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On a problem in simultaneous Diophantine approximation: Littlewood's conjecture

by

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Dedicated to Bill Parry on his 65th birthday

1. Introduction

1.1. Background: elementary number theory

Before stating the problem we recall a few fundamental results from the theory of Diophantine approximation. Given a real number x we use the standard notation $\|x\|$ to denote the distance of x to the nearest integer, and throughout I will denote the unit interval $[0,1]$. The classical result of Dirichlet states:

DIRICHLET'S THEOREM (1842). *For any $\alpha\in I:=[0,1]$, there exist infinitely many $q\in\mathbf{N}$ such that*

$$\|q\alpha\|\leqslant q^{-1}.$$

A consequence of Hurwitz's theorem is that the right-hand side of the above inequality cannot be improved by an arbitrary positive constant ε. More precisely, for $\varepsilon<1/\sqrt{5}$ there exist real numbers $\alpha\in I$ for which the inequality $\|q\alpha\|\leqslant\varepsilon q^{-1}$ has at most a finite number of solutions. These α are the badly approximable numbers, and we will denote by **Bad** the set of all such numbers; that is,

$$\mathbf{Bad}:=\{\alpha\in I: \text{there exist } c(\alpha)>0 \text{ so that } \|q\alpha\|>c(\alpha)q^{-1} \text{ for all } q\in\mathbf{N}\}.$$

We now briefly describe the beautiful connection between **Bad** and the theory of continued fractions. Let $\alpha=[a_1,a_2,a_3,...]$ represent the regular continued fraction expansion of α, and as usual let $p_n/q_n:=[a_1,a_2,a_3,...,a_n]$ denote its nth convergent. It is

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easy to verify that

$$\frac{1}{a_{n+1}+2} \leqslant q_n \|q_n \alpha\| \leqslant \frac{1}{a_{n+1}}. \tag{1}$$

This instantly gives a proof of Dirichlet's theorem, and together with the fact that the convergents are the best approximates (that is, $\|q\alpha\| \geqslant \|q_n \alpha\|$ for any $q \leqslant q_n$) implies that

$$\alpha \in \mathbf{Bad} \quad \Leftrightarrow \quad \text{the partial quotients } a_i \text{ are bounded from above.}$$

Thus all quadratic irrationals are in **Bad** since their continued fraction expansions are periodic. In fact, it is conjectured that these are the only algebraic irrationals in **Bad**. Also notice that if $\alpha \in \mathbf{Bad}$ then

$$\frac{q_n}{q_{n-1}} = a_n + [a_{n-1}, a_{n-2}, \ldots, a_1] \geqslant K(\alpha) > 1, \tag{2}$$

where $K(\alpha)$ is an absolute constant; that is, the denominators of the convergents form a *lacunary* sequence and

$$q_n \|q_n \alpha\| \asymp 1; \tag{3}$$

that is, the left-hand side is bounded from above and below by constants independent of n. We will make use of all these elementary facts later. For further details and proofs see [4], [11], [12].

It is clear from the above discussion that **Bad** is uncountable. A simple consequence of a fundamental result due to Khintchine is that **Bad** is a set of zero Lebesgue measure.

KHINTCHINE'S THEOREM (1924). *Let ψ be a real positive function and let $W(\psi) := \{x \in I : \|qx\| \leqslant \psi(q)$ for infinitely many $q \in \mathbf{N}\}$ denote the set of ψ-well approximable numbers. If $q\psi(q)$ is decreasing, then*

$$|W(\psi)| = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} \psi(q) < \infty, \\ 1 & \text{if } \sum_{q=1}^{\infty} \psi(q) = \infty. \end{cases}$$

Now let $\psi(q) = 1/q \log q$. Then the sum in Khintchine's theorem diverges and $|W(\psi)| = 1$, which implies that $|I \setminus W(\psi)| = 0$. Clearly $\mathbf{Bad} \subset I \setminus W(\psi)$, and so **Bad** is a set of Lebesgue measure zero. In terms of dimension, however, the set of badly approximable numbers is maximal in that it has the same dimension as the unit interval. A result of Jarník (1928) states that $\dim \mathbf{Bad} = 1$, where $\dim X$ denotes the Hausdorff dimension of the set X (see §2).

Notation. To simplify notation the symbols $\ll$ and $\gg$ will be used to indicate an inequality with an unspecified positive multiplicative constant. If $a \ll b$ and $a \gg b$ we write $a \asymp b$, and say that the quantities a and b are comparable.

1.2. The problem: Littlewood's conjecture

For any pair of real numbers $(\alpha,\beta)\in I$, there exist infinitely many $q\in\mathbf{N}$ such that $\|q\alpha\|\,\|q\beta\|\leqslant q^{-1}$. This is a simple consequence of Dirichlet's theorem and the trivial fact that $\|x\|<1$ for any x. For any arbitrary $\varepsilon>0$, the problem of whether or not the statement remains true by replacing the right-hand side of the inequality by εq^{-1} now arises. This is precisely the content of Littlewood's conjecture.

LITTLEWOOD'S CONJECTURE. *For any pair* $(\alpha,\beta)\in I$,

$$\liminf_{q\to\infty} q\,\|q\alpha\|\,\|q\beta\|=0.$$

In view of Hurwitz's theorem, the analogous conjecture in the one-dimensional setting is clearly false. In the simultaneous situation, however, very little seems to be known. We make the following simple observations:

(i) *The conjecture is true for pairs* (α,β) *when either* α *or* β *lie in a set of full Lebesgue measure.* This follows at once from Khintchine's theorem. In fact, one has that for all α and almost all $\beta\in I$,

$$q\,\|q\alpha\|\,\|q\beta\|\leqslant\frac{1}{\log q}\quad\text{infinitely often.}$$

(Gallagher [3] has shown that for almost all pairs (α,β) the right-hand side of the above inequality can be replaced by $1/(\log q)^2$.)

(ii) *The conjecture is true for pairs* (α,β) *when either* α *or* β *are not in* **Bad**. Suppose $\beta\notin\mathbf{Bad}$ and consider its convergents p_n/q_n. It follows from the right-hand side of inequality (1) that $q_n\,\|q_n\alpha\|\,\|q_n\beta\|\leqslant 1/a_{n+1}$ for all n. Since β is not badly approximable the partial quotients a_i are unbounded, and the conjecture follows.

In view of (ii) *we assume without loss of generality that both* α *and* β *are in* **Bad**.

To our knowledge the following is the only known 'deep' result regarding Littlewood.

CASSELS AND SWINNERTON-DYER (1955). *If* α,β *are both cubic irrationals in the same cubic field then the conjecture is true.*

This result was subsequently sharpened by Peck [8] who showed that for α,β both cubic irrationals (in the same cubic field) the inequality $q\,\|q\alpha\|\,\|q\beta\|\leqslant 1/\log q$ is satisfied infinitely often. As mentioned in the previous section, it is conjectured that the only algebraic irrationals which are badly approximable are the quadratic irrationals. Of course, if this conjecture is true then the Cassels and Swinnerton-Dyer result follows immediately. In any case, given our current state of knowledge the result of Cassels and Swinnerton-Dyer sheds no light on the following simple and natural question.

Question. Given $\alpha \in \mathbf{Bad}$, are there any independent $\beta \in \mathbf{Bad}$ so that Littlewood's conjecture is true for the pair (α, β)?

Recall that two real numbers α and β are said to be *independent* if $1, \alpha, \beta$ are linearly independent over the field of rationals. It is easy to show that if α and β are not independent then the inequality $q\|q\alpha\|\,\|q\beta\| \ll 1/q$ is satisfied infinitely often, and in this case Littlewood's conjecture is obviously true. We show that the answer to the above question is 'yes', and moreover that in terms of dimension the choice of β is in fact maximal.

THEOREM 1. *Given $\alpha \in \mathbf{Bad}$, there exists a subset $\mathbf{G}(\alpha)$ of $\mathbf{Bad}$ with $\dim \mathbf{G}(\alpha) = 1$, such that for any $\beta \in \mathbf{G}(\alpha)$,*

$$q\,\|q\alpha\|\,\|q\beta\| \leqslant \frac{1}{\log q} \quad \textit{infinitely often.}$$

A simple consequence of this is

COROLLARY 1. *For $\beta \in \mathbf{G}(\alpha)$, the pair (α, β) satisfy Littlewood's conjecture.*

It is worth pointing out that the theorem gives rise to pairs (α, β) which satisfy the conjecture with an explicit 'rate of approximation' or 'error' function of $1/\log q$. The corollary is of course obvious once the theorem has been established. We have stated it separately, however, since there is a short, independent proof which we give towards the end of the paper.

The main strategy behind the proof of the theorem is simple enough. Given $\alpha \in \mathbf{Bad}$ we consider the sequence q_n of denominators arising from its convergents. In view of (3), we always have that $q_n\|q_n\alpha\| \asymp 1$. We then show that there are $\beta \in \mathbf{Bad}$ for which $\|q_n\beta\| \leqslant 1/\log q_n$ for infinitely many n, and that the set of such β is of full dimension.

As our next result shows, there is a disadvantage with the above strategy.

THEOREM 2. *Given $\alpha \in \mathbf{Bad}$ and $\lambda \in (0,1)$, there exists a subset $\mathbf{B}_\lambda(\alpha)$ of $\mathbf{Bad}$ with $\dim \mathbf{B}_\lambda(\alpha) = \lambda$, such that for any $\beta \in \mathbf{B}_\lambda(\alpha)$,*

$$\|q_n\beta\| \geqslant \delta \quad \textit{for all } n \in \mathbf{N},$$

where $\delta = \delta(\alpha, \lambda) > 0$ is a constant, and q_n is the denominator of the n-th convergent of α.

The upshot of this is that there is absolutely no hope of proving Littlewood's conjecture by simply looking at the convergents. Apparently, this has been known for some time, as M. Dodson has informed us, and the proof of this fact alone is not too difficult. Nevertheless, the properties of the convergents turn out to be enough for establishing our main result—Theorem 1.

The proof of Theorem 2 follows from the fact that the sequence of denominators q_n associated with a badly approximable number is lacunary (see equation (2)). Given any lacunary sequence $t_n/t_{n-1}\geqslant k>1$ and $\lambda\in(0,1)$, we can find a set $\mathbf{B}'_\lambda(k)$ of dimension at least λ for which $\|t_n\beta\|\geqslant\delta(k,\lambda)>0$ for every $\beta\in\mathbf{B}'_\lambda(k)$. This is proved in [9]. By slightly modifying the proof there one can also ensure that the numbers β are badly approximable. This is a straightforward intervals construction and we omit the details.

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2. Material required for the proof

The proof of Theorem 1 makes essential use of particular measures constructed by Kaufman [6] which are supported on natural Cantor subsets of **Bad**. Basically, these measures have the property that their Fourier transforms vanish at infinity. We begin, however, with a short section on Hausdorff measure and dimension in order to establish some notation.

2.1. Hausdorff measure and dimension

The Hausdorff dimension of a non-empty subset X of k-dimensional Euclidean space $\mathbf{R}^k$ is an aspect of the size of X that can discriminate between sets of Lebesgue measure zero.

For $\varrho>0$, a countable collection $\{C_i\}$ of Euclidean cubes in $\mathbf{R}^k$ with sidelength $l(C_i)\leqslant\varrho$ for each i such that $X\subset\bigcup_i C_i$ is called a ϱ-cover for X. Let s be a non-negative number and define

$$\mathcal{H}^s_\varrho(X):=\inf\left\{\sum_i l^s(C_i):\{C_i\}\text{ is a }\varrho\text{-cover of }X\right\},$$

where the infimum is taken over all possible ϱ-covers of X. The *s-dimensional Hausdorff measure* $\mathcal{H}^s(X)$ of X is defined by

$$\mathcal{H}^s(X)=\lim_{\varrho\to0}\mathcal{H}^s_\varrho(X)=\sup_{\varrho>0}\mathcal{H}^s_\varrho(X),$$

and the *Hausdorff dimension* $\dim X$ of X by

$$\dim X=\inf\{s:\mathcal{H}^s(X)=0\}=\sup\{s:\mathcal{H}^s(X)=\infty\}.$$

Strictly speaking, in the standard definition of Hausdorff measure the ϱ-cover by cubes is replaced by non-empty subsets in $\mathbf{R}^k$ with diameter at most ϱ. It is easy to check that the resulting measure is comparable to $\mathcal{H}^s$ defined above, and thus the Hausdorff dimension is the same in both cases. Further details and alternative definitions of Hausdorff measure and dimension can be found in [2], [7].

We will need to calculate lower bounds for the dimension of various subsets of **Bad**. A general and classical method for obtaining a lower bound for the Hausdorff dimension of an arbitrary set is the following mass distribution principle which is essentially the easy half of Frostman's lemma [7].

LEMMA (mass distribution principle). *Let m be a probability measure supported on a subset X of $\mathbf{R}^k$. Suppose that there are positive constants c, s and l_0 such that*

$$m(C) \leqslant c l^s(C),$$

for any cube C with sidelength $l(C) \leqslant l_0$. If F is a subset of X with $m(F)=\lambda>0$ then $\mathcal{H}^s(F) \geqslant \lambda/c$. In particular, $\dim F \geqslant s$.

Proof. If $\{C_i\}$ is a ϱ-cover of F with $\varrho \leqslant l_0$ then

$$\lambda = m(F) = m\Big(\bigcup_i C_i\Big) \leqslant \sum_i m(C_i) \leqslant c \sum_i l^s(C_i).$$

It follows that $\mathcal{H}^s_\varrho(F) \geqslant \lambda/c$ for any $\varrho \leqslant l_0$. On letting $\varrho \to 0$, the quantity $\mathcal{H}^s_\varrho(F)$ increases, and so we obtain the required result. Since $\mathcal{H}^s(F)>0$, the last part, $\dim F \geqslant s$, follows immediately from the definition of Hausdorff dimension. □

We will also work with the Fourier transform of a measure. The Fourier transform of a measure m supported on a subset X of $\mathbf{R}$ is defined by

$$\widehat{m}(t) := \int_X \exp(2\pi i t x)\, dm(x), \quad t \in \mathbf{R}.$$

The decay rate of the transform is related to lower bounds for the dimension of X. We will not require the relationship, but for completeness we mention that if $|\widehat{m}(t)| \leqslant c|t|^{-\eta}$ for some $\eta>0$ then $\dim X \geqslant \min\{1, 2\eta\}$. Further details and references can be found in [2], [7].

2.2. The set F_N and Kaufman's measure

Let F_N denote the set of real numbers in the unit interval with partial quotients bounded above by $N \in \mathbf{N}$. Thus

$$F_N := \{x \in I : x = [a_1, a_2, \ldots] \text{ with } a_i \leqslant N \text{ for all } i\}.$$

By definition, the set F_N is a subset of **Bad**. Let $I^{(n)}$ denote those numbers in I whose first n partial quotients $a_1, ..., a_n$ are bounded above by N. Then $F_N = \bigcap_{n=1}^{\infty} I^{(n)}$. Thus F_N is a Cantor-type set and it is this structure which is utilized to determine the dimension of F_N. Let $d(N)$ denote the Hausdorff dimension of F_N. The following estimates are due to Jarník [5]. For $N > 8$,

$$1 - \frac{4}{N \log 2} \leqslant d(N) \leqslant 1 - \frac{1}{8N \log N}.$$

A simple consequence of this is that $\dim \mathbf{Bad} = 1$ since $1 \geqslant \dim \mathbf{Bad} \geqslant d(N) \to 1$ as $N \to \infty$.

The important point for us is the existence of a probability measure μ supported on F_N with the two key properties:

(P1) For $s < d(N)$, there exist positive constants c and l_0 such that

$$\mu(C) \leqslant c l^s(C),$$

for all intervals C with length $l(C) \leqslant l_0$.

(P2) For $N \geqslant 3$ the Fourier transform of μ satisfies

$$|\hat{\mu}(t)| \leqslant c|t|^{-\eta} \quad \text{for some } \eta > 0.$$

By making use of the Cantor-type structure of F_N, measures with property (P1) are relatively easy to construct. The existence of a measure with both properties is due to R. Kaufman [6], and in his construction he shows that one may take $\eta = 0.0007$. We will refer to this measure μ as the *Kaufman measure*.

Remark. In his paper, Kaufman points out that it is quite likely that the measure μ also exists for the set F_2. His argument requires, however, that $d(N) > \frac{2}{3}$, which is the case if $N \geqslant 3$. For $N = 2$, it is known that $d(N) = 0.53$ to two decimal places, and so Kaufman's argument fails.

3. The proof of Theorem 1

Given $\alpha \in \mathbf{Bad}$, let q_n denote the denominator of its nth convergent. For $N \geqslant 3$ let

$$\mathbf{G}_N(\alpha) := \{\beta \in F_N : \|q_n \beta\| \leqslant 1/\log q_n \text{ for infinitely many } n \in \mathbf{N}\}.$$

Since $q_n \|q_n \alpha\| \asymp 1$ (see (3)), it follows that for any $\beta \in \mathbf{G}_N(\alpha)$,

$$q_n \|q_n \alpha\| \, \|q_n \beta\| \leqslant \frac{1}{\log q_n} \quad \text{infinitely often;}$$

that is, the pair (α,β) satisfies Littlewood's conjecture with the rate of approximation as stated in the theorem.

For the moment let us assume that $\mu(\mathbf{G}_N(\alpha))>0$, where μ is the Kaufman measure. Then property (P1) of the measure together with the mass distribution principle implies that $\dim \mathbf{G}_N(\alpha)\geqslant s$ for any $s<d(N)$. By continuity, on letting $s\to d(N)$ from below we obtain the lower bound result

$$\dim \mathbf{G}_N(\alpha)\geqslant d(N).$$

Trivially, $\dim \mathbf{G}_N(\alpha)\leqslant d(N)$ since $\mathbf{G}_N(\alpha)\subset F_N$. Hence $\dim \mathbf{G}_N(\alpha)=d(N)$. Now let

$$\mathbf{G}(\alpha):=\{\beta\in\mathbf{Bad}:\|q_n\beta\|\leqslant 1/\log q_n \text{ for infinitely many } n\in\mathbf{N}\}.$$

By Jarník's estimates for $d(N)$ (see §2.2) it follows that for $N>8$,

$$1\geqslant \dim \mathbf{G}(\alpha)\geqslant \dim \mathbf{G}_N(\alpha)\geqslant 1-\frac{4}{N\log 2}.$$

On letting $N\to\infty$ we conclude that $\dim \mathbf{G}(\alpha)=1$, and this completes the proof of the theorem assuming of course that $\mu(\mathbf{G}_N(\alpha))>0$—this we now prove.

3.1. Proof of the claim $\mu(\mathbf{G}_N(\alpha))>0$

Let ψ be a real positive decreasing function such that $q\psi(q)\to 0$ as $q\to\infty$. For $q\in\mathbf{N}$ let

$$E_q(\psi):=[0,\psi(q))\cup\bigcup_{p=1}^{q-1}B(p/q,\psi(q))\cup(1-\psi(q),1],$$

where $B(c,r)$ is an interval centered at c with radius r. We now estimate the Kaufman measure of $E_q(\psi)$. The following result shows that if the interval width determined by the function ψ is sufficiently large then the Kaufman measure of $E_q(\psi)$ is essentially equal to the Lebesgue measure of $E_q(\psi)$, that is, to the total length of the disjoint intervals defining $E_q(\psi)$.

LEMMA 1. *For $N\geqslant 3$,*

$$\mu(E_q(\psi))=2q\psi(q)+O(q^{-\eta/2}).$$

Proof. We fix q and for the sake of clarity put $\delta:=\psi(q)$. Let $\chi_\delta:\mathbf{R}\to\mathbf{R}$ be the characteristic function defined by

$$\chi_\delta(x)=\begin{cases}1 & \text{if } \|x\|\leqslant\delta,\\ 0 & \text{if } \|x\|>\delta,\end{cases}$$

and let $\chi^+_{\delta,\varepsilon}\colon \mathbf{R}\to\mathbf{R}$ be the continuous approximation of χ_δ given by

$$\chi^+_{\delta,\varepsilon}(x)=\begin{cases}1 & \text{if } \|x\|\leqslant\delta,\\ 1+(\delta-\|x\|)/\varepsilon & \text{if } \delta<\|x\|\leqslant\delta+\varepsilon,\\ 0 & \text{if } \|x\|>\delta+\varepsilon,\end{cases}\tag{4}$$

where $0<\varepsilon\leqslant\delta$. The function $\chi^+_{\delta,\varepsilon}$ is normally referred to as the 'upper smoothed' characteristic function, and obviously $\chi^+_{\delta,\varepsilon}(x)\geqslant\chi_\delta(x)$ for all x in $\mathbf{R}$. Clearly, $\chi^+_{\delta,\varepsilon}$ is a periodic function with period 1. Next consider the function $W^+_{\delta,\varepsilon}$ defined by

$$W^+_{\delta,\varepsilon}(x):=\left(\sum_{p=0}^{q-1}\delta_{p/q}(x)\right)*\chi^+_{\delta,\varepsilon}(x),\tag{5}$$

where as usual $*$ denotes convolution and δ_a is the Dirac delta-function. It is easily verified that

$$W^+_{\delta,\varepsilon}(x)=\sum_{p=0}^{q-1}\chi^+_{\delta,\varepsilon}(x-p/q),$$

and so it follows that

$$\mu(E_q(\psi))\leqslant\int_0^1 W^+_{\delta,\varepsilon}(x)\,d\mu(x).$$

We now proceed to evaluate the integral by considering the Fourier series expansion of $W^+_{\delta,\varepsilon}$. For $k\in\mathbf{Z}$, let $\widehat{\chi}^+_{\delta,\varepsilon}(k)$ and $\widehat{W}^+_{q,\delta}(k)$ denote the kth Fourier coefficient of $\chi^+_{\delta,\varepsilon}$ and $W^+_{\delta,\varepsilon}$ respectively. A straightforward calculation shows that

$$\widehat{\chi}^+_{\delta,\varepsilon}(k)=\begin{cases}2\delta+\varepsilon & \text{if } k=0,\\ \dfrac{\cos(2\pi k\delta)-\cos(2\pi k(\delta+\varepsilon))}{2\pi^2k^2\varepsilon} & \text{if } k\neq0.\end{cases}\tag{6}$$

Since $W^+_{\delta,\varepsilon}$ is defined via convolution, we have that

$$\widehat{W}^+_{q,\delta}(k):=\sum_{p=0}^{q-1}\hat{\delta}_{p/q}(k)\cdot\widehat{\chi}^+_{\delta,\varepsilon}(k).$$

Trivially, $\hat{\delta}_{p/q}(k)=\exp(-2\pi ikp/q)$. Thus it follows from (6) that for $k\neq0$,

$$\widehat{W}^+_{q,\delta}(k)=\begin{cases}\dfrac{q(\cos(2\pi k\delta)-\cos(2\pi k(\delta+\varepsilon)))}{2\pi^2k^2\varepsilon} & \text{if } q\mid k,\\ 0 & \text{if } q\nmid k,\end{cases}\tag{7}$$

and for $k=0$,

$$\widehat{W}^+_{q,\delta}(0)=2\delta q+q\varepsilon.\tag{8}$$

Clearly $\sum_{k\in\mathbf{Z}}|\widehat{W}^+_{q,\delta}(k)|<\infty$, and so the Fourier series

$$\sum_{k\in\mathbf{Z}}\widehat{W}^+_{q,\delta}(k)\exp(2\pi ikx)$$

converges uniformly to $W^+_{\delta,\varepsilon}(x)$ for all x. An immediate consequence of the uniform convergence is a version of Parseval's identity for measures:

$$\int_0^1 W^+_{\delta,\varepsilon}(x)\,d\mu(x)=\sum_{k\in\mathbf{Z}}\widehat{W}^+_{q,\delta}(k)\,\hat{\mu}(-k)=2\delta q+q\varepsilon+\sum_{k\in\mathbf{Z}\setminus\{0\}}\widehat{W}^+_{q,\delta}(k)\,\hat{\mu}(-k).$$

The last equality follows from (8) together with the fact that $\hat{\mu}(0)=1$. Now property (P2) of the Kaufman measure together with (7) implies that

$$\Bigg|\sum_{k\in\mathbf{Z}\setminus\{0\}}\widehat{W}^+_{q,\delta}(k)\,\hat{\mu}(-k)\Bigg|\leqslant\frac{2c}{\pi^2q^{1+\eta}\varepsilon}\sum_{m=1}^{\infty}\frac{1}{m^{2+\eta}}<\frac{c}{3q^{1+\eta}\varepsilon}.$$

Hence we obtain the upper bound inequality

$$\mu(E_q(\psi))\leqslant\int_0^1 W^+_{\delta,\varepsilon}(x)\,d\mu(x)\leqslant 2\delta q+q\varepsilon+\frac{c}{3q^{1+\eta}\varepsilon}. \tag{9}$$

To obtain a lower bound estimate we consider the 'lower smoothed' characteristic function $\chi^-_{\delta,\varepsilon}$ given by

$$\chi^-_{\delta,\varepsilon}(x)=\begin{cases}1 & \text{if } \|x\|\leqslant\delta-\varepsilon,\\ (\delta-\|x\|)/\varepsilon & \text{if } \delta-\varepsilon<\|x\|\leqslant\delta,\\ 0 & \text{if } \|x\|>\delta.\end{cases}$$

The function $W^-_{\delta,\varepsilon}$ is defined in the obvious way:

$$W^-_{\delta,\varepsilon}(x):=\bigg(\sum_{p=0}^{q-1}\delta_{p/q}(x)\bigg)*\chi^-_{\delta,\varepsilon}(x).$$

It can be readily verified that for $k\neq0$ the corresponding Fourier coefficients are

$$\widehat{W}^-_{q,\delta}(k)=\begin{cases}\dfrac{q(\cos(2\pi k(\delta-\varepsilon))-\cos(2\pi k\delta))}{2\pi^2k^2\varepsilon} & \text{if } q\mid k,\\ 0 & \text{if } q\nmid k,\end{cases}$$

and for $k=0$,

$$\widehat{W}^-_{q,\delta}(0)=2\delta q-q\varepsilon.$$

The same argument as before now leads to the lower bound

$$\mu(E_q(\psi)) \geqslant \int_0^1 W^-_{\delta,\varepsilon}(x)\,d\mu(x) \geqslant 2\delta q - q\varepsilon - \frac{c}{3q^{1+\eta}\varepsilon}.$$

This together with (9) implies that

$$\mu(E_q(\psi)) = 2\delta q + O(q\varepsilon + c/q^{1+\eta}\varepsilon). \tag{10}$$

The lemma now follows on setting $\varepsilon = 1/q^{1+\eta/2}$. □

We now put $q=q_n$ (the denominator of the nth convergent of α) and $\psi(q_n) = 1/q_n \log q_n$. We will often make use of the fact that

$$K_1 \leqslant \frac{q_n}{q_{n-1}} \leqslant K_2, \tag{11}$$

where $K_2 > K_1 > 1$ are constants. This follows from the fact that the sequence q_n is lacunary and that it satisfies the recurrence $q_n = a_n q_{n-1} + q_{n-2}$ for $n \geqslant 2$. Also let

$$E_n := E_{q_n}(\psi(q_n)) = \bigcup_{p=0}^{q_n} B(p/q_n, 1/q_n \log q_n) \cap I.$$

By definition, $\mathbf{G}_N(\alpha)$ is precisely the set of real numbers in F_N which lie in infinitely many of the sets E_n; that is,

$$\mathbf{G}_N(\alpha) = F_N \cap \limsup_{n\to\infty} E_n := F_N \cap \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} E_n.$$

Recall that our aim is to show that $\mu(\mathbf{G}_N(\alpha)) > 0$. Note that since μ is supported on F_N we trivially have that

$$\mu(\mathbf{G}_N(\alpha)) = \mu\big(\limsup_{n\to\infty} E_n\big).$$

By Lemma 1,

$$\mu(E_n) \asymp \frac{1}{\log q_n} \asymp \frac{1}{n} \tag{12}$$

since $K_1^n \leqslant q_n \leqslant K_2^n$—just iterate (11) n times. Hence

$$\sum_{n=1}^{\infty} \mu(E_n) \asymp \sum_{n=1}^{\infty} \frac{1}{n} = \infty. \tag{13}$$

This is a good sign in that if the above sum was to converge, then a simple consequence of the Borel–Cantelli lemma from probability theory is that $\mu(\mathbf{G}_N(\alpha)) = 0$. The divergent sum alone, however, is not enough to ensure positive measure. We require independence of some sort. The following quasi-independence on average turns out to be sufficient.

LEMMA 2 (quasi-independence on average). *There exists a constant $C>1$ such that for Q sufficiently large,*

$$\sum_{m,n=1}^{Q} \mu(E_m \cap E_n) \leqslant C\Big(\sum_{n=1}^{Q} \mu(E_n)\Big)^2.$$

We shall prove the lemma in the next subsection. In order to complete the proof of the claim, we require the following result ([12, p. 17]), which is a generalization of the divergent part of the standard Borel–Cantelli lemma.

PROPOSITION. *Let (X, A, m) be a probability space, and let $A_n \in A$ be a sequence of measurable sets such that $\sum_{n=1}^{\infty} m(A_n)=\infty$. Then*

$$m\big(\limsup_{n\to\infty} A_n\big) \geqslant \limsup_{Q\to\infty} \frac{(\sum_{n=1}^{Q} m(A_n))^2}{\sum_{m,n=1}^{Q} m(A_m \cap A_n)}.$$

In our situation, the proposition together with the divergent sum (13) and the quasi-independence on average result implies that

$$\mu(\mathbf{G}_N(\alpha)) \geqslant 1/C > 0.$$

This completes the proof of the claim assuming of course the quasi-independence on average result—this we now prove.

3.2. Proof of Lemma 2: quasi-independence on average

In view of (12) and (13), it is sufficient to prove that for Q sufficiently large,

$$\sum_{1\leqslant m<n\leqslant Q} \mu(E_m \cap E_n) \ll (\log Q)^2. \tag{14}$$

We begin by proving the easier analogous inequality for Lebesgue measure λ.

LEMMA 3. *There exists a constant $C>1$ such that for Q sufficiently large,*

$$\sum_{1\leqslant m<n\leqslant Q} \lambda(E_m \cap E_n) \leqslant C(\log Q)^2.$$

Proof. Recall that E_m is a disjoint union of intervals within $I:=[0,1]$, centered at rationals with denominator q_m and radius $\psi(q_m):=1/q_m \log q_m$. Thus, for any $m<n$,

$$\lambda(E_m \cap E_n) = \sum_{p=0}^{q_m} \lambda(B(p/q_m, \psi(q_m)) \cap E_n) \leqslant \sum_{p=0}^{q_m} \sum_{a\in A(p,n)} \lambda(B(a/q_n, \psi(q_n))),$$

where $A(p,n):=\{0\leqslant a\leqslant q_n : B(p/q_m,\psi(q_m))\cap B(a/q_n,\psi(q_n))\neq\varnothing\}$. The trivial fact that the distance between any two consecutive rationals with denominator q is $1/q$ implies that the cardinality of $A(p,n)$ is at most $q_n\lambda(B(p/q_m,\psi(q_m)))+2$. So, for any $m<n$ we obtain the upper bound

$$\lambda(E_m\cap E_n)\leqslant\left(\frac{2q_n}{q_m\log q_m}+2\right)\frac{2}{q_n\log q_n}(q_m+1)=\frac{8}{\log q_m\log q_n}+\frac{8q_m}{q_n\log q_n}.$$

In particular, if $q_n>q_m\log q_m$ then

$$\lambda(E_m\cap E_n)\leqslant\frac{16}{\log q_m\log q_m}\asymp\frac{1}{mn}. \tag{15}$$

Suppose for the moment that m is fixed, and choose t to be the unique integer such that $q_{m+t-1}\leqslant q_m\log q_m<q_{m+t}$. On iterating backwards t times the fact that $K_1\leqslant q_{m+t}/q_{m+t-1}\leqslant K_2$ (this is just (11)), one finds that $t\asymp\log m$. This implies that there is a constant c such that if $n>m+c\log m$ then $q_n>q_m\log q_m$, and so (15) holds. For n in the range $m<n\leqslant m+c\log m$ we make use of the trivial estimate that

$$\lambda(E_m\cap E_n)\leqslant\lambda(E_n)=\frac{2}{\log q_n}\asymp\frac{1}{n}.$$

It now follows that

$$\sum_{1\leqslant m<n\leqslant Q}\lambda(E_m\cap E_n)\ll\sum_{m=1}^{Q}\sum_{n=m}^{m+c\log m}\frac{1}{n}+\sum_{m=1}^{Q}\sum_{n=1}^{Q}\frac{1}{mn}\ll(\log Q)^2,$$

as required. □

Remark. We will apply the lemma to sets tE_n obtained via scaling E_n by a positive factor t; that is, $tE_n:=\bigcup_{p=0}^{q_n}B(p/q_n,t/q_n\log q_n)\cap I$. The above lemma is easily seen to hold for the 'scaled' sets tE_n. The constant C will of course depend on t.

Before proceeding with the proof of (14) we introduce some further notation. For $n\in\mathbf{N}$, let $\delta_n:=1/q_n\log q_n$ and write

$$W_{\delta_n}:=W^+_{\delta_n,\delta_n},$$

where $W^+_{\delta_n,\delta_n}$ is given by (5); that is, we put $\delta=\varepsilon=\delta_n$ in the general definition of $W^+_{\delta,\varepsilon}$ and of course $q=q_n$. In view of (7) and (8) we have that for $k\neq0$,

$$\widehat{W}_{\delta_n}(k)=\begin{cases}\dfrac{q_n(\cos(2\pi k\delta_n)-\cos(2\pi k(2\delta_n)))}{2\pi^2k^2\delta_n} & \text{if } q_n\mid k,\\ 0 & \text{if } q_n\nmid k,\end{cases} \tag{16}$$

and for $k=0$,

$$\widehat{W}_{\delta_n}(0)=3\delta_n q_n=\frac{3}{\log q_n}. \tag{17}$$

By definition, for any pair of natural numbers m, n we have that

$$\mu(E_m\cap E_n)\leqslant\int_0^1 W_{\delta_m}(x)W_{\delta_n}(x)\,d\mu(x). \tag{18}$$

Set $W_{\delta_m;\delta_n}(x):=W_{\delta_m}(x)W_{\delta_n}(x)$. Our aim is to obtain a sufficiently strong upper bound for the above integral. This we do by considering the Fourier series expansion of the function $W_{\delta_m;\delta_n}$. It is easily verified that for any $k\in\mathbf{Z}$,

$$\widehat{W}_{\delta_m;\delta_n}(k):=\int_0^1 W_{\delta_m}(x)W_{\delta_n}(x)\exp(-2\pi ikx)\,dx=\sum_{j\in\mathbf{Z}}\widehat{W}_{\delta_m}(j)\widehat{W}_{\delta_n}(k-j), \tag{19}$$

and moreover that the Fourier series $\sum_{k\in\mathbf{Z}}\widehat{W}_{\delta_m;\delta_n}(k)\exp(2\pi ikx)$ converges uniformly to $W_{\delta_m;\delta_n}(x)$ for all x. Thus, by Parseval's identity,

$$\begin{aligned}\int_0^1 W_{\delta_m}(x)W_{\delta_n}(x)\,d\mu(x)&=\sum_{k\in\mathbf{Z}}\widehat{W}_{\delta_m;\delta_n}(k)\,\hat{\mu}(-k)\\&=\widehat{W}_{\delta_m;\delta_n}(0)+\sum_{k\in\mathbf{Z}\setminus\{0\}}\widehat{W}_{\delta_m;\delta_n}(k)\,\hat{\mu}(-k).\end{aligned} \tag{20}$$

We consider the two terms of (20) separately. By definition,

$$\widehat{W}_{\delta_m;\delta_n}(0):=\int_0^1 W_{\delta_m}(x)W_{\delta_n}(x)\,dx\leqslant\lambda(2E_m\cap 2E_n),$$

where λ is Lebesgue measure. Hence, by Lemma 3 (if necessary, see the remark straight after its proof) we have that

$$\sum_{1\leqslant m<n\leqslant Q}\widehat{W}_{\delta_m;\delta_n}(0)\ll(\log Q)^2. \tag{21}$$

Regarding the second term of (20), it follows from (16), (17) and (19) that

$$\begin{aligned}\sum_{k\in\mathbf{Z}\setminus\{0\}}\widehat{W}_{\delta_m;\delta_n}(k)\,\hat{\mu}(-k)&=\sum_{\substack{s,t\in\mathbf{Z}\\ sq_m+tq_n\neq 0}}\widehat{W}_{\delta_m}(sq_m)\widehat{W}_{\delta_n}(tq_n)\,\hat{\mu}(-(sq_m+tq_n))\\&=3\delta_m q_m\sum_{t\in\mathbf{Z}\setminus\{0\}}\widehat{W}_{\delta_n}(tq_n)\,\hat{\mu}(-tq_n)\\&\quad+3\delta_n q_n\sum_{s\in\mathbf{Z}\setminus\{0\}}\widehat{W}_{\delta_m}(sq_m)\,\hat{\mu}(-sq_m)\\&\quad+\sum_{\substack{s,t\in\mathbf{Z}\setminus\{0\}\\ sq_m+tq_n\neq 0}}\widehat{W}_{\delta_m}(sq_m)\widehat{W}_{\delta_n}(tq_n)\,\hat{\mu}(-(sq_m+tq_n)).\end{aligned} \tag{22}$$

Now let A_{mn}, B_{mn} and C_{mn} denote the first, second and third terms of (22) respectively. Property (P2) of the Kaufman measure together with (11) and (16) give rise to the estimates

$$|A_{mn}| \leqslant \frac{c\log q_n}{\log q_m q_n^\eta} \ll \frac{n}{m(K_1^\eta)^n}, \quad |B_{mn}| \leqslant \frac{c\log q_m}{\log q_n q_m^\eta} \ll \frac{m}{n(K_1^\eta)^m},$$

where $K_1>1$, and

$$|C_{mn}| \ll \sum_{\substack{s,t\in\mathbf{Z}\setminus\{0\}\\ sq_m+tq_n\neq 0}} \frac{m}{s^2}\frac{n}{t^2}|sq_m+tq_n|^{-\eta}. \tag{23}$$

Notice that in the above estimates the difference of the cosines in (16) are being estimated trivially. Hence,

$$\sum_{1\leqslant m<n\leqslant Q} |A_{mn}| \ll \log Q \quad \text{and} \quad \sum_{1\leqslant m<n\leqslant Q} |B_{mn}| \ll \log Q. \tag{24}$$

We now deal with C_{mn}. Suppose for the moment that in (23) we only sum over s and t with the same sign. Then $|sq_m+tq_n|\geqslant 2(sq_mtq_n)^{1/2}$, and the corresponding sum over $1<m<n\leqslant Q$ is easily seen to be bounded above by a constant. The upshot of this is that we now only need to consider the sum

$$\sum_{\substack{s,t\in\mathbf{N}\\ sq_m-tq_n\neq 0}} \frac{m}{s^2}\frac{n}{t^2}|sq_m-tq_n|^{-\eta}.$$

Fix $0<\varepsilon<1$, and write the above sum as

$$\sum_{\substack{s,t\in\mathbf{N}:\,st<(mn)^{2+\varepsilon}\\ sq_m-tq_n\neq 0}} \frac{m}{s^2}\frac{n}{t^2}|sq_m-tq_n|^{-\eta} + \sum_{\substack{s,t\in\mathbf{N}:\,st\geqslant(mn)^{2+\varepsilon}\\ sq_m-tq_n\neq 0}} \frac{m}{s^2}\frac{n}{t^2}|sq_m-tq_n|^{-\eta}. \tag{25}$$

The latter sum is estimated as

$$\sum_{\substack{s,t\in\mathbf{N}:\,st\geqslant(mn)^{2+\varepsilon}\\ sq_m-tq_n\neq 0}} \frac{m}{s^2}\frac{n}{t^2}|sq_m-tq_n|^{-\eta} \leqslant \frac{1}{mn}\sum_{s,t\in\mathbf{N}}(st)^{-1-\varepsilon} \ll (mn)^{-1},$$

since trivially

$$|sq_m-tq_n|^\eta \geqslant 1 > (mn)^{2+\varepsilon}(st)^{-1}.$$

Thus the second sum in (25) when summed over $1\leqslant m<n\leqslant Q$ is bounded above by $(\log Q)^2$. We now deal with the first sum in (25), which we denote by D_{mn}. Then

$$D_{mn} \leqslant \sum_{s=1}^{(mn)^{2+\varepsilon}} \sum_{\substack{t=1\\ sq_m-tq_n\neq 0}}^{(mn)^{2+\varepsilon}/s} \frac{m}{s^2}\frac{n}{t^2}|sq_m-tq_n|^{-\eta}.$$

Since we will eventually be summing over $1 \leqslant m < n \leqslant Q$ we may as well assume that $m<n$. In addition, assume for the moment that $q_n \geqslant n^6 q_m$. Then under the conditions of the double sum above we have that $|sq_m - tq_n| \geqslant \frac{1}{2} tq_n$. Using the same type of argument as in the proof of Lemma 3, we find that there is a positive constant c such that if $n > m + c\log m$ then the inequality $q_n \geqslant n^6 q_m$ is satisfied. Hence

$$\sum_{\substack{1 \leqslant m < n \leqslant Q: \\ n > m + c \log m}} D_{mn} \ll \sum_{1 \leqslant m < n \leqslant Q} \frac{n^2}{(K_1^\eta)^n} \ll 1,$$

and in view of the above discussion we have that

$$\sum_{\substack{1 \leqslant m < n \leqslant Q: \\ n > m + c \log m}} |C_{mn}| \ll (\log Q)^2.$$

This together with (18), (20), (21), (22) and (24) implies that

$$\sum_{\substack{1 \leqslant m < n \leqslant Q: \\ n > m + c \log m}} \mu(E_m \cap E_n) \leqslant (\log Q)^2. \tag{26}$$

For n in the range $m < n \leqslant m + c \log m$ we use the trivial estimate that

$$\mu(E_m \cap E_n) \leqslant \mu(E_n) \asymp \frac{1}{n}.$$

Thus,

$$\sum_{\substack{1 \leqslant m < n \leqslant Q: \\ n < m + c \log m}} \mu(E_m \cap E_n) \leqslant \sum_{m=1}^{Q} \sum_{n=m}^{m + c \log m} \frac{1}{n} \ll (\log Q)^2,$$

which together with (26) gives (14), and so completes the proof of Lemma 2. □

3.3. A few comments: further developments

There are numerous ways in which aspects of this paper could be developed and refined. Here, we mention just two. Let ψ be any real positive decreasing function such that $q\psi(q) \to 0$ as $q \to \infty$, and consider the set

$$\mathbf{G}_N(\alpha, \psi) := \{\beta \in F_N : \|q_n \beta\| \leqslant q_n \psi(q_n) \text{ for infinitely many } n \in \mathbf{N}\}.$$

So with $\psi(q) = 1/q \log q$, this is precisely the set $\mathbf{G}_N(\alpha)$ considered above. It is almost certain that a more careful analysis during the proof of Lemma 2 would lead to a Khintchine-type theorem for the set $\mathbf{G}_N(\alpha, \psi)$ with respect to the Kaufman measure μ supported on F_N. More precisely, let $E_{q_n}(\psi) := \bigcup_{p=0}^{q_n} B(p/q_n, \psi(q_n)) \cap I$. Then

$$\mathbf{G}_N(\alpha, \psi) = F_N \cap \limsup_{n \to \infty} E_{q_n}(\psi).$$

We claim that one should be able to prove a strong enough independence on average result for the sets $E_{q_n}(\psi)$, which would lead to the following zero–one law:

$$\mu(\mathbf{G}_N(\alpha,\psi)) = \begin{cases} 0 & \text{if } \sum_{n=1}^{\infty} \mu(E_{q_n}(\psi)) < \infty, \\ 1 & \text{if } \sum_{n=1}^{\infty} \mu(E_{q_n}(\psi)) = \infty. \end{cases}$$

As usual the convergence case follows immediately. The argument used in this paper only yields positive measure in the divergence case.

Assuming that one is able to establish the above-mentioned Khintchine-type result then it is likely that one can also obtain the following quantitative refinement. For $\beta \in F_N$ and $M \in \mathbf{N}$ let

$$R(\beta, M) := \#\{n \leqslant M : \|q_n\beta\| \leqslant q_n \psi(q_n)\}.$$

Suppose that $\sum_{n=1}^{\infty} \mu(E_{q_n}(\psi)) = \infty$. Then for μ-almost all $\beta \in F_N$,

$$R(\beta, M) \sim \sum_{n=1}^{M} \mu(E_{q_n}(\psi)).$$

Given $\alpha \in \mathbf{Bad}$, such a result would imply that for μ-almost all $\beta \in F_N$,

$$\#\{q \leqslant M : q\,\|q\alpha\|\,\|q\beta\| \leqslant 1/\log q\} \gg \log\log M; \tag{27}$$

or in the language of Theorem 1, there exists a subset $\mathbf{G}(\alpha)$ of $\mathbf{Bad}$ with $\dim \mathbf{G}(\alpha) = 1$ such that for any $\beta \in \mathbf{G}(\alpha)$ inequality (27) is satisfied. We intend to pursue these and related problems in a forthcoming article.

4. An independent proof of Corollary 1

COROLLARY 1. *Given* $\alpha \in \mathbf{Bad}$, *there exists a subset* $\mathbf{G}(\alpha)$ *of* $\mathbf{Bad}$ *with* $\dim \mathbf{G}(\alpha) = 1$, *such that for any* $\beta \in \mathbf{G}(\alpha)$,

$$\liminf_{q \to \infty} q\,\|q\alpha\|\,\|q\beta\| = 0.$$

In this section we give a short, simple proof of the above corollary by applying the following well-known result in the theory of uniform distribution (see [10]). Given a real number x we write $\{x\}$ for the fractional part of x.

THEOREM (Davenport, Erdös and LeVeque). *Let* μ *be a probability measure supported on a subset* X *of* I. *If*

$$\sum_{N=1}^{\infty} \frac{1}{N^3} \sum_{m,n=1}^{N} \hat{\mu}(h(s_m - s_n)) < \infty \tag{28}$$

for all integers $h \neq 0$, *then the sequence* $\{s_n x\}$ *is uniformly distributed for* μ*-almost all* $x \in X$.

To prove the corollary we apply the Davenport–Erdös–LeVeque theorem to the Kaufman measure μ supported on F_N, and take for our sequence s_n the sequence q_n of denominators associated with the convergents of α. Then provided (28) holds for every integer $h\neq 0$ we have that $\{q_n x\}$ is uniformly distributed for μ-almost all $x\in F_N$. But for such x, a simple consequence of uniform distribution is that $\liminf_{n\to\infty}\|q_n x\|=0$. Now let

$$\mathbf{G}_N(\alpha):=\{\beta\in F_N:\liminf_{n\to\infty}\|q_n\beta\|=0\}.$$

Then the above argument shows that

$$\mu(\mathbf{G}_N(\alpha))=1,$$

and clearly for any $\beta\in\mathbf{G}_N(\alpha)$ one has that $\liminf_{n\to\infty} q_n\|q_n\alpha\|\,\|q_n\beta\|=0$. Recall that in proving Theorem 1, the main part was in showing that $\mu(\mathbf{G}_N(\alpha))$ is strictly positive. Here, this turns out to be a simple consequence of the Davenport–Erdös–LeVeque theorem. The corollary now follows on using exactly the same arguments as in the proof of the theorem. It remains to verify that (28) holds for every integer $h\neq 0$.

By property (P2) of the Kaufman measure, for every integer $h\neq 0$,

$$\begin{aligned}\sum_{m,n=1}^{N}\hat{\mu}(h(q_m-q_n)) &:= N+\sum_{\substack{m,n=1\\ m\neq n}}^{N}\hat{\mu}(h(q_m-q_n))\\ &\leqslant N+c\sum_{\substack{m,n=1\\ m\neq n}}^{N}|h(q_m-q_n)|^{-\eta}\\ &\leqslant N+2c\sum_{m=2}^{N}\sum_{n=1}^{m-1}|q_m-q_n|^{-\eta}.\end{aligned}$$

Since the sequence q_m of denominators is a lacunary sequence, for $m>n\geqslant 1$,

$$q_m-q_n\geqslant q_m-q_{m-1}=q_m(1-q_{m-1}/q_m)\geqslant c_1q_m,\quad c_1>0.$$

Another simple consequence of lacunary growth is the existence of a positive constant $K>1$ such that $q_m\geqslant K^{m-1}$. Thus

$$\begin{aligned}\sum_{m,n=1}^{N}\hat{\mu}(h(q_m-q_n)) &\leqslant N+2cc_1^{-\eta}\sum_{m=2}^{N}\sum_{n=1}^{m-1}q_m^{-\eta}\\ &\leqslant N+2cc_1^{-\eta}N\sum_{m=1}^{N-1}(K^m)^{-\eta}\leqslant c_2N.\end{aligned}$$

Hence,

$$\sum_{N=1}^{\infty} \frac{1}{N^3} \sum_{m,n=1}^{N} \hat{\mu}(h(q_m - q_n)) \leqslant c_2 \cdot \tfrac{1}{6}\pi^2,$$

and this shows that (28) holds for every integer $h \neq 0$. □

Remark. Notice that in the course of proving the corollary the only fact that we use regarding the sequence q_n is that it is lacunary. We do not require the fact that the 'denominators' q_n of a badly approximable number also satisfy the upper bound inequality $q_n/q_{n-1} \leqslant K_2$ where $K_2 > 1$ is a constant—see (11). This latter fact is, however, necessary in proving the stronger result (Theorem 1); namely to ensure that $\sum \mu(E_n)$ diverges—see (13).

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