . . . . . . . . 78
2.2. Reduction to a linear system of equations . . . . . . . . . . 79
2.2.1. Derivatives of null-splines at knots . . . . . . . . . . 79
2.2.2. The matrices $B, B', C$ . . . . . . . . . . 81
2.2.3. Linear systems for $z_\nu$ . . . . . . . . . . 82
2.3. First estimates for $z_\nu$ . . . . . . . . . . 84
2.3.1. Total positivity of the matrices $A, B, C$ . . . . . . . . . . 84
2.3.2. First estimate for $z_0$ . . . . . . . . . . 84
2.3.3. First estimate for $z_\nu$ . . . . . . . . . . 85
2.4. Properties of the matrices $C$ . . . . . . . . . . 87
2.5. Second estimates for $z_\nu$ . . . . . . . . . . 90

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On leave from Computing Center, Novosibirsk, Russia.

3. Proof of Theorem Z: final estimates for $z_\nu$ . . . 91
3.1. Preliminary remarks . . . 91
3.2. The matrices $S$ and $A$ . . . 92
3.2.1. The matrix $S$ . . . 92
3.2.2. The matrix $A$ . . . 94
3.3. The matrices $Q$ . . . 96
3.4. A further strategy . . . 98
3.5. Minimal and maximal paths . . . 100
3.6. Characterization of $\mathbf{E}_{[\beta,i]}$ . . . 104
3.7. Relation between the minors of $Q$ and $C$ . . . 112
3.8. Index relations . . . 113
3.8.1. The statement . . . 113
3.8.2. Proof: the case $l=0$ . . . 114
3.8.3. Proof: the case $l\neq 0$ . . . 118
3.9. Completion of the proof of Theorem Z . . . 119
3.10. Last but not least . . . 120
4. Comments . . . 120
4.1. A survey of earlier and related results . . . 120
4.1.1. $L_2$-projector onto finite-element spaces . . . 121
4.1.2. A general spline-interpolation problem . . . 122
4.1.3. A problem for the multivariate $D^k$-splines . . . 123
4.2. On de Boor's Lemma 1.2.4 . . . 123
4.2.1. Gram matrix and de Boor's Lemma 1.2.4 . . . 123
4.2.2. On the choice of the null-spline $\sigma$ . . . 124
4.3. Simplifications in particular cases . . . 126
4.4. Additional facts . . . 128
4.4.1. Orthogonality of $\phi\in\mathbf{S}_k(\Delta)$ to $S_{k-1}(\Delta)$ . . . 128
4.4.2. Null-splines with Birkhoff boundary conditions at $t_0$ . . . 129
4.4.3. Further properties of the matrices $C$ . . . 130
4.5. On the constant $c_k$ . . . 132
References . . . 136

# 0. Introduction

## 0.1. Preface

(1) *Preface.* In this paper we prove de Boor's conjecture concerning the $L_2$-spline projector. The exact formulation is given in §0.2. Since the proof is rather long, it is divided into three chapters, with an outline given in §0.3. For the same reason, all the comments (historical notes, motivations, analysis of other methods, etc.) are moved to the end of the paper. The proof is almost self-contained, we cite (without proof) only some basic spline properties and determinant identities, and two somewhat more special lemmas (accompanied by known simple proofs).

(2) *Notation.* There is some mixture of notations. We use the familiar $i, j$ both as single and multivariate indices, and we use $p$ as $p{:=}k-2$ when dealing with $k$, the order

of the splines, while in other cases $p$ is just an integer.

(3) *Acknowledgements.* I am grateful to W. Dahmen for giving me the opportunity to work at the RWTH in Aachen, and for his constant inspiring encouragement of my studies. Thanks are extended to H. Esser, who took a lively part in discussions and provided many constructive suggestions. It is a pleasure to acknowledge that C. de Boor, in spite of some consequences for his finances, took an active part at all stages of the proof's evolution. To him I am obliged for a lot of hints and remarks, in particular, for essential simplification of some of my arguments and notations.

## 0.2. Formulation of Theorem I

(1) For an integer $k>0$, and a partition

$$\Delta := \Delta_N := \{a = t_0 < t_1 < \dots < t_N = b\},$$

denote by

$$\mathbf{S} := \mathbf{S}_k(\Delta) := \mathbf{P}_k(\Delta) \cap C^{k-2}[a,b]$$

the space of polynomial splines of order $k$ (i.e., of degree $<k$) with the knot sequence $\Delta$ satisfying $k-1$ continuity conditions at each interior knot.

Consider $P_{\mathbf{S}}$, the orthoprojector onto $\mathbf{S}$ with respect to the ordinary inner product $(f,g) := \int_a^b fg$, i.e.,

$$(f,s) = (P_{\mathbf{S}}(f), s) \quad \text{for all } s \in \mathbf{S}.$$

We are interested in $P_{\mathbf{S}}$ as an operator from $L_\infty$ to $L_\infty$, i.e., in bounds for its norm

$$\|P_{\mathbf{S}}\|_\infty := \sup_f \frac{\|P_{\mathbf{S}}(f)\|_\infty}{\|f\|_\infty}.$$

In this paper we prove the following fact.

THEOREM I. *For any $k$, the $L_\infty$-norm of the $L_2$-projector $P$ onto the spline space $\mathbf{S}_k(\Delta)$ is bounded independently of $\Delta$, i.e.,*

$$\sup_\Delta \|P_{\mathbf{S}_k(\Delta)}\|_\infty \leqslant c_k. \tag{0.2.1}$$

This theorem proves the conjecture of de Boor of 1972 made in [B2], see also §3.10 for details.

Earlier the mesh-independent bound (0.2.1) was proved for $k=2,3,4$.

For $k>4$ all previously known results proved boundedness of $\|P_S\|_\infty$ only under certain restrictions on the mesh $\Delta$. (See §4.1 for a survey of earlier and related results.)

(2) Some of the earlier restrictions on $\Delta$ included spline spaces with multiple and/or (bi)infinite knot sequences. Therefore two corollaries of Theorem I are worthwhile to be mentioned.

The first extends the result to the splines with a lower smoothness, the so-called splines with multiple knots. For $k$ and $\Delta=(t_i)_0^N$ as given above, we introduce a sequence of smoothness parameters $\mathbf{m}:=(m_i)_0^N$ where $0\leqslant m_i\leqslant k-1$, and denote by $\mathbf{S}_k(\Delta,\mathbf{m})$ the space of polynomial splines of order $k$ with the knot sequence $\Delta$ which, for every $i$, have $m_i-1$ continuous derivatives in a neighbourhood of $t_i$. If all $m_i$ are equal to $m$, then

$$\mathbf{S}_{k,m}(\Delta):=\mathbf{S}_k(\Delta,(m,\dots,m))=\mathbf{P}_k(\Delta)\cap C^{m-1}[a,b], \quad \mathbf{S}_k(\Delta)=\mathbf{S}_{k,k-1}(\Delta).$$

COROLLARY I. *For any* $k$,

$$\sup_{\Delta,\mathbf{m}} \|P_{\mathbf{S}_k(\Delta,\mathbf{m})}\|_\infty \leqslant c_k. \tag{0.2.2}$$

The second corollary extends Theorem I to the splines with (bi)infinite knot sequence $\Delta_\infty:=(t_i)$ and with smoothness parameters $\mathbf{m}_\infty:=(m_i)$. We denote the space of these splines by $\mathbf{S}_k(\Delta_\infty,\mathbf{m}_\infty)$.

COROLLARY II. *For any* $k$,

$$\sup_{\Delta_\infty,\mathbf{m}_\infty} \|P_{\mathbf{S}_k(\Delta_\infty,\mathbf{m}_\infty)}\|_\infty \leqslant c_k. \tag{0.2.3}$$

### 0.3. Outline of the proof

The proof is divided into three parts.

(1) The first part (Chapter 1) describes the main ingredients of the proof.

Let $(M_\nu)$, $(N_\nu)$ be the $L_1$- and the $L_\infty$-normalized B-spline basis of $\mathbf{S}_k(\Delta)$, respectively (see §1.1). Our starting point (§1.3) is the observation that if $\phi$ is a spline such that

($A_0$) $\phi\in\mathbf{S}_k(\Delta)$;

($A_1$) $(-1)^\nu \operatorname{sign}(\phi,M_\nu)=\text{const}$ for all $\nu$;

($A_2$) $|(\phi,M_\nu)|\geqslant c_{\min}$ for all $\nu$;

($A_3$) $\|\phi\|_\infty\leqslant c_{\max}$;

then

$$\|P_{\mathbf{S}_k(\Delta)}\|_\infty \leqslant d_k\cdot\frac{c_{\max}}{c_{\min}}.$$

This is an analytic version of de Boor's rather simple algebraic lemma (§1.2) on the inverse of a totally positive matrix applied to the Gram matrix $\{(M_\nu,N_\lambda)\}$.

Our main idea (§1.4) is the choice

$$\phi := \sigma^{(k-1)}, \quad \sigma \in \mathbf{S}_{2k-1}(\Delta), \tag{0.3.1}$$

where $\sigma$ is the null-spline of the even degree $2k-2$ such that

$$\begin{aligned} \sigma(t_\nu) &= 0, \quad \nu = 0, \ldots, N; \\ \sigma^{(l)}(t_0) = \sigma^{(l)}(t_N) &= 0, \quad l = 1, \ldots, k-2; \\ \frac{1}{(k-1)!}\sigma^{(k-1)}(t_N) &= 1. \end{aligned} \tag{0.3.2}$$

The main claim, Theorem $\Phi$ of §1.4, is that $\phi$ so defined satisfies the properties ($\mathrm{A}_0$)–($\mathrm{A}_3$) given above.

As we show in §§1.6–1.8, the choice (0.3.1) makes the most problematic property ($\mathrm{A}_1$) almost automatically fulfilled and provides also ($\mathrm{A}_2$) quite easily. To prove ($\mathrm{A}_3$), we use for the components of the vector

$$z_\nu = (z_\nu^{(1)}, \ldots, z_\nu^{(2p+1)}), \quad z_\nu^{(l)} := \frac{1}{l!}\sigma^{(l)}(t_\nu)\cdot|h_\nu|^{l-1-p}, \quad p := k-2, \tag{0.3.3}$$

(where $|h_\nu| := t_{\nu+1} - t_\nu$), the estimate

$$|z_\nu^{(l)}| \leqslant c_k, \quad \text{if } l \geqslant p+1,\ \nu \leqslant N-k. \tag{0.3.4}$$

This estimate forms the content of Theorem Z in §1.9. The rest of the proof (Chapters 2 and 3) consists of deriving (0.3.4).

(2) In Chapter 2, we show that, for each $\nu$, the vector $z_\nu$ in (0.3.3) is a solution to a certain system of linear equations and provide intermediate estimates for it.

The known linear equations (§2.2) connecting derivatives $z_\nu$ of a null-spline at the neighbouring knots are of the form

$$z_{\nu+1} = -D(\varrho_\nu)Az_\nu, \quad \nu = 0, \ldots, N.$$

Here $\varrho_\nu := h_\nu / h_{\nu+1}$ is the local mesh ratio, $D(\varrho)$ and $A$ are some special matrices. For a fixed $\nu$, this gives the equations

$$B' z_\nu = z_0, \quad C z_\nu = z_N,$$

with the matrices $B', C$ being products of $A$ and $D(\varrho_s)$ in certain combinations. Our choice (0.3.2) of the null-spline $\sigma$ provides the boundary conditions

$$z_0 := (\underbrace{0, \ldots, 0}_{p}, z_0^{(p+1)}, \ldots, z_0^{(2p+1)}), \quad z_N := (\underbrace{0, \ldots, 0, 1}_{p+1}, z_N^{(p+2)}, \ldots, z_N^{(2p+1)}).$$

They allow us to determine the vector $z_\nu$ as a solution of the linear system of equations

$$Mz_\nu = (\underbrace{0,\dots,0,1}_{2p+1})^T, \quad M = \begin{array}{|c|} \hline B'[\mathbf{p},:] \\ \hline C[\mathbf{p+1},:] \\ \hline \end{array},$$

where the matrix $M$ is composed of the first $p$ rows of $B'$ and the first $p+1$ rows of $C$ (see §2.2). We solve this system explicitly by Cramer's rule,

$$z_\nu^{(l)} = (-1)^{2p+1+l} \frac{\det M^{(l)}}{\det M},$$

and then apply the Laplace expansion by minors of $B'$ and $C$ to both determinants. Some elementary inequalities yield then (§2.3) the first estimates:

$$|z_\nu^{(l)}| \leqslant \max_{i \in \mathbf{J}^l} \frac{C(\mathbf{p}, i^l)}{C(\mathbf{p+1}, i')}, \quad l = 1, \dots, 2p+1. \tag{0.3.5}$$

Here, $\mathbf{J}, \mathbf{J}^l$ are the sets of (multi)indices of the form

$$\mathbf{J} := \{i \in \mathbf{N}^p : 1 \leqslant i_1 < \dots < i_p \leqslant 2p+1\}, \quad \mathbf{J}^l := \{i \subset \mathbf{J} : i_s \neq l\};$$

bold $\mathbf{n}$ stands for the index $(1, 2, \dots, n)$; $i'$ and $i^l$ are two different complements to $i \in \mathbf{J}^l$,

$$i \cup i' = \mathbf{2p+1}, \quad i \cup i^l = (\mathbf{2p+1}) \setminus \{l\},$$

and $C(i, j)$ are the corresponding minors (see §2.1 for detailed notation).

The orders of the minors on the right-hand side of (0.3.5) differ by one. We use some relations to equalize them and obtain (§2.5) the second estimate:

$$|z_\nu^{(l)}| \leqslant c_p \max_{i \in \mathbf{J}^l} \frac{C(\mathbf{p}, i^l)}{C(\mathbf{p}, i^*)}, \quad l = 1, \dots, 2p+1. \tag{0.3.6}$$

Here $i^* \in \mathbf{J}$ is the index symmetric to $i \in \mathbf{J}^l$, i.e., $i_s^* = 2p+2-i_{p+1-s}$.

(3) In Chapter 3, in §§3.3–3.7, we find a necessary and sufficient condition on the indices $i, j$ denoted

$$i \preceq j, \quad i, j \in \mathbf{J},$$

for the inequality

$$C(\mathbf{p}, i) \leqslant c_p C(\mathbf{p}, j).$$

In §3.8 we verify that depending on $l$ the indices $i^l$ and $i^*$ satisfy this condition, namely that

$$i^{l_2} \preceq i^* \preceq i^{l_1}, \quad l_1 \leqslant p+1 \leqslant l_2,$$

which gives

$$C(\mathbf{p}, i^l) \leqslant c_p C(\mathbf{p}, i^*), \quad l \geqslant p+1.$$

Combined with (0.3.6) this proves (0.3.4) and hence Theorem I.

This part of the proof is a bit long and technical, and it would be interesting to find simpler arguments (see §§4.3–4.4 for a discussion).

## 1. Main ingredients of the proof

### 1.1. B-splines and their properties

As before, for $k, N \in \mathbf{N}$, and a knot sequence

$$\Delta = \{a = t_0 < t_1 < \dots < t_N = b\},$$

the notation

$$\mathbf{S}_k(\Delta) := \mathbf{P}_k(\Delta) \cap C^{k-2}[a,b]$$

stands for the space of polynomial splines of order $k$ (i.e., of degree $<k$) on $\Delta$.

The subintervals of $\Delta$ and their lengths will be denoted by

$$I_j := (t_j, t_{j+1}), \quad |h_j| := t_{j+1} - t_j.$$

Let $\Delta^{(k)} = (t_i)_{i=-k+1}^{N+k-1}$ be an extended knot sequence such that

$$a = t_{-k+1} = \dots = t_0 < t_1 < \dots < t_N = \dots = t_{N+k-1} = b.$$

By $(N_j)_{j=-k+1}^{N-1}$ we denote the B-spline sequence of order $k$ on $\Delta^{(k)}$ forming a partition of unity, i.e.,

$$N_j(x) := N_{j,k}(x) := ([t_{j+1}, \dots, t_{j+k}] - [t_j, \dots, t_{j+k-1}])(\cdot - x)_+^{k-1},$$

and by $(M_j)$ the same sequence normalized with respect to the $L_1$-norm:

$$M_j(x) := M_{j,k}(x) := k[t_j, \dots, t_{j+k}](\cdot - x)_+^{k-1} := \frac{k}{t_{j+k} - t_j} N_j(x).$$

The following lemmas are well-known.

LEMMA 1.1.1 [B4, (4.2)–(4.5)]. *For any $k$ and any $\Delta^{(k)}$, one has*

$$\operatorname{supp} N_j = [t_j, t_{j+k}], \quad N_j \geqslant 0, \quad \sum N_j = 1, \tag{1.1.1}$$

$$M_j(x) = \frac{k}{t_{j+k} - t_j} N_j(x), \quad \int_{t_j}^{t_{j+k}} M_j(t)\, dt = 1. \tag{1.1.2}$$

LEMMA 1.1.2 [B4, Theorem 3.1]. *The B-spline sequence $(N_i)$ forms a basis for* $\mathbf{S}_k(\Delta)$.

LEMMA 1.1.3 [B4, Theorem 5.2]. *For any $k$, there exists a constant $\varkappa_k$, the so-called B-spline basis condition number, such that, for any $a=(a_j)$ and any $\Delta$,*

$$\varkappa_k^{-1}\|a\|_{l_\infty} \leqslant \Big\|\sum_j a_j N_j\Big\|_{L_\infty} \leqslant \|a\|_{l_\infty}. \tag{1.1.3}$$

LEMMA 1.1.4 [Schu, Theorem 4.53]. *Any spline $s\in\mathbf{S}_k(\Delta_N)$ has at most $N+k-2$ zeros counting multiplicities.*

LEMMA 1.1.5 [B4, (4.6)].

$$M_{i,1}(x)=\frac{1}{t_{i+1}-t_i}, \quad x\in[t_i,t_{i+1}),\ i=0,\dots,N-1; \tag{1.1.4}$$

$$M'_{i,k}(x)=\frac{k}{t_{i+k}-t_i}[M_{i,k-1}(x)-M_{i+1,k-1}(x)], \quad i=-k+1,\dots,N-1. \tag{1.1.5}$$

We will need two more lemmas.

LEMMA 1.1.6. *Let $M_i\in\mathbf{S}_k(\Delta)$ be the $L_1$-normalized B-spline. Then*

$$\operatorname{sign} M_i^{(k-1)}\big|_{(t_{i+\nu-1},t_{i+\nu})}=(-1)^{\nu-1}, \quad \nu=1,\dots,k. \tag{1.1.6}$$

*Proof.* Follows by induction from (1.1.4) and (1.1.5).

LEMMA 1.1.7. *Let $I_{i'}$ be a largest subinterval of* $\operatorname{supp} M_i=[t_i,t_{i+k}]$. *Then*

$$|M_i^{(k-1)}(x)|=\text{const}\geqslant|h_{i'}|^{-k}, \quad x\in(t_{i'},t_{i'+1}). \tag{1.1.7}$$

*Proof.* By induction. For $k=1$ due to (1.1.4) the lemma is true. Let $x\in I_{i'}$. From (1.1.5) and (1.1.6) we obtain

$$\begin{aligned}|M_{i,k}^{(k-1)}(x)|&=\frac{k}{t_{i+k}-t_i}|M_{i,k-1}^{(k-2)}(x)-M_{i+1,k-1}^{(k-2)}(x)|\\&=\frac{k}{t_{i+k}-t_i}(|M_{i,k-1}^{(k-2)}(x)|+|M_{i+1,k-1}^{(k-2)}(x)|)\\&\geqslant\frac{1}{|h_{i'}|}\cdot|h_{i'}|^{-(k-1)}=|h_{i'}|^{-k}.\end{aligned}$$

## 1.2. $L_2$-projector and the inverse of the B-spline Gramian

Consider $P_{\mathbf{S}}$, the orthogonal projector onto $\mathbf{S}_k(\Delta)$ with respect to the ordinary inner product, i.e.,

$$(f,s)=(P_{\mathbf{S}}(f),s) \quad \text{for all } s\in\mathbf{S}_k(\Delta).$$

For $N'=N+k-1$, let $G$ be the $(N'\times N')$-matrix

$$G=\{(M_i,N_j)\}_{i,j=-k+1}^{N-1}.$$

LEMMA 1.2.1 [B1]. *For any $k, \Delta$, one has*

$$\|P_{\mathbf{S}_k(\Delta)}\|_{L_\infty} \leqslant \|G^{-1}\|_{l_\infty}.$$

*Proof.* Let $f \in L_\infty$, and $P_{\mathbf{S}}(f) = \sum_j a_j(f) N_j$, so that for $a = (a_i(f))$,

$$(Ga)_i := \sum_j (M_i, N_j) a_j(f) = (f, M_i) =: b_i(f).$$

By (1.1.3),

$$\|P_{\mathbf{S}}(f)\|_{L_\infty} \leqslant \|a(f)\|_{l_\infty},$$

and by (1.1.1)–(1.1.2),

$$\|b(f)\|_{l_\infty} := \max_i |(f, M_i)| \leqslant \|f\|_{L_\infty} \cdot \max_i \|M_i\|_{L_1} = \|f\|_{L_\infty}.$$

Thus

$$\|P_{\mathbf{S}}\|_\infty = \sup_f \frac{\|P_S(f)\|_{L_\infty}}{\|f\|_{L_\infty}} \leqslant \sup_f \frac{\|a(f)\|_{l_\infty}}{\|b(f)\|_{l_\infty}} = \sup_f \frac{\|G^{-1}b(f)\|_{l_\infty}}{\|b(f)\|_{l_\infty}} \leqslant \|G^{-1}\|_\infty$$

as claimed. □

LEMMA 1.2.2 [B1]. *The matrix $G$ is totally positive, i.e.,*

$$G\begin{pmatrix} i_1, \ldots, i_p \\ j_1, \ldots, j_p \end{pmatrix} \geqslant 0.$$

LEMMA 1.2.3 [B1]. *The matrix $G^{-1} := (g_{ij}^{(-1)})$ is checkerboard, i.e.,*

$$|g_{ij}^{(-1)}| = (-1)^{i+j} g_{ij}^{(-1)}.$$

*Proof.* Let $G_{ji}$ be the algebraic adjoint to $g_{ji}$. By Cramer's rule,

$$g_{ij}^{(-1)} = (-1)^{i+j} \frac{\det G_{ji}}{\det G},$$

and by Lemma 1.2.2 both determinants $\det G, \det G_{ji}$ are non-negative. □

LEMMA 1.2.4 [B1]. *Let $H^{-1}$ be a checkerboard matrix, and let $a, b \in \mathbf{R}^N$ be vectors such that $Ha = b$ and*

($a_1$) $(-1)^i \operatorname{sign} b_i = \text{const}$ *for all* $i$;

($a_2$) $\min_i |b_i| \geqslant c_{\min}$;

($a_3$) $\|a\|_\infty \leqslant c_{\max}$.

*Then*

$$\|H^{-1}\|_\infty \leqslant \frac{c_{\max}}{c_{\min}}.$$

*Proof.* Let $a, b$ satisfy ($a_1$)–($a_3$), and let

$$H^{-1} := (h_{ij}^{(-1)}), \quad |h_{ij}^{(-1)}| = (-1)^{i+j} h_{ij}^{(-1)}.$$

Then

$$|a_i| = |(H^{-1}b)_i| := \Big|\sum_j h_{ij}^{(-1)} b_j\Big| = \sum_j |h_{ij}^{(-1)} b_j| \geqslant \min_j |b_j| \cdot \sum_j |h_{ij}^{(-1)}|.$$

Therefore,

$$\|a\|_\infty := \max_i |a_i| \geqslant \min_j |b_j| \cdot \max_i \sum_j |h_{ij}^{(-1)}| = \min_j |b_j| \cdot \|H^{-1}\|_\infty. \qquad \square$$

### 1.3. Analytic version of de Boor's Lemma 1.2.4

Let $a \in \mathbf{R}^{N'}$ and let $\phi \in \mathbf{S}_k(\Delta)$ be a spline of order $k$ on $\Delta$ that has the expansion

$$\phi = \sum_j a_j N_j.$$

Then, since $G := \{(M_i, N_j)\}$, one obtains

$$b_i := (Ga)_i = \sum_j (M_i, N_j) a_j = (M_i, \phi).$$

By Lemma 1.1.3, we also have

$$\|a\|_{l_\infty} \leqslant \varkappa_k \|\phi\|_{L_\infty},$$

where $\varkappa_k$ is the B-spline basis condition number.

Using these two facts, Lemma 1.2.4 applied to the matrix $G$ combined with Lemma 1.2.1 implies

LEMMA 1.3.1. *Let $\phi$ be any spline such that*

($A_0$) $\phi \in \mathbf{S}_k(\Delta)$;

($A_1$) $(-1)^i \operatorname{sign}(\phi, M_i) = \mathrm{const}$ *for all* $i$;

($A_2$) $|(\phi, M_i)| \geqslant c_{\min}(k)$ *for all* $i$;

($A_3$) $\|\phi\|_\infty \leqslant c_{\max}(k)$.

*Then*

$$\|P_{\mathbf{S}_k(\Delta)}\|_\infty \leqslant \varkappa_k \frac{c_{\max}(k)}{c_{\min}(k)}.$$

### 1.4. Main idea: definition of $\phi$ via a null-spline $\sigma$. Formulation of Theorem $\Phi$

*Definition* 1.4.1. Define the spline $\sigma$ as the spline of the even degree $2k-2$ on $\Delta$, i.e.,

$$\sigma \in \mathbf{S}_{2k-1}(\Delta), \tag{1.4.1}$$

that satisfies the conditions

$$\sigma(t_i) = 0, \quad i = 0, \ldots, N; \tag{1.4.2}$$

$$\sigma^{(l)}(t_0) = \sigma^{(l)}(t_N) = 0, \quad l = 1, \ldots, k-2; \tag{1.4.3}$$

$$\frac{1}{(k-1)!} \sigma^{(k-1)}(t_N) = 1. \tag{1.4.4}$$

The spline $\sigma$ defined by (1.4.1)–(1.4.4) exists and is unique, see [Schu, Theorem 4.67]. This fact will follow also from our further considerations where we show that $\sigma$ results from the solution of a system of linear equations with some non-singular matrix.

Our main idea is to define $\phi$ as follows.

*Definition* 1.4.2. Set

$$\phi(x) := \sigma^{(k-1)}(x). \tag{1.4.5}$$

*Example* 1.4.3. For $k=2$, $\sigma$ is a parabolic null-spline, and its first derivative $\phi = \sigma'$ is the broken line that alternates between $+1$ and $-1$ at the knots, i.e.,

$$\phi = \sum (-1)^i N_i, \quad k = 2.$$

Our main result is

THEOREM $\Phi$. *For any $k$ there exist constants $c_{\max}(k)$, $c_{\min}(k)$ such that for any $\Delta_N$ with $N \geqslant 2k$ the spline $\phi$ defined via* (1.4.5) *satisfies the relations*

($A_0$) $\phi \in \mathbf{S}_k(\Delta_N)$;

($A_1$) $(-1)^i \operatorname{sign}(\phi, M_i) = \text{const}$ *for all $i$*;

($A_2$) $|(\phi, M_i)| > c_{\min}(k)$ *for all $i$*;

($A_3$) $\|\phi\|_{L_\infty[t_i, t_{i+1}]} < c_{\max}(k)$ *for all $i$*.

*Remark.* The restrictions $N \geqslant 2k$ is needed only in the proof of ($A_3$).

*Proof of* ($A_0$). Since $\sigma \in \mathbf{S}_{2k-1}(\Delta)$, clearly $\phi := \sigma^{(k-1)} \in \mathbf{S}_k(\Delta)$.

## 1.5. Proof of Theorem I and its corollaries

*Proof of Theorem* I. From Theorem $\Phi$, by Lemma 1.3.1,

$$\|P_{\mathbf{S}_k(\Delta_N)}\|_\infty \leqslant c_k, \quad N \geqslant 2k.$$

To complete the proof, it remains to cover the case $N < 2k$. As is known (see, e.g., [S1]),

$$\|P_{\mathbf{S}_k(\Delta_N)}\|_\infty \leqslant c(k, N),$$

hence

$$\|P_{\mathbf{S}_k(\Delta_N)}\|_\infty \leqslant c'_k, \quad N < 2k,$$

and finally

$$\|P_{\mathbf{S}_k(\Delta)}\|_\infty \leqslant c''_k \quad \text{for all } \Delta. \qquad \square$$

*Proof of Corollary* I. Let $(M_i)$, $(N_i)$ be the B-spline sequences for the space $\mathbf{S}_k(\Delta, \mathbf{m})$ of splines with multiple knots defined on the extended knot sequence

$$(\tau_0, \ldots, \tau_{N'}) := (\underbrace{t_0, \ldots, t_0}_{k-m_0}, \ldots, \underbrace{t_i, \ldots, t_i}_{k-m_i}, \ldots, \underbrace{t_N, \ldots, t_N}_{k-m_N}).$$

Further, let $(M_i^{(n)})$, $(N_i^{(n)})$ be the B-spline sequences on the knot sequences $\Delta^{(n)} = (t_j^{(n)})$ chosen so that

$$t_j^{(n)} < t_{j+1}^{(n)}, \quad \lim_{n\to\infty} t_j^{(n)} = \tau_j.$$

Then, as is known,

$$\lim_{n\to\infty} (M_i^{(n)}, N_j^{(n)}) = (M_i, N_j),$$

whence, for the corresponding Gramians, we have

$$\|G^{-1}\|_\infty = \lim_{n\to\infty} \|(G^{(n)})^{-1}\|_\infty \leqslant c_k,$$

where that last inequality is due to Theorem I. Thus,

$$\|P_{\mathbf{S}_k(\Delta,\mathbf{m})}\|_\infty \leqslant \|G^{-1}\|_\infty \leqslant c_k. \qquad \square$$

*Proof of Corollary* II. Let $(M_i)$, $(N_i)$ be the B-spline sequences for the space $\mathbf{S}_k(\Delta_\infty, \mathbf{m}_\infty)$ of splines with multiple (bi)infinite knot sequence. Then also

$$\|P_{\mathbf{S}_k(\Delta_\infty,\mathbf{m}_\infty)}\|_\infty \leqslant \|G_{\Delta_\infty}^{-1}\|_\infty,$$

where $G_{\Delta_\infty} := (M_i, N_j)$ is the corresponding (bi)infinite Gram matrix. By Corollary I, all of its finite principal submatrices $G_{\Delta_N}$ are boundedly invertible. This implies that $G_{\Delta_\infty}$ is invertible, too, and

$$\|G_{\Delta_\infty}^{-1}\|_\infty \leqslant \lim_{N\to\infty} \|G_{\Delta_N}^{-1}\|_\infty \leqslant c_k. \qquad \square$$

### 1.6. Proof of Theorem $\Phi$: proof of $(\mathbf{A}_1)$

LEMMA 1.6.1. *The spline $\sigma$ changes its sign exactly at the points $(t_i)_{i=1}^{N-1}$, i.e.,*

$$(-1)^i \operatorname{sign} \sigma|_{(t_{i-1},t_i)} = \text{const}, \quad i = 1, ..., N.$$

*Proof.* By the definitions (1.4.2) and (1.4.3), the spline $\sigma \in \mathbf{S}_{2k-1}(\Delta)$ has at least $N+1+2(k-2)$ zeros counting multiplicities, and by Lemma 1.1.4 any spline from $\mathbf{S}_{2k-1}(\Delta)$ has at most $N+(2k-1)-2$ such zeros. Therefore, $\sigma$ has no zeros different from (1.4.2) and (1.4.3). $\square$

PROPERTY $(\mathrm{A}_1)$. *Let $\phi$ be the spline* (1.4.5). *Then*

$$(-1)^i \operatorname{sign}(\phi, M_i) = \text{const} \quad \textit{for all } i.$$

*Proof of* $(\mathrm{A}_1)$. Integration by parts yields

$$\begin{aligned}(\phi, M_i) &:= \int_{t_i}^{t_{i+k}} \sigma^{(k-1)}(t) M_i(t)\, dt \qquad (1.6.1)\\ &= (-1)^{k-1} \int_{t_i}^{t_{i+k}} \sigma(t) M_i^{(k-1)}(t)\, dt + \sum_{l=1}^{k-1} (-1)^{l+1} \sigma^{(k-1-l)}(x) M_i^{(l-1)}(x)\Big|_{t_i}^{t_{i+k}}.\end{aligned}$$

At the point $x = t_i$ we have

$$\begin{aligned}\sigma^{(k-1-l)}(t_i) &= 0, \quad t_i = t_0,\ l = 1, ..., k-1;\\ M_i^{(l-1)}(t_i) &= 0, \quad t_i > t_0,\ l = 1, ..., k-1;\end{aligned}$$

and similarly for $x=t_{i+k}$,

$$\begin{aligned} \sigma^{(k-1-l)}(t_{i+k}) &= 0, \quad t_{i+k}=t_N,\ l=1,\dots,k-1;\\ M_i^{(l-1)}(t_{i+k}) &= 0, \quad t_{i+k}<t_N,\ l=1,\dots,k-1. \end{aligned}$$

Thus, the sum in (1.6.1) vanishes and

$$(\phi, M_i) := \int_{t_i}^{t_{i+k}} \sigma^{(k-1)}(t) M_i(t)\, dt = (-1)^{k-1} \int_{t_i}^{t_{i+k}} \sigma(t) M_i^{(k-1)}(t)\, dt. \tag{1.6.2}$$

Since both $\sigma(t)$ and $M_i^{(k-1)}(t)$ alternate in sign on the sequence of subintervals of $[t_i, t_{i+k}]$, we have

$$\begin{aligned} (-1)^i \operatorname{sign}(\phi, M_i) &= (-1)^i \cdot (-1)^{k-1} \operatorname{sign} \sigma|_{(t_i,t_{i+1})} \operatorname{sign} M_i^{(k-1)}|_{(t_i,t_{i+1})}\\ &= (-1)^i \cdot (-1)^{k-1} \cdot (-1)^i \mathrm{const} \cdot 1\\ &= (-1)^{k-1} \cdot \mathrm{const}. \end{aligned}$$

Hence,

$$(-1)^i \operatorname{sign}(\phi, M_i) = \mathrm{const}, \quad i=-k+1,\dots,N-1. \qquad \square$$

### 1.7. An invariant

For the proof of ($A_2$) and for some further use in §2.4, we will need the following considerations.

*Definition* 1.7.1. For two functions $f, g$ and $n \in \mathbf{N}$, set

$$G(f,g;x) := \sum_{l=0}^{n+1} (-1)^l f^{(l)}(x) g^{(n+1-l)}(x),$$

whenever the right-hand side makes sense.

LEMMA 1.7.2. *Let $p, q$ be two polynomials of degree $n+1$ on $I$. Then*

$$G(p,q;x) = \mathrm{const}(p,q) \quad \textit{for all } x \in I.$$

*Proof.* It is readily seen that $G'(p,q;x)=0$ for all $x \in \mathbf{R}$, hence the statement. $\square$

LEMMA 1.7.3. *Let $s_1, s_2$ be two null-splines of degree $n+1$ on $\Delta$, i.e.,*

$$s_1, s_2 \in \mathbf{S}_{n+2}(\Delta), \quad s_1(t_i) = s_2(t_i) = 0, \quad i = 0, ..., N. \tag{1.7.1}$$

*Then*

$$G(s_1, s_2; x) = \mathrm{const}(s_1, s_2), \quad x \in [a, b]. \tag{1.7.2}$$

*Proof.* By Lemma 1.7.2 the function $G(s_1, s_2)$ is piecewise constant.

On the other hand, since the continuity conditions on $s_1, s_2 \in \mathbf{S}_{n+2}(\Delta)$ imply the inclusion $s_1, s_2 \in C^n[a, b]$, we have

$$s_1^{(l)} s_2^{(n+1-l)}\big|_{t_i-0} = s_1^{(l)} s_2^{(n+1-l)}\big|_{t_i+0}, \quad l = 1, ..., n,$$

and due to the null-values of $s_1, s_2$ on $\Delta$ also

$$s_1^{(l)} s_2^{(n+1-l)}\big|_{t_i-0} = s_1^{(l)} s_2^{(n+1-l)}\big|_{t_i+0} = 0, \quad l = 0, \ l = n+1,$$

i.e., the function $G(s_1, s_2)$ is continuous. □

As a corollary, we obtain

LEMMA 1.7.4. *Let $\sigma \in \mathbf{S}_{2k-1}(\Delta)$ be the null-spline defined in* (1.4.1)–(1.4.3). *Then*

$$H(x) := [\sigma^{(k-1)}(x)]^2 + 2\sum_{l=1}^{k-1} (-1)^l \sigma^{(k-1-l)}(x)\,\sigma^{(k-1+l)}(x) = (k-1)!^2. \tag{1.7.3}$$

*Proof.* The function $H$ is obtained from $G(s_1, s_2)$ if we set $s_1 = s_2 = \sigma$ and $n+1 = 2k-2$, precisely

$$H(x) = (-1)^{k-1} G(\sigma, \sigma; x).$$

Therefore, by (1.7.2), it is a constant function.

The boundary conditions on $\sigma$ at $t_N$ are

$$\sigma^{(l)}(t_N) = 0, \quad l \leqslant k-2,$$
$$\sigma^{(k-1)}(t_N) = (k-1)!,$$

and therefore for $x = t_N$ the sum in (1.7.3) vanishes, i.e.,

$$H(t_N) = [\sigma^{(k-1)}(t_N)]^2 := (k-1)!^2.$$

Thus,

$$H(x) = H(t_N) = (k-1)!^2 \quad \text{for all } x \in [a, b].$$ □

LEMMA 1.7.5. *We have*

$$\frac{1}{(k-1)!}\,|\sigma^{(k-1)}(t_0)| = 1. \tag{1.7.4}$$

*Proof.* The boundary conditions (1.4.3) on $\sigma$ at $t_0$ are

$$\sigma^{(l)}(t_0) = 0, \quad l \leqslant k-2.$$

Therefore, for $x = t_0$, the sum in (1.7.3) vanishes, i.e.,

$$H(t_0) = [\sigma^{(k-1)}(t_0)]^2.$$

On the other hand, by (1.7.3),

$$H(t_0) = (k-1)!^2. \qquad \square$$

## 1.8. Proof of Theorem $\Phi$: proof of ($\mathrm{A}_2$)

For the proof of ($\mathrm{A}_2$), we need the following estimate.

LEMMA 1.8.1. *There exists a positive constant $c_k$ such that the inequality*

$$\|\sigma\|_{L_1[t_i,t_{i+1}]} \geqslant c_k |h_i|^k \tag{1.8.1}$$

*holds uniformly in $i$.*

*Proof.* By (1.7.3), we have

$$\begin{aligned}(k-1)!^2 &= H(t_i)\\ &:= [\sigma^{(k-1)}(t_i)]^2 + 2\sum_{m=1}^{k-2}(-1)^m \sigma^{(k-1-m)}(t_i)\,\sigma^{(k-1+m)}(t_i)\\ &= [\sigma^{(k-1)}(t_i)]^2 + 2\sum_{m=1}^{k-2}(-1)^m [\sigma^{(k-1-m)}(t_i)\cdot|h_i|^{-m}]\cdot[\sigma^{(k-1+m)}(t_i)\cdot|h_i|^m].\end{aligned}$$

From the latter equality follows that

$$\max_{|m|\leqslant k-2} |\sigma^{(k-1+m)}(t_i)|\cdot|h_i|^m \geqslant c_k,$$

or, equivalently,

$$\max_{1\leqslant l\leqslant 2k-3} |\sigma^{(l)}(t_i)|\cdot|h_i|^{l+1} \geqslant c_k|h_i|^k. \tag{1.8.2}$$

By the Markov inequality for polynomials,

$$\|\sigma\|_{L_1[t_i,t_{i+1}]} \geqslant c_l |h_i|^{l+1}\,\|\sigma^{(l)}\|_{L_\infty[t_i,t_{i+1}]} \quad \text{for all } l,$$

so that, making use of (1.8.2), we obtain

$$\|\sigma\|_{L_1[t_i,t_{i+1}]} \geqslant c_k' |h_i|^k. \qquad \square$$

PROPERTY ($A_2$). *There exists a positive constant $c_{\min}(k)$ depending only on $k$ such that, for any $\Delta$, the spline $\phi$ defined in* (1.4.5) *satisfies the relation*

$$|(\phi, M_i)| \geqslant c_{\min}(k), \quad i = -k+1, \dots, N-1.$$

*Proof of* ($A_2$). Let $I_{i'}$ be a largest subinterval of $\operatorname{supp} M_i := [t_i, t_{i+k}]$. Since

$$\operatorname{sign} \sigma(t) \cdot \operatorname{sign} M_i^{(k-1)}(t) = \text{const}, \quad t \in [t_i, t_{i+k}],$$

we have

$$\begin{aligned} |(\phi, M_i)| &:= \left| \int_{t_i}^{t_{i+k}} \sigma^{(k-1)}(t) M_i(t)\, dt \right| \\ &= \left| \int_{t_i}^{t_{i+k}} \sigma(t) M_i^{(k-1)}(t)\, dt \right| \quad \text{(by (1.6.2))} \\ &= \int_{t_i}^{t_{i+k}} |\sigma(t) M_i^{(k-1)}(t)|\, dt \\ &\geqslant \int_{t_{i'}}^{t_{i'+1}} |\sigma(t) M_i^{(k-1)}(t)|\, dt \\ &= |M_i^{(k-1)}(x_{i'})| \cdot \|\sigma\|_{L_1[t_{i'}, t_{i'+1}]}, \end{aligned}$$

and due to (1.8.1) and (1.1.7),

$$|(\phi, M_i)| \geqslant c_k c_k' =: c_{\min}(k). \qquad \square$$

## 1.9. Vectors $z_\nu$. Formulation of Theorem Z

Theorem Z formulated below enables us to verify in the next section the last condition ($A_3$) of Theorem $\Phi$.

*Definition* 1.9.1. Set

$$z_i := (z_i^{(1)}, \dots, z_i^{(2k-3)}) \in \mathbf{R}^{2k-3}, \quad i = 0, \dots, N-1, \tag{1.9.1}$$

with

$$z_i^{(l)} := \frac{1}{l!} \sigma^{(l)}(t_i) \cdot |h_i|^{l-k+1}, \quad l = 1, \dots, 2k-3. \tag{1.9.2}$$

In the rest of the paper we are going to prove

THEOREM Z. *There exists a constant $c_k$ depending only on $k$ such that, for $N \geqslant k$, the estimates*

$$|z_i^{(l)}| \leqslant c_k, \quad l \geqslant k-1, \ i=0,\dots,N-k, \tag{1.9.3}$$

*hold uniformly in $i$ and $l$.*

This theorem almost evidently implies the estimate

$$\|\phi\|_{L_\infty[t_i,t_{i+1}]} := \|\sigma^{(k-1)}\|_{L_\infty[t_i,t_{i+1}]} \leqslant c'_k, \quad i \leqslant N-k,$$

which coincides with ($A_3$) except for the indices $i > N-k$. In the next section we prove this implication and show how to cover for $N \geqslant 2k$ the case $i > N-k$ of ($A_3$).

## 1.10. Proof of Theorem $\Phi$: proof of ($A_3$)

PROPERTY ($A_3$). *There exists a constant $c_{\max}(k)$ depending only on $k$ such that, for any $\Delta_N$ with $N \geqslant 2k$, the spline $\phi$ defined in (1.4.5) satisfies the relation*

$$\|\phi\|_{L_\infty[t_i,t_{i+1}]} \leqslant c_{\max}(k) \quad \textit{for all } i. \tag{1.10.1}$$

*Proof of* ($A_3$). (1) *The case $N \geqslant 2k$, $i \leqslant N-k$.* In this case, by (1.9.3) of Theorem Z, and by the definitions (1.9.2) and (1.4.5), we have

$$\begin{aligned} \frac{1}{m!}|\phi^{(m)}(t_i)|\cdot|h_i|^m &= \frac{1}{m!}|\sigma^{(k-1+m)}(t_i)|\cdot|h_i|^m \\ &= \frac{(k-1+m)!}{m!}|z_i^{(k-1+m)}| \leqslant c'_k, \quad m=0,\dots,k-2. \end{aligned}$$

On $[t_i, t_{i+1}]$ the spline $\phi := \sigma^{(k-1)}$ is an algebraic polynomial of degree $k-1$, and by Taylor expansion,

$$\phi(t_{i+1}) = \sum_{m=0}^{k-1} \frac{1}{m!}\phi^{(m)}(t_i)\cdot|h_i|^m.$$

Hence,

$$|\phi^{(k-1)}(t_i)|\cdot|h_i|^{k-1} \leqslant |\phi(t_{i+1})| + \sum_{m=0}^{k-2} \frac{1}{m!}|\phi^{(m)}(t_i)|\cdot|h_i|^m \leqslant k\cdot c'_k,$$

and finally

$$\begin{aligned} \|\phi(x)\|_{L_\infty[t_i,t_{i+1}]} &\leqslant \sum_{m=0}^{k-1} \frac{1}{m!}|\phi^{(m)}(t_i)|\cdot|h_i|^m \\ &\leqslant (2k-1)\cdot c'_k =: c_{\max}(k), \quad i \leqslant N-k, \ N \geqslant 2k. \end{aligned}$$

(2) *The case* $N \geqslant 2k$, $i \geqslant N-k$. Let $\tilde{\sigma}$ be the null-spline that is defined by the same interpolation and boundary conditions (1.4.2)–(1.4.3) as $\sigma$, but with the normalization at the left endpoint

$$\frac{1}{(k-1)!}\tilde{\sigma}(t_0) = 1.$$

Accordingly, we set

$$\tilde{\phi} = \tilde{\sigma}^{(k-1)}.$$

Then, due to symmetry, by Theorem Z applied to $\tilde{\sigma}$, we obtain

$$\|\tilde{\phi}\|_{L_\infty[t_i, t_{i+1}]} \leqslant c_{\max}(k), \quad i \geqslant k.$$

On the other hand, we established in (1.7.4) that

$$\frac{1}{(k-1)!}\sigma(t_0) = \pm 1.$$

This implies the equality

$$\tilde{\phi} = \pm\phi,$$

and, correspondingly, the estimate

$$\|\phi\|_{L_\infty[t_i, t_{i+1}]} \leqslant c_{\max}(k), \quad i \geqslant k.$$

If $N \geqslant 2k$, then $N-k \geqslant k$, and thus

$$\|\phi\|_{L_\infty[t_i, t_{i+1}]} < c_{\max}(k), \quad i \geqslant N-k, \ N \geqslant 2k. \qquad \square$$

This completes the proof of Theorem $\Phi$.

*Remark.* The size and the structure of the proof of Theorem Z (i.e., of $(A_3)$) given in the next two chapters are in a sharp contrast with the short proofs of $(A_1)$ and $(A_2)$ given above. We conclude this chapter with a conjecture which probably could be useful in finding a simpler proof of $(A_3)$.

CONJECTURE 1.10.1. *Let* $\phi := \sigma^{(k-1)}$ *be the spline* (1.4.5). *Then it takes its maximal absolute values at the endpoints, i.e.,*

$$|\phi(x)| \leqslant |\phi(a)| \ (= |\phi(b)| = (k-1)!) \quad \textit{for all } x \in [a, b].$$

*In particular, the sum in* (1.7.3) *is always non-negative, and zero only if* $x$ *is a knot of high multiplicity.*

## 2. Proof of Theorem Z: intermediate estimates for $z_\nu$

### 2.1. Notation and auxiliary statements

Let $U$ be any $(n\times n)$-matrix. We denote by

$$U[\alpha,\beta]:=U\begin{bmatrix}\alpha_1,\dots,\alpha_p\\ \beta_1,\dots,\beta_q\end{bmatrix}$$

the submatrix of $U$ (not necessarily square) whose $(s,t)$-entry is $U[\alpha_s,\beta_t]$ with $\alpha$ and $\beta$ sequences (indices) with increasing entries. The default sequence $(:)$ stands for the sequence of all possible entries. So, $U[\alpha,:]$ is the matrix made up from rows $\alpha_1,\dots,\alpha_p$ of $U$. The sequence $(\backslash s)$ stands for all entries but one numbered $s$. For example, $U[\backslash 1,\backslash l{+}1]$ is the matrix made up from rows $2,\dots,n$ and columns $1,\dots,l,l{+}2,\dots,n$ of $U$.

The notation

$$U(\alpha,\beta):=\det U\begin{bmatrix}\alpha_1,\dots,\alpha_p\\ \beta_1,\dots,\beta_p\end{bmatrix}:=U\begin{pmatrix}\alpha_1,\dots,\alpha_p\\ \beta_1,\dots,\beta_p\end{pmatrix}$$

(now with $\#\alpha=\#\beta$) stands for the corresponding subdeterminant.

A matrix $U$ is called totally positive (TP) if

$$U(\alpha,\beta)\geqslant 0 \quad \text{for all } \alpha,\beta.$$

As was already mentioned, by indices we mean sequences with increasing entries. For convenience we will also view indices as sets when writing, e.g., $\alpha\subset\beta$ to express that the components of $\alpha$ appear also in $\beta$.

For $n\in\mathbf{N}$, the bold $\mathbf{n}$ denotes the index $(1,2,\dots,n)$. Further,

$$\mathbf{I}_{p,n}:=\{i\subset\mathbf{n}:\#i=p\}:=\{(i_s)_{s=1}^p:1\leqslant i_1<\dots<i_p\leqslant n\}.$$

For the special case $\mathbf{n=2p+1}$ we set

$$\mathbf{J}:=\mathbf{I}_{p,2p+1},\quad \mathbf{J}^l:=\{i\in\mathbf{J}:\{l\}\notin i\},\quad l=1,\dots,2p{+}1.$$

For $i\in\mathbf{I}_{p,n}$, its complement $i'$ and its conjugate index $i^*$ are given, respectively, by

$$\begin{aligned} &i'\in\mathbf{I}_{n-p,n}, && i':=\mathbf{n}\backslash i,\\ &i^*\in\mathbf{I}_{p,n}, && i^*:=(n{+}1{-}i_p,\dots,n{+}1{-}i_1).\end{aligned}$$

For $i\in\mathbf{J}^l$, we define also the $l$-complement

$$i^l\in\mathbf{J}^l,\quad i^l:=i'\backslash\{l\}.$$

Finally, for two indices $i,j\in\mathbf{I}_{p,n}$, we denote

$$i\leqslant j \;\Leftrightarrow\; i_s\leqslant j_s \text{ for all } s,\quad |i|:=\sum_s i_s.$$

The following lemmas will be used frequently (see [K, pp. 1–6]).

LEMMA 2.1.1 (Cauchy–Binet formula). *If* $U, V, W \in \mathbf{R}^{n\times n}$ *and* $U = VW$, *then for any* $i, j \in \mathbf{I}_{p,n}$,

$$U(i,j) = \sum_{\alpha\in\mathbf{I}_{p,n}} V(i,\alpha)W(\alpha,j).$$

This relation will be referred to as 'the CB-formula' for short.

LEMMA 2.1.2 (inverse determinants). *If* $V = U^{-1}$, *then for any* $i, j \in \mathbf{I}_{p,n}$ *we have*

$$V(i,j) = (-1)^{|i+j|} \frac{U(j',i')}{\det U}.$$

LEMMA 2.1.3 (Laplace expansion by minors). *For any fixed index* $i \in \mathbf{I}_{p,n}$, *we have*

$$\det U = \sum_{\alpha\in\mathbf{I}_{p,n}} (-1)^{|i+\alpha|} U(i,\alpha)U(i',\alpha').$$

We will also use the following estimate.

LEMMA 2.1.4. *Let* $q \in \mathbf{N}$ *and* $a_s, b_s, c_s \geqslant 0$. *Then*

$$\min_s \frac{b_s}{c_s} \leqslant \frac{\sum_{s=1}^q a_s b_s}{\sum_{s=1}^q a_s c_s} \leqslant \max_s \frac{b_s}{c_s}. \tag{2.1.1}$$

*Proof.* Let

$$\min_s \frac{b_s}{c_s} = \underline{\varepsilon}, \quad \max_s \frac{b_s}{c_s} = \bar{\varepsilon}.$$

Then $\underline{\varepsilon} c_s \leqslant b_s \leqslant \bar{\varepsilon} c_s$ and

$$\underline{\varepsilon} \sum_{s=1}^q a_s c_s \leqslant \sum_{s=1}^q a_s b_s \leqslant \bar{\varepsilon} \sum_{s=1}^q a_s c_s. \qquad \square$$

## 2.2. Reduction to a linear system of equations

2.2.1. *Derivatives of null-splines at knots.* Let $q$ be a null-spline on $\Delta$ of degree $n+1$, i.e.,

$$q \in \mathbf{S}_{n+2}(\Delta), \quad q(t_\nu) = 0 \quad \text{for all } \nu.$$

Set

$$q_\nu := (q_\nu^{(1)}, ..., q_\nu^{(n)}) \in \mathbf{R}^n, \quad q_\nu^{(l)} := \frac{1}{l!} q^{(l)}(t_\nu), \quad l = 0, ..., n+1.$$

On $[t_\nu, t_{\nu+1}]$, $q$ is an algebraic polynomial, and by Taylor expansion of $q$ at $x = t_\nu$ we obtain

$$\frac{1}{i!} q^{(i)}(t_{\nu+1}) = \frac{1}{i!} \sum_{j=i}^{n+1} \frac{1}{(j-i)!} q^{(j)}(t_\nu)\cdot|h_\nu|^{j-i} = \sum_{j=i}^{n+1} \left(\frac{j!}{i!\,(j-i)!}\right) \frac{1}{j!} q^{(j)}(t_\nu)\cdot|h_\nu|^{j-i},$$

i.e.,

$$q_{\nu+1}^{(i)} \cdot |h_\nu|^i = \sum_{j=i}^{n+1} \binom{j}{i} q_\nu^{(j)} \cdot |h_\nu|^j.$$

Since $q_\nu^{(0)} = q_{\nu+1}^{(0)} = 0$, we have

$$q_\nu^{(n+1)} \cdot |h_\nu|^{n+1} = -\sum_{j=1}^{n} q_\nu^{(j)} \cdot |h_\nu|^j,$$

and hence

$$q_{\nu+1}^{(i)} \cdot |h_\nu|^i = \sum_{j=i}^{n} \left[ \binom{j}{i} - \binom{n+1}{i} \right] q_\nu^{(j)} \cdot |h_\nu|^j, \quad i = 1, \dots, n.$$

For the vectors $q_\nu$ we have therefore the equality

$$D_0(h_\nu) q_{\nu+1} = -A D_0(h_\nu) q_\nu, \tag{2.2.1}$$

where $A$ is the $(n \times n)$-matrix given by

$$A = \left\{ \binom{n+1}{i} - \binom{j}{i} \right\}_{i,j=1}^{n} \tag{2.2.2}$$

and

$$D_0(h) = \operatorname{diag} \lceil h, h^2, \dots, h^n \rfloor. \tag{2.2.3}$$

By Taylor expansion of $q$ at $x = t_{\nu+1}$, we conclude that

$$D_0(-h_\nu) q_\nu = -A D_0(-h_\nu) q_{\nu+1},$$

so that in view of (2.2.1)

$$A^{-1} = D_0 A D_0, \tag{2.2.4}$$

with

$$D_0 := D_0(-1) = \operatorname{diag} \lceil -1, 1, -1, 1, \dots \rfloor. \tag{2.2.5}$$

It is more convenient to employ another scaling of $q_\nu$ in (2.2.1), namely by the matrix

$$\begin{aligned} D_h := D(h) &:= h^{-n/2-1/2} D_0(h) \\ &= \operatorname{diag} \lceil h^{-n/2+1/2}, h^{-n/2+3/2}, \dots, h^{n/2-1/2} \rfloor, \end{aligned} \tag{2.2.6}$$

which satisfies

$$\det D(h) = 1.$$

Then we also have the equality

$$D(h_\nu)q_{\nu+1} = -AD(h_\nu)q_\nu,$$

which may be rewritten as

$$D(h_{\nu+1})q_{\nu+1} = -D(h_{\nu+1}/h_\nu)AD(h_\nu)q_\nu. \tag{2.2.7}$$

2.2.2. *The matrices* $B, B', C$. Set

$$\begin{aligned} y_\nu &:= D(h_\nu)q_\nu, \quad \nu < N, \\ y_N &:= D(h_{N-1})q_N, \end{aligned}$$

i.e., for a null-spline $q \in \mathbf{S}_{n+2}(\Delta)$, we define the vectors

$$y_\nu := (y_\nu^{(1)}, \dots, y_\nu^{(n)}) \in \mathbf{R}^n,$$

with the components

$$\begin{aligned} y_\nu^{(l)} &:= \frac{1}{l!} q^{(l)}(t_\nu) \cdot |h_\nu|^{l-(n+1)/2}, \quad \nu = 0, \dots, N-1, \\ y_N^{(l)} &:= \frac{1}{l!} q^{(l)}(t_\nu) \cdot |h_{N-1}|^{l-(n+1)/2}. \end{aligned}$$

Set also

$$\varrho_\nu := \frac{h_{\nu+1}}{h_\nu}.$$

Then from (2.2.7) follows that the vectors $y_\nu$ are connected by the rules

$$\begin{aligned} y_{\nu+1} &= -D(\varrho_\nu)Ay_\nu, \quad \nu = 0, \dots, N-2, \\ y_N &= -Ay_{N-1}, \end{aligned}$$

and

$$\begin{aligned} y_{\nu-1} &= -D_0AD(1/\varrho_{\nu-1})D_0y_\nu, \quad \nu = 1, \dots, N-1, \\ y_{N-1} &= -D_0AD_0y_N. \end{aligned}$$

Now fix an index $\nu$. Then we have two systems of equations

$$Cy_\nu = (-1)^{N-\nu}y_N, \quad B'y_\nu = (-1)^\nu y_0, \tag{2.2.8}$$

with
$$
\begin{aligned}
C &:= C_{N-\nu} := AD(\varrho_{N-1})AD(\varrho_{N-2})\dots AD(\varrho_\nu)A,\\
B &:= B_\nu := AD(1/\varrho_0)AD(1/\varrho_1)\dots AD(1/\varrho_{\nu-1}),\\
B' &:= B'_\nu := D_0 B D_0.
\end{aligned}
\tag{2.2.9}
$$

2.2.3. *Linear system for $z_\nu$.* Now we rewrite formula (2.2.8) for our special null-spline $\sigma \in \mathbf{S}_{2k-2}(\Delta)$ defined in (1.4.1)–(1.4.4). For the sake of brevity, set

$$p := k-2.$$

Then the corresponding vectors are

$$z_\nu := (z_\nu^{(1)}, \dots, z_\nu^{(2p+1)}) \in \mathbf{R}^{2p+1}, \quad \nu = 0, \dots, N,$$

with

$$
\begin{aligned}
z_\nu^{(l)} &:= \frac{1}{l!}\sigma^{(l)}(t_\nu)\cdot|h_\nu|^{l-(p+1)}, \quad \nu = 0, \dots, N-1,\\
z_N^{(l)} &:= \frac{1}{l!}\sigma^{(l)}(t_N)\cdot|h_{N-1}|^{l-(p+1)}, \quad \nu = N.
\end{aligned}
$$

Moreover, by definition (1.4.2)–(1.4.4) of $\sigma$, we know that

$$
\begin{aligned}
z_0 &= \big(\underbrace{0, \dots, 0}_{p=k-2}, z_0^{(p+1)}, z_0^{(p+1)}, \dots, z_0^{(2p+1)}\big),\\
z_N &= \big(\underbrace{0, \dots, 0}_{p=k-2}, 1, z_n^{(p+1)}, \dots, z_n^{(2p+1)}\big).
\end{aligned}
$$

By (2.2.8), we have two systems of equations,

$$B' z_\nu = (-1)^\nu z_0, \quad C z_\nu = (-1)^{N-\nu} z_N,$$

or in view of the prescribed values of the first components of $z_0, z_N$,

$$
B' z_\nu = (-1)^\nu \begin{pmatrix} \left.\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}\right\} p=k-2 \\ z_0^{(p+1)} \\ z_0^{(p+2)} \\ \vdots \\ z_0^{(2p+1)} \end{pmatrix}, \qquad
C z_\nu = (-1)^{N-\nu} \begin{pmatrix} \left.\begin{matrix} 0 \\ \vdots \\ 0 \\ 1 \end{matrix}\right\} p+1=k-1 \\ z_N^{(p+2)} \\ \vdots \\ z_N^{(2p+1)} \end{pmatrix}, \qquad \nu > 0.
$$

According to the notation introduced in §2.1 the upper half of these equations could be written as

$$B'[\mathbf{p},:]\times z_\nu(:)=(-1)^\nu \left.\begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix}\right\}p, \qquad C[\mathbf{p+1},:]\times z_\nu(:)=(-1)^{N-\nu}\left.\begin{pmatrix}0\\ \vdots\\ 0\\ 1\end{pmatrix}\right\}p+1.$$

For $\nu=0$ we have

$$C_N z_0=(-1)^N z_N,$$

or

$$Cz_0:=C\times\begin{pmatrix}\left.\begin{matrix}0\\ \vdots\\ 0\end{matrix}\right\}p=k-2\\ z_0^{(p+1)}\\ z_0^{(p+2)}\\ \vdots\\ z_0^{(2p+1)}\end{pmatrix}=(-1)^N\begin{pmatrix}\left.\begin{matrix}0\\ \vdots\\ 0\\ 1\end{matrix}\right\}p+1=k-1\\ z_N^{(p+2)}\\ \vdots\\ z_N^{(2p+1)}\end{pmatrix}, \qquad \nu=0.$$

In terms of the unknowns $\tilde{z}_0:=\big(z_0^{(p+1)},z_0^{(p+2)},...,z_0^{(2p+1)}\big)$ and in our notation, the upper half of this system is equivalent to

$$C[\mathbf{p+1},\mathbf{p}']\times\tilde{z}_0=(-1)^N\left.\begin{pmatrix}0\\ \vdots\\ 0\\ 1\end{pmatrix}\right\}p+1.$$

In summary, we can form one system with a known right-hand side and obtain the following result.

THEOREM 2.2.1. *Let*

$$z_\nu:=\big(z_\nu^{(1)},...,z_\nu^{(2p+1)}\big),\quad z_\nu^{(l)}:=\frac{1}{l!}\sigma^{(l)}(t_\nu)\cdot|h_\nu|^{l-(p+1)},\quad \tilde{z}_0:=\big(z_0^{(p+1)},z_0^{(p+2)},...,z_0^{(2p+1)}\big).$$

*Then, the vector* $z_\nu\in\mathbf{R}^{2p+1}$ *is a solution to the system*

$$Mz_\nu=(-1)^{N-\nu}(\underbrace{0,...,0}_{p},\underbrace{0,...,0,1}_{p+1}),\quad M:=\begin{bmatrix}B'[\mathbf{p},:]\\ C[\mathbf{p+1},:]\end{bmatrix},\quad \nu>0, \tag{2.2.10}$$

*and the vector* $\tilde{z}_0\in\mathbf{R}^{p+1}$ *is a solution to the system*

$$M_0\tilde{z}_0=(-1)^N(\underbrace{0,...,0,1}_{p+1}),\quad M_0:=C[\mathbf{p+1},\mathbf{p}']. \tag{2.2.11}$$

### 2.3. First estimates for $z_\nu$

2.3.1. *Total positivity of the matrices $A, B, C$.* By definition (2.2.9),

$$\begin{aligned} C &:= C_{N-\nu} := AD_{\gamma_1} AD_{\gamma_2} \dots AD_{\gamma_{N-\nu}} A, \\ B &:= B_\nu := AD_{\delta_1} AD_{\delta_2} \dots AD_{\delta_\nu}, \\ B' &:= B'_\nu := D_0 B D_0, \end{aligned}$$

where $\gamma_s, \delta_s$ are some positive numbers.

LEMMA 2.3.1. *The matrix $A$ is totally positive.*

*Proof.* See e.g. [BS]. We present another proof in §3.2.2. □

LEMMA 2.3.2. *The matrices $B$ and $C$ are totally positive.*

*Proof.* By Lemma 2.3.1, the matrix $A$ is totally positive, and so is $D(\gamma)$, as a diagonal matrix with positive entries. By the CB-formula, the product of TP-matrices is a TP-matrix. □

LEMMA 2.3.3. *For any $\nu \in \mathbf{N}$, we have*

$$B'_\nu(i,j) = (-1)^{|i+j|} B_\nu(i,j). \tag{2.3.1}$$

*Proof.* By definition, we have

$$D_0 := \operatorname{diag}\lceil (-1)^l \rfloor,$$

and thus, by the CB-formula,

$$B'_\nu(i,j) = D_0(i,i) B_\nu(i,j) D_0(j,j).$$

But since

$$D_0(i,i) = (-1)^{|i|}, \quad D_0(j,j) = (-1)^{|j|},$$

the statement follows. □

2.3.2. *First estimate for $z_0$.*

THEOREM 2.3.4. *The solution $\tilde z_0 = \big(z_0^{(p+1)}, \dots, z_0^{(2p+1)}\big)^T$ to the problem*

$$M_0 \tilde z_0 = (\underbrace{0, \dots, 0, 1}_{p+1})^T, \quad M_0 := C[\mathbf{p+1}, \mathbf{p}'] \tag{2.3.2}$$

*satisfies the relation*

$$|z_0^{(l)}| = \frac{C(\mathbf{p}, \mathbf{p}^l)}{C(\mathbf{p+1}, \mathbf{p}')}, \quad l = p+1, \dots, 2p+1. \tag{2.3.3}$$

*Proof.* From (2.3.2) we infer

$$\tilde{z}_0 = \big(z_0^{(p+1)}, \dots, z_0^{(2p+1)}\big) = M_0^{-1} \cdot (\underbrace{0, \dots, 0, 1}_{p+1})^T = M_0^{-1}[\,:, p+1],$$

i.e., $\tilde{z}_0$ coincides with the last column of $M_0^{-1}$. By Cramer's rule, we obtain

$$z_0^{(l)} = \tilde{z}_0^{(l-p)} = M_0^{-1}[l-p, p+1] = (-1)^{l+1} \frac{\det M_0^{(l-p)}}{\det M_0},$$

where $M_0^{(l-p)}$ is the algebraic adjoint to the element $M_0[p+1, l-p]$. The formulas

$$\begin{aligned} &\det M_0^{(l-p)} := M_0(\backslash p+1, \backslash l-p) := M_0(\mathbf{p}, \backslash l-p) := C(\mathbf{p}, \mathbf{p}^l), \\ &\det M_0 := C(\mathbf{p+1}, \mathbf{p}') \end{aligned}$$

follow from definitions and prove the theorem. □

2.3.3. *First estimate for $z_\nu$.*

THEOREM 2.3.5. *The solution $z_\nu \in \mathbf{R}^{2p+1}$ to the problem*

$$M z_\nu = (\underbrace{0, \dots, 0, 1}_{2p+1})^T, \quad M := \begin{bmatrix} B'[\mathbf{p}, :] \\ C[\mathbf{p+1}, :] \end{bmatrix} \in \mathbf{R}^{(2p+1)\times(2p+1)} \tag{2.3.4}$$

*admits the estimate*

$$|z_\nu^{(l)}| \leqslant \max_{j \in \mathbf{J}^l} \frac{C(\mathbf{p}, j^l)}{C(\mathbf{p+1}, j')}. \tag{2.3.5}$$

*Proof.* (1) First we derive an expression for $z_\nu$. Note that

$$M := \begin{bmatrix} B'[\mathbf{p}, :] \\ C[\mathbf{p+1}, :] \end{bmatrix} := \begin{bmatrix} \left.\begin{matrix} B'[\mathbf{p}, :] \\ C[\mathbf{p}, :] \end{matrix}\right\} =: M[\mathbf{2p}, :] \\ C[p+1, :] \end{bmatrix}. \tag{2.3.6}$$

From (2.3.4) we infer that

$$z_\nu = M^{-1} \cdot (0, \dots, 0, 1)^T = M^{-1}[\,:, 2p+1],$$

i.e., the vector $z_\nu$ is equal to the last column of $M^{-1}$. By Cramer's rule we obtain

$$z_\nu^{(l)} = M^{-1}[l, 2p+1] = (-1)^{2p+1+l} \frac{\det M^{(l)}}{\det M}, \tag{2.3.7}$$

where $M^{(l)}$ is the algebraic adjoint to the element $M[2p+1, l]$, i.e.,

$$\det M^{(l)} := M(\backslash 2p+1, \backslash l) = M(\mathbf{2p}, \backslash l).$$

(2) Next we estimate $\det M^{(l)}$. Expanding the determinant $M(\mathbf{2p}, \backslash l)$ in (2.3.6) by Laplace expansion by minors (Lemma 2.1.3) of $B'(\mathbf{p}, \backslash l)$ and $C(\mathbf{p}, \backslash l)$, we obtain

$$\det M^{(l)} := M(\mathbf{2p}, \backslash l) = \sum_{j \in \mathbf{J}^l} (-1)^{\varepsilon_l(j)} B'(\mathbf{p}, j) C(\mathbf{p}, j^l),$$

where $\varepsilon_l(j)$ are some integers. From (2.3.1) it follows that

$$B'(\mathbf{p}, j) = (-1)^{\varepsilon(j)} B(\mathbf{p}, j)$$

for some integer $\varepsilon(j)$. Therefore

$$|\det M^{(l)}| \leqslant \sum_{j \in \mathbf{J}^l} B(\mathbf{p}, j) C(\mathbf{p}, j^l). \tag{2.3.8}$$

(3) We also need an expression for $\det M$. Expanding the determinant $\det M$ in (2.3.6) by Laplace expansion by minors (Lemma 2.1.3) of $B'$ and $C$, and using (2.3.1), we find

$$\begin{aligned} \det M &= \sum_{j \in \mathbf{J}} (-1)^{|\mathbf{p}+j|} M(\mathbf{p}, j) M(\mathbf{p}', j') \\ &:= \sum_{j \in \mathbf{J}} (-1)^{|\mathbf{p}+j|} B'(\mathbf{p}, j) C(\mathbf{p}+\mathbf{1}, j') \\ &= \sum_{j \in \mathbf{J}} B(\mathbf{p}, j) C(\mathbf{p}+\mathbf{1}, j'), \end{aligned}$$

i.e.,

$$\det M = \sum_{j \in \mathbf{J}} B(\mathbf{p}, j) C(\mathbf{p}+\mathbf{1}, j'). \tag{2.3.9}$$

(4) Now we are able to bound $z_\nu$. From (2.3.7)–(2.3.9), it follows that

$$|z_\nu^{(l)}| = \frac{|\det M^{(l)}|}{|\det M|} \leqslant \frac{\sum_{j \in \mathbf{J}^l} B(\mathbf{p}, j) C(\mathbf{p}, j^l)}{\sum_{j \in \mathbf{J}} B(\mathbf{p}, j) C(\mathbf{p}+\mathbf{1}, j')} \leqslant \frac{\sum_{j \in \mathbf{J}^l} B(\mathbf{p}, j) C(\mathbf{p}, j^l)}{\sum_{j \in \mathbf{J}^l} B(\mathbf{p}, j) C(\mathbf{p}+\mathbf{1}, j')}.$$

Applying Lemma 2.1.4 to the latter ratio we obtain

$$|z_\nu^{(l)}| \leqslant \max_{j \in \mathbf{J}^l} \frac{C(\mathbf{p}, j^l)}{C(\mathbf{p}+\mathbf{1}, j')}. \qquad \square$$

## 2.4. Properties of the matrices $C$

The orders of the minors of $C$ on the right-hand side of (2.3.3) and (2.3.5) differ by one. In this section we establish some relations between minors of $C$ which allow us to equalize these orders.

*Definition* 2.4.1. Define $F \in \mathbf{R}^{n\times n}$ as an anti-diagonal matrix with the only non-zero elements

$$F[i, n+1-i] = \binom{n+1}{i}^{-1}.$$

Recall that by definition (2.2.5)

$$D_0 := \lceil -1, +1, \dots \rfloor.$$

LEMMA 2.4.2. *There holds the equality*

$$C^{-1} = (D_0 F)^{-1} C^* (D_0 F). \tag{2.4.1}$$

*Proof.* Consider two null-splines $s_1, s_2$ of degree $n+1$ on $\Delta$,

$$s_1, s_2 \in S_{n+2}(\Delta), \quad s_1(t_\nu) = s_2(t_\nu) = 0 \quad \text{for all } t_\nu \in \Delta,$$

and the vectors $x_\nu, y_\nu \in \mathbf{R}^n$ of their normalized successive derivatives,

$$x_\nu^{(l)} := \frac{1}{l!} s_1^{(l)}(t_\nu) \cdot |h_\nu|^{l-n/2+1}, \quad y_\nu^{(l)} := \frac{1}{l!} s_2^{(l)}(t_\nu) \cdot |h_\nu|^{l-n/2+1}. \tag{2.4.2}$$

We proved in Lemma 1.7.3 the equality

$$G(s_1, s_2; x) := \sum_{l=0}^{n+1} (-1)^l s_1^{(l)}(x) s_2^{(n+1-l)}(x) = \mathrm{const}(s_1, s_2), \quad x \in [a, b]. \tag{2.4.3}$$

It follows, in particular, that

$$G(s_1, s_2; t_\nu) = G(s_1, s_2; t_N). \tag{2.4.4}$$

Notice that due to the null-values of $s_1, s_2$ on $\Delta$ we can omit in the sum (2.4.3) the terms corresponding to $l=0$ and $l=n+1$, i.e., we have

$$G(s_1, s_2; t_\nu) = \sum_{l=1}^{n} (-1)^l s_1^{(l)}(t_\nu) s_2^{(n+1-l)}(t_\nu).$$

Using the equalities (2.4.2) we may rewrite the latter expression in terms of the vectors $x, y$ as

$$\frac{1}{(n+1)!} G(s_1, s_2; t_\nu) = \sum_{l=1}^{n} (-1)^l \binom{n+1}{l}^{-1} x_\nu^{(l)} y_\nu^{(n+1-l)}. \tag{2.4.5}$$

With the help of matrices $D_0$ and $F$ one obtains

$$(-1)^l \binom{n+1}{l}^{-1} = (D_0 F)_{l,n+1-l}.$$

Hence,

$$(-1)^l \binom{n+1}{l} y_\nu^{(n+1-l)} = (D_0 F y_\nu)^{(l)},$$

so that (2.4.5) becomes

$$\frac{1}{(n+1)!} G(s_1, s_2; t_\nu) = (x_\nu, D_0 F y_\nu).$$

Now, from (2.4.4) we conclude that

$$(x_\nu, D_0 F y_\nu) = (x_N, D_0 F y_N). \tag{2.4.6}$$

Recall that we defined the matrix $C$ in (2.2.8)–(2.2.9) through the relations

$$(-1)^{N-\nu} x_N = C x_\nu, \quad (-1)^{N-\nu} y_N = C y_\nu.$$

Thus, from (2.4.6) it follows that

$$(x_\nu, D_0 F y_\nu) = (C x_\nu, D_0 F C y_\nu) = (x_\nu, C^* D_0 F C y_\nu).$$

Since we have not made any assumptions on $x_\nu, y_\nu$, the latter equality holds for any $x_\nu, y_\nu \in \mathbf{R}^n$. Hence,

$$D_0 F = C^* D_0 F C,$$

and therefore

$$C^{-1} = (D_0 F)^{-1} C^* (D_0 F). \qquad \square$$

LEMMA 2.4.3. *For any $i, j \in \mathbf{I}_{p,n}$, we have the equality*

$$C(i', j') = f[i, j] \cdot C(i^*, j^*), \tag{2.4.7}$$

*where*

$$f[i, j] := \frac{F(i, i^*)}{F(j, j^*)} := \frac{\prod_{s=1}^{p} \binom{n+1}{j_s}}{\prod_{s=1}^{p} \binom{n+1}{i_s}}.$$

*Proof.* From

$$C^{-1} = (D_0 F)^{-1} C^* (D_0 F) \tag{2.4.8}$$

it follows that $\det C = \det C^* = \det C^{-1}$, and since $C$ is a TP-matrix, we have

$$\det C = 1.$$

Therefore, by the inverse determinants identity (Lemma 2.1.2), we obtain

$$C(i', j') = (-1)^{|i+j|} C^{-1}(j, i). \tag{2.4.9}$$

To estimate the minor $C^{-1}(j,i)$ we apply the CB-formula to the right-hand side of (2.4.8). Since the matrix $D_0$ (resp. $F$) is diagonal (resp. anti-diagonal), it follows that

$$\begin{aligned} D_0(\alpha, \beta) \neq 0 \quad &\text{if and only if} \quad \alpha = \beta, \\ F(\alpha, \beta) \neq 0 \quad &\text{if and only if} \quad \alpha = \beta^*. \end{aligned}$$

Thus, the CB-formula gives the equality

$$C^{-1}(j,i) = F^{-1}(j, j^*) D_0^{-1}(j^*, j^*) C^*(j^*, i^*) D_0(i^*, i^*) F(i^*, i).$$

Due to the relations

$$\begin{aligned} D_0(\alpha^*, \alpha^*) &= (-1)^{|\alpha^*|} := (-1)^{(n+1)p - |\alpha|}, \\ F^{-1}(\alpha, \alpha^*) &= [F(\alpha, \alpha^*)]^{-1} = [F(\alpha^*, \alpha)]^{-1}, \\ C^*(\alpha, \beta) &= C(\beta, \alpha), \end{aligned}$$

the latter formula for $C^{-1}(j,i)$ is reduced to

$$C^{-1}(j,i) = (-1)^{-|i| - |j|} \frac{F(i, i^*)}{F(j, j^*)} C(i^*, j^*).$$

Combining this expression with (2.4.9) gives (2.4.7). □

LEMMA 2.4.4. *For any $p, n \in \mathbf{N}$ we have*

$$C(\mathbf{n} - \mathbf{p}, \mathbf{p}') = C(\mathbf{p}, \mathbf{p}^*), \tag{2.4.10}$$

*and there exist constants $c_n, c_n'$ such that*

$$c_n C(\mathbf{p}, j^*) \leqslant C(\mathbf{n} - \mathbf{p}, j') \leqslant c_n' C(\mathbf{p}, j^*) \quad \text{for all } j \in \mathbf{I}_{p,n}. \tag{2.4.11}$$

*Proof.* By definition,

$$\mathbf{p} := (1, ..., p) = (n-p+1, ..., n)^*,$$
$$\mathbf{n}-\mathbf{p} := (1, ..., n-p) = (n-p+1, ..., n)'.$$

Thus by (2.4.7) we obtain

$$C(\mathbf{n}-\mathbf{p}, j') = f[\mathbf{p}, j]\,C(\mathbf{p}, j^*). \tag{2.4.12}$$

The equality (2.4.10) follows now if we take $j=\mathbf{p}$, since $f[\mathbf{p},\mathbf{p}]=1$. The inequalities (2.4.11) follow with

$$c_n := \min\{f[\mathbf{p}, j] : 1 \leqslant p \leqslant n,\ j \in \mathbf{I}_{p,n}\},$$
$$c_n' := \max\{f[\mathbf{p}, j] : 1 \leqslant p \leqslant n,\ j \in \mathbf{I}_{p,n}\}. \qquad \square$$

With $n=2p+1$, Lemma 2.4.4 takes the following form.

LEMMA 2.4.5. *For any $p$ with $n=2p+1$ we have*

$$C(\mathbf{p+1}, \mathbf{p}') = C(\mathbf{p}, \mathbf{p}^*), \tag{2.4.13}$$

*and there exist constants $c_p, c_p'$ such that*

$$c_p C(\mathbf{p}, j^*) \leqslant C(\mathbf{p+1}, j') \leqslant c_p' C(\mathbf{p}, j^*) \quad \text{for all } j \in \mathbf{J}. \tag{2.4.14}$$

### 2.5. Second estimates for $z_\nu$

THEOREM 2.5.1. *The components of the vector $z_\nu$ satisfy the relations*

$$|z_0^{(l)}| = \frac{C(\mathbf{p}, \mathbf{p}^l)}{C(\mathbf{p}, \mathbf{p}^*)}, \quad l = p+1, ..., 2p+1, \tag{2.5.1}$$

$$|z_\nu^{(l)}| \leqslant c_p \max_{j \in \mathbf{J}^l} \frac{C(\mathbf{p}, j^l)}{C(\mathbf{p}, j^*)}, \quad l = 1, ..., 2p+1. \tag{2.5.2}$$

*Remark.* Since for $l=p+1$ we have $\mathbf{p}^l = \mathbf{p}^*$, it follows that

$$|z_0^{(p+1)}| = \frac{C(\mathbf{p}, \mathbf{p}^{p+1})}{C(\mathbf{p}, \mathbf{p}^*)} = 1,$$

in accordance with (1.7.4).

*Proof.* By Theorem 2.3.4 we have

$$|z_0^{(l)}| = \frac{C(\mathbf{p}, \mathbf{p}^l)}{C(\mathbf{p+1}, \mathbf{p}')}, \quad l = p+1, ..., 2p+1,$$

and by (2.4.13),

$$C(\mathbf{p+1}, \mathbf{p}') = C(\mathbf{p}, \mathbf{p}^*),$$

which implies the first equality (2.5.1).

Similarly, by Theorem 2.3.5 we have

$$|z_\nu^{(l)}| \leqslant \max_{j \in \mathbf{J}^l} \frac{C(\mathbf{p}, j^l)}{C(\mathbf{p+1}, j')}, \quad l = 1, ..., 2p+1,$$

and by (2.4.14),

$$C(\mathbf{p+1}, j') \geqslant c_p C(\mathbf{p}, j^*),$$

which leads to the second inequality. □

## 3. Proof of Theorem Z: final estimates for $z_\nu$

### 3.1. Preliminary remarks

To estimate the ratio

$$\frac{C(\mathbf{p}, i)}{C(\mathbf{p}, j)}$$

for specific $i, j \in \mathbf{J}$, in particular, for those given in (2.5.2), we may split the whole product

$$C := \prod_{r=1}^{N-\nu} [AD_{\gamma_r}] \cdot A$$

into two arbitrary parts,

$$C = KR_q, \quad R_q := \prod_{r=1}^{q} [AD_{\gamma_r}] \cdot A, \tag{3.1.1}$$

and use the CB-formula keeping the total positivity of the matrices involved in mind. This gives

$$\frac{C(\mathbf{p}, i)}{C(\mathbf{p}, j)} \leqslant \max_{\alpha \in \mathbf{J}} \frac{R_q(\alpha, i)}{R_q(\alpha, j)}, \tag{3.1.2}$$

so that it is sufficient to estimate $R_q(\alpha, i)/R_q(\alpha, j)$ for some $q$. Clearly, the smaller the number $q$ of the factors of $R_q$ in (3.1.1), the simpler the work to be done. It would be ideal if we could take

$$q = 0, \quad R_0 = A.$$

Unfortunately, $A$, though totally positive, is not strictly totally positive, i.e.,

$$A(\alpha,\beta)=0 \quad \text{for quite a lot of indices } \alpha,\beta\in\mathbf{J}.$$

But fortunately, $A$ is an oscillation matrix, and we prove in the next §3.2 that

$$A(\alpha,\beta)>0 \quad \text{if and only if} \quad \alpha_s\leqslant\beta_{s+1}. \tag{3.1.3}$$

As we show in §3.3 this implies that

$$R_{p-1}(\beta,i)>0 \quad \text{for all } \beta,i\in\mathbf{J}.$$

Thus, it suffices to estimate the ratio

$$\frac{Q(\beta,i)}{Q(\beta,j)}, \quad Q:=R_{p-1}:=\prod_{r=1}^{p-1}[AD_{\gamma_r}]\cdot A.$$

This will be done in §§3.6–3.8.

### 3.2. The matrices $S$ and $A$

3.2.1. *The matrix $S$.*

*Definition* 3.2.1. Set

$$S:=S_{n+2}:=\left\{\binom{j}{i}\right\}_{i,j=0}^{n+1}:=\left\{\binom{j-1}{i-1}\right\}_{i,j=1}^{n+2}. \tag{3.2.1}$$

*Example* 3.2.2.

$$S_2=\begin{pmatrix}1&1\\0&1\end{pmatrix},\quad S_3=\begin{pmatrix}1&1&1\\0&1&2\\0&0&1\end{pmatrix},\quad S_4=\begin{pmatrix}1&1&1&1\\0&1&2&3\\0&0&1&3\\0&0&0&1\end{pmatrix}.$$

LEMMA 3.2.3. *The matrix $S$ in (3.2.1) is a TP-matrix, i.e.,*

$$S(\alpha,\beta)\geqslant 0 \quad \textit{for all } \alpha,\beta\in\mathbf{I}_{p,n}. \tag{3.2.2}$$

*Moreover, we have*

$$S(\alpha,\beta)>0 \quad \textit{if and only if} \quad \alpha\leqslant\beta. \tag{3.2.3}$$

*Proof.* The first part (3.2.2) of the lemma, i.e., the total positivity of $S$, was already proved by Schoenberg [Scho]. We present an alternative proof by induction which gives (3.2.3) as well.

(1) Let $S_n$ be a TP-matrix (as it is for $n=2$). Since

$$\sum_{j'=2}^{j} \binom{j'-2}{i-2} = \binom{j-1}{i-1},$$

it follows that

$$S_{n+1} := \left\{ \binom{j-1}{i-1} \right\}_{i,j=1}^{n+1} = S'_{n+1} \cdot I_{n+1}, \tag{3.2.4}$$

where

$$S'_{n+1} = \left[ \begin{array}{c|c} 1 & 0 \ldots\ldots 0 \\ \hline 0 & \\ \vdots & S_n := \left\{ \binom{j'-2}{i-2} \right\}_{i,j'=2}^{n+1} \\ 0 & \end{array} \right], \qquad I_{n+1} = \begin{bmatrix} 1 & 1 & \ldots & 1 \\ 0 & 1 & \ldots & 1 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & 1 \end{bmatrix}. \tag{3.2.5}$$

The matrix $I_n$ is totally positive (all its minors are either 0 or 1), and hence, by the CB-formula and the induction hypothesis, the total positivity of $S_{n+1}$ follows.

(2) Let us prove (3.2.3).

(A) If

$$\alpha_s > \beta_s \quad \text{for some } s \in \{1, \ldots, p\},$$

then the entries of the matrix

$$T := S[\alpha, \beta],$$

which is a $(p \times p)$-submatrix of the lower triangular matrix $S$, satisfy

$$T[\lambda, \mu] = S[\alpha_\lambda, j_\mu] = 0, \quad \lambda \geqslant s \geqslant \mu.$$

Hence the rows $\{T[\lambda, :]\}_{\lambda=s}^{p}$ are linearly dependent, i.e.,

$$\det T := S(\alpha, \beta) = 0.$$

(B) Suppose that for any $\gamma, \delta \in \mathbf{I}_{p,n}$ we have the equivalence

$$S_n(\gamma, \delta) > 0 \quad \text{if and only if} \quad \gamma \leqslant \delta.$$

Now let

$$\alpha, \beta \in \mathbf{I}_{p,n+1}, \quad \alpha_s \leqslant \beta_s \quad \text{for all } s = 1, \ldots, p. \tag{3.2.6}$$

We assume also that $p \leqslant n$, since for $p=n+1$ by definition we have $\det S_{n+1}=1$. From (3.2.4) and (3.2.5), by the CB-formula, we conclude that

$$S_{n+1}\binom{\alpha_1,\dots,\alpha_p}{\beta_1,\dots,\beta_p} = \sum_\delta S'_{n+1}\binom{\alpha_1,\dots,\alpha_p}{\delta_1,\dots,\delta_p} I_{n+1}\binom{\delta_1,\dots,\delta_p}{\beta_1,\dots,\beta_p}. \tag{3.2.7}$$

We distinguish two cases.

(B1) If $\alpha_1>1$, then, by (3.2.6) we also have $\beta_1>1$. Hence

$$S'_{n+1}\binom{\alpha_1,\dots,\alpha_p}{\beta_1,\dots,\beta_p} = S_n\binom{\alpha_1-1,\dots,\alpha_p-1}{\beta_1-1,\dots,\beta_p-1}.$$

Taking from the sum (3.2.7) only one term with $\delta=\beta$ we obtain

$$\begin{aligned} S_{n+1}\binom{\alpha_1,\dots,\alpha_p}{\beta_1,\dots,\beta_p} &\geqslant S_n\binom{\alpha_1-1,\dots,\alpha_p-1}{\beta_1-1,\dots,\beta_p-1} I_{n+1}\binom{\beta_1,\dots,\beta_p}{\beta_1,\dots,\beta_p} \\ &= S_n\binom{\alpha_1-1,\dots,\alpha_p-1}{\beta_1-1,\dots,\beta_p-1} > 0, \end{aligned}$$

where the last inequality holds by the induction hypothesis.

(B2) If $\alpha_1=1$, then

$$S'_{n+1}\binom{1,\alpha_2,\dots,\alpha_p}{\beta_1,\beta_2,\dots,\beta_p} = \begin{cases} 0, & \text{if } \beta_1>1, \\ S_n\binom{\alpha_2-1,\dots,\alpha_p-1}{\beta_2-1,\dots,\beta_p-1}, & \text{if } \beta_1=1. \end{cases}$$

In this case taking from the sum (3.2.7) the term with

$$\begin{aligned} &\delta_1=1, \\ &\delta_s=\beta_s, \quad s\geqslant 2, \end{aligned}$$

we obtain

$$\begin{aligned} S_{n+1}\binom{1,\alpha_2,\dots,\alpha_p}{\beta_1,\beta_2,\dots,\beta_p} &\geqslant S_n\binom{\alpha_2-1,\dots,\alpha_p-1}{\beta_2-1,\dots,\beta_p-1} I_{n+1}\binom{1,\beta_2,\dots,\beta_p}{\beta_1,\beta_2,\dots,\beta_p} \\ &= S_n\binom{\alpha_2-1,\dots,\alpha_p-1}{\beta_2-1,\dots,\beta_p-1} > 0. \end{aligned}$$

□

3.2.2. *The matrix A.* The matrix $A$ was defined in (2.2.2). We recall this definition.

*Definition* 3.2.4. Set

$$A:=A_n:=(a_{ij})_{i,j=1}^n, \quad a_{ij}:=\binom{n+1}{i}-\binom{j}{i}. \tag{3.2.8}$$

*Example* 3.2.5.

$$A_2=\begin{pmatrix}2&1\\3&2\end{pmatrix}, \quad A_3=\begin{pmatrix}3&2&1\\6&5&3\\4&4&3\end{pmatrix}, \quad A_4=\begin{pmatrix}4&3&2&1\\10&9&7&4\\10&10&9&6\\5&5&5&4\end{pmatrix}.$$

LEMMA 3.2.6. *The matrix $A$ in (3.2.8) is a TP-matrix, i.e.,*

$$A(\alpha,\beta)\geqslant 0 \quad \textit{for all } \alpha,\beta\in \mathbf{I}_{p,n}. \tag{3.2.9}$$

*Moreover,*

$$A(\alpha,\beta)>0 \quad \textit{if and only if} \quad \alpha_s\leqslant \beta_{s+1} \textit{ for all } s=1,\dots,p-1. \tag{3.2.10}$$

*Proof.* The following considerations are due to [BS]. For the matrix $S$ defined in (3.2.1), consider the matrix $S^-$ obtained from $S$ by subtracting the last column of $S$ from all other columns. We have

$$S^- := \left\{\binom{j}{i}\right\}_{i,j=0}^{n+1} - \left\{\binom{n+1}{i}\right\}_{i=0}^{n+1} = \left(\begin{array}{c|c|c} 0 & 0 \;\cdots\cdots\cdots\; 0 & 1 \\ \hline \vdots & \left\{\binom{j}{i}-\binom{n+1}{i}\right\}_{i,j=1}^{n} =: -A_n & \vdots \\ \hline \vdots & \cdots\cdots\cdots\cdots & \vdots \end{array}\right).$$

This implies that for $\alpha,\beta\in\mathbf{I}_{p,n}$,

$$\begin{aligned} S\begin{pmatrix} 0,\alpha_1,\dots,\alpha_{p-1},\alpha_p \\ \beta_1,\beta_2,\dots,\beta_p,n+1\end{pmatrix} &= S^-\begin{pmatrix} 0,\alpha_1,\dots,\alpha_{p-1},\alpha_p \\ \beta_1,\beta_2,\dots,\beta_p,n+1\end{pmatrix} \\ &= (-1)^{(p+1)+1}\det(-A[\alpha,\beta]) \\ &= (-1)^{(p+1)+1}(-1)^pA(\alpha,\beta) \\ &= A(\alpha,\beta), \end{aligned}$$

i.e.,

$$S\begin{pmatrix} 0,\alpha_1,\dots,\alpha_{p-1},\alpha_p \\ \beta_1,\beta_2,\dots,\beta_p,n+1\end{pmatrix} = A\begin{pmatrix}\alpha_1,\dots,\alpha_p \\ \beta_1,\dots,\beta_p\end{pmatrix}.$$

By (3.2.2), $S$ is totally positive, and by (3.2.3) one has

$$S\begin{pmatrix} 0,\alpha_1,\dots,\alpha_{p-1},\alpha_p \\ \beta_1,\beta_2,\dots,\beta_p,n+1\end{pmatrix} > 0 \quad \text{if and only if} \quad \begin{cases} 0\leqslant\beta_1, \\ \alpha_s\leqslant\beta_{s+1} & \text{for all } s=1,\dots,p-1, \\ \alpha_p\leqslant n+1. \end{cases}$$

This is equivalent to (3.2.10), since the condition $\alpha,\beta\in\mathbf{I}_{p,n}$ implies that $\beta_1\geqslant 1$ and $\alpha_p\leqslant n$. □

### 3.3. The matrices $Q$

*Definition* 3.3.1. Set

$$Q_\gamma := AD_{\gamma_1} AD_{\gamma_2} \dots AD_{\gamma_{p-1}} A = \prod_{r=1}^{p-1} [AD_{\gamma_r}] \cdot A \tag{3.3.1}$$

where

$$A = A_{2p+1}, \quad D_{\gamma_r} := D(\gamma_r) := \operatorname{diag} \lceil |\gamma_r|^{-p}, \dots, |\gamma_r|^p \rfloor, \quad A, D_\gamma \in \mathbf{R}^{(2p+1)\times(2p+1)}. \tag{3.3.2}$$

In this section we establish a relation between indices $\beta, i, j \in \mathbf{J}$ of the form

$$\mathbf{E}_{[\beta,i]} \subset \mathbf{E}_{[\beta,j]},$$

which implies the estimate

$$Q_\gamma(\beta, i) \leqslant c_p Q_\gamma(\beta, j) \quad \text{for all } \gamma = (\gamma_1, \dots, \gamma_{p-1}) \in \mathbf{R}^{p-1}.$$

Here $c_p$ is a constant that is *independent* of $\gamma$, i.e., *independent* of the knot sequence (we recall that in (3.3.1) $\gamma_r$ stands for the local mesh ratio $\varrho_\nu = h_\nu / h_{\nu+1}$ with some $\nu$).

Let

$$\alpha^{(r)} \in \mathbf{J}, \quad r = 0, \dots, p,$$

be a sequence of indices with

$$\alpha^{(0)} := \beta, \quad \alpha^{(p)} := i.$$

From (3.3.1) and the CB-formula, we infer

$$Q_\gamma(\alpha^{(0)}, \alpha^{(p)}) = \sum_{\alpha^{(1)},\dots,\alpha^{(p-1)} \in \mathbf{J}} \left[ \prod_{r=1}^{p-1} A(\alpha^{(r-1)}, \alpha^{(r)}) D_{\gamma_r}(\alpha^{(r)}, \alpha^{(r)}) \right] A(\alpha^{(p-1)}, \alpha^{(p)}). \tag{3.3.3}$$

Since by definition (3.3.2) we have

$$D_{\gamma_r}(\alpha^{(r)}, \alpha^{(r)}) = \gamma_r^{\sum_{s=1}^p [\alpha_s^{(r)} - (p+1)]} = \gamma_r^{-p(p+1)} \cdot \gamma_r^{|\alpha^{(r)}|},$$

we may rewrite (3.3.3) as

$$\begin{aligned} Q_\gamma(\alpha^{(0)}, \alpha^{(p)}) \cdot \prod_{r=1}^{p-1} \gamma_r^{p(p+1)} &= \sum_{\alpha^{(1)},\dots,\alpha^{(p-1)} \in \mathbf{J}} \left[ \prod_{r=1}^{p-1} A(\alpha^{(r-1)}, \alpha^{(r)}) \gamma_r^{|\alpha^{(r)}|} \right] A(\alpha^{(p-1)}, \alpha^{(p)}) \\ &= \sum_{\alpha^{(1)},\dots,\alpha^{(p-1)} \in \mathbf{J}} \prod_{r=1}^{p} A(\alpha^{(r-1)}, \alpha^{(r)}) \prod_{r=1}^{p-1} \gamma_r^{|\alpha^{(r)}|}. \end{aligned} \tag{3.3.4}$$

By Lemma 3.2.6 the condition

$$A(\alpha^{(r-1)}, \alpha^{(r)}) > 0$$

is equivalent to the inequalities

$$\alpha_s^{(r-1)} \leqslant \alpha_{s+1}^{(r)}, \quad s = 1, \dots, p-1. \tag{3.3.5}$$

This means that in (3.3.4) we could restrict the sum to the non-vanishing minors of $A$, i.e., to the sequence of indices that satisfy (3.3.5) for all $r = 1, \dots, p$ simultaneously.

Set

$$c_\gamma := \prod_{r=1}^{p-1} |\gamma_r|^{p(p+1)}.$$

This is the factor on the left-hand side of (3.3.4) that is independent of $\beta$ and $i$. Then from (3.3.4) we obtain

$$c'_p \sum_{\alpha^{(1)}, \dots, \alpha^{(p-1)} \in \mathbf{J}_{[\beta,i]}} \prod_{r=1}^{p-1} |\gamma_r|^{|\alpha^{(r)}|} \leqslant c_\gamma Q_\gamma(\beta, i) \leqslant c''_p \sum_{\alpha^{(1)}, \dots, \alpha^{(p-1)} \in \mathbf{J}_{[\beta,i]}} \prod_{r=1}^{p-1} |\gamma_r|^{|\alpha^{(r)}|}, \tag{3.3.6}$$

where for a fixed $\beta =: \alpha^{(0)}$ and $i =: \alpha^{(p)}$, the sum is taken over the set $\mathbf{J}_{[\beta,i]}$ of sequences $(\alpha^{(r)})_{r=1}^{p-1}$ of indices $\alpha^{(r)} \in \mathbf{J}$ which satisfy the condition (3.3.5) simultaneously.

Precisely, we formulate

*Definition* 3.3.2. For given $\beta, i \in \mathbf{J}$, we set

$$\alpha^{(0)} := \beta, \quad \alpha^{(p)} := i.$$

Further, we write

$$\alpha := (\alpha^{(r)})_{r=1}^{p-1} \in \mathbf{J}_{[\beta,i]},$$

and we say that the sequence $\alpha$ is admissible for the pair $[\beta, i]$ if

$$\begin{gathered} \alpha^{(r)} \in \mathbf{J}, \quad r = 1, \dots, p-1, \\ \alpha_{s-1}^{(r-1)} \leqslant \alpha_s^{(r)}, \quad r = 1, \dots, p, \ s = 2, \dots, p. \end{gathered} \tag{3.3.7}$$

*Definition* 3.3.3. For given $\beta, i \in \mathbf{J}$, we write

$$\varepsilon := (\varepsilon_1, \dots, \varepsilon_{p-1}) \in \mathbf{E}_{[\beta,i]},$$

and we say that the path $\varepsilon$ is admissible for $[\beta, i]$, if there exists a sequence of indices

$$\alpha = (\alpha^{(1)}, \dots, \alpha^{(p-1)}) \in \mathbf{J}_{[\beta,i]}$$

such that

$$\varepsilon_r = |\alpha^{(r)}|, \quad r = 1, \ldots, p-1.$$

With such a definition, (3.3.6) becomes

$$c'_p \sum_{\varepsilon \in \mathbf{E}_{[\beta,i]}} \prod_{r=1}^{p-1} |\gamma_r|^{\varepsilon_r} \leqslant c_\gamma Q_\gamma(\beta, i) \leqslant c''_p \sum_{\varepsilon \in \mathbf{E}_{[\beta,i]}} \prod_{r=1}^{p-1} |\gamma_r|^{\varepsilon_r}, \tag{3.3.8}$$

where the sum is taken over all different paths $\varepsilon \in \mathbf{E}_{[\beta,i]}$.

Set

$$Q_{[\beta,i]}(\gamma) := \sum_{\varepsilon \in \mathbf{E}_{[\beta,i]}} \prod_{r=1}^{p-1} |\gamma_r|^{\varepsilon_r}. \tag{3.3.9}$$

The next lemma follows immediately.

LEMMA 3.3.4. *There exists a constant $c_p$ such that if*

$$\mathbf{E}_{[\beta,i]} \subset \mathbf{E}_{[\beta,j]}, \quad \beta, i, j \in \mathbf{J}, \tag{3.3.10}$$

*then for any $\gamma = (\gamma_1, \ldots, \gamma_{p-1})$ we have*

$$Q_{[\beta,i]}(\gamma) \leqslant Q_{[\beta,j]}(\gamma),$$

*and consequently*

$$Q_\gamma(\beta, i) \leqslant c_p Q_\gamma(\beta, j).$$

### 3.4. A further strategy

(1) The function

$$Q_{[\beta,i]}(\gamma) := \sum_{\varepsilon \in \mathbf{E}_{[\beta,i]}} \prod_{r=1}^{p-1} |\gamma_r|^{\varepsilon_r}$$

defined in (3.3.9) is a multivariate polynomial in $\gamma$. All the coefficients of this polynomial are equal to 1. We want to find whether, for special $i, j \in \mathbf{J}$, the inequality

$$Q_{[\beta,i]}(\gamma) \leqslant c_p Q_{[\beta,j]}(\gamma) \tag{3.4.1}$$

holds *for all* $\gamma \in \mathbf{R}_+^{p-1}$ (all $\gamma$'s are positive). The condition (3.3.10) in Lemma 3.3.4 provides, of course, this inequality, but we need to find a way to check its validity.

(2) A trivial necessary condition for the inequality (3.4.1) to be true is that

(A) the minimal degree of $Q_{[\beta,i]}(\gamma) \geqslant$ the minimal degree of $Q_{[\beta,j]}(\gamma)$,

(B) the maximal degree of $Q_{[\beta,i]}(\gamma) \leqslant$ the maximal degree of $Q_{[\beta,j]}(\gamma)$.

This gives rise to the minimal and the maximal paths which we define in §3.5. These paths are nothing but the corresponding degrees of the monomials in $Q_{[\beta,i]}$.

As we show in §3.5, the set of admissible paths $\varepsilon \in \mathbf{E}_{[\beta,i]}$ (i.e., the set of monomials of the polynomial $Q_{[\beta,i]}(\gamma)$) has the properties:

(a) the minimal path (degree) $\underline{\varepsilon}^{[\beta]}$ depends only on $\beta$,

(b) the maximal path (degree) $\bar{\varepsilon}^{[i]}$ depends only on $i$.

Hence, among the conditions (A) and (B), only (B) will remain under consideration.

(3) For two arbitrary multivariate polynomials, the condition (B) is not sufficient to provide (3.4.1). For example, for

$$P_1(x,y) := 1+x^2y, \quad P_2(x,y) := 1+x^3y^2,$$

$P_1$ can not be bounded by $(\text{const} \cdot P_2)$ for all positive values $x, y$. Therefore, we will prove in §3.6 that for our particular polynomials the condition (B) for the maximal degrees, or equivalently the condition

(B′) the maximal path $\bar{\varepsilon}^{[i]} \leqslant$ the maximal path $\bar{\varepsilon}^{[j]}$

for the maximal paths, implies that

$$\{\text{the set of all monomials of } Q_{[\beta,i]}\} \subset \{\text{the set of all monomials of } Q_{[\beta,j]}\}.$$

In the path terminology it looks like

$$\bar{\varepsilon}^{[i]} \leqslant \bar{\varepsilon}^{[j]} \quad \Rightarrow \quad \mathbf{E}_{[\beta,i]} \subset \mathbf{E}_{[\beta,j]}.$$

Then, by (3.3.10), the inequality (3.4.1) trivially follows.

(4) To prove the last implication, we establish in §3.6 a criterion for the inclusion

$$\gamma^\varepsilon := \gamma_1^{\varepsilon_1} \dots \gamma_r^{\varepsilon_r} \in Q_{[\beta,i]}(\gamma), \quad \text{or equivalently,} \quad \varepsilon \in \mathbf{E}_{[\beta,i]}.$$

With $Q_{[\beta,\omega]}$ being the polynomial of the highest maximal degree $\omega$ (with the highest maximal path $\varepsilon^{[\omega]}$), the criterion is

$$\gamma^\varepsilon \in Q_{[\beta,\omega]}(\gamma), \ \varepsilon \leqslant \bar{\varepsilon}^{[i]} \quad \Leftrightarrow \quad \gamma^\varepsilon \in Q_{[\beta,i]}(\gamma).$$

In words, a monomial $\gamma^\varepsilon$ belongs to the polynomial $Q_{[\beta,i]}(\gamma)$ if and only if

(i) it belongs to the highest polynomial $Q_{[\beta,\omega]}(\gamma)$,

(ii) its degree $\varepsilon$ does not exceed the maximal degree $\bar{\varepsilon}^{[i]}$ of the polynomial $Q_{[\beta,i]}(\gamma)$.

In the path terminology this can be rephrased as

$$\varepsilon \in \mathbf{E}_{[\beta,\omega]},\ \varepsilon \leqslant \bar{\varepsilon}^{[i]} \quad \Leftrightarrow \quad \varepsilon \in \mathbf{E}_{[\beta,i]}.$$

Only sufficiency needs to be proved, i.e. the implication $\Rightarrow$.

(5) The latter will be proved by the iterative use of the following "elementary" step: for any $i'$ which differs from $i$ only in one component $i_m$, the same implication holds:

$$\varepsilon \in \mathbf{E}_{[\beta,i']},\ \varepsilon \leqslant \bar{\varepsilon}^{[i]} \quad \Rightarrow \quad \varepsilon \in \mathbf{E}_{[\beta,i]}.$$

All of §3.6 is devoted to the proof of this latter statement.

(a) We have a path $\varepsilon' \in \mathbf{E}_{[\beta,i']}$ (a monomial $\gamma^{\varepsilon'} \in Q_{[\beta,i']}(\gamma)$) with $\varepsilon' \leqslant \bar{\varepsilon}^{[i]}$.

(b) It is defined by a sequence $(\alpha'^{(r)}) \in \mathbf{J}_{[\beta,i']}$ with $|\alpha'^{(r)}| = \varepsilon'_r$.

(c) Since $i' \geqslant i$, this sequence may not be admissible for $[\beta, i]$.

(d) But we can modify it to a sequence $(\alpha''^{(r)})$ such that $(\alpha''^{(r)}) \in \mathbf{J}_{[\beta,i]}$ and $|\alpha''^{(r)}| = \varepsilon'_r$.

These modifications are treated in Lemmas 3.6.1–3.6.3. The statements of these lemmas are summarized then in Lemmas 3.6.4–3.6.5.

### 3.5. Minimal and maximal paths

In this section we define the minimal and the maximal admissible sequences $\underline{\alpha}^{(r)}, \bar{\alpha}^{(r)} \in \mathbf{J}_{[\beta,j]}$, and respectively the minimal and the maximal paths $\underline{\varepsilon}^{[\beta]}, \bar{\varepsilon}^{[j]} \in \mathbf{E}_{[\beta,j]}$.

We start with examples of what the admissible sequences $(\alpha^{(r)}) \in \mathbf{J}_{[\beta,i]}$ look like. According to definition (3.3.7) we have two strings of inequalities,

$$\begin{gathered} 1 \leqslant \alpha^{(r)}_{s-1} < \alpha^{(r)}_s < 2p+1, \quad r = 1, \dots, p-1,\ s = 2, \dots, p, \\ \alpha^{(r-1)}_{s-1} \leqslant \alpha^{(r)}_s, \quad r = 1, \dots, p,\ s = 2, \dots, p. \end{gathered}$$

In order to analyse these strings, we will frequently express them in the following matrix form.

*Example* 3.5.1. (1) $p=2$, $(\alpha^{(1)}) \in \mathbf{J}_{[\beta,i]}$:

$$\begin{bmatrix} & & \overset{\alpha^{(1)}}{\downarrow} & & i_1 \\ & & \alpha^{(1)}_1 & \leqslant & i_2 \\ \beta_1 & \leqslant & \alpha^{(1)}_2 & & \\ \beta_2 & & & & \end{bmatrix}.$$

(2) $p=3$, $(\alpha^{(1)},\alpha^{(2)})\in\mathbf{J}_{[\beta,i]}$:

$$\begin{bmatrix}
 & & & & \overset{\alpha^{(2)}}{\downarrow} & & i_1 \\
 & & \overset{\alpha^{(1)}}{\downarrow} & & \alpha_1^{(2)} & \leqslant & i_2 \\
 & & \alpha_1^{(1)} & \leqslant & \alpha_2^{(2)} & \leqslant & i_3 \\
\beta_1 & \leqslant & \alpha_2^{(1)} & \leqslant & \alpha_3^{(2)} & & \\
\beta_2 & \leqslant & \alpha_3^{(1)} & & & & \\
\beta_3 & & & & & &
\end{bmatrix}.$$

(3) Arbitrary $p$, $(\alpha^{(1)},\dots,\alpha^{(p-1)})\in\mathbf{J}_{[\beta,i]}$:

$$\begin{bmatrix}
 & & & & & & & & & & \overset{\alpha^{(p-1)}}{\downarrow} & & i_1 \\
 & & & & & & & & \overset{\alpha^{(p-2)}}{\downarrow} & & \alpha_1^{(p-1)} & \leqslant & i_2 \\
 & & & & \overset{\alpha^{(2)}}{\downarrow} & & & & \alpha_1^{(p-2)} & \leqslant & \alpha_2^{(p-1)} & \leqslant & i_3 \\
 & & \overset{\alpha^{(1)}}{\downarrow} & & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
 & & \alpha_1^{(1)} & \leqslant & \alpha_2^{(2)} & \leqslant & \dots & \leqslant & \alpha_{p-2}^{(p-2)} & \leqslant & \alpha_{p-1}^{(p-1)} & \leqslant & i_p \\
\beta_1 & \leqslant & \alpha_2^{(1)} & \leqslant & \alpha_3^{(2)} & \leqslant & \dots & \leqslant & \alpha_{p-1}^{(p-2)} & \leqslant & \alpha_p^{(p-1)} & & \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & & & \\
\beta_{p-2} & \leqslant & \alpha_{p-1}^{(1)} & \leqslant & \alpha_p^{(2)} & & & & & & & & \\
\beta_{p-1} & \leqslant & \alpha_p^{(1)} & & & & & & & & & & \\
\beta_p & & & & & & & & & & & &
\end{bmatrix}.$$

In such a representation, each column is an index from $\mathbf{J}$, i.e., the following "vertical" inequalities are also valid:

$$1\leqslant\alpha_1^{(r)}<\dots<\alpha_p^{(r)}\leqslant 2p+1. \tag{3.5.1}$$

In particular, it follows that

$$s\leqslant\alpha_s^{(r)}\leqslant p+1+s, \quad s=1,\dots,p. \tag{3.5.2}$$

LEMMA 3.5.2. *For any $\beta, i \in \mathbf{J}$ the set $\mathbf{J}_{[\beta,i]}$ is non-empty.*

*Proof.* The following sequence $(\alpha^{(r)})$ is always admissible:

$$
\left[\begin{array}{ccccccccccccc}
 & & & & & & & & & & \alpha^{(p-1)} & & \\
 & & & & & & & & & & \downarrow & & i_1 \\
 & & & & & & & & \alpha^{(p-2)} & & & & \\
 & & & & & & & & \downarrow & & 1 & < & i_2 \\
 & & & & \alpha^{(2)} & & & & & & & & \\
 & & & & \downarrow & & & & 1 & < & 2 & < & i_3 \\
 & & \alpha^{(1)} & & & & & & & & & & \\
 & & \downarrow & & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
 & & 1 & < & 2 & < & \ldots & < & p-2 & < & p-1 & < & i_p \\
\beta_1 & < & p+3 & < & p+4 & < & \ldots & < & 2p & < & 2p+1 & & \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & & & \\
\beta_{p-2} & < & 2p & < & 2p+1 & & & & & & & & \\
\beta_{p-1} & < & 2p+1 & & & & & & & & & & \\
\beta_p & & & & & & & & & & & &
\end{array}\right].
$$

□

LEMMA 3.5.3. *For any $\beta, i \in \mathbf{J}$, and any $(\alpha^{(r)}) \in \mathbf{J}_{[\beta,i]}$, we have*

$$\alpha^{(r)} \leqslant \bar{\alpha}^{(r)}, \tag{3.5.3}$$

*where*

$$\bar{\alpha}_s^{(r)} = \begin{cases} \min(i_{p-r+s}, p+1+s), & s \leqslant r, \ r = 1, \ldots, p-1, \\ p+1+s, & s > r, \ r = 1, \ldots, p-1. \end{cases} \tag{3.5.4}$$

*Proof.* In view of (3.5.2), Table 1 presents the admissible sequence $(\bar{\alpha}^{(r)})$ whose entries take the maximal possible values. □

LEMMA 3.5.4. *For any $\beta, i \in \mathbf{J}$, and any $(\alpha^{(r)}) \in \mathbf{J}_{[\beta,i]}$, we have*

$$\underline{\alpha}^{(r)} \leqslant \alpha^{(r)}, \tag{3.5.5}$$

*where*

$$\underline{\alpha}_s^{(r)} = \begin{cases} s, & s \leqslant r, \ r = 1, \ldots, p-1, \\ \max(\beta_{s-r}, s), & s > r, \ r = 1, \ldots, p-1. \end{cases} \tag{3.5.6}$$

*Proof.* In view of (3.5.2), Table 2 presents the admissible sequence $(\underline{\alpha}^{(r)})$ whose entries take the minimal possible values. □

$$
\left[
\begin{array}{cccccccccc}
 & & & & & & & & \overset{\bar{\alpha}^{(p-1)}}{\downarrow} & i_1 \\
 & & & & & & \overset{\bar{\alpha}^{(p-2)}}{\downarrow} & & \min(i_2,p+2) & \leqslant i_2 \\
 & & & & \overset{\bar{\alpha}^{(2)}}{\downarrow} & & \min(i_3,p+2) & \leqslant & \min(i_3,p+3) & \leqslant i_3 \\
 & & \overset{\bar{\alpha}^{(1)}}{\downarrow} & & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots & \cdots\cdots & \cdots \\
 & & \min(i_p,p+2) & \leqslant & \min(i_p,p+3) & \leqslant \ldots \leqslant & \min(i_p,2p-1) & \leqslant & \min(i_p,2p) & \leqslant i_p \\
\beta_1 & < & p+3 & < & p+4 & < \ldots < & 2p & < & 2p+1 & \\
\cdots & \cdots & \cdots\cdots & \cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & & & \\
\beta_{p-2} & < & 2p & < & 2p+1 & & & & & \\
\beta_{p-1} & < & 2p+1 & & & & & & & \\
\beta_p & & & & & & & & &
\end{array}
\right]
$$

Table 1. The maximal sequence $(\bar{\alpha}^{(r)})$.

$$
\left[
\begin{array}{cccccccccc}
 & & & & & & & & \overset{\underline{\alpha}^{(p-1)}}{\downarrow} & i_1 \\
 & & & & & & \overset{\underline{\alpha}^{(p-2)}}{\downarrow} & & 1 & < i_2 \\
 & & & & \overset{\underline{\alpha}^{(2)}}{\downarrow} & & 1 & < & 2 & < i_3 \\
 & & \overset{\underline{\alpha}^{(1)}}{\downarrow} & & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots & \cdots\cdots & \cdots \\
 & & 1 & < & 2 & < \ldots < & p-2 & < & p-1 & < i_p \\
\beta_1 & \leqslant & \max(\beta_1,2) & \leqslant & \max(\beta_1,3) & \leqslant \ldots \leqslant & \max(\beta_1,p-1) & \leqslant & \max(\beta_1,p) & \\
\cdots & \cdots & \cdots\cdots & \cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & & & \\
\beta_{p-2} & \leqslant & \max(\beta_{p-2},p-1) & \leqslant & \max(\beta_{p-2},p) & & & & & \\
\beta_{p-1} & \leqslant & \max(\beta_{p-1},p) & & & & & & & \\
\beta_p & & & & & & & & &
\end{array}
\right]
$$

Table 2. The minimal sequence $(\underline{\alpha}^{(r)})$.

*Definition* 3.5.5. For $\beta, i \in \mathbf{J}$ define the maximal path $\bar{\varepsilon}^{[i]}$ and the minimal path $\underline{\varepsilon}^{[\beta]}$ as follows:

$$\bar{\varepsilon}^{[i]} \in \mathbf{E}_{[\beta,i]}, \quad \bar{\varepsilon}_r^{[i]} := |\bar{\alpha}^{(r)}| = \sum_{s=1}^{r} \min(i_{p-r+s}, p+1+s) + \sum_{s=r+1}^{p} (p+1+s), \tag{3.5.7}$$

$$\underline{\varepsilon}^{[\beta]} \in \mathbf{E}_{[\beta,i]}, \quad \underline{\varepsilon}_r^{[\beta]} := |\underline{\alpha}^{(r)}| = \sum_{s=1}^{r} s + \sum_{s=r+1}^{p} \max(\beta_{s-r}, s). \tag{3.5.8}$$

LEMMA 3.5.6. *For any $\beta, i \in \mathbf{J}$, we have*

$$\underline{\varepsilon}^{[\beta]} \leqslant \varepsilon \leqslant \bar{\varepsilon}^{[i]} \quad \textit{for all } \varepsilon \in \mathbf{E}_{\beta,i}. \tag{3.5.9}$$

*Proof.* Follows directly from Lemmas 3.5.3–3.5.4 and Definition 3.5.5. □

### 3.6. Characterization of $\mathbf{E}_{[\beta,i]}$

Here we will prove the equality

$$\mathbf{E}_{[\beta,i]} = \{\varepsilon \in \mathbf{E}_{[\beta,\omega]} : \varepsilon \leqslant \bar{\varepsilon}^{[i]}\} \quad \text{for all } \beta, i \in \mathbf{J},$$

where $\omega := (p+2, ..., 2p+1)$ is the index from $\mathbf{J}$ with maximal possible entries. The latter will be proved by the iterative use of the following "elementary" step: for any $i'$ that differs from $i$ only in one component $i_m$, the same implication holds:

$$\varepsilon \in \mathbf{E}_{[\beta,i']}, \ \varepsilon \leqslant \bar{\varepsilon}^{[i]} \quad \Rightarrow \quad \varepsilon \in \mathbf{E}_{[\beta,i]}.$$

In this section exclusively, for $i \in \mathbf{J}$ we denote by $i', i'' \in \mathbf{J}$ some modifications of $i$ which have nothing to do with unfortunately the same notation for the complementary index.

LEMMA 3.6.1. *For any given $m \in \{1, ..., p\}$, let $i, i' \in \mathbf{J}$ be such that*

$$\begin{aligned} i'_s &= i_s, \quad s \neq m, \\ i'_m &= i_m + 1. \end{aligned}$$

*If for a given $\beta \in \mathbf{J}$ we have*

$$\varepsilon' \in \mathbf{E}_{[\beta,i']}, \quad \varepsilon' \leqslant \bar{\varepsilon}^{[i]},$$

*then for the same $\beta$ there exists a path $\varepsilon$, and a number $l \in \{1, ..., p\}$, such that*

$$\varepsilon \in \mathbf{E}_{[\beta,i]}, \quad \varepsilon_r = \begin{cases} \varepsilon'_r, & r = 1, ..., l-1, \\ \varepsilon'_r - 1, & r = l, ..., p-1. \end{cases} \tag{3.6.1}$$

*Proof.* Let

$$\varepsilon' \in \mathbf{E}_{[\beta,i']}, \quad \varepsilon' \leqslant \bar{\varepsilon}^{[i]}.$$

By definition, there exists a sequence $\alpha' \in \mathbf{J}_{[\beta,i']}$ which satisfies the inequalities

$$\begin{bmatrix} & & & & & & & & i_1 \\ & & & & & & & \alpha_1'^{(p-1)} \leqslant & i_2 \\ & & & & & & \cdots & \cdots & \cdots \\ & & & & & \alpha_1'^{(p-m+2)} \leqslant & \ldots \leqslant & \alpha_m'^{(p-1)} \leqslant & i_{m-1} \\ & & & & \alpha_1'^{(p-m+1)} \leqslant & \alpha_2'^{(p-m+2)} \leqslant & \ldots \leqslant & \alpha_{m-1}'^{(p-1)} \leqslant & i_m+1 \\ & & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & \alpha_1'^{(1)} \leqslant & \ldots \leqslant & \alpha_{p-m+1}'^{(p-m+1)} \leqslant & \alpha_{p-m+2}'^{(p-m+2)} \leqslant & \ldots \leqslant & \alpha_{p-1}'^{(p-1)} \leqslant & i_p & \\ \beta_1 \leqslant & \alpha_2'^{(1)} \leqslant & \ldots \leqslant & \alpha_{p-m+2}'^{(p-m+1)} \leqslant & \alpha_{p-m+3}'^{(p-m+2)} \leqslant & \ldots \leqslant & \alpha_p'^{(p-1)} & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & & & & \\ \beta_{p-1} \leqslant & \alpha_p'^{(1)} & & & & & & & \\ \beta_p & & & & & & & & \end{bmatrix}, \tag{3.6.2}$$

and moreover

$$|\alpha'^{(r)}| = \varepsilon' \leqslant \varepsilon_r^{[i]}, \quad r = 1, \ldots, p-1.$$

To produce a required sequence $\alpha \in \mathbf{J}_{[\beta,i]}$, we change the values of the components of $\alpha' \in \mathbf{J}_{[\beta,i']}$ only in the $m$th row:

$$\alpha_1'^{(p-m+1)} \leqslant \ldots \leqslant \alpha'^{(p-1)} \leqslant i_m' := i_m + 1.$$

For $\alpha'$'s in this row we have two possible relations.

(1) The first one is the inequality

$$\alpha_{m-1}'^{(p-1)} < i_m + 1.$$

Then

$$\alpha_1'^{(p-m+1)} \leqslant \ldots \leqslant \alpha_{m-1}'^{(p-1)} \leqslant i_m.$$

Therefore, $\alpha' \in \mathbf{J}_{[\beta,i]}$, hence

$$\varepsilon' \in \mathbf{E}_{[\beta,i]},$$

and (3.6.1) is satisfied with $l=p$, i.e., we do not have to do anything.

(2) The second possibility is that for some $t \in \{1, \ldots, m-1\}$ we have the relations

$$\alpha_1'^{(p-m+1)} \leqslant \ldots \leqslant \alpha_{m-1}'^{(p-m+t-1)} < \alpha_t'^{(p-m+t)} = \ldots = \alpha_{m-1}'^{(p-1)} = i_m + 1. \tag{3.6.3}$$

In this case we set

$$\begin{aligned} \alpha_s^{(p-m+s)} &:= \alpha_s'^{(p-m+s)} - 1 = i_m, \quad s = t, \ldots, m-1, \\ \alpha_s^{(r)} &:= \alpha_s'^{(r)}, \quad \text{otherwise}; \end{aligned} \tag{3.6.4}$$

thus, changing by $-1$ only the last $m-t$ entries of the $m$th row.

(2a) By such a definition, the second part of (3.6.1) holds evidently with $l=p-m+t$.

(2b) To show that $\varepsilon \in \mathbf{E}_{[\beta,i]}$, we need to prove that

$$\alpha \in \mathbf{J}_{[\beta,i]}.$$

Since the changes are restricted to the $m$th row we need to care only about the inequalities where the changed values are involved, i.e., about the inequalities

$$\begin{array}{ccccccccc} & & \alpha_{t-1}^{(p-m+t)} & & \cdots & & \alpha_{m-2}^{(p-1)} & & i_{m-1} \\ & & \wedge & & \wedge & & \wedge & & \\ \alpha_{t-1}^{(p-m+t-1)} & \leqslant & \alpha_t^{(p-m+t)} & \leqslant & \cdots & \leqslant & \alpha_{m-1}^{(p-1)} & \leqslant & i_m. \end{array} \tag{3.6.5}$$

(2c) From (3.6.3) and (3.6.4) it follows that in the $m$th row we have

$$\alpha_{t-1}^{(p-m+t-1)} \leqslant \alpha_t^{(p-m+t)} = \ldots = \alpha_{k-1}^{(p-1)} = i_m,$$

i.e., the "horizontal" inequalities in (3.6.5) are valid.

(2d) In the columns $(\alpha^{(p-m+s)})_{s=t}^{m-1}$ we have

$$\alpha_{s-1}^{(p-m+s)} := \alpha_{s-1}'^{(p-m+s)} \leqslant i_{m-1} < i_m =: \alpha_s^{(p-m+s)},$$

i.e., the "vertical" inequalities in (3.6.5) are also true. □

LEMMA 3.6.2. *For some $l \in \{1, \ldots, p-1\}$, let $\varepsilon$ be a path such that*

$$\varepsilon \in \mathbf{E}_{[\beta,i]}, \quad \varepsilon_r := \begin{cases} \varepsilon_r' \leqslant \bar{\varepsilon}_r^{[i]}, & r = 1, \ldots, l-1, \\ \varepsilon_r' - 1 < \bar{\varepsilon}_r^{[i]}, & r = l, \ldots, p-1. \end{cases} \tag{3.6.6}$$

*Then there exists an $l'' > l$ and a path*

$$\varepsilon'' \in \mathbf{E}_{[\beta,i]} \tag{3.6.7}$$

*such that*

$$\varepsilon_r'' = \begin{cases} \varepsilon_r' \leqslant \bar{\varepsilon}_r^{[i]}, & r = 1, \ldots, l''-1, \\ \varepsilon_r' - 1 < \bar{\varepsilon}_r^{[i]}, & r = l'', \ldots, p-1. \end{cases} \tag{3.6.8}$$

*Proof.* By definition, there exists a sequence $\alpha=\{\alpha^{(r)}\}$ such that

$$\alpha\in\mathbf{J}_{[\beta,i]},\quad |\alpha^{(r)}|=\varepsilon_r:=\begin{cases}\varepsilon_r'\leqslant\bar\varepsilon_r^{[i]}, & r=1,\dots,l-1,\\ \varepsilon_r'-1<\bar\varepsilon_r^{[i]}, & r=l,\dots,p-1.\end{cases}$$

We will change now by $+1$ a non-zero number $q+1$ of successive elements of $\alpha\in\mathbf{J}_{[\beta,i]}$ in a certain row starting from an element $\alpha_{s^*}^{(l)}$ in the $l$th column.

(A) By such a change the equality (3.6.8) holds automatically.

(B) The task is to find a starting element so that the new sequence $\alpha''$ would still be in $\mathbf{J}_{[\beta,i]}$. Since the changes are restricted to a certain row we need to care only about the inequalities where the changed values are involved, i.e., about the inequalities

$$\begin{array}{ccccccccc}\alpha_{s^*}''^{(l)} & \leqslant & \alpha_{s^*+1}''^{(l+1)} & \leqslant & \dots & \leqslant & \alpha_{s^*+q}''^{(l+q)} & \leqslant & \alpha_{s^*+q+1}^{(l+q+1)}\\ \wedge & & \wedge & & \wedge & & \wedge & & \\ \alpha_{s^*+1}''^{(l)} & & \alpha_{s^*+2}''^{(l+1)} & & \dots & & \alpha_{s^*+q+1}''^{(l+q)}. & & \end{array}\tag{3.6.9}$$

Consider the index $\alpha^{(l)}$. Since

$$\alpha_s^{(l)}\leqslant\bar\alpha_s^{(l)},\quad s=1,\dots,p,$$

and by assumption (3.6.6),

$$\sum_{s=1}^{p}\alpha_s^{(l)}:=|\alpha^{(l)}|<\bar\varepsilon_l^{[i]}:=|\bar\alpha^{(l)}|:=\sum_{s=1}^{p}\bar\alpha_s^{(l)},$$

there exists a number $s'$ such that

$$\alpha_{s'}^{(l)}<\bar\alpha_{s'}^{(l)}.$$

Set

$$s^*:=\max\{s\in\{1,\dots,p\}:\alpha_s^{(l)}<\bar\alpha_s^{(l)}\}.\tag{3.6.10}$$

(1) If $s^*=p$, then we set

$$\alpha_p^{(l)}=\alpha_p^{(l)}+1,$$

and the lemma is proved with $l''=l+1$.

(2) Let $s^*<p$. Then, by definition of $s^*$,

$$\alpha_{s^*}^{(l)}<\bar\alpha_{s^*}^{(l)}<\bar\alpha_{s^*+1}^{(l)}=\alpha_{s^*+1}^{(l)},$$

i.e.,

$$\alpha_{s^*}^{(l)}+1<\alpha_{s^*+1}^{(l)}.\tag{3.6.11}$$

Set

$$l'' = \max\{l+t \in \{l, \ldots, p-1\} : \alpha_{s^*}^{(l)} = \alpha_{s^*+t}^{(l+t)}\} + 1,$$

and let

$$l'' =: l+q+1, \quad q \in \{0, \ldots, p-1-l\}.$$

Then we have the following three possibilities for the position of $l''$ in the table.

(a) The case $l+q<p-1$, $s^*+q<p$:

$$\begin{bmatrix} \ldots \leqslant & \alpha_{s^*}^{(l)} & = \alpha_{s^*+1}^{(l+1)} & = \ldots = & \alpha_{s^*+q}^{(l+q)} & < \alpha_{s^*+q+1}^{(l+q+1)} \leqslant \ldots \\ \ldots \leqslant & \alpha_{s^*+1}^{(l)} & \leqslant \alpha_{s^*+2}^{(l+1)} & \leqslant \ldots \leqslant & \alpha_{s^*+q+1}^{(l+q)} & \leqslant \quad \ldots \end{bmatrix}.$$

(b) The case $(l+q)<(p-1)$, $s^*+q=p$:

$$\begin{bmatrix} \ldots \leqslant & \alpha_{s^*}^{(l)} & = \alpha_{s^*+1}^{(l+1)} & = \ldots = & \alpha_{p-1}^{(l+q-1)} & = \alpha_p^{(l+q)} \\ \ldots \leqslant & \alpha_{s^*+1}^{(l)} & \leqslant \alpha_{s^*+2}^{(l+1)} & \leqslant \ldots \leqslant & \alpha_p^{(l+q-1)} & \end{bmatrix}.$$

(c) The case $(l+q)=(p-1)$ (then $s^*+q=m-1<p$):

$$\begin{bmatrix} \ldots \leqslant & \alpha_{s^*}^{(l)} & = \alpha_{s^*+1}^{(l+1)} & = \ldots = & \alpha_{m-1}^{(p-1)} & \leqslant i_m \\ \ldots \leqslant & \alpha_{s^*+1}^{(l)} & \leqslant \alpha_{s^*+2}^{(l+1)} & \leqslant \ldots \leqslant & \alpha_m^{(p-1)} & \end{bmatrix}.$$

Set

$$\begin{aligned} \alpha_{s^*+t}^{\prime\prime(l+t)} &= \alpha_{s^*+t}^{(l+t)} + 1, \quad t = 0, \ldots, q, \\ \alpha_s^{\prime\prime(r)} &= \alpha_s^{(r)}, \quad \text{otherwise}; \end{aligned} \tag{3.6.12}$$

thus, increasing by $+1$ the elements in the upper row of the above subtables.

(2.1) Let us verify the "vertical" inequalities in (3.6.9). Since, by (3.6.11),

$$\alpha_{s^*}^{(l)} + 1 < \alpha_{s^*+1}^{(l)},$$

and since, for the upper and lower row of the above subtables, the relations

$$\alpha_{s^*+t}^{(l+t)} + 1 = \alpha_{s^*}^{(l)} + 1, \quad \alpha_{s^*+1}^{(l)} \leqslant \alpha_{s^*+t+1}^{(l+t)}, \quad t = 0, \ldots, q,$$

are valid, we have

$$\alpha_{s^*+t}^{(l+t)} + 1 = \alpha_{s^*}^{(l)} + 1 < \alpha_{s^*+1}^{(l)} \leqslant \alpha_{s^*+t+1}^{(l+t)},$$

i.e.,

$$\alpha_{s^*+t}^{(l+t)} + 1 < \alpha_{s^*+t+1}^{(l+t)}.$$

According to the definition (3.6.12), this gives

$$\alpha''^{(l+t)}_{s^*+t} := \alpha^{(l+t)}_{s^*+t} + 1 < \alpha^{(l+t)}_{s^*+t+1} =: \alpha''^{(l+t)}_{s^*+t+1}, \quad t = 0, \dots, q,$$

i.e.,

$$\alpha''^{(l+t)}_{s^*+t} < \alpha''^{(l+t)}_{s^*+t+1}, \quad t = 0, \dots, q.$$

This proves the "vertical" inequalities in (3.6.9).

(2.2) Let us prove the "horizontal" inequalities in (3.6.9). It is clear that, due to the equalities

$$\alpha^{(l)}_{s^*} = \alpha^{(l+1)}_{s^*+1} = \dots = \alpha^{(l+q)}_{s^*+q},$$

the definition (3.6.12) implies

$$\alpha''^{(l)}_{s^*} = \alpha''^{(l+1)}_{s^*+1} = \dots = \alpha''^{(l+q)}_{s^*+q}.$$

Also in the case (a) we have

$$\alpha''^{(l+q)}_{s^*+q} := \alpha^{(l+q)}_{s^*+q} + 1 \leqslant \alpha^{(l+q+1)}_{s^*+q+1} =: \alpha''^{(l+q+1)}_{s^*+q+1},$$

and that completes the "horizontal" part of (3.6.9) for this case.

Further, since by definition (3.6.10) we have

$$\alpha^{(l)}_{s^*} + 1 < \bar{\alpha}^{(l)}_{s^*},$$

it follows that

$$\alpha^{(l+t)}_{s^*+t} + 1 = \alpha^{(l)}_{s^*} + 1 \leqslant \bar{\alpha}^{(l)}_{s^*} \leqslant \bar{\alpha}^{(l+t)}_{s^*+t}.$$

This implies

$$\alpha''^{(l+t)}_{s^*+t} := \alpha^{(l+t)}_{s^*+t} + 1 \leqslant \bar{\alpha}^{(l+t)}_{s^*+t}.$$

i.e., the values of the modified $\alpha''$ lie in the admissible intervals. In particular, in the case (b),

$$\alpha''^{(l+q)}_{p} \leqslant \bar{\alpha}^{(l+q)}_{p} = 2p+1,$$

and in the case (c),

$$\alpha''^{(p-1)}_{m-1} \leqslant \bar{\alpha}^{(p-1)}_{m-1} = \min(p+m, i_m) \leqslant i_m.$$

This finishes the proof of the "horizontal" part of (3.6.9), and of the lemma. □

LEMMA 3.6.3. *For some $l\in\{1,...,p-1\}$, let $\varepsilon$ be a path such that*

$$\varepsilon\in\mathbf{E}_{[\beta,i]},\quad \varepsilon_r:=\begin{cases}\varepsilon'_r\leqslant\bar\varepsilon_r^{[i]}, & r=1,...,l-1,\\ \varepsilon'_r-1<\bar\varepsilon_r^{[i]}, & r=l,...,p-1.\end{cases}$$

*Then*

$$\varepsilon'\in\mathbf{E}_{[\beta,i]}.$$

*Proof.* An iterative use of Lemma 3.6.2. □

We summarize Lemmas 3.6.1–3.6.3 in the following two statements.

LEMMA 3.6.4. *For any given $m\in\{1,...,p\}$, let $i,i'\in\mathbf{J}$ be such that*

$$\begin{aligned} i'_s&=i_s,\quad s\neq m,\\ i'_m&=i_m+1.\end{aligned}$$

*If $\varepsilon'$ is a path such that*

$$\varepsilon'\in\mathbf{E}_{[\beta,i']},\quad \varepsilon'\leqslant\bar\varepsilon^{[i]},$$

*then*

$$\varepsilon'\in\mathbf{E}_{[\beta,i]}. \tag{3.6.13}$$

*Proof.* By Lemma 3.6.1, for such a path $\varepsilon'$, there exists a path $\varepsilon$, and a number $l\in\{1,...,p-1\}$, such that

$$\varepsilon\in\mathbf{E}_{[\beta,i]},\quad \varepsilon_r=\begin{cases}\varepsilon'_r, & r=1,...,l-1,\\ \varepsilon'_r-1, & r=l,...,p-1.\end{cases}$$

And, by Lemma 3.6.3, we have then the inclusion (3.6.13). □

LEMMA 3.6.5. *For any given $m\in\{1,...,p\}$, let $i,i'\in\mathbf{J}$ be such that*

$$\begin{aligned} i'_s&=i_s,\quad s\neq m,\\ i'_m&=i_m+1.\end{aligned}$$

*Then*

$$\mathbf{E}_{[\beta,i]}=\{\varepsilon\in\mathbf{E}_{[\beta,i']}:\varepsilon\leqslant\bar\varepsilon^{[i]}\}.$$

*Proof.* For $i,i'$ so defined, the inclusion

$$\{\varepsilon\in\mathbf{E}_{[\beta,i']}:\varepsilon\leqslant\bar\varepsilon^{[i]}\}\subset\mathbf{E}_{[\beta,i]}$$

is just a reformulation of Lemma 3.6.4. On the other hand, since $i \leqslant i'$, it is clear that

$$\mathbf{E}_{[\beta,i]} \subset \mathbf{E}_{[\beta,i']},$$

and it remains to recall that, by (3.5.9), for $\varepsilon \in \mathbf{E}_{[\beta,i]}$ we have $\varepsilon \leqslant \bar{\varepsilon}^{[i]}$. □

Set

$$\omega := (p+2, ..., 2p+1), \quad \omega \in \mathbf{J}.$$

Then $\omega$ is the index of $\mathbf{J}$ with the maximal possible entries, i.e.,

$$i \leqslant \omega \quad \text{for all } i \in \mathbf{J}.$$

PROPOSITION 3.6.6. *For any $\beta, i \in \mathbf{J}$, we have*

$$\mathbf{E}_{[\beta,i]} = \{\varepsilon \in \mathbf{E}_{[\beta,\omega]} : \varepsilon \leqslant \bar{\varepsilon}^{[i]}\}.$$

*Proof.* Since $i \leqslant \omega$, i.e.,

$$i_s \leqslant \omega_s, \quad s = 1, ..., p-1,$$

there exists a number $N$, a sequence of indices $(i^{(\nu)})_{\nu=0}^N$, and a sequence of numbers $(m_\nu)_{\nu=1}^N$, such that

$$i^{(0)} = i, \quad i^{(N)} = \omega,$$

and

$$i_s^{(\nu)} = \begin{cases} i_s^{(\nu-1)}, & s \neq m_\nu, \\ i_s^{(\nu-1)}+1, & s = m_\nu. \end{cases}$$

Since

$$i \leqslant i^{(1)} \leqslant ... \leqslant i^{(N-1)} \leqslant \omega,$$

we have

$$\bar{\varepsilon}^{[i]} \leqslant \bar{\varepsilon}^{[i^{(1)}]} \leqslant ... \leqslant \bar{\varepsilon}^{[i^{(N-1)}]} \leqslant \bar{\varepsilon}^{[\omega]},$$

and, by iterative use of Lemma 3.6.5, we obtain

$$\begin{aligned} \mathbf{E}_{[\beta,i]} &= \{\varepsilon \in \mathbf{E}_{[\beta,i^{(1)}]} : \varepsilon \leqslant \bar{\varepsilon}^{[i]}\} \\ &= \{\varepsilon \in \mathbf{E}_{[\beta,i^{(2)}]} : \varepsilon \leqslant \bar{\varepsilon}^{[i^{(1)}]}, \varepsilon \leqslant \bar{\varepsilon}^{[i]}\} \\ &= \{\varepsilon \in \mathbf{E}_{[\beta,i^{(2)}]} : \varepsilon \leqslant \bar{\varepsilon}^{[i]}\} \\ &= ... \\ &= \{\varepsilon \in \mathbf{E}_{[\beta,i^{(N-1)}]} : \varepsilon \leqslant \bar{\varepsilon}^{[i]}\} \\ &= \{\varepsilon \in \mathbf{E}_{[\beta,\omega]} : \varepsilon \leqslant \bar{\varepsilon}^{[i]}\}. \end{aligned}$$ □

PROPOSITION 3.6.7. *If*

$$\bar{\varepsilon}^{[i]} \leqslant \bar{\varepsilon}^{[j]}, \quad i,j \in \mathbf{J},$$

*then*

$$\mathbf{E}_{[\beta,i]} \subset \mathbf{E}_{[\beta,j]} \quad \textit{for all } \beta \in \mathbf{J}.$$

*Proof.* By Proposition 3.6.6, we have

$$\mathbf{E}_{[\beta,i]} = \{\varepsilon \in \mathbf{E}_{[\beta,\omega]} : \varepsilon \leqslant \bar{\varepsilon}^{[i]}\}, \quad \mathbf{E}_{[\beta,j]} = \{\varepsilon \in \mathbf{E}_{[\beta,\omega]} : \varepsilon \leqslant \bar{\varepsilon}^{[j]}\},$$

and it is clear that

$$\bar{\varepsilon}^{[i]} \leqslant \bar{\varepsilon}^{[j]} \quad \Rightarrow \quad \{\varepsilon \in \mathbf{E}_{[\beta,\omega]} : \varepsilon \leqslant \bar{\varepsilon}^{[i]}\} \subset \{\varepsilon \in \mathbf{E}_{[\beta,\omega]} : \varepsilon \leqslant \bar{\varepsilon}^{[j]}\}. \qquad \square$$

## 3.7. Relation between the minors of $Q$ and $C$

*Definition* 3.7.1. For $i,j \in \mathbf{J}$, we write

$$i \preceq j \quad \Leftrightarrow \quad \bar{\varepsilon}^{[i]} \leqslant \bar{\varepsilon}^{[j]}, \tag{3.7.1}$$

or, equivalently,

$$i \preceq j \quad \Leftrightarrow \quad \sum_{s=1}^{p-t} \min(i_{s+t}, p+1+s) \leqslant \sum_{s=1}^{p-t} \min(j_{s+t}, p+1+s), \quad t=1,\dots,p-1. \tag{3.7.2}$$

Let us show the equivalence. By Definition 3.5.7,

$$\bar{\varepsilon}^{[i]} \leqslant \bar{\varepsilon}^{[j]} \quad \Leftrightarrow \quad \sum_{s=1}^{r} \min(i_{p-r+s}, p+1+s) \leqslant \sum_{s=1}^{r} \min(j_{p-r+s}, p+1+s), \quad r=1,\dots,p-1. \tag{3.7.3}$$

To see that the inequalities (3.7.2) and (3.7.3) are equivalent, one should set $r=p-t$.

PROPOSITION 3.7.2. *For any $p \in \mathbf{N}$, there exists a constant $c_p$ such that if*

$$i,j \in \mathbf{J}, \quad i \preceq j,$$

*then*

$$Q(\beta,i) \leqslant c_p Q(\beta,j) \quad \textit{for all } \beta \in \mathbf{J}. \tag{3.7.4}$$

*Proof.* By Definition 3.7.1, by Lemma 3.6.7 and by Lemma 3.3.4, we have the implications

$$i \preceq j \;\Rightarrow\; \bar{\varepsilon}^{[i]} \leqslant \bar{\varepsilon}^{[j]} \;\Rightarrow\; \mathbf{E}_{[\beta,i]} \subset \mathbf{E}_{[\beta,j]} \;\Rightarrow\; Q(\beta,i) \leqslant c_p Q(\beta,j) \quad \text{for all } \beta \in \mathbf{J}. \qquad \square$$

PROPOSITION 3.7.3. *For any $p\in\mathbf{N}$, there exists a constant $c_p$ such that if*

$$i,j\in\mathbf{J},\quad i\preceq j, \tag{3.7.5}$$

*then for any $\nu\leqslant N-p+1$ we have*

$$C_{N-\nu}(\mathbf{p},i)\leqslant c_p C_{N-\nu}(\mathbf{p},j).$$

*Proof.* If $\nu\leqslant N-p+1$, then $N-1\geqslant\nu+p-2$ and we find that

$$C_{N-\nu}:=\prod_{s=\nu}^{N-1}[AD(\varrho_s)]\cdot A=K\cdot\prod_{s=\nu}^{\nu+p-2}[AD(\varrho_s)]\cdot A=K\cdot\prod_{s=1}^{p-1}[AD(\varrho_{\nu+s-1}]\cdot A=:K\cdot Q,$$

with some totally positive matrix $K$. By the CB-formula, making use of (3.7.4), we obtain

$$C_{N-\nu}(\mathbf{p},i)=\sum_{\beta\in\mathbf{J}}K(\mathbf{p},\beta)Q(\beta,i)\leqslant c_p\sum_{\beta\in\mathbf{J}}K(\mathbf{p},\beta)Q(\beta,j)=c_pC_{N-\nu}(\mathbf{p},j).\qquad\square$$

### 3.8. Index relations

3.8.1. *The statement.* Recall the definitions from §2.1:

$$\mathbf{2p+1}:=(1,\ldots,2p+1),\quad \mathbf{J}:=\{j\subset\mathbf{2p+1}:\#j=p\},$$
$$\mathbf{J}^l:=\{j\in\mathbf{J}:\{l\}\notin j\},\quad l=1,\ldots,2p+1.$$

For $i\in\mathbf{J}^l$ we defined its $l$-complement $i^l$ and its conjugate index $i^*$ as

$$i^l\in\mathbf{J}^l,\quad i^l=\mathbf{2p+1}\setminus\{l\}\setminus i,$$
$$i^*\in\mathbf{J},\quad i^*=(2p+2-i_p,\ldots,2p+2-i_1).$$

In this section we will prove

PROPOSITION 3.8.1. *Let $i\in\mathbf{J}^l$. Then*

$$i^{l_2}\preceq i^*\preceq i^{l_1},\quad l_1\leqslant p+1\leqslant l_2,$$

*or, equivalently,*

$$\sum_{s=1}^{p-t}\min(i^l_{s+t},p+1+s)\leqslant\sum_{s=1}^{p-t}\min(i^*_{s+t},p+1+s),\quad t=1,\ldots,p-1,\ l\geqslant p+1, \tag{3.8.1}$$

$$\sum_{s=1}^{p-t}\min(i^l_{s+t},p+1+s)\geqslant\sum_{s=1}^{p-t}\min(i^*_{s+t},p+1+s),\quad t=1,\ldots,p-1,\ l\leqslant p+1. \tag{3.8.2}$$

We will prove this statement in another equivalent formulation. It is clear that we may compare the sums of the shifted values

$$\min(\hat{\jmath}_{s+t}, s), \quad \hat{\jmath}_s := j_s - (p+1).$$

We define, therefore, the sets of the shifted indices

$$\pi_p := (-p, ..., p), \quad \mathbf{J}_p := \{j \subset \pi_p : \#j = p\},$$
$$\mathbf{J}_p^l := \{j \in \mathbf{J} : \{l\} \notin j\}, \quad l = -p, ..., p.$$

For $j \in \mathbf{J}_p^l$ its $l$-complement and conjugate index are defined respectively as

$$j^l \in \mathbf{J}_p^l, \quad j^l := \pi_p \backslash \{l\} \backslash j \qquad (3.8.3)$$
$$j^* \in \mathbf{J}_p, \quad j^* := -j.$$

For $j \in \mathbf{J}_p$ we set also

$$|j| := \sum_{s=1}^{p} j_s. \qquad (3.8.4)$$

Thus, Proposition 3.8.1 follows from

PROPOSITION 3.8.2. *Let* $i \in \mathbf{J}_p^l$. *Then*

$$i^{l_2} \preceq i^* \preceq i^{l_1}, \quad l_1 \leqslant 0 \leqslant l_2,$$

*or, equivalently,*

$$\sum_{s=1}^{p-t} \min(i_{s+t}^l, s) \leqslant \sum_{s=1}^{p-t} \min(i_{s+t}^*, s), \quad t = 0, ..., p-1, \ l \geqslant 0, \qquad (3.8.5)$$

$$\sum_{s=1}^{p-t} \min(i_{s+t}^l, s) \geqslant \sum_{s=1}^{p-t} \min(i_{s+t}^*, s), \quad t = 0, ..., p-1, \ l \leqslant 0. \qquad (3.8.6)$$

*Remark* 3.8.3. We have added also the inequalities with $t=0$.

Now we start with the proof of Proposition 3.8.2.

3.8.2. *Proof*: *the case* $l=0$.

*Definition* 3.8.4. Let any $p \in \mathbf{N}$ and any $j \in \mathbf{J}_p$ be given. For $t=0, ..., p-1$ define the indices

$$\begin{aligned} j^{[t]} &\in \mathbf{J}_{p-t}, \quad j_s^{[t]} := \min(j_{s+t}, s), && s = 1, ..., p-t, \\ j^{[-t]} &\in \mathbf{J}_{p-t}, \quad j_s^{[-t]} := \max(j_s, -(p-t)+(s-1)), && s = 1, ..., p-t. \end{aligned} \qquad (3.8.7)$$

$$\left[\begin{array}{llllll}
j^{[0]} := ( & j_1, & j_2, & \ldots, & j_{p-2}, & j_{p-1}, \quad j_p) \\
j^{[1]} := ( & \underbrace{\min(j_2,1)}_{j_1^{[1]}}, & \underbrace{\min(j_3,2)}_{j_2^{[1]}}, & \ldots, & \underbrace{\min(j_{p-1},p-2)}_{j_{p-2}^{[1]}}, & \underbrace{\min(j_p,p-1)}_{j_{p-1}^{[1]}}\,) \\
j^{[2]} := ( & \underbrace{\min(j_3,1)}_{j_1^{[2]}}, & \underbrace{\min(j_4,2)}_{j_2^{[2]}}, & \ldots, & \underbrace{\min(j_p,p-2)}_{j_{p-2}^{[2]}}\,) & \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
j^{[p-1]} := ( & \underbrace{\min(j_p,1)}_{j_1^{[p-1]}}\,) & & & &
\end{array}\right]$$

Table 3. The indices $j^{[t]}$.

Since the components of $j\in\mathbf{J}_p$ satisfy

$$-p+(s-1)\leqslant j_s\leqslant s, \tag{3.8.8}$$

we have

$$j^{[-0]}=j^{[0]}=j.$$

For $s=1,\ldots,p-t$, due to (3.8.8), we also have

$$-(p-t)\leqslant \min(j_{s+t},s)\leqslant p-t,$$
$$-(p-t)\leqslant \max(j_s,-(p-t)+(s-1))\leqslant p-t,$$

i.e., the inclusion $j^{[t]},j^{[-t]}\in\mathbf{J}_{p-t}$ in (3.8.7) really takes place.

Tables 3 and 4 show what the indices $j^{[t]}$ and $j^{[-t]}$ look like.

In notation (3.8.7) and (3.8.4), we have the equality

$$\sum_{s=1}^{p-t}\min(j_{s+t},s)=:|j^{[t]}|,$$

so that (for $l=0$) the statement (3.8.5) to be proved is

$$|(i^0)^{[t]}|=|(i^*)^{[t]}|,\quad t=0,\ldots,p-1,\ \text{for all } i\in\mathbf{J}_p^0. \tag{3.8.9}$$

LEMMA 3.8.5. *For any* $j\in\mathbf{J}_p$,

$$j^{[t+1]}=(j^{[t]})^{[1]},\quad j^{[-t-1]}=(j^{[-t]})^{[-1]},\quad t=0,\ldots,p-2. \tag{3.8.10}$$

*Proof.* Clear from Tables 3 and 4. □

$$
\left[\begin{array}{llllll}
j^{[0]} := (j_1, & j_2, & j_3, & \dots, & j_{p-1}, & j_p \quad ) \\
j^{[-1]} := & \big(\underbrace{\max(j_1,-p+1)}_{j_1^{[-1]}}, & \underbrace{\max(j_2,-p+2)}_{j_2^{[-1]}}, & \dots, & \underbrace{\max(j_{p-2},-2)}_{j_{p-2}^{[-1]}}, & \underbrace{\max(j_{p-1},-1)}_{j_{p-1}^{[-1]}}\big) \\
j^{[-2]} := & & \big(\underbrace{\max(j_1,-p+2)}_{j_1^{[-2]}}, & \dots, & \underbrace{\max(j_{p-3},-2)}_{j_{p-3}^{[-2]}}, & \underbrace{\max(j_{p-2},-1)}_{j_{p-2}^{[-2]}}\big) \\
\cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots & \cdots\cdots & \cdots\cdots \\
j^{[-p]} := & & & & & \big(\underbrace{\max(j_1,-1)}_{j_1^{[-p]}}\big)
\end{array}\right]
$$

Table 4. The indices $j^{[-t]}$.

LEMMA 3.8.6. *For any given $p$ and any $i \in \mathbf{J}_p^0$, we have*

$$i^{[-1]} \in \mathbf{J}_{p-1}^0, \tag{3.8.11a}$$

$$(i^{[-1]})^0 = (i^0)^{[1]}, \tag{3.8.11b}$$

$$(i^{[-1]})^* = (i^*)^{[1]}. \tag{3.8.11c}$$

*Proof.* We prove first the equalities (3.8.11a) and (3.8.11b). By definition, for $i \in \mathbf{J}_p^0$ we have

$$\#i = \#i^0 = p, \quad i \cup i^0 = \pi_p \backslash \{0\}, \quad i \cap i^0 = \varnothing.$$

Let

$$i_p =: q, \quad i_1^0 + p =: r.$$

Then we have two cases:

(1) $i_p = -1$,

(2) $i_p > 0$.

*Case* 1: $i_p = -1$. In this case $i_1^0 = 1$ and the only possible entries of $i$ and $i^0$ are

| $i_1$ | $i_2$ | ... | $i_p$ | | $i_1^0$ | ... | $i_{p-1}^0$ | $i_p^0$ |
|---|---|---|---|---|---|---|---|---|
| $-p$ | $-p+1$ | ... | $-1$ | 0 | 1 | ... | $p-1$ | $p$ |

.

In this case we have

$$i^{[-1]} = (-p+1, \dots, -1), \quad (i^0)^{[1]} = (1, \dots, p-1),$$

and the equalities (3.8.11a) and (3.8.11b) are evident.

*Case* 2: $i_p>0$. In this case $i_1^0<0$ and the entries of $i, i^0$ are located as follows:

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $i_1$ | ... | $i_r$ | $i_1^0$ | ... | | ... | $i_p$ | $i_{q+1}^0$ | ... | $i_p^0$ |
| $-p$ | ... | $-p+r-1$ | $-p+r$ | ... | $0$ | ... | $q$ | $q+1$ | ... | $p$ |

In this case,

$$i_s^{[-1]} := \max(i_s, -p+s) = \begin{cases} -p+s, & s=1,\dots,r, \\ i_s, & s=r+1,\dots,p-1; \end{cases}$$

$$(i^0)_s^{[1]} := \min(i_{s+1}, s) = \begin{cases} i_{s+1}^0, & s=1,\dots,q-1, \\ s, & s=q,\dots,p-1. \end{cases}$$

Briefly, it can be written as

$$i^{[-1]} = i \cup \{i_1^0\} \setminus \{-p\} \setminus \{i_p\}, \quad (i^0)^{[1]} = i^0 \cup \{i_p\} \setminus \{p\} \setminus \{i_1^0\}.$$

It follows that

$$i^{[-1]} \cap (i^0)^{[1]} = \varnothing, \quad i^{[-1]} \cup (i^0)^{[1]} = \pi_{p-1} \setminus \{0\},$$

what is equivalent to (3.8.11a) and (3.8.11b).

The equality (3.8.11c) is straightforward:

$$\begin{aligned}(i^{[-1]})_s^* &:= -i_{p-s}^{[-1]} := -\max(i_{p-s}, -(p-1)+(p-s-1)) = -\max(i_{p-s}, -s) \\ &= \min(-i_{p-s}, s) = \min(-i_{p+1-(s+1)}, s) =: \min(i_{s+1}^*, s) \\ &=: (i^*)_s^{[1]}.\end{aligned}$$

□

LEMMA 3.8.7. *For any* $p\in\mathbf{N}$, *any* $i\in\mathbf{J}_p^0$ *and any* $t=0,\dots,p-1$, *we have*
(a) $i^{[-t]}\in\mathbf{J}_{p-t}^0$,
(b) $(i^{[-t]})^0=(i^0)^{[t]}$,
(c) $(i^{[-t]})^*=(i^*)^{[t]}$.

*Proof.* Follows from Lemmas 3.8.5 and 3.8.6. □

LEMMA 3.8.8. *For any* $p\in\mathbf{N}$, *and any* $j\in\mathbf{J}_p^0$,

$$|j^0| = |j^*|.$$

*Proof.* Since $j\cup j^0=\pi_p\setminus\{0\}$, and $j^*=-j$, we have

$$|j|+|j^0| = |\pi_p| = 0, \quad |j|+|j^*| = 0,$$

i.e., $|j^0|=|j^*|$. □

Now we are ready to prove the case $l=0$ of Proposition 3.8.2.

LEMMA 3.8.9. *For any* $i\in\mathbf{J}_p^0$,

$$i^0 \asymp i^*, \tag{3.8.12}$$

*or, equivalently,*

$$\sum_{s=1}^{p-t}\min(i^0_{s+t},s)=\sum_{s=1}^{p-t}\min(i^*_{s+t},s), \quad t=0,\dots,p-1. \tag{3.8.13}$$

*Proof.* By Lemma 3.8.7, for any $i\in\mathbf{J}_p^0$ and any $t=0,\dots,p-1$, the index $j:=i^{[-t]}$ satisfies the relations

$$(i^0)^{[t]}=j^0, \quad (i^*)^{[t]}=j^*, \quad j\in\mathbf{J}^0_{p-t}.$$

By Lemma 3.8.8, we have

$$|j^*|=|j^0| \quad \text{for all } j\in\mathbf{J}^0_{p-t}.$$

Thus

$$|(i^0)^{[t]}|=|(i^*)^{[t]}|, \quad t=0,\dots,p-1,$$

and that is equivalent to (3.8.13). □

This finishes the proof of Proposition 3.8.2 for $l=0$.

3.8.3. *Proof*: *the case* $l\neq 0$. It is clear that the following implications are valid:
(a) $i\leqslant j \Rightarrow i\preceq j$,
(b) $i=j \Rightarrow i\asymp j$.

*Case* 1: $i\in\{\mathbf{J}_p^l\cap\mathbf{J}_p^0\}$. This is the case if $\{0\}\notin i$. Since for $i\in\mathbf{J}_p^l$ by definition (3.8.3) we have

$$i^l:=\pi_p\backslash i\backslash\{l\},$$

it is easy to see that

$$i^{l_2}\leqslant i^0\leqslant i^{l_1} \quad \text{if } l_1<0<l_2,$$

and respectively

$$i^{l_2}\prec i^0\prec i^{l_1} \quad \text{if } l_1<0<l_2.$$

Since $i\in\mathbf{J}_p^0$, we have by Lemma 3.8.9

$$i^0\asymp i^*,$$

and therefore,

$$i^{l_2}\prec i^*\prec i^{l_1} \quad \text{if } l_1<0<l_2,\ i\in\{\mathbf{J}_p^0\cap\mathbf{J}_p^{l_\nu}\}.$$

*Case* 2: $i\in\mathbf{J}_p^l$, $i\notin\mathbf{J}_p^0$. This is the case if $\{0\}\in i$. Then we have the inclusions

$$i^l\in\mathbf{J}_p^l,\quad i^l\in\mathbf{J}_p^0.$$

Set

$$j:=i\setminus\{0\}\cup\{l\}.$$

Then

$$(1)\begin{cases} j\in\mathbf{J}_p^0, \\ j^0=i^l, \end{cases}\qquad (2)\begin{cases} j<i, & l<0, \\ i<j, & l>0. \end{cases}\tag{3.8.14}$$

From the first part of these relations, by Lemma 3.8.9, it follows that

$$i^l\asymp j^0\asymp j^*.$$

From the second part one obtains

$$\begin{cases} i^*<j^* & \text{if } l<0, \\ j^*<i^* & \text{if } l>0, \end{cases}\quad\Rightarrow\quad\begin{cases} i^*\preceq j^* & \text{if } l<0, \\ j^*\preceq i^* & \text{if } l>0. \end{cases}$$

Thus,

$$i^{l_2}\preceq i^*\preceq i^{l_1}\quad \text{if } l_1<0<l_2,\ i\in\mathbf{J}_p^{l_\nu},\ i\notin\mathbf{J}_p^0.$$

Proposition 3.8.2, and hence Proposition 3.8.1, are proved.

### 3.9. Completion of the proof of Theorem Z

THEOREM Z (§1.9). *There exists a constant $c_p$ depending only on $p$ such that the inequalities*

$$\frac{1}{l!}|\sigma^{(l)}(t_\nu)|=:|z_\nu^{(l)}|\leqslant c_p,\quad l=p+1,\dots,2p+1,\ \nu=0,\dots,N-p+1,$$

*hold uniformly in $\nu,l$.*

*Proof.* By Theorem 2.5.1, we have

$$|z_\nu^{(l)}|\leqslant\max_{j\in\mathbf{J}^l}\frac{C_{N-\nu}(\mathbf{p},j^l)}{C_{N-\nu}(\mathbf{p},j^*)},\quad l=1,\dots,2p+1.$$

By Proposition 3.8.1,

$$j^l\preceq j^*\quad\text{if } l\geqslant p+1,\ j\in\mathbf{J}^l,$$

and by Proposition 3.7.3, this implies

$$C_{N-\nu}(\mathbf{p},j^l)\leqslant c_pC_{N-\nu}(\mathbf{p},j^*)\quad\text{if }\nu\leqslant N-p+1.\qquad\square$$

### 3.10. Last but not least

In [B2] C. de Boor wrote:

> "... I offer the modest sum of $m-1972$ ten-dollar bills to the first person who communicates to me a proof or a counterexample (but not both) of his or her making for the following conjecture (known to be true when $k=2$ or $k=3$):
>
> CONJECTURE. *For a given $n$ and $\underline{t}$, let $(\lambda_i\phi_j)$ be the $(n\times n)$-matrix whose entries are given by*
> $$\lambda_i\phi_j = k\int \frac{N_{ik}N_{jk}}{t_{i+k}-t_i}.$$
> *Then*
> $$\sup_{n,\underline{t}} \|(\lambda_i\phi_j)^{-1}\|_\infty < \infty.$$
>
> Here $m$ is the year A.D. of such communication."

*Added in proof.* The cheque has been received. With $m=1999$, and, to a nice surprise, doubled, the modest sum turned out to be not that modest. Regarding the origin of the factor 2, C. de Boor replied: "... well, about 5–6 years ago, I stated at some occasion that, given inflation and all that, I was doubling that rate. In fact, Jia was kind enough to remind me of that."

## 4. Comments

### 4.1. A survey of earlier and related results

Earlier the mesh-independent bound (0.2.1) was proved for $k=2,3,4$ (the case $k=1$ is trivial). For $k>4$ all previously known results proved boundedness of $\|P_S\|_\infty$ only under certain restrictions on the mesh $\Delta$. This included, in particular, meshes with multiple knots which correspond to the spline spaces

$$\mathbf{S}_{k,m}(\Delta) := \mathbf{P}_k(\Delta)\cap C^{m-1}[a,b], \quad \mathbf{S}_k(\Delta) := \mathbf{S}_{k,k-1}(\Delta)$$

We summarize these results in two theorems. The number in the square brackets indicates the year of the result.

THEOREM A. *Let $K$ be one of the mesh classes given below. Then*

$$\sup_{\Delta\subset K}\sup_{m}\|P_{\mathbf{S}_{k,m}(\Delta)}\|_\infty < c_k(K) \quad \textit{for all } k\in\mathbf{N}.$$

| | | | |
|---|---|---|---|
| $(K_1)$ | quasi-uniform | $\frac{h_{\max}}{h_{\min}} \leqslant M$ or like | Domsta [72], Douglas–Dupont–Wahlbin [$75_1$], de Boor [$76_3$], Demko [77] |
| $(K_1')$ | quasi-geometric | $\frac{h_{i\pm1}}{h_i} < 1+\varepsilon_k$ | de Boor [$76_3$] |
| $(K_2)$ | strictly geometric | $\frac{h_{i+1}}{h_i} = \varrho,\ \varrho>0$ | Feng–Kozak [81], Höllig [81], Mityagin [83], Jia [87] |

THEOREM B. *If $k,m$ are as given below, then*

$$\sup_{\Delta}\|P_{\mathbf{S}_{k,m}(\Delta)}\|_\infty < c_k.$$

| | | |
|---|---|---|
| $m=k-1$ | $k=2$ | Ciesielski [63] |
| $m=k-1$ | $k=3,4$ | de Boor [68], [79] |
| $m=0$ | $k\geqslant 1$ | trivial |
| $m=1$ | $k\geqslant 2$ | de Boor [$76_3$], Zmatrakov–Subbotin [83] |
| $m=2,3$ | $k>(m+1)^2$ | Shadrin [98] |

4.1.1. *$L_2$-projector onto finite-element spaces.* The arguments used by Douglas, Dupont, Wahlbin [DDW1], de Boor [B3] and Demko [Dem] for proving the boundedness of $\|P_{\mathbf{S}}\|_\infty$ for the quasi-uniform meshes revealed that such a boundedness has nothing to do with the particular spline nature. The essential structural requirements on a subspace $S$ needed for these proofs can be summarized as follows:

($\mathrm{B}_0$) $S=\operatorname{span}\{\phi_i\}$,

($\mathrm{B}_1$) $\operatorname{supp}\phi_i<\infty$, $\#\{\phi_j:\phi_j\phi_i\not\equiv 0\}\leqslant k$,

($\mathrm{B}_2$) the local condition number $\varkappa(\Phi)$ of $\Phi:=\{\phi_i\}$ is bounded, i.e., $\varkappa(\Phi)\leqslant d$ for some $d$,

($\mathrm{B}_3$) partition of the domain is quasi-uniform.

A general result (for quasi-uniform partitions) including also the multivariate case was proved by Douglas, Dupont and Wahlbin in [DDW2], and in fact in an earlier paper by Descloux [Des].

To this end, a natural question is whether the mesh-independent bound of $P_{\mathbf{S}}$ could be extended to (and perhaps more simply derived for) general finite-element spaces. The answer is no.

More precisely, denote by $S_{k,d}$ the set of all finite-element spaces $S$ that satisfy $(\mathrm{B}_0)$–$(\mathrm{B}_2)$. Then, for $k=2$ and any $d>36$, we have

$$\sup_{S\in S_{k,d}} \|P_S\|_p = \infty, \qquad \left|\frac{1}{p}-\frac{1}{2}\right| > \frac{3}{\sqrt{d}}.$$

This result shows that the mesh-independent $L_\infty$-boundedness of the $L_2$-spline projector is based on some peculiarities of the spline nature.

On the other hand, one can show that, for any $k\in\mathbf{N}$, $d\in\mathbf{R}$, $d>k$,

$$\sup_{S\in S_{k,d}} \|P_S\|_p < c(k,d), \qquad \left|\frac{1}{p}-\frac{1}{2}\right| < \frac{1}{2kd^2\ln d},$$

i.e., the $L_p$-boundedness of the spline projector $P_{\mathbf{S}}$ for $p$ in some neighbourhood of $p=2$ (proved earlier in [S2]) is not something extraordinary.

See [S5] for details.

4.1.2. *A general spline-interpolation problem.* C. de Boor's problem is a particular case of a general problem concerned with spline interpolation.

For $p\in[1,\infty]$, and $f$ from the Sobolev space $W_p^l[a,b]$, let $s:=s_{2k,\Delta}(f)$ be a spline of the odd degree $2k-1$ which interpolates $f$ on $\Delta$, i.e.,

$$s\in\mathbf{S}_{2k}(\Delta), \quad s|_\Delta = f|_\Delta.$$

To obtain uniqueness, one should add some boundary conditions, e.g.,

$$s^{(l)}(x)|_{x=a,b} = f^{(l)}(x)|_{x=a,b}, \quad l=1,\dots,k-1.$$

A general problem is to estimate the $L_q$-norm of such a spline-interpolation operator, i.e., to find

$$L(k,l,m,p,q,K) := \sup_{\Delta\subset K} \sup_{\|f^{(l)}\|_p\leqslant 1} \|f^{(m)} - s_{2k,\Delta}^{(m)}(f)\|_q,$$

where $K$ is a class of meshes, see [B7], [H], [S1], [Ma].

A particular problem is to determine whether the value

$$L^*(k,l,p) := \sup_{\Delta} \sup_{\|f^{(l)}\|_p\leqslant 1} \|s_{2k,\Delta}^{(l)}(f)\|_p \tag{4.1.1}$$

is bounded (independently of the mesh). A necessary condition was found to be

$$L^*(k,l,p) < \infty \quad \Rightarrow \quad W_p^l \in \{W_\infty^{k-1}, W_p^k, W_1^{k+1}\}. \tag{4.1.2}$$

It was conjectured that this is also a sufficient condition. For $l=k$ this particular problem is known to be equivalent to de Boor's conjecture, since

$$s_{2k,\Delta}^{(k)}(f) = P_{\mathbf{S}_k(\Delta)}[f^{(k)}]. \tag{4.1.3}$$

Now, by our Theorem I, due to (4.1.3), a particular converse of (4.1.2) follows:

$$W_p^l = W_p^k \quad \Rightarrow \quad L^*(k,l,p) < \infty.$$

The question whether such a converse is also true for two other cases in (4.1.2),

$$W_p^l \in \{W_\infty^{k-1}, W_1^{k+1}\} \quad \stackrel{?}{\Rightarrow} \quad L^*(k,l,p) < \infty,$$

remains open.

4.1.3. *A problem for the multivariate $D^k$-splines.* The univariate splines can be defined through a variational approach. Now the question is that perhaps the variational nature of splines determines the mesh-independent boundedness of the spline orthoprojector. The answer is no, too.

For another class of variational splines, the so-called multivariate $D^k$-splines on a domain of $\mathbf{R}^n$, the analogue of de Boor's conjecture is false, see [S4], [Ma]. In particular, in terms of the previous subsection, we have

$$L^*(k,l,p) < \infty \quad \Leftrightarrow \quad l=k, \ p=2, \quad \text{if } n>4.$$

## 4.2. On de Boor's Lemma 1.2.4

4.2.1. *Gram matrix and de Boor's Lemma* 1.2.4. A simple intermediate estimate

$$\|P_{\mathbf{S}}\|_\infty \leqslant \|G^{-1}\|_\infty$$

stated in Lemma 1.2.1 is a kind of folklore and has been used in most (but not all) papers on the subject cited in Theorems A and B above. C. de Boor [B2] proved that the converse (not so simple) inequality

$$\|G^{-1}\|_\infty \leqslant c_k \|P_{\mathbf{S}}\|_\infty$$

is also valid, i.e., to quote [B6], "in bounding $\|P_{\mathbf{S}}\|$ in the uniform norm, we are bounding $\|G^{-1}\|_\infty$, whether we want to or not".

For $k=2$, $G$ is strictly diagonally dominant, and the direct estimate by Ciesielski [C] was

$$\|G^{-1}\|_\infty \leqslant 3. \tag{4.2.1}$$

For $k>2$, $G$ fails to be diagonally dominant, so a different argument has to be used.

For $k=3,4$, de Boor [B1], [B6] proved the boundedness of $G^{-1}$ making use of his Lemma 1.2.4. Namely, he found that the following "comparatively simple" choice of the vector $(a_i)$ works:

$$\begin{aligned} k=3, &\quad (-1)^i a_i := 1+\frac{(t_{i+2}-t_{i+1})^2}{(t_{i+2}-t_i)(t_{i+3}-t_{i+1})}, &\quad \operatorname{supp} M_i = [t_i, t_{i+3}], \\ k=4, &\quad (-1)^i a_i := 3+4\frac{(t_{i+3}-t_{i+1})^2}{(t_{i+3}-t_i)(t_{i+4}-t_{i+1})}, &\quad \operatorname{supp} M_i = [t_i, t_{i+4}]. \end{aligned} \tag{4.2.2}$$

This choice clearly provides the fulfillment of

$(\mathrm{a}_3)$ $\|a\|_\infty < c_{\max}$,

but makes the verification of $(\mathrm{a}_1)$ and $(\mathrm{a}_2)$ "comparatively" problematic. (The proof of $k=4$ announced in 1979 has never been published.)

In this sense our proof is of an opposite nature. We offer a construction which gives a simple proof of $(\mathrm{A}_1)$ and $(\mathrm{A}_2)$, but encounter the problems with $(\mathrm{A}_3)$ instead.

4.2.2. *On the choice of the null-spline $\sigma$.* The main difficulty in using Lemma 1.2.4 for estimating $\|G^{-1}\|_\infty$ is the problem of finding a vector $a=(a_i)$ satisfying the condition $(\mathrm{a}_1)$ of this lemma, or, respectively, the problem of finding a spline $\phi=\sum a_i N_i$ satisfying the condition $(\mathrm{A}_1)$ of Lemma 1.3.1.

(1) Since the Gram matrix $G$ is an oscillation matrix, a candidate for the vector $a$ could be the eigenvector corresponding to the minimal eigenvalue. (By a theorem of Gantmacher–Krein such an eigenvector is sign-alternating.)

(2) Consider

$$\delta^{(k)} = \{t_{-k+1} = \dots = t_0 = 0 < 1 = t_1 = \dots = t_k\},$$

the mesh with the so-called Bernstein knots. In this case the B-spline basis reduces to the polynomials

$$\binom{k-1}{i} x^i (1-x)^{k-1-i}.$$

For the Bernstein Gramian $G_\delta$ the explicit expression for the "minimal" eigenvector is available, namely

$$a = (a_i)_{i=1}^k, \quad a_i = (-1)^i \binom{k-1}{i-1}.$$

Also, it is known that the corresponding polynomial $\psi(x) := \sum a_i N_i(x)$ is the Legendre polynomial

$$\psi = c\Psi^{(k-1)}, \quad \Psi(x) := [x(1-x)]^{k-1},$$

i.e., the $(k-1)$st derivative of the null-spline $\Psi$ of degree $2k-2$.

Our null-spline $\sigma$ may be viewed as a generalization of $\Psi$.

(3) However, it turned out that the coefficients of the spline $\phi := \sigma^{(k-1)}$ have nothing to do (and could not have something to do, see below) with the "minimal" eigenvector. Nevertheless, this choice provides the fulfillment of $(\mathrm{A}_1)$ in a simple and natural way.

(4) *Remark in retrospect.* The "minimal" eigenvector $(a_i)$ of $G$ *can not* be used in de Boor's lemma. Recall that in order to use this lemma, one should have the relations

$$b = Ga, \quad \max_{i,j} |a_i/b_j| < c_k.$$

For the "minimal" eigenvector $(a_i)$ of $G$ they should therefore be

$$|a_{\max}/a_{\min}| \overset{?}{<} c_k'.$$

This is, however, not true, as the following lemma shows.

LEMMA 4.2.1. *Let $(a_i)$ be the eigenvector of $G_\Delta$ corresponding to the minimal eigenvalue. Then, for $k>2$,*

$$\sup_{\Delta} |a_{\max}/a_{\min}| = \infty.$$

*Proof.* Let $\Delta = (t_i)_{i=0}^N$ and $h_i = t_{i+1} - t_i$. Then, e.g., for $k=3$,

$$G^* := \lim_{h_{N-1}\to 0} \lim_{h_{N-2}\to 0} \cdots \lim_{h_1 \to 0} G_\Delta = \frac{1}{10} \begin{bmatrix} 6 & 4 & & & & \\ & \ddots & \ddots & & & \\ & & & 6 & 4 & \\ & & & 6 & 3 & 1 \\ & & & 3 & 4 & 3 \\ & & & 1 & 3 & 6 \end{bmatrix},$$

the limit minimal eigenvalue is $\lambda^*_{\min} = \frac{1}{10}$, and the corresponding limit eigenvector is

$$a^* = ((-x)^{N-1}, (-x)^{N-2}, \ldots, x^2, -x, 1, -2, 1), \quad 6x - 4 = x, \quad x = \tfrac{4}{5}.$$

Thus,

$$\sup_{\#\Delta = N} |a_{\max}/a_{\min}| \geqslant 2 \cdot \left(\tfrac{5}{4}\right)^{N-1}. \qquad \square$$

### 4.3. Simplifications in particular cases

The most elaborate part of the proof of Theorem I, viz. Chapter 3, is concerned with the estimate

$$\max_{\alpha\in\mathbf{J}}\frac{R_q(\alpha,i)}{R_q(\alpha,j)}<c_p,\quad R_q:=\prod_{r=1}^{q}[AD_{\gamma_r}]\cdot A,$$

with $q=p-1$. The analysis would be simpler if we could take

$$q=0,\quad R_0=A, \tag{4.3.1}$$

but we were forced to take $q=p-1$, since $A$ in general has vanishing minors.

We indicate here the cases when considerations from Chapter 3 starting with §§3.3–3.5 can be omitted.

In Cases 1 and 2 below, the choice (4.3.1) works. Case 3 uses $q=p-1$, but the only ingredient taken from Chapter 3 is non-emptiness of the set $\mathbf{J}_{[\beta,i]}$, proved in §3.5.

*Case* 1. *Knots with multiplicity $k-m$ with $m\leqslant\frac{1}{2}(k+1)$.* Consider

$$\mathbf{S}_{k,m}(\Delta):=\mathbf{P}_k(\Delta)\cap C^{m-1}[a,b],$$

the spline space with the B-spline basis defined on the knot sequence $\Delta$ with knot multiplicity $k-m$. The following particular case of Theorem I does not rely on the analysis made in §§3.3–3.6.

PROPOSITION 4.3.1. *If $m\leqslant\frac{1}{2}(k+1)$, then*

$$\sup_{\Delta}\|P_{\mathbf{S}_{k,m}(\Delta)}\|_\infty<c_k.$$

*The last step of the proof.* For this space, the null-spline $\sigma$ is a spline with $(k-m)$-multiple zeros on $\Delta$. The matrix $A$ which connects the vectors $z_\nu$ of the non-zero derivatives of $\sigma$ at $t_\nu$ by the rule $z_{\nu+1}=Az_\nu$ has the lower order

$$A\in\mathbf{R}^{(2m-1)\times(2m-1)}.$$

It could be obtained from the matrix $S$ by $k-m$ successive transformations similar to those in §3.2.2. This gives the criterion

$$A\begin{pmatrix}\alpha_1,\dots,\alpha_q\\ \beta_1,\dots,\beta_q\end{pmatrix}>0\quad\text{if and only if}\quad\alpha_s\leqslant\beta_{s+k-m},\quad s=1,\dots,q-(k-m). \tag{4.3.2}$$

Here $\alpha,\beta$ are indices from $\mathbf{I}_{q,2m-1}$; in particular, we have

$$s\leqslant\alpha_s\leqslant(2m-1)-(q-s). \tag{4.3.3}$$

If $k-m \geqslant q$, then the condition on $\alpha, \beta$ in (4.3.2) is void. Now let

(i) $k-m \leqslant q-1$,

(ii) $m \leqslant \frac{1}{2}(k+1)$.

Then

$$\alpha_s \overset{(4.3.3)}{\leqslant} (2m-1)-(q-s) \overset{\text{(ii)}}{\leqslant} k-q+s \overset{\text{(i)}}{\leqslant} s+m-1 \overset{\text{(ii)}}{\leqslant} s+k-m \overset{(4.3.3)}{\leqslant} \beta_{s+k-m},$$

i.e., the condition on $\alpha, \beta$ in (4.3.2) is fulfilled. Thus,

$$A(\alpha,\beta) > 0 \quad \text{for all } \alpha, \beta, \text{ if } m \leqslant \tfrac{1}{2}(k+1),$$

and accordingly,

$$|z_\nu^{(l)}| \leqslant \max_{i \in \mathbf{J}} \frac{C(\mathbf{p}, i^l)}{C(\mathbf{p}, i^*)} \leqslant \max_{\alpha,\beta,\gamma,\delta} \frac{A(\alpha,\beta)}{A(\gamma,\delta)} \leqslant c_p. \qquad \square$$

*Case* 2. *The estimate of* $z_0$. For $\nu=0$, the estimate $|z_0^{(l)}| < c_p$ of Theorem Z (see §3.9) can also be proved without analysis of §§3.3–3.8, but with making use of properties of the matrix $A$ only.

LEMMA 4.3.2. *There exists a constant* $c_p$, *depending only on* $p$, *such that the inequalities*

$$\frac{1}{l!}|\sigma^{(l)}(t_\nu)| =: |z_\nu^{(l)}| \leqslant c_p, \quad l = p+1, \ldots, 2p+1, \ \nu = 0,$$

*hold uniformly in* $l$.

*Proof.* From (2.5.1), making use of the CB-formula we obtain

$$|z_0^{(l)}| = \frac{C(\mathbf{p}, \mathbf{p}^l)}{C(\mathbf{p}, \mathbf{p}^*)} \leqslant \max_{\alpha \in \mathbf{J}} \frac{A(\alpha, \mathbf{p}^l)}{A(\alpha, \mathbf{p}^*)}, \quad l = p+1, \ldots, 2p+1. \tag{4.3.4}$$

The criterion (see Lemma 3.2.9)

$$A(\alpha, i) > 0 \quad \text{if and only if} \quad \alpha_s \leqslant i_{s+1} \text{ for all } s$$

easily gives the implication

$$i \leqslant j \quad \Rightarrow \quad A(\alpha, i) \leqslant c_p A(\alpha, j) \text{ for all } \alpha \in \mathbf{J}. \tag{4.3.5}$$

It is not hard to see that, for two different $l$-complements of $i \in \mathbf{J}$, we have

$$i^{l_2} \leqslant i^{l_1} \quad \text{if } l_1 < l_2,$$

in particular,

$$\mathbf{p}^l \leqslant \mathbf{p}^{p+1} = \mathbf{p}^* \quad \text{if } l \geqslant p+1. \tag{4.3.6}$$

Altogether, (4.3.4)–(4.3.6) proves

$$|z_0^{(l)}| \leqslant c_p, \quad l = p+1, \ldots, 2p+1. \qquad \square$$

*Case* 3. *The estimate in terms of a local mesh ratio.* The next particular case of Theorem I does not need more than non-emptiness of the set $\mathbf{J}_{[\beta,i]}$, proved in §3.5.

PROPOSITION 4.3.3. *Let $L(M)$ be the class of meshes with the bounded local mesh ratio, i.e.,*

$$L(M) := \{\Delta : \max_{|\nu-\mu|=1} h_\nu/h_\mu \leqslant M\}. \tag{4.3.7}$$

*Then*

$$\sup_{\Delta\in L(M)} \|P_{\mathbf{S}_k(\Delta)}\|_\infty < c_k(M).$$

*The last step of the proof.* In §3.3 we proved the inequalities (3.3.6):

$$c_p \sum_{\alpha\in\mathbf{J}_{[\beta,i]}} \prod_{r=1}^{p-1} |\gamma_r|^{|\alpha^{(r)}|} \leqslant c_\gamma Q_\gamma(\beta,i) \leqslant c_p' \sum_{\alpha\in\mathbf{J}_{[\beta,i]}} \prod_{r=1}^{p-1} |\gamma_r|^{|\alpha^{(r)}|}.$$

We recall that $\gamma_r$ stands for the local mesh ratio $\varrho_\nu$ with some $\nu$, i.e.,

$$\gamma_r := \varrho_\nu := h_\nu/h_{\nu+1},$$

that $c_\gamma$ is a constant independent of $\beta$ and $i$, and that the set $\mathbf{J}_{[\beta,i]}$ is always non-empty (see §3.5). On account of (4.3.7), this yields the estimate

$$c_1(M,p) \leqslant c_\gamma Q_\gamma(\beta,i) \leqslant c_2(M,p) \quad \text{for all } \beta, i \in \mathbf{J},$$

i.e.,

$$\max_{\alpha\in\mathbf{J}} \frac{Q(\alpha,i)}{Q(\alpha,j)} < c_p(M) \quad \text{for all } i,j\in\mathbf{J}. \qquad \square$$

### 4.4. Additional facts

Here we present some additional facts which we have not used at all in our proof of Theorem I, but which could be useful in finding a simpler proof.

4.4.1. *Orthogonality of $\phi\in\mathbf{S}_k(\Delta)$ to $\mathbf{S}_{k-1}(\Delta)$.* For the Bernstein knots, $\phi$ being the Legendre polynomial of degree $k-1$ is orthogonal to the polynomials of smaller degree. The following lemma generalizes this property to any $\Delta$.

LEMMA 4.4.1. *The spline $\phi$ of degree $k-1$ on $\Delta$ defined via* (1.4.1)–(1.4.5) *is orthogonal to all splines of degree $k-2$ on $\Delta$, i.e.,*

$$(\phi, s) = 0 \quad \textit{for all } s\in\mathbf{S}_{k-1}(\Delta).$$

*Up to a constant factor, $\phi$ is the unique spline from $\mathbf{S}_k(\Delta)$ which possesses this property.*

*Proof.* It can be shown (e.g., by integration by parts) that if any function $f \in W_1^{k-1}[a, b]$ satisfies the conditions

$$\begin{aligned} f(t_\nu) &= 0, \quad \nu = 0, \ldots, N, \\ f^{(l)}(t_0) = f^{(l)}(t_N) &= 0, \quad l = 1, \ldots, k-2, \end{aligned} \tag{4.4.1}$$

then

$$(f^{(k-1)}, s) = 0 \quad \text{for all } s \in \mathbf{S}_{k-1}(\Delta).$$

Since $\sigma$ satisfies (4.4.1) (they are the same as (1.4.2) and (1.4.3)), and since $\phi := \sigma^{(k-1)}$, the statement follows. □

4.4.2. *Null-splines with Birkhoff boundary conditions at $t_0$.* Let $i \in \mathbf{J}$ be any index, and let $\hat{\sigma} \in \mathbf{S}_{2k-1}(\Delta)$ be the null-spline that satisfies the conditions

$$\begin{aligned} \hat{\sigma}(t_\nu) &= 0, \quad \nu = 0, \ldots, N, \\ \hat{\sigma}^{(i_s)}(t_0) = \hat{\sigma}^{(s)}(t_N) &= 0, \quad s = 1, \ldots, k-2, \\ \frac{1}{(k-1)!} \hat{\sigma}^{(k-1)}(t_N) &= 1. \end{aligned} \tag{4.4.2}$$

In comparison with the null-spline $\sigma$ defined in (1.4.2)–(1.4.4) we have changed at the left endpoint $t_0$ the Hermite boundary conditions (1.4.3) into Birkhoff boundary conditions. The spline $\hat{\sigma}$ also exists and is unique.

LEMMA 4.4.2. *We have the equalities*

$$\frac{1}{l!} |\hat{\sigma}^{(l)}(t_0)| \cdot |h_0|^{l-k+1} =: \hat{z}_0^{(l)} = \frac{C(\mathbf{p}, i^l)}{C(\mathbf{p+1}, i')}, \quad \{l\} \notin i. \tag{4.4.3}$$

*Proof.* Let $p := k-2$, and let

$$i := (i_1, \ldots, i_p)$$

be the index whose components are the orders of the derivatives involved in (4.4.2). Then we can find $\hat{z}_0$ as a solution to the system of linear equations similar to (2.2.11), and, as in the proof of Theorem 2.3.5, one obtains

$$|\hat{z}_0^{(l)}| = \frac{C(\mathbf{p}, i^l)}{C(\mathbf{p+1}, i')}. \qquad \square$$

Lemma 4.4.2 is of some interest for the following reasons. In Theorem 2.3.5 we established that

$$|z_\nu^{(l)}| \leqslant \max_{i \in \mathbf{J}_l} \frac{C(\mathbf{p}, i^l)}{C(\mathbf{p+1}, i')}.$$

Therefore, by (4.4.3), we have the estimate

$$|z_\nu^{(l)}| \leqslant \max_{\hat{\sigma}} |\hat{z}_0^{(l)}|,$$

where the maximum is taken over all null-splines $\hat{\sigma}$ with various Birkhoff boundary conditions in (4.4.2). Maybe it is possible to obtain an easier proof of the inequality

$$|\hat{z}_0^{(l)}| < c_p, \quad l \geqslant p+1,$$

for the left endpoint, as was the case for $|z_0^{(l)}|$ in Lemma 4.3.2.

4.4.3. *Further properties of the matrices $C$.* For $x=(x^{(l)})\in\mathbf{R}^n$, $S^-(x)$ and $S^+(x)$ denote respectively the minimal and maximal number of sign changes in the sequence $x$.

LEMMA 4.4.3. *For any $\nu$, the matrix $C:=C_{N-\nu}$ is similar to its inverse.*

*Proof.* By (2.4.1), we have $C^{-1}=(D_0F)^{-1}C^*(D_0F)$. □

The fact that $C$ is an oscillation matrix permits the following conclusion.

LEMMA 4.4.4. *For any $\nu$, the spectrum of $C_{N-\nu}\in\mathbf{R}^{2p+1}$ consists of $2p+1$ different positive numbers*

$$0 < \lambda_1 < \ldots < \lambda_{2p+1}.$$

*Moreover, by Lemma 4.4.3,*

$$\lambda_s = \frac{1}{\lambda_{2p+2-s}}, \quad \lambda_{p+1} = 1.$$

*If $\{u_{\nu,s}\}$ is a corresponding sequence of eigenvectors of $C_{N-\nu}$, then*

$$S^-(u_{\nu,s}) = S^+(u_{\nu,s}) = s-1, \quad s=1,\ldots,2p+1.$$

The fact that, for any $\nu$, a solution $z_\nu$ of the equations

$$C_{N-\nu} z_\nu = z_N$$

remains bounded at least in the second half of its components indicates that in the expansion

$$z_\nu = \sum_{s=1}^{2p+1} a_s u_{\nu,s}$$

the eigenvector $u_{\nu,p+1}$ corresponding to the eigenvalue 1 dominates in a sense. Here is one more evidence for this "dominance".

LEMMA 4.4.5. *For any $\nu$, we have*

$$S^-(z_\nu)=S^+(z_\nu)=p\ [=S(u_{\nu,p+1})].$$

*Proof.* By the Budan–Fourier Theorem for Splines [BS], with $p:=k-2$ we obtain

$$\begin{aligned} Z_\sigma(a,b)\leqslant Z_{\sigma^{(2p+2)}}(a,b)+S^-[\sigma(a+),\ldots,\sigma^{(2p+2)}(a+)]\\ -S^+[\sigma(b-),\ldots,\sigma^{(2p+2)}(b-)],\end{aligned}\tag{4.4.4}$$

where $Z_f(a,b)$ stands for the number of zeros of $f$ on the interval $(a,b)$ counting multiplicities. Also, by Lemma 1.6.1,

$$Z_\sigma(t_\nu,t_\mu)=Z_{\sigma^{(2p+2)}}(t_\nu,t_\mu)\quad \text{for all } \nu,\mu,$$

and the boundary conditions (1.4.2)–(1.4.3) say that

$$S^-[\sigma(t_0+),\ldots,\sigma^{(2p+2)}(t_0+)]\leqslant p+1\leqslant S^+[\sigma(t_N-),\ldots,\sigma^{(2p+2)}(t_N-)].$$

Taking now (4.4.4) with
(1) $a=t_0$, $b=t_N$,
(2) $a=t_0$, $b=t_\nu$,
(3) $a=t_\nu$, $b=t_N$,
successively, we obtain

$$S^+[\sigma(t_\nu-0),\ldots,\sigma^{(2p+2)}(t_\nu-0)]=S^-[\sigma(t_\nu+0),\ldots,\sigma^{(2p+2)}(t_\nu+0)]=p+1\quad \text{for all } \nu.$$

Since

$$\sigma^{(l)}(t_\nu-0)=\sigma^{(l)}(t_\nu+0),\quad l=1,\ldots,2p+1,$$

and since

$$\sigma(t_\nu-0)=\sigma(t_\nu+0)=0,\quad \operatorname{sign}\sigma^{(2p+2)}(t_\nu-0)=-\operatorname{sign}\sigma^{(l)}(t_\nu+0),$$

we conclude that

$$S[\sigma'(t_\nu),\ldots,\sigma^{(2p+1)}(t_\nu)]=p\quad \text{for all } \nu.$$

This, in view of the relations

$$z_\nu^{(l)}=\text{const}\cdot\sigma^{(l)}(t_\nu),\quad l=1,\ldots,2p+1,$$

proves the statement. □

### 4.5. On the constant $c_k$

There are two constants in de Boor's problem:

(a) the norm of the orthoprojector

$$c_k[P] := \sup_\Delta c_{k,\Delta}[P], \quad c_{k,\Delta}[P] := \|P_{\mathbf{S}_k(\Delta)}\|_\infty,$$

(b) the norm of the inverse of the B-spline Gramian

$$c_k[G] := \sup_\Delta c_{k,\Delta}[G], \quad c_{k,\Delta}[G] := \|G_\Delta^{-1}\|_\infty.$$

Our method based on properties of the spline $\phi := \phi_\Delta := \sum_j a_j(\phi_\Delta) N_j$ provides also

(c) the constant

$$c_k[\phi] := \sup_\Delta c_{k,\Delta}[\phi], \quad c_{k,\Delta}[\phi] := \max_{i,j} \frac{|a_j(\phi_\Delta)|}{|(M_i, \phi_\Delta)|}.$$

These constants are related by the inequalities

$$c_k[P] \leqslant c_k[G] \leqslant c_k[\phi], \tag{4.5.1}$$

and we proved in Theorem I that

$$c_k[\phi] \leqslant c_k.$$

It is possible of course to estimate all the constants involved in the proof, hence the final constant $c_k$, but we find it more useful to give a comparative analysis of the constants in (4.5.1).

(1) *Lower bounds for $c_k[G]$ and $c_k[\phi]$.* Consider

$$\delta^{(k)} := \{t_{-k+1} = \ldots = t_0 = 0 < 1 = t_1 = \ldots = t_k\},$$

the mesh $\delta$ with the Bernstein knots. In this case the corresponding B-splines are simply the polynomials

$$N_i(x) = \binom{k-1}{i} x^i (1-x)^{k-1-i}, \quad M_i(x) = k N_i(x),$$

and the Gram matrix $G_\delta$ is given by

$$G_\delta := \{(M_i, N_j)\} = (g_{ij})_{i,j=0}^{k-1}, \quad g_{ij} = \frac{k}{2k-1} \cdot \frac{\binom{k-1}{i} \cdot \binom{k-1}{j}}{\binom{2k-2}{i+j}}.$$

The first values for the constants are as follows:

| $k$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|
| $c_{k,\delta}[G]$ | 3 | 13 | $41\frac{2}{3}$ | 171 | $583\frac{4}{5}$ | $2\,364\frac{1}{5}$ | $8\,373\frac{6}{7}$ | $33\,737\frac{2}{7}$ |
| $c_{k,\delta}[\phi]$ | 3 | 20 | 105 | 756 | 4 620 | 34 320 | 225 225 | 1 701 700 |

(4.5.2)

They satisfy the relations

$$c_{k,\delta}[G] \sim k^{-1/2}4^k, \quad \frac{1}{2}\binom{2k}{k} < c_{k,\delta}[G] < \binom{2k}{k},$$
$$c_{k,\delta}[\phi] \sim k^{-1}8^k, \quad c_{k,\delta}[\phi] = \binom{2k-1}{k-1} \cdot \binom{k-1}{[\frac{1}{2}(k-1)]}.$$

To find $c_{k,\delta}[\phi]$, we have used the formula

$$c_{k,\delta}[\phi] = \lambda_{\min}^{-1} \cdot \max_{i,j} \frac{a_i}{a_j},$$

where $\lambda_{\min}$ is the minimal eigenvalue of $G_\delta$, and

$$(a_i) := \left((-1)^i \binom{k-1}{i-1}\right)$$

is the corresponding eigenvector.

The first values and the two-sided estimates for $c_{k,\delta}[G]$ were obtained with the help of the MAPLE package. It is possible to find an explicit expression for this constant, too.

(2) *Lower bound for* $c_k[P]$. For the Bernstein knots, $P_{\mathbf{S}}$ is simply the orthoprojector onto the space $\mathbf{P}_k$ of polynomials, and in this case

$$c_{2,\delta}[P] = 1\tfrac{2}{3}, \quad c_{k,\delta}[P] \sim \sqrt{k}\,.$$

For $k=2$, K. Oskolkov [O] improved the lower bound $1\frac{2}{3}$, and showed that

$$c_2[P] \geqslant 3. \tag{4.5.3}$$

His method is easily extended for arbitrary $k$.

LEMMA 4.5.1. *For any* $k$,

$$c_k[P] \geqslant 2k-1. \tag{4.5.4}$$

*Proof.* For $f \in L_\infty$, let its orthoprojection $P_{\mathbf{S}}(f)$ onto $\mathbf{S}_k(\Delta_N)$ have the expansion

$$P_{\mathbf{S}}(f,x) = \sum_{j=1}^{N'} a_j(f,\Delta_N) N_j(x).$$

Then, the value of $P(f,x)$ at the left endpoint $x=t_1$ of $\Delta_N$ is equal to the first coefficient of this expansion, i.e.,

$$P_{\mathbf{S}}(f,t_1) = a_1(f,\Delta_N).$$

Therefore,

$$\|P_{\mathbf{S}}(f)\|_\infty \geqslant |a_1(f,\Delta_N)|,$$

and it follows that

$$\|P_{\mathbf{S}_k(\Delta_N)}\| \geqslant K(\Delta_N), \quad K(\Delta_N) := \sup_{\|f\|_\infty \leqslant 1} |a_1(f,\Delta_N)|.$$

Now let

$$\Delta_N = (t_i)_1^N, \quad \Delta_{N+1} = \{t_0\} \cup \Delta_N, \quad h := t_1 - t_0.$$

Then, for the corresponding Gramians $G_N$ and $G_{N+1}$ we have the relation

$$\lim_{h\to 0} G_{N+1} = \left[\begin{array}{c|c|c|c|c} b_1 & b_2 & 0 & \dots & 0 \\ \hline 0 & & & & \\ \hline \vdots & & G_N & & \\ \hline 0 & & & & \end{array}\right].$$

In the same way as in [O], one can prove the inequality

$$\lim_{h\to 0} K(\Delta_{N+1}) \geqslant \frac{1}{b_1} + \frac{b_2}{b_1} K(\Delta_N).$$

This implies the estimate

$$K_{N+1} \geqslant \frac{1}{b_1} + \frac{b_2}{b_1} K_N, \quad K_N := \sup_{\#\Delta_N = N} K(\Delta_N),$$

and as a consequence

$$\lim_{N\to\infty} K_N \geqslant \frac{1}{b_1} \sum_{s=0}^{\infty} \left(\frac{b_2}{b_1}\right)^s = \frac{1/b_1}{1-b_2/b_1} = \frac{1}{b_1-b_2}.$$

For any $k$, the corresponding values $b_1, b_2$ are easily computed as

$$b_1 = k\int_0^1 x^{k-1}x^{k-1}\,dx = \frac{k}{2k-1}, \quad b_2 = 1-b_1 = \frac{k-1}{2k-1},$$

so that

$$\lim_{N\to\infty} K_N \geqslant 2k-1. \qquad \square$$

(3) *Upper bounds.* For $k=2$, the exact values of all constants are known:

$$k=2, \quad c_2[P]=c_2[G]=c_2[\phi]=3.$$

Two further estimates of de Boor are available:

$$k=3, \quad c_3[G]\leqslant 30,$$
$$k=4, \quad c_4[G]\leqslant 81\tfrac{2}{3}.$$

(4) *Expectations.* Symbolic computations with MAPLE for $k,N\leqslant 5$ give evidence that

$$c_k[G]=c_{k,\delta}[G], \quad c_k[\phi]=c_{k,\delta}[\phi].$$

These relations are also supported by theoretical estimates for the classes

$$\Delta_\varrho := \{\Delta : h_\nu/h_{\nu+1}=\varrho \text{ for all } \nu\in\mathbf{N}\}$$

of strictly geometric meshes. They are [H]

$$2k-1=\lim_{\varrho\to\infty} c_{k,\Delta_\varrho}[G] < c_{k,\Delta_\varrho}[G] \leqslant \lim_{\varrho\to 1} c_{k,\Delta_\varrho}[G] \sim \left(\tfrac{1}{2}\pi\right)^{2k}.$$

In view of these inequalities and (4.5.4) it is plain to make the following

CONJECTURE. *For any $k\in\mathbf{N}$,*

$$\sup_\Delta \|P_{\mathbf{S}_k(\Delta)}\|_\infty = \inf_\Delta \left\|G^{-1}_{\mathbf{S}_k(\Delta)}\right\|_\infty = 2k-1.$$

## References


[B1] BOOR, C. DE, On the convergence of odd-degree spline interpolation. *J. Approx. Theory*, 1 (1968), 452–463.

[B2] — The quasi-interpolant as a tool in elementary polynomial spline theory, in *Approximation Theory* (Austin, TX, 1973), pp. 269–276. Academic Press, New York, 1973.

[B3] — A bound on the $L_\infty$-norm of $L_2$-approximation by splines in terms of a global mesh ratio. *Math. Comp.*, 30 (1976), 765–771.

[B4] — Splines as linear combinations of B-splines. A survey, in *Approximation Theory*, II (Austin, TX, 1976), pp. 1–47. Academic Press, New York, 1976.

[B5] — On local linear functionals which vanish at all B-splines but one, in *Theory of Approximation with Applications* (Calgary, AB, 1975), pp. 120–145. Academic Press, New York, 1976.

[B6] — On a max-norm bound for the least-squares spline approximant, in *Approximation and Function Spaces* (Gdańsk, 1979), pp. 163–175. North-Holland, Amsterdam–New York, 1981.

[B7] — On bounding spline interpolation. *J. Approx. Theory*, 14 (1975), 191–203.

[BS] BOOR, C. DE & SCHOENBERG, I. J., Cardinal interpolation and spline functions, VIII. The Budan–Fourier theorem for splines and applications, in *Spline Functions* (Karlsruhe, 1975), pp. 1–79. Lecture Notes in Math., 501. Springer-Verlag, Berlin, 1976.

[C] CIESIELSKI, Z., Properties of the orthonormal Franklin system. *Studia Math.*, 23 (1963), 141–157.

[DDW1] DOUGLAS, J., DUPONT, T. & WAHLBIN, L., Optimal $L_\infty$-error estimate for Galerkin approximations to solutions of two-point boundary value problems. *Math. Comp.*, 29 (1975), 475–483.

[DDW2] — The stability in $L^q$ of the $L^2$-projection into finite element function spaces. *Numer. Math.*, 23 (1974/75), 193–197.

[Dem] DEMKO, S., Inverses of band matrices and local convergence of spline projections. *SIAM J. Numer. Anal.*, 14 (1977), 616–619.

[Des] DESCLOUX, J., On finite element matrices. *SIAM J. Numer. Anal.*, 9 (1972), 260–265.

[Do] DOMSTA, J., A theorem on B-splines. *Studia Math.*, 41 (1972), 291–314.

[FK] FENG, Y. Y. & KOZAK, J., On the generalized Euler–Frobenius polynomial. *J. Approx. Theory*, 32 (1981), 327–338.

[H] HÖLLIG, K., $L_\infty$-boundedness of $L_2$-projections on splines for a geometric mesh. *J. Approx. Theory*, 33 (1981), 318–333.

[J] JIA, R., $L_\infty$-boundedness of $L_2$-projections on splines for a multiple geometric mesh. *Math. Comp.*, 48 (1987), 675–690.

[K] KARLIN, S., *Total Positivity*. Stanford Univ. Press, Stanford, CA, 1968.

[Ma] MATVEEV, O. V., Spline interpolation of functions in several variables and bases in Sobolev spaces. *Trudy Mat. Inst. Steklov.*, 198 (1992), 125–152 (Russian); English translation in *Proc. Steklov Inst. Math.*, 198 (1994), 119–146.

[Mi] MITYAGIN, B., Quadratic pencils and least-squares piecewise-polynomial approximation. *Math. Comp.*, 40 (1983), 283–300.

[O] OSKOLKOV, K. I., The upper bound of the norms of orthogonal projections onto subspaces of polygonals, in *Approximation Theory* (Warsaw, 1975), pp. 177–183. Banach Center Publ., 4. PWN, Warsaw, 1979.

[S1] SHADRIN, A. YU., On the approximation of functions by interpolating splines defined on nonuniform nets. *Mat. Sb.*, 181 (1990), 1236–1255; English translation in *Math. USSR-Sb.*, 71 (1992), 81–99.

[S2] — On $L_p$-boundedness of the $L_2$-projector onto splines. *J. Approx. Theory*, 77 (1994), 331–348.

[S3] — On a problem of de Boor for multivariate $D^m$-splines. *Trudy Mat. Inst. Steklov.*, 219 (1997), 420–452; English translation in *Proc. Steklov Inst. Math.*, 219 (1997), 413–446.

[S4] — On $L_\infty$-boundedness of the $L_2$-projector onto splines with multiple knots. IGPM Preprint, 157. Technische Hochschule, Aachen, 1998.

[S5] — On $L_p$-boundedness of the $L_2$-projector onto finite element spaces. Manuscript.

[Scho] SCHOENBERG, I. J., Zur Abzählung der reellen Wurzeln algebraischer Gleichungen. *Math. Z.*, 38 (1934), 546–564.

[Schu] SCHUMAKER, L. L., *Spline Functions: Basic Theory.* Wiley, New York, 1981.

[ZS] ZMATRAKOV, N. & SUBBOTIN, YU. N., Multiple interpolating splines of degree $2k+1$ with deficiency $k$. *Trudy Mat. Inst. Steklov.*, 164 (1983), 75–99 (Russian).

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*Received February, 28, 2000*