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Complex geometry of convex domains that cover varieties

by

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1. Introduction

The main result of this paper is:

THEOREM 1. *Let Ω be a convex hyperbolic domain in $\mathbf{C}^n$ and suppose there is a subgroup $\Gamma\subset\mathrm{Aut}(\Omega)$ such that*

(Γ1) Γ *is discrete and acts freely* (*each $\gamma\in\Gamma$ is fixed-point free*),

(Γ1) Γ *is co-compact* (*in* Ω).

Then Ω is biholomorphic to a bounded symmetric domain.

The hypothesis that Γ acts freely can now be removed, see the comment on Lemma 11.8.

This confirms a conjecture cited by Yau in [36], p. 140. The hypotheses are equivalent to saying that there is a compact complex manifold M whose universal cover is a convex hyperbolic domain Ω in $\mathbf{C}^n$. Thus, it is a type of uniformization theorem. Regarding the notion of hyperbolicity, see Proposition 2.8, and for a generalization weakening the convexity condition considerably see Theorem 2.6 which follows from Theorem 1 and the results in § 7.

The first part of this paper introduces a new method that given a non-compact automorphism group acting on a domain Ω produces continuous families of automorphisms. One needs some mild regularity hypothesis on the boundary, unless the automorphism group is co-compact, in this case convexity suffices, see Theorem 2.4. The general idea is to use boundary localization, involving rescaling, an idea used by Kuiper and Benzecri, in the context of affine and projective geometry, in the fifties, and many complex analysts in the seventies, including Pincuk, Greene and Krantz and I. Graham. Gromov and Mostow have used it in other contexts. We survey these and other subjects in [6]. Our new contributions here are firstly to prove a distortion theorem for convex holomorphic embeddings and to use this to reduce a complex analysis problem to one in affine geometry. Secondly, we exploit rescaling to produce a continuous family of automorphisms. Thus our particular *technique* of boundary localization is very different from what went before, we dub it the *rescale blow-up* and we believe that it can be applied to attack a very broad class of problems in the geometry of domains. We will elaborate this approach in future articles. A second application of the rescale blow-up is Theorem 7.1.

The second part of this paper does not involve the rescale blow-up technique. The techniques there include a lot of structure theory of Lie groups, group co-homology, as well as complex differential geometry of compact manifolds. One may jump to the conclusion that there is an easy proof of the next theorem, we caution the reader that the hypothesis that Γ is discrete is essential here, and that one must use some fairly deep global ideas to exploit this. It is an open problem to generalize this to the (irreducible, pseudo-convex but) non-convex case. The main result of the second part is:

THEOREM 2. *Let Ω be a convex hyperbolic domain in $\mathbf{C}^n$ and suppose there is a subgroup $\Gamma\subset\mathrm{Aut}(\Omega)$ such that*

(Γ1) *Γ is discrete and acts freely,*

(Γ2) *Γ is co-compact (in Ω).*

Then if $\mathrm{Aut}_0(\Omega)$ is non-trivial,

(1) *Ω has a factor Ω_1 that is biholomorphic to a bounded symmetric domain, i.e. $\Omega=\Omega_1\times\Omega_2$ where Ω_1 is non-trivial and is biholomorphic to a bounded symmetric domain.*

(2) *Γ has a finite index normal subgroup Γ' such that $\Gamma'=\Gamma_1'\times\Gamma_2'$ where $\Gamma'\subset \mathrm{Aut}(\Omega_1)\times\mathrm{Aut}(\Omega_2)$ and $\Gamma_j'=\Gamma'\cap\mathrm{Aut}(\Omega_j)$.*

This paper is divided into three parts, § 1, § 2 comprise 'Part 0'. In § 1 we quickly survey results relating compact, complex manifolds and bounded domains in several complex variables, and relate them to our new results. In § 2 we introduce some notation, review some background material, and state our main results with some comments on the necessity of each hypothesis.

Part 1 includes § 3 to § 8, and introduces the rescale blow-up technique. § 3 explains the idea, and subsequent sections provide detailed proofs.

This technique has many applications; to boundary regularity of holomorphic maps, to non-convexity of embeddings of Teichmüller spaces in $\mathbf{C}^n$, to classifying automorphism group actions on domains, to constructing canonical embeddings of domains, and to estimates of boundary asymptotics of intrinsic metrics. In this paper we confine ourselves to what is perhaps the simplest application; given a non-compact automorphism group acting on a convex domain Ω we produce continuous families of automorphisms.

Part 2 continues the study of cocompact group actions, but from an intrinsic, geometric point of view, as opposed to Part 1 where we relate complex structure to affine structure.

Compact complex manifolds and bounded domains

Complex analysis in several variables has many features that distinguish it from the theory of one complex variable. In complex dimension n, with $n \geqslant 2$, the analytic theory of *compact* varieties is not based on the geometry of domains in $\mathbf{C}^n$, as it is in one dimension, via the uniformization theorem. On the other hand, there are very important examples of domains that cover varieties; the bounded symmetric domains, and Teichmüller spaces (of punctured Riemann surfaces, these have *finite-volume* quotients).

In this paper we solve the following problem: what bounded convex domains in $\mathbf{C}^n$ cover compact complex manifolds? The answer is: only the bounded symmetric domains. In fact our technique leads to a stronger result (see Theorem 2.6, Definition 7.3).

The general study of domains that cover compact varieties commenced in the early fifties; H. Cartan and others [2] constructed automorphic forms to show that quotients of bounded domains are projective algebraic. Baily's extension of Kodaira's embedding theorem to V-manifolds implies that any quotient of a bounded domain Ω by a uniform lattice Γ (a discrete, co-compact group of holomorphic automorphisms) is projective algebraic, [20, 1]. Ω is automatically pseudo-convex by a result of C. L. Siegel. Our conclusion, that Ω is symmetric relies on a stronger hypothesis; that $\partial\Omega$ is h-convex at some point. The relationship of pseudo-convexity and h-convexity has been a subject of interest since the well known paper of Kohn and Nirenberg [23].

In the mid-seventies B. Wong [34] showed that if Ω has smooth boundary and admits a co-compact group of automorphisms (i.e. Ω has a compact quotient), then it is biholomorphically the ball, B^n. J. Vey [32] showed that if Ω is a generalized Siegel domain and admits a discrete co-compact group of automorphisms, Γ, then it is symmetric. The hypotheses in these theorems are quite special; essentially Wong assumes that Ω osculates the ball to second derivatives at a boundary point, and Vey's hypothesis guarantees that $\mathrm{Aut}_0(\Omega)$ is large. In fact, it seems our result is the first to classify automorphism groups on a domain Ω without assuming, in some sense, that Ω is canonically embedded, or close to a canonically embedded domain.

The techniques we develop in this paper may be used to attack a wide class of problems in the geometry of complex domains, their embeddings and automorphisms. However we restrict the focus here to the aforementioned problem and leave other applications to subsequent papers.

The results in this paper (in the complex 2-dimensional case) were obtained as part of the author's doctoral dissertation [5]. The proof has been reorganized slightly in this version and generalized to the n-dimensional case. This subsequent work was done at Columbia University and during one semester which I had the pleasure of spending at MSRI. I would like to thank NSERC for supporting me as a graduate student, Harvard's math department for their hospitality while I did the initial work, and Stanford's for their continuing help.

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2. Convex hyperbolic domains and their automorphism groups

In this section we present some basic definitions and notation, and references for some basic results, then we state our main result and provide some context for the hypotheses.

In this paper D, Ω, will always denote open connected subsets of $\mathbf{C}^n$. We call these sets *domains*. Ω should be thought of as a complex manifold, and more precisely as the natural domain of definition of a ring of holomorphic functions.

Definition 2.1. If $f\colon \Omega_1 \to \Omega_2$ is holomorphic and $\exists g\colon \Omega_2 \to \Omega_1$ such that $g = f^{-1}$, then we say that f is a biholomorphism and Ω_1 is biholomorphic to Ω_2. We also write $\Omega_1 \sim \Omega_2$, or $\Omega_1 \overset{f}{\sim} \Omega_2$. If Ω_2 is Ω_1 then we call f an automorphism. The group of automorphisms of Ω is denoted $\mathrm{Aut}(\Omega)$.

Ω will always satisfy the following additional properties:

(1) Ω is convex, i.e.

$$\forall x, y \in \Omega, t \in [0,1] \quad \Rightarrow \quad tx+(1-t)y \in \Omega,$$

(2) Ω is hyperbolic. (We always intend *hyperbolic* in the sense of Kobayashi.) We refer to [17] for a full treatment of this subject. The non-specialist reader can substitute the stronger condition that Ω is bounded (but see Lemma 2).

Kobayashi metrics

We briefly review some basic facts about Kobayashi metrics, automorphism groups, and affine groups. If a domain is hyperbolic then the Kobayashi distance $k: \Omega\times\Omega\to\Omega$ is a *metric* on Ω.

$$k(x, y) = \inf \sum_i d(f_i^{-1} x_i, f_i^{-1} x_{i+1})$$

where $x_i \in \Omega$, $f_i: \{z: |z|<1\} \to \Omega$ is holomorphic, and 'd' denotes the Poincaré metric on $\{z: |z|<1\}$. The infimum is taken over all sequences x_i such that $x_0=x$, $x_N=y$, N arbitrarily large, and all holomorphic maps f_i, $i=1, \dots, N$.

We write $k(x, y; \Omega)$ or $k(x, y)$. It is an *intrinsic* metric, i.e. it depends only on the complex structure of Ω, not the embedding to $\mathbf{C}^n$. It follows that k is $\mathrm{Aut}(\Omega)$ invariant, i.e. $k(\gamma x, \gamma y)=k(x, y)$. We let $K(x, r)$ denote Kobayashi balls of radius r, $\{y: k(x, y)<r\}$. If Ω is convex then $K(x, r)$ is convex *as a subset of* $\mathbf{C}^n$ (not in the $k(x, y)$-intrinsic sense). The Kobayashi metric satisfies a simple comparison property:

$$\Omega_1 \subset \Omega_2 \quad \Rightarrow \quad k(x, y; \Omega_1) \geq k(x, y; \Omega_2).$$

If k is a complete metric then we say Ω is *complete hyperbolic*. In particular, if Ω is convex or if $\mathrm{Aut}(\Omega)$ is co-compact (see below), then Ω is complete hyperbolic.

There are two important intrinsic *Kähler* metrics on a bounded domain Ω, the Bergman metric and the Einstein-Kähler metric [12, 3, 26]. The latter is complete if Ω is pseudo-convex, in particular if Ω is convex or if $\mathrm{Aut}(\Omega)$ is co-compact.

Automorphism groups

References for the material below on $\mathrm{Aut}(\Omega)$ are [27, 18]. The most natural topology on $\mathrm{Aut}(\Omega)$ is the *compact-open topology*. Given an intrinsic Kähler metric on Ω, let $\mathcal{F}\Omega$ denote the bundle of unitary frames on Ω, and π the projection to Ω. Fixing $f \in \mathcal{F}\Omega$

determines a canonical embedding $f\colon \mathrm{Aut}(\Omega)\to\mathscr{F}\Omega$, essentially $\gamma\mapsto d\gamma\cdot f$. This induces the compact-open topology on $\mathrm{Aut}(\Omega)$ (independent of the choice of f). $\mathrm{Aut}(\Omega)$ *is a Lie group in the compact-open topology*. In particular it is closed and locally connected. Furthermore any closed subgroup G is locally connected. $N(G)$ denotes the *normalizer* of G. If $\Omega=\Omega_1\times\Omega_2$ and $\Omega_1\neq\Omega_2$, then $\mathrm{Aut}(\Omega)=\mathrm{Aut}(\Omega_1)\times\mathrm{Aut}(\Omega_2)$ (if $\Omega_1=\Omega_2$ then we must extend $\mathrm{Aut}(\Omega_1)\times\mathrm{Aut}(\Omega_2)$ by Z_2-permutations of the factors, in this paper we can ignore this point, because we are free to pass to finite index subgroups).

A subgroup $\Gamma\subset\mathrm{Aut}(\Omega)$ is *discrete* if it has no accumulation points in $\mathrm{Aut}(\Omega)$, and *co-compact* if it has a compact fundamental domain in Ω. (Occasionally we may consider $\Gamma\subset G$ *co-compact in a group* G, which means Γ has a compact fundamental domain in G [31].) One usually also assumes a group action is *properly discontinuous* so that the quotient is a manifold. In the case of isometric actions, discrete implies properly discontinuous, so this is redundant.

The *affine group* $\mathscr{A}(n)$ acts on $\mathbf{C}^n$ such that

$$\mathscr{A}(n)=\{A\colon \mathbf{C}^n\to\mathbf{C}^n\colon Az=Mz+b\}$$

with $M\in\mathrm{GL}(n,\mathbf{C})$, and $b\in\mathbf{C}^n$. We will discuss affine geometry in more detail in § 4.

Bounded symmetric domains

The Riemann mapping theorem fails for $n>1$, in fact there are infinite-dimensional families of holomorphically distinct domains, see [33]. In every dimension there are a finite number of domains whose group of holomorphic automorphisms satisfies:

(1) $\mathrm{Aut}(\Omega)$ is transitive on Ω,

(2) $\mathrm{Aut}(\Omega)$ is a semi-simple Lie group.

These are called bounded symmetric domains (BSD), see [12]. If $\Omega\sim B$ where B is a BSD, we say Ω is *symmetric*. The reader should recall that every BSD has canonical convex embeddings; as a bounded domain the Harish-Chandra embedding and as unbounded domains the Siegel domains of types 1, 2 and 3, see [30]. Intuitively speaking these embeddings are canonical because a certain class of one dimensional subspaces (extremal discs for the Kobayashi metric) are realized as affine discs or upper-half-planes. The canonical bounded and unbounded embeddings are related by the Cayley transforms. The technique we introduce here (rescale blow-up), provides an elementary geometric construction of the Cayley transform, using $\mathrm{Aut}(\Omega)$, and $\mathscr{A}(n)$. In fact, rescale blow-up provides canonical embeddings for a much larger class of domains.

Statement of theorems

We apply the rescale blow-up to prove the following theorem:

THEOREM 2.2. *Let Ω be a convex hyperbolic domain in $\mathbf{C}^n$ and suppose there is a subgroup* $\Gamma\subset\mathrm{Aut}(\Omega)$ *such that*

(Γ1) Γ *is discrete and acts freely,*

(Γ2) Γ *is co-compact (in Ω).*

Then Ω is biholomorphic to a bounded symmetric domain.

The development of the tools we need occupies all of this paper, at the end of § 13 we provide the proof of Theorem 2.2.

Remark 2.3. (1) By a well-known result of Borel every BSD admits a discrete, co-compact group of automorphisms.

(2) If we drop the hypothesis (Γ1) then the result fails, Piiateski-Shapiro constructed homogeneous bounded domains that are not symmetric. By results of Borel, Hano and Koszul, a homogeneous bounded domain admitting a discrete co-finite volume group of automorphisms must be symmetric. There are examples due to Katz–Vinberg and later, W. Goldman, of convex hyperbolic domains that are not homogeneous, but with co-compact automorphism groups, [7], [16]. The fixed-point free hypothesis should be removable. The main problem is in extending Lemma 11.8 to this case.

(3) If we drop the hypothesis that Ω is convex then the conclusion fails, because the universal covers of Kodaira surfaces [21] embed in $\mathbf{C}^2$ by the Griffiths uniformization technique (essentially Ber's simultaneous uniformization construction in this case). However, our result implies that the Kodaira surfaces, as well as the surfaces of Mostow and Siu [25] *have no convex uniformization.*

(4) We can weaken the convexity hypothesis to a local condition on a single point at the boundary, see Definition 7.3, similarly we can strengthen the assertion of non-convexity of the last remark to a local property: no boundary point can be h-convex.

In part one we prove:

THEOREM 2.4. *Let Ω be a convex hyperbolic domain in $\mathbf{C}^n$ and suppose there is a subgroup* $\Gamma\subset\mathrm{Aut}(\Omega)$ *such that*

(Γ2) Γ *is co-compact (in Ω).*

Then $\mathrm{Aut}_0(\Omega)$ *is non-trivial, in fact there is a convex holomorphic embedding $w\colon\Omega\to\mathbf{C}^n$ such that $w(\Omega)$ is invariant under a 1-parameter group of translations.*

The proof is at the end of § 6, it relies on the work from § 4 to § 6, the idea is sketched in § 3.

In part two of this paper, to begin, we will reduce the problem to the case where Ω/Γ is *essentially irreducible,* i.e. no finite covering has a decomposition as a product of complex manifolds.

In the second part of this paper we prove the following result and apply it to the proof of Theorem 2.2.

THEOREM 2.5. *Let Ω be a convex hyperbolic domain in $\mathbf{C}^n$ and suppose there is a subgroup* $\Gamma \subset \mathrm{Aut}(\Omega)$ *such that*

(Γ1) Γ *is discrete and acts freely,*

(Γ2) Γ *is co-compact (in Ω).*

If $\mathrm{Aut}_0(\Omega)$ *is non-trivial then Ω has a non-trivial factor which is biholomorphic to a bounded symmetric domain.*

We provide the proof of Theorem 2.5 in § 13.

Actually one only needs convexity near one boundary point p (in the hypothesis of Theorem 2.2) and we are free to choose any local holomorphic chart to make $\partial\Omega \cap B(p, \varepsilon)$ convex. We introduce the notion of *h-convexity* in § 7 so that we may state our result in proper generality:

THEOREM 2.6. *Let Ω be a bounded domain in a Stein space and suppose some $p \in \partial\Omega$ is h-convex, suppose there is a subgroup* $\Gamma \subset \mathrm{Aut}(\Omega)$ *such that*

(Γ1) Γ *is discrete and acts freely,*

(Γ2) Γ *is co-compact (in Ω).*

Then Ω is biholomorphic to a bounded symmetric domain.

This follows directly from Corollary 7.2 and Theorem 2.2.

Appendix

We show that for most of our purposes, we can replace the hypothesis of hyperbolicity by something simpler.

Definition 2.7. We say a domain $\Omega \subset \mathbf{C}^n$ is affine hyperbolic if any complex affine-linear embedding of the complex line into Ω,

$$z \mapsto z \cdot A + B$$

for $A, B \in \mathbf{C}^n$ is a constant map. Equivalently, given a complex (affine-linear) line L in $\mathbf{C}^n$, $L - L \cap \Omega$ contains at least two points.

PROPOSITION 2.8. *The following are equivalent*:

(1) Ω *is convex and hyperbolic.*

(2) Ω *is convex and affine hyperbolic.*

(3) Ω *is convex, and there is a bounded holomorphic embedding* $w: \Omega \to \mathbf{C}^n$.

(4) *For all* L *(as above)*, $L \cap \Omega$ *is convex and* $L - L \cap \Omega \neq \emptyset$.

Proof. The second and fourth items are trivially equivalent. (3)⇒(1)⇒(2) is obvious, so we show (2)⇒(3).

We will produce co-ordinates $(z_1, \ldots, z_n)$ such that Ω is bounded away from the co-ordinate hyperplanes:

$$\{z \in \mathbf{C}^n : \exists i, z_i = 0\} \cap [B(0, \varepsilon) + \Omega] = \emptyset.$$

Then the co-ordinates $(w_1, \ldots, w_n)$ where $w_i = 1/z_i$ provide the bounded embedding. Note that $B(0, \varepsilon)$ is a priori a distance ball in the z-co-ordinates, i.e. in the Euclidean metric $\Sigma |dz_i|^2$, however any pair of affinely equivalent Euclidean co-ordinates determine uniformly equivalent Euclidean metrics. Thus, we work in fixed co-ordinates $(x_1, \ldots, x_n)$, and with the associated Euclidean metric $d(x, y)$.

It suffices to find complex hyperplanes H_i, $i = (1, \ldots, n)$ intersecting transversally, such that $d(H_i, \Omega) > \varepsilon$.

We construct the H_i by induction; given complex hyperplanes H_i, $i = (1, \ldots, k)$ intersecting transversally, such that $d(H_i, \Omega) > \varepsilon$, it suffices to find an H_{k+1} satisfying the inductive hypothesis.

Let $L \subset \cap H_i$ be a complex line, and L_x the line parallel to L that passes through a point $x \in \Omega$. Since Ω is affine hyperbolic, there is a point $p \in L_x \cap \partial\Omega$ and a supporting hyperplane H_p to $\partial\Omega$ at p. Note that $H_p \cap L_x$ transversally ($x \in \Omega$, but $H_p \cap \Omega = \emptyset$).

Let $H_{k+1} = H_p + v$, where $v \perp H_p$, $\|v\| > \varepsilon$ and v translates H_p away from Ω. This completes the inductive step.

COROLLARY 2.9. *The Bergman metric and the Einstein-Kähler metric are well-defined on any convex hyperbolic domain.*

3. Sketch and discussion of the proof

The philosophy of our approach is that certain problems in complex analysis involving automorphisms and maps of convex domains can be directly reduced to problems in affine geometry by an elementary technique involving localization near a boundary point. In this work we develop this technique called *rescale blow-up* far enough to show that the domain Ω of Theorem 2.2 admits a 1-parameter group of automorphisms $\sigma_t \in \mathrm{Aut}_0(\Omega)$.

We can push the technique much farther and it is quite possible that one can use techniques relating affine geometry and complex analysis to complete the proof, however we have chosen a different route in this paper; we use intrinsic metrics together with structure theory of Lie groups and some differential geometry to show that the closed group generated by Γ and σ_t is semi-simple and (at least if Ω or Γ is irreducible) transitive.

The reasons we chose this latter approach are: (i) It is probably simpler than developing the rescale blow-up technique, though it relies heavily on the fact that Γ is discrete. (ii) It yields a result which is interesting in itself, Theorem 2.5, and techniques that are pertinent to the structure theory of negatively curved compact manifolds. For such a result, see the paper of P. Eberlein in Acta Math., 149 (1982), 41–69. (iii) There should be some generalization to the bounded non-convex case.

The main new technique developed here is named 'rescale blow-up', but should not be confused with the like-named process in algebraic geometry, it is closely related to the rescale blow-up construction of tangent spaces (or cones) in geometric measure theory, or Gromov's theory of Hausdorff convergence [9]. (This is clear from equation (2) below.) It connects complex geometry to affine geometry near the boundary of a domain in a fundamental way. It is a simple application of one-variable techniques, which contrasts sharply with the more sophisticated techniques from several complex variables, that were previously applied to this type of problem.

The second half of our proof is differential-geometric. Various points in our proof rely on the existence of intrinsic metrics: the Kobayashi, Bergman, and Calabi-Yau or Einstein-Kähler metrics. We never use estimates of the boundary asymptotics of these metrics. One expects that the rescale blow-up can be used to rederive such estimates with very weak regularity hypotheses on the boundary.

We will sketch and discuss part one here, relegating the sketch of part two to the body of that chapter.

As motivation, we ask; given Ω as in Theorem 2.2, how can one construct an embedding $w\colon \Omega \to \mathbf{C}^n$ such that $B = w(\Omega)$ is a canonically embedded bounded symmetric domain? The answer is surprisingly simple; essentially we construct such a map w, with

$$w = \lim A_i \gamma_i, \quad A_i \in \mathscr{A}(n), \quad \gamma_i \in \mathrm{Aut}(\Omega). \tag{1}$$

A_i, γ_i must be chosen carefully, see § 6. It follows that

$$w(\Omega) = \lim A_i \Omega, \tag{2}$$

in fact there is a $p \in \partial\Omega$ such that

$$w(\Omega) = \lim A_i(\Omega \cap B(p, \varepsilon))$$

for arbitrarily small ε. Thus we call $\hat{\Omega}=w(\Omega)$ the *rescale blow-up* of Ω at p.

In this paper we don't carry the analysis of $w(\Omega)$ far enough to show that $w(\Omega)$ is a canonical embedding. In fact this follows as a consequence of Theorem 2.2 and a further development of the rescale blow-up technique, (to appear elsewhere).

If we impose an additional hypothesis, $w(\Omega)$ is easily seen to be canonically embedded, namely, suppose that Ω osculates a bounded symmetric domain B to 'sufficiently high order' at a boundary point $p \in \partial\Omega \cap \partial B$ (this is essentially the point in B. Wongs theorem as well as extensions by Green and Krantz). The significance of our approach is that *we don't require such strong hypotheses* (furthermore, the above works use $\bar{\partial}$-estimates to control boundary behavior, whereas *we only use the* 1-*variable Schwarz lemma*). There *is* a possibility of proving Theorem 2.2 by using affine geometry to classify convex domains with co-compact automorphism groups, but this looks quite difficult, at the same time it would give a more general result (see Remark 2.3).

The choice of A_i is determined in two different ways; firstly by

$$A_i z = [d\gamma_i(z_0^i)]^{-1}(z - x_i) \tag{3}$$

as follows from equation (1) (see Definition 6.1), and secondly by equation (2) which implies that A_i are related to the shape of $\partial\Omega$ at p. (Actually it is only certain asymptotic properties of the sequence that are uniquely determined by the geometry.) It is the interaction of these two considerations that enables us to relate the affine geometry of $\partial\Omega$ to the complex structure (and intrinsic geometry) of Ω.

The first step in our proof is to show that $w_i = A_i \gamma_i$ has a convergent subsequence. The necessary estimates come from observing that the affine-structure-function $\phi: \Omega \times \Omega \to \Omega$ associated to a holomorphic convex embedding w by

$$\phi(x, y) = w^{-1}\left(\frac{1}{2}[w(x) + w(y)]\right)$$

form a normal family, since Ω is hyperbolic.

It is easy to show that convergence of w_i implies convergence of $w_i(\Omega)$. In exploiting equation (2) we rely heavily on the facts that $w(\Omega)$ is open and hyperbolic (not too small, not too big). The analysis proceeds from here entirely on the level of affine geometry. We conclude that $w(\Omega)$ is preserved by a 1-parameter group of translations $\sigma_t \in \mathrm{Aut}_0(\Omega)$.

4. Convergence of holomorphic embeddings

In this section we discuss two notions of convergence for holomorphic embeddings. Given a sequence of holomorphic maps $f_i\colon \Omega \to \mathbf{C}^n$ of a domain Ω, $\lim f_i = f$ or $f_i \to f$ should always be interpreted in the sense of uniform convergence on compact subsets of Ω.

We now introduce a notion of convergence of subsets of $\mathbf{C}^n$ essential to this work. It is based on the well-known notion of *Hausdorff distance*; given two sets, $S_i \subset \mathbf{C}^n$(we take for granted a euclidean or hermitian metric on $\mathbf{C}^n$)

$$d(p, S_i) = \inf_{q \in S_i} d(p, q),$$

$$d(S_1, S_0) = \sup_{i=0,1} \sup_{p \in S_{1-i}} d(p, S_i).$$

The *Hausdorff R-seminorm* is

$$d_R(S_1, S_0) = d(S_1 \cap B(0,R), S_0 \cap B(0,R))$$

and we say $S_i \to S$ (as $i \to \infty$) or $\lim S_i = S$ if

$$\forall R \gg 0,\ d_R(S_i, S) \to 0$$

and we call this *convergence of sets*.

Remark 4.1. (1) We only apply this notion of convergence to convex sets, including affine linear subspaces of $\mathbf{C}^n$. (Convergence of sets induces the standard topology on Grassmannians.)

(2) The distance of a domain S to its closure $\bar{S}$ is zero. Since there is a $1-1$ correspondence of open convex domains to their closures, this generally causes no confusion. Occasionally we must specify whether sets S_i are open or closed.

(3) For all $S, d(S, \varnothing) = \infty$.

(4) The Blaschke selection theorem proves convergence of Cauchy sequences of convex sets, but we don't need it in this paper.

The important point for this paper is that convergence of a sequence of convex holomorphic embeddings $w_i \to w$ of a hyperbolic domain Ω, implies convergence of their images, $w_i(\Omega) \to w(\Omega)$. If we drop the convexity hypothesis this fails, even for $n=1$;

Example 4.2. Let D be the open unit disc in C, and $D+t$ its translate.

$$D \cap D+t = \varnothing \quad \Leftrightarrow \quad \|t\| \geqslant 2.$$

Let $E_t = D \cup D_t$ and $f_t: D \to E_t$ be the biholomorphic map such that $f_t(0)=0$, and $f_t'(0) \in \mathbf{R}$. Clearly $\lim_{t\to 2} f_t = f$ is the identity map, $f(x)=x$, so $f(D)=D$. On the other hand, $\lim_{t\to 2} f_t(D) = E_2$.

We call this a *bubbling-off phenomenon* because of its similarity to like-named ideas in harmonic maps. We will use the rescale blow-up technique introduced in this paper to analyze bubbling-off in the several variable context in a future paper.

We conclude with two basic lemmas, useful in analyzing convergent sequences of convex sets;

LEMMA 4.3. *Given convex sets D_i^j such that $D_i^j \to D_0^j$, $j=1,2$.*
(1) *If all D_i^j are closed, then*

$$\lim(D_i^1 \cap D_i^2) \subset D_0^1 \cap D_0^2.$$

(2) *If all D_i^j are open, then*

$$\lim(D_i^1 \cap D_i^2) = D_0^1 \cap D_0^2.$$

The proof is trivial. There are easy examples where the first inclusion is strict.

Subspaces of $\mathbf{C}^n$ defined by systems of equations of the type

$$\{x: \langle x, v_i \rangle = t_i\}$$

with $v_i \in \mathbf{C}^n$, $t_i \in \mathbf{R}$ are referred to as *affine linear subspaces*. We use the notation $A+B$ to denote $\{a+b: a \in A, b \in B\}$.

LEMMA 4.4. *Suppose Ω_i are convex domains and $\Omega_i \to \Omega_0$. If P_i are affine linear subspaces, and $P_i \to P_0$, and there is a compact $K \subset \Omega_0$ such that $\forall i \gg 0, K \cap P_i \neq \varnothing$, then*

$$\lim(\Omega_i \cap P_i) = \Omega_0 \cap P_0.$$

Proof. Suppose $K+B_\eta \subset \Omega_0$ and fix R very large. (By our definition of convergence). It suffices to prove the lemma assuming $\Omega_i \subset B_R$. The key point is that $\Omega_i \cap P_i \cap K \neq \emptyset \Rightarrow \forall \varepsilon\, \exists \delta$ such that

$$(P_i+B_\delta) \cap \Omega_i \subset (P_i \cap \Omega_i)+B_\varepsilon$$

where δ is independent of i, in fact $\delta < \varepsilon\eta/R$ suffices. (To see this the reader should consider supporting hyperplanes H to Ω_i at $p \in \partial\Omega_i \cap P_i$ noting $H \cap (K+B_\eta)=\emptyset$. One constructs a cone C on p, such that $\Omega_i \cap C=\emptyset$.)

Now P_i+B_δ is an open set, and the previous lemma may easily be applied.

Remark 4.5. Given P_i as above (affine linear subspaces such that $K \cap P_i \neq \emptyset$), there is always a convergent sub-sequence, by compactness of the Grassmannian.

5. Normalized convex embeddings

Definition 5.1. Given a compact set $K' \subset \Omega$. A family $\mathcal{F}$ of embeddings of a domain Ω is normalized at K' with parameters K, R or (K', K, R)-normalized, if there is a compact $K \subset \mathbf{C}^n$ and $R>0$ such that $\forall w \in \mathcal{F}\, \exists x \in K'$ satisfying

(1) $w(x) \in K$,

(2) $\|[dw(x)]\|, \|[dw(x)]^{-1}\| \leq R$.

The simplest choice of parameters is $R=1, K=\{0\}$, and $K'=\{x_0\}$. In this case we say that $\mathcal{F}$ is *simply normalized.*

Remark 5.2. There are many essentially equivalent ways of defining norms on tensors $T=T_\alpha^\beta$ such as dw, ∇dw. In the case at hand, $w\colon \Omega \to \mathbf{C}^n$ (there are given coordinates in the domain and range), so one simply lets $\|T\|=\sup\|T_\beta^\alpha\|$, the maximum entry. Similarly, one can use the Euclidean metrics on domain and range to define Euclidean metrics on all tensors in a natural way. The maximum of $\|T(x)\|$, for $x \in K$, is denoted $\|T\|_K$.

Definition 5.3. Given a convex embedding $w\colon \Omega \to \mathbf{C}^n$, the affine-structure-function $\phi\colon \Omega \times \Omega \to \Omega$ is defined by

$$\phi(x, y) = w^{-1}\left(\frac{1}{2}[w(x)+w(y)]\right).$$

Note that if w is holomorphic then so is ϕ.

The following lemma is well-known, but we reprove it using the affine-structure-function, (the same proof works for a broad class of metrics).

LEMMA 5.4. *If Ω is a convex hyperbolic domain in $\mathbf{C}^n$ then the Kobayashi distance-balls $K(x,R)$ are convex (in the euclidean structure).*

Proof. Recall,

$$k((x_1,y_1),(x_2,y_2);\Omega_1\times\Omega_2)=\sup\{k(x_1,x_2),k(y_1,y_2\}. \tag{4}$$

Thus the Kobayashi balls in $\Omega_1\times\Omega_2$ are of the form $K(x,r,\Omega_1)\times K(x,r,\Omega_2)$. In the case at hand $\Omega_1=\Omega_2=\Omega$, and $x,y\in K(x_0,r)\Rightarrow(x,y)\in K((x_0,x_0),r)$.

Since ϕ is distance decreasing, $k(\phi(x_0,x_0),\phi(x,y))<k((x_0,x_0),(x,y))$, which, for $x,y\in K(x_0,r)$ implies, $\phi(x,y)\in K(x_0,r)\Rightarrow\frac{1}{2}(x+y)\in K(x,r)$. But this easily implies $K(x,r)$ is convex.

COROLLARY 5.5. *If $\Omega\subset\mathbf{C}^n$ is a convex domain and $\Omega=\Omega_1\times\Omega_2$ (as complex manifolds) then the submanifolds (x,Ω_2) and (Ω_1,y) are linearly embedded in $\mathbf{C}^n$.*

Proof. Applying the preliminary comment of the proof of the previous lemma together with the conclusion of the previous lemma and the fact that a holomorphic subvariety of the boundary of a convex set is linearly embedded in $\mathbf{C}^n$, proves the corollary.

THEOREM 5.6. *Let $w_i\colon\Omega\to\mathbf{C}^n$ be a (K',K,R)-normalized family of convex holomorphic embeddings (Definition 5.1).*

(1) *There is a subsequence w_{i_j} which converges, $w_{i_j}\to w$ (uniformly on compact sets).*

(2) *w is a (K',K,R)-normalized convex holomorphic embedding.*

(3) $\lim w_{i_j}(\Omega)=w(\Omega)$ *(as sets).*

Proof. Let us assume at first that the family is simply normalized, the general case will easily follow.

To each w_i associate the affine-structure-function ϕ_i. Since ϕ_i maps to Ω and Ω is hyperbolic, there are constants $C(K,\Omega)$ independent of i, such that

$$\|\nabla d\phi_i\|_K<C(K,\Omega).$$

(K is an arbitrary compact set in Ω. We will follow the custom of using '$\ldots<C$' to mean *there is a constant* such that $\ldots<C$, allowing C to change throughout the proof.)

Now $2w_i(\phi_i(x,y))=w_i(x)+w_i(y)$, and we proceed to differentiate twice by x at a point on the diagonal $\{(x,y): x=y\}$ (note $\phi(x,x)=x$);

$$2w_{i,\alpha}(x)\phi^{\alpha}_{i,\lambda}(x,x)=w_{i,\lambda}(x) \quad\Rightarrow\quad \phi^{\alpha}_{i,\lambda}=\frac{1}{2}\delta^{\alpha}_{\lambda}$$

(the Kronecker delta)

$$2w_{i,\alpha,\beta}(x)\phi^{\alpha}_{i,\lambda}(x,x)\,\phi^{\beta}_{i,\lambda}(x,x)+2w_{i,\alpha}(x)\,\phi^{\alpha}_{i,\lambda,\lambda}(x,x)=w_{i,\lambda,\lambda}(x)$$

and by the last two formulae

$$\frac{1}{2}w_{i,\lambda,\lambda}(x)+w_{i,\alpha}(x)\,\phi^{\alpha}_{i,\lambda,\lambda}(x,x)=w_{i,\lambda,\lambda}(x)$$

and we conclude

$$\|\nabla dw\|_K < C\|dw\|_K.$$

The normalization, Definition 5.1(2), and a comparison theorem for O.D.E's implies

$$\|dw\|_K < C$$

which with Definition 5.1(1) gives

$$\|w\|_K < C$$

and the first item of the theorem follows.

Now $\det(dw(x))$ is a holomorphic function, and (supposing without loss of generality that $w_i \to w$) $\det(dw_i(x)) \to \det(dw(x))$. But $\det(dw_i(x)) \neq 0$ since w_i are embeddings, and $\det(dw(x_0)) \neq 0$ by Definition 5.1(b), so Hurwitz's theorem implies $\det(dw(x)) \neq 0$. In particular,

$$\|[dw]^{-1}\|_K < C.$$

Together with the fact that

$$w_i(K) \to w(K) \tag{5}$$

as sets, for any K compact, this easily implies that $[w_i]^{-1} \to u$ is well-defined on $w(K)$, and $u^{-1}=w$ on K. Since K is arbitrary, w is an embedding. This argument is essentially in a paper of Fornaess and Sibony.

Recall that $\forall i, R$ the Kobayashi balls $w_i(K(x,R))$ are convex. By equation (5) $w(K(x,R))$ is convex so $w(\Omega)$ is convex. This finishes the proof of the second item of the theorem.

By equation (5)

$$D = \lim w_i(\Omega) \supset w(\Omega).$$

The reverse inclusion is a consequence of convexity, we prove it by contradiction; Suppose $p \in D - \overline{w(\Omega)}$. $D, w(\Omega)$ are convex so, denoting the convex-hull of a set S by $Q(S)$,

$$Q(w(\Omega) \cup B(p,\varepsilon)) \subset D.$$

In particular, $\exists U$ open such that

$$l = \{tp + (1-t)x_0 : t \in [0,1]\} \subset U \subset w_i(\Omega), \quad \forall i \gg 0.$$

By the comparison theorem for Kobayashi metrics

$$k(x_0, p; w_i(\Omega)) < C, \quad \forall i \gg 0$$

and because Ω is complete hyperbolic, $p \in w(K(x_0, C+1)) \subset w(\Omega)$, contradicting $p \in D - \overline{w(\Omega)}$. The last item of the theorem follows by the contradiction.

Remark 5.7. An immediate corollary of the estimates in the proof, is that the generalized normalization (at K') *implies* normalization at x_0, but with different choices of K, R. Hence the theorem above is valid with the more general type of normalization. In fact one just has to compose each w_i on the left by an appropriate $A_i \in \mathscr{A}(n)$ to deduce the necessary estimates.

6. Rescale blow-ups

We will employ the following convention systematically when dealing with sequences s_i of various types of objects; if s_i has a convergent subsequence s_{i_j}, we will automatically relabel the latter as s_i. To alert the reader when we do this we write 'without loss of generality'. This indicates that the subsequence satisfies all the conditions we have imposed on the sequence s_i.

Definition 6.1 Given a compact set $K' \subset \Omega$, $z_0^i \in K'$ and $\gamma_i \in \mathrm{Aut}(\Omega)$ such that

$$x_i = \gamma_i(z_0^i) \to p \in \partial\Omega,$$

let $A_i \in \mathscr{A}(n)$ be defined by

$$A_i z = [d\gamma_i(z_0^i)]^{-1}(z - x_i).$$

Then the embeddings

$$w_i = A_i \cdot \gamma_i$$

satisfy the hypotheses of Theorem 5.6 and there is a convergent subsequence without loss of generality

$$w_i \to w.$$

We call $w(\Omega)$ the rescale blow-up of Ω at p by γ_i or, the rescale blow-up of Ω by A_i, noting that

$$w(\Omega) = \lim A_i \Omega.$$

Note that when we extracted the subsequence of w_i we did the same with γ_i, A_i.

When $\mathrm{Aut}(\Omega)$ is co-compact we can choose γ_i such that the x_i have particularly nice properties; in fact given any $x_i \to p$ and a compact fundamental domain K one can choose γ_i such that $\gamma_i^{-1} x_i \in K$.

Definition 6.2. We say the rescale blow-up of Ω defined above is *very regular* if the sequence x_i satisfies the following property (we continue with the notation of the previous definition): there is an affine complex line $L \subset \mathbf{C}^n$ such that $p \in L \cap \partial\Omega$ and

(1) $D = L \cap \Omega$ has a unique supporting line $l \subset L$ at p. In this case ∂D is differentiable at p.

(2) $x_i \in L$ and $x_i \to p$ *radially* in D, i.e. the x_i lie in a(n affine real) line through p and perpendicular to l.

In this situation we also use the following notation: let t be an arclength parameter for l such that $t=0$ at p, let v be the constant vector-field on $\mathbf{C}^n$ such that $v = \partial_t$ on l, and let $\sigma_t \in \mathscr{A}(n)$ represent the 1-parameter flow determined by v, $\sigma_t z = z + tv$.

Remark 6.3. (1) A_i conjugates the group σ_t to another translation group σ_t^i. The latter have a convergent subsequence, and we will see that $w(\Omega)$ is invariant under the limiting group.

(2) The notion of 'admissible approach' is connected to the rescale blow-up in a natural way, but we defer this topic to another article. The next definition is a simple

manifestation of this phenomenon, a special case that will suffice for the purposes of this paper.

The following notation will be employed for the rest of this section: given a set X such that $\lim A_i X$ exists, we denote the limit by $\hat{X}$. Thus $w(\Omega)=\hat{\Omega}$, however in general $\hat{X}$ is not the same as $w(X)$.

A very regular rescale blow-up is obtained as follows.

LEMMA 6.4. *Let Ω be a convex domain, then almost every boundary point $p\in\partial\Omega$ has a unique supporting hyperplane.*

This well known fact follows from an analysis of the Gauss map on $\partial\Omega$. In fact, in this paper we only need this fact for curves in $\mathbf{R}^2$, cross-sections of Ω. (Increasing functions are a.e. differentiable.)

We call such p *differentiable points*. Choosing such a p, we can easily get L, x_i that satisfy conditions 1 and 2 of the definition of 'very regular'. If $\mathrm{Aut}(\Omega)$ is co-compact with compact fundamental domain K', we can choose γ_i such that $[\gamma_i]^{-1}(x_i)\in K'$.

Now there are two simple lemmas that will give us the desired automorphisms of Ω.

Definition 6.5. We say a convex hyperbolic domain $\Omega\subset\mathbf{C}^n$ is *large* if there is a real 1-dimensional affine-linear subspace (i.e. a real line) $R\subset\mathbf{C}^n$ such that $R\subset\Omega$. (To be careful, one may prefer saying that a particular convex embedding is large.) We associate to a large domain the constant vector field v_R on $\mathbf{C}^n$ (tangent to R), and the associated flow σ^R.

LEMMA 6.6. *If $\hat{\Omega}$ is a very regular rescale blow-up of Ω then it is large.*

LEMMA 6.7. *If $\hat{\Omega}$ is large then it is invariant under the associated flow, $\sigma_t^R\hat{\Omega}=\hat{\Omega}$ for all $t\in\mathbf{R}$, i.e. $z\in\hat{\Omega}, t\in\mathbf{R}\Rightarrow z+tv\in\hat{\Omega}$.*

Proof (of Lemma 6.6). By the construction, $0\in A_iL$ for all i, so without loss of generality $A_iL\to\hat{L}$ and by Lemma 4.4

$$\hat{L}\cap\hat{\Omega}=\lim(A_iL\cap A_i\Omega)=\lim A_i(L\cap\Omega).$$

Now by projecting into $\hat{L}$ this reduces to a problem of (complex) affine geometry in one complex dimension. We state it as a separate lemma.

LEMMA 6.8. *Given a convex domain $D\subset\mathbf{C}$ with a differentiable point $p\in\partial D$, as* *well as $x_i \in D$ such that $x_i \to p$ radially, and $A_i \in \mathscr{A}(1)$ such that $A_i x_i = 0$. If $A_i D \to \hat{D}$ where $\hat{D}$ is an (open) domain in* $\mathbf{C}$, *then $\hat{D}$ is large.*

Proof. In the 1-dimensional case $A_i z = \alpha_i z + \beta_i$, $\alpha_i, \beta_i \in \mathbf{C}$. In particular

$$\|A_i x - A_i y\| = \|\alpha_i\| \, \|x - y\|.$$

Inscribe a triangle (q_i^1, p, q_i^2) in D such that $q_i^j \in \partial D$ and the segment $[q_i^1, q_i^2]$ is parallel to l, and $x_i \in [q_i^1, q_i^2]$. By uniqueness of l and radial approach of the x_i, we see

$$\frac{\|q_i^j - x_i\|}{\|p - x_i\|} \to \infty.$$

In fact the vertex angle at p tends to π.

But $A_i D \to \hat{D} \supset B(0, r)$ implies

$$\|A_i(p - x_i)\| > r, \quad \forall i \gg 0,$$

thus

$$\|A_i(q_i^j - x_i)\| \to \infty$$

and we are finished.

Proof (of Lemma 6.7). Given $x \in \hat{\Omega}$ and $R \subset \hat{\Omega}$ as above, the convex-hull $Q(B(x, \varepsilon) \cup R) \subset \hat{\Omega}$ is invariant under σ^R.

Remark 6.9. (1) The vector field of the flow σ_t is holomorphic on Ω.

(2) It is very possible that one can apply the rescale blow-up technique to study domains with non-compact automorphism group, especially with some regularity hypothesis on $\partial\Omega$.

(3) The relationship of the sequence $\gamma_i \in \mathrm{Aut}(\Omega)$ to σ_t is a little subtle, which accounts to some extent for the difficulty in exploiting σ_t in part 2 of the proof. In this regard one should note that the group of automorphisms obtainable by rescale blow-up of the upper-half-plane, with $\gamma_i \in \mathscr{A}(1)$ is not transitive. However with regularity hypotheses on $\partial\Omega$ one could use affine geometry to produce different types of complete holomorphic vector fields on $w(\Omega)$ in a systematic way.

Outline of the proof of Theorem 2.4

We produce a holomorphic embedding of Ω via the rescale blow-up as in Definition 6.1, and subject to the additional hypotheses in Definition 6.2. The lemmas from Definition 6.5 to Lemma 6.8 guarantee that this embedding is invariant under a one parameter group of translations.

7. H-convexity

Slight refinements of the rescale blow-up technique allow us to weaken the convexity hypothesis in our main theorems to convexity near one boundary point. Roughly, $\partial\Omega$ is *h-convex* at p, if $\partial\Omega\cap B(p,\varepsilon)$ is convex in some holomorphic chart for $B(p,\varepsilon)$.

We assume throughout this section that Ω *is a bounded subdomain of a Stein space* V with some hermitian metric e on V, both of complex dimension n, and that $p\in\partial\Omega$. If $V=\mathbf{C}^n$ then e is the euclidean metric. Clearly Ω is Kobayashi hyperbolic.

The main goal of this section is to prove

THEOREM 7.1. *If there exists* $\gamma_i\in\mathrm{Aut}(\Omega)$ *and* $z\in\Omega$ *such that* $\gamma_i(z)\to p$, *and if* $\partial\Omega$ *is h-convex at* p, *then* Ω *is biholomorphic to a convex hyperbolic domain in* $\mathbf{C}^n$.

COROLLARY 7.2. *If* $\mathrm{Aut}(\Omega)$ *is co-compact in* Ω, *and if* $\partial\Omega$ *is h-convex at* p, *then* Ω *is biholomorphic to a convex hyperbolic domain in* $\mathbf{C}^n$.

Proof (of corollary). One uses the comment after Definition 6.1 to see that there exists $\gamma_i\in\mathrm{Aut}(\Omega)$ and $z^i\in K\subset\Omega$ such that $\gamma_i(z)\to p$.

Definition 7.3. (Distances and balls B are with respect to the hermitian metric e.) $p\in\partial\Omega\subset\mathbf{C}^n$ is *peak-convex* if there exists $\varepsilon>0$ such that

(1) $S=\partial\Omega\cap B(p,\varepsilon)$ is convex and

(2) for all complex affine linear $H\subset\mathbf{C}^n$, such that p is an interior point of $S\cap H$ in the topology of H, $S\cap H\subset\subset S$ (i.e. complex linear subdomains of S do not extend to ∂S).

Given $p\in\partial\Omega$ with Ω a bounded subdomain of a Stein space, $\partial\Omega$ is *h-convex* at p, if there exists $\varepsilon>0$ such that $u(\partial\Omega\cap B(p,\varepsilon))$ is peak-convex in some holomorphic chart $u\colon B(p,\varepsilon)\to\mathbf{C}^n$.

There is a unique maximal set of the form $S\cap H$; since S is convex, the convex hulls of the union of two such sets is also such a set.

For the rest of this section we use p,ε,S,H *etc. as in Definition* 7.3 *with* H

maximal as in the remark. The condition involving H guarantees that $\partial\Omega$ is strictly convex in a coarse sense, and that the rescale blow-up of Ω will be hyperbolic.

We will use an idea of Rosay to prove

LEMMA 7.4. *If $K \subset\subset \Omega$ is compact, $\gamma_i \in \mathrm{Aut}(\Omega)$ and $z \in K$ such that $\gamma_i(z) \to p$, then $\gamma_i(K) \subset\subset B(p, \varepsilon), \forall i \gg 0$.*

One feature of our approach here is that we avoid the usual use of global peak-functions for this type of boundary localization. We feel this is worthwhile because it frees us from considerations of boundary regularity. We prepare for the proof with:

Let $D = \{z \in \mathbf{C}: |z| < 1\}$. Given any two points $x, y \in \Omega$ there is a holomorphic $f: D \to \Omega$ such that $x, y \in f(D)$. In fact x, y can be joined by a real analytic curve, which can be analytically continued to provide such an f. We can always suppose that $f(0) = x$.

Now let

$$l(x, y) = \inf d(f^{-1}x, f^{-1}y)$$

where $f: D \to \Omega$ is holomorphic, and 'd' denotes the Poincaré metric on D. The infimum is taken over all holomorphic maps f. It is irrelevant for our purposes here whether $l(x, y)$ is a distance, we only care that it is intrinsic, $l(\gamma x, \gamma y) = l(x, y)$ for $\gamma \in \mathrm{Aut}(\Omega)$.

We let $L(x, r)$ denote 'balls' of radius r; $\{y: l(x, y) < r\}$. Given any $K \subset\subset \Omega$ compact, there is clearly an $L(x, r) \supset K$.

LEMMA 7.5. *Suppose $f: D \to \bar{\Omega}$ such that $f(0) = p$, where Ω is h-convex at p. Then $f: D \to S \cap H$.*

This is clear for $f: D_\delta \to \bar{\Omega}$ where D_δ is a small neighborhood of 0. (One can use a local peak function for $S \cap H$ to see this.) By analytic continuation it is true for D.

Proof of Lemma 7.4. Given $p \in \partial\Omega$ h-convex. Any sequence of holomorphic $\psi_i: D \to \bar{\Omega}$ has a convergent subsequence, without loss of generality $\psi_i \to \psi: D \to \bar{\Omega}$ converges. If $\psi_i(0) = x_i \to p$, then $\psi(D) \subset\subset S \cap H$ by Lemma 7.5. Using standard facts on compactness and uniform convergence, it follows that, $\forall \varepsilon_1 \ll 1, r\ \exists\delta$ such that if $x \in B(p, \delta)$ and $l(x, y) < r$ then $y \in B(S \cap H, \varepsilon_1) \subset B(p, \varepsilon)$. Consequently, if $K \subset\subset \Omega$ is compact, $\gamma_i \in \mathrm{Aut}(\Omega)$ and $z \in K$ such that $\gamma_i(z) \to p$, then $\gamma_i(K) \subset\subset B(p, \varepsilon), \forall i \gg 0$.

Proof of Theorem 7.1. Choose ε, u as in Definition 7.3. We construct the rescale blow-up of Ω at p by γ_i as before, with the following modifications; choose an exhaustion of Ω by compacta $K_j \subset\subset K_{j+1}$. For each j, and all i (in view of Lemma 7.4) large enough that $\gamma_i K_j \subset B(p, \varepsilon)$, construct the embeddings $w_i = A_i u \gamma_i$ as in § 6, normalized at some $x \in K_0$. Note $w_i: K_j \to \mathbf{C}^n$.

We claim the embeddings w_i have convergent subsequences, in fact by the definition of *h-convexity* the affine-structure-function (Definition 5.3) $\phi_i: K_j \times K_j \to \Omega$ is well defined if $i \gg 0$. Thus the estimates of Theorem 5.6 apply to $K_{j-1} \subset\subset K_j$, and the w_i have a subsequence, convergent on K_{j-1}, for all j.

Diagonalizing, we get an embedding w such that for all j, we have $w: K_j \to \mathbf{C}^n$, hence $w: \Omega \to \mathbf{C}^n$.

It remains to show that $w(\Omega)$ is convex. The affine-structure-functions ϕ_i form a normal family, so without loss of generality,

$$\phi_i \to \phi = w^{-1}\left(\frac{1}{2}[w(x)+w(y)]\right): K_j \times K_j \to \Omega \quad \text{for all } j,$$

hence $\phi: \Omega \times \Omega \to \Omega$. Since ϕ is well defined on all of Ω, w must be a convex embedding (the midpoint of any pair of points in Ω is in Ω). This finishes the proof.

One can also show that $w(\Omega)$ is the Hausdorff distance limit of $A_i(u(\Omega \cap B(p, \varepsilon))$, where $u(\Omega \cap B(p, \varepsilon)$ is peak convex at p. The argument is analogous to that in Theorem 5.6. It would be interesting to develop an intrinsic notion of h-convexity.

8. Part 2 of the proof, introduction

In Part 1 of this paper we produced a one-parameter subgroup $\sigma \subset \mathrm{Aut}_0(\Omega)$. In this, Part 2 of the paper, we analyze the interaction of σ with the discrete co-compact group Γ. Our goal is to prove Theorem 9.1, which, given σ implies Theorem 2.2. The second part of this paper does not involve the rescale blow-up technique. It is more in the spirit of complex differential geometry of compact manifolds.

The philosophy of our proof is easy to describe; the continuous group of automorphisms gives complete holomorphic vector fields, pushing them forward by $\gamma \in \Gamma$ we generate a large group, $\mathcal{G}$, which is normalized by Γ. $\mathcal{G}$ orbits in Ω descend to compact submanifolds in $\Omega/\Gamma = M$. We study the geometry of these submanifolds to show (i) that $\mathcal{G}$ is semi-simple, and (ii) that $\mathcal{G}$ is transitive on a factor of Ω. The techniques we apply combine Lie group theory, homology theory and differential geometry, including some easy Bochner-Weitzenboch formulae. We apply results from Lie group theory such as the Iwasawa decomposition, the Levi-Malcev decomposition, the Borel density theorem, the Margulis-Zassenhaus lemma, and basic theory of semi-simple lie groups and symmetric spaces, for which we provide references. We must also apply the existence theorems for Einstein-Kähler metrics on bounded domains, and basic facts about Kobayashi metrics in our use of differential geometry. Finally, the machinery of group cohomology is applied in the proof of Theorem 11.2. Thus, in contrast to Part 1, Part 2 is far from self-contained, but the basic ideas are still quite intuitive, and the analysis is essentially 'soft'. It is reasonable to believe that this is not the ideal proof of the theorem, and that quite different approaches could work better. On the other hand, many approaches that, at first glance, appear easy, do not work, and we are convinced that the result is far from trivial. In the author's thesis a simpler proof was given, but only for the complex two dimensional case, the main difference is that it avoided the machinery in § 10 by using an argument involving characteristic classes, this also appeared as an appendix to the last section in the preprint version of this paper.

The main work in the proofs is in Sections 10, 11. The fixed-point theorem in § 12 is applied in § 11.

B. Wong has studied the interaction of Γ and $\mathrm{Aut}_0(\Omega)$, viewing the orbit structure as a fibration, but his techniques are unrelated to ours.

9. Notation and basic facts

This section contains general facts necessary for the proofs in subsequent sections. The material in this section is mostly a reiteration of some well known theory.

The general notation we use for groups and their actions is as follows:

$H+J$ denotes the group generated by H, J in $\mathrm{Aut}(\Omega)$.
Γ_H denotes $H\cap\Gamma$
$[H]$ denotes the closure of H in $\mathrm{Aut}(\Omega)$.
H_0 denotes the maximal connected subgroup of H.
$\mathcal{N}(H)$ denotes the normalizer of H.
$\mathcal{C}(H)$ denotes the centralizer of H.
Hx denotes the H-orbit of x in Ω. (x denotes a point in Ω.)
H_x denotes the isotropy of x in H, i.e. $\{h\in H: hx=x\}$.
H^* denotes the fixed-point set of H in Ω.

We identify elements X of Lie algebras with complete holomorphic vector fields $X(x)$ on Ω throughout the rest of this paper. Note that $X(x)=d\,\mathrm{Exp}(tX)x\cdot\partial_t$, at $t=0$. We use γ_* to denote the 'push forward' of a vector, or a tensor, by the differential $d\gamma$, when this makes sense.

The main result of the second part is

THEOREM 9.1. *Let Ω be a convex hyperbolic domain and suppose there is a subgroup $\Gamma\subset\mathrm{Aut}(\Omega)$ such that*

(Γ1) Γ *is discrete,*

(Γ2) Γ *is co-compact* (*in* Ω),

(Γ3) Γ *acts freely* (*in* Ω).

Suppose there is a non-trivial one-parameter subgroup $\sigma\subset\mathrm{Aut}_0(\Omega)$, and let $\mathscr{G}'=[\Gamma+\{\sigma_t\colon t\in\mathbf{R}\}]$, and $\mathscr{G}$ denote the maximal connected subgroup of $\mathscr{G}'$. Let $\Omega_1=\mathscr{G}x$ and $\Omega_2=\mathscr{G}_x^*$.

(1) *$\mathscr{G}$ is semi-simple.*

(2) *$\Omega=\Omega_1\times\Omega_2$ holomorphically, in particular both factors are complex manifolds, furthermore Ω_1 is non-trivial and is biholomorphic to a bounded symmetric domain.*

(3) *Γ has a finite index normal subgroup Γ' such that $\Gamma'=\Gamma_1'\times\Gamma_2'$ where $\Gamma'\subset\mathrm{Aut}(\Omega_1)\times\mathrm{Aut}(\Omega_2)$ and $\Gamma_j'=\Gamma'\cap\mathrm{Aut}(\Omega_j)$.*

Item 1 is proved in § 10. Items 2 and 3 are proved in § 11. Lemma 9.2 contains some basic preparatory material. Lemma 9.4 essentially reduces Theorem 9.1 to the case where Ω/Γ is irreducible. We recap the proof in § 13.

We let $M=\Omega/\Gamma$ and $\pi\colon\Omega\to M$ be the projection, (or covering map). (We always let $\mathrm{Aut}(\Omega)$ act on the left, hence we really should write $\Gamma\backslash\Omega$, nevertheless we use the more conventional Ω/Γ except in the rare instances where this would be confusing, such as $\Gamma\backslash G/K$.)

In the following lemma we let Ω,Γ and $\mathscr{G}$ be the entities defined in Theorem 9.1.

LEMMA 9.2. (1) $\Gamma\subset\mathcal{N}(\mathscr{G})$.

(2) *The orbits $\mathscr{G}x\subset\Omega$ and $\mathscr{G}'x\subset\Omega$ are closed, locally connected, properly embedded, smooth submanifolds of Ω.*

(3) *$\pi(\mathscr{G}x)=\pi(\mathscr{G}'x)$, in fact $\pi^{-1}(\pi(\mathscr{G}x))=\mathscr{G}'x$. It follows that $\pi(\mathscr{G}x)\subset M$ is a compact, locally connected, properly embedded* (*no boundary*), *smooth submanifold of M.*

(4) *$\mathscr{G}\cap\Gamma$ is co-compact in $\mathscr{G}$. In fact $x\in\Omega$ determines a canonical, proper map $x\colon\mathscr{G}/\Gamma\to\pi(\mathscr{G}x)$.*

Proof. (1) By construction of $\mathscr{G}$.

(2) Let $\mathscr{F}\Omega$ be the unitary frame bundle of Ω, (with respect to the Bergman metric). Note that $\mathrm{Aut}(\Omega)$ acts on $\mathscr{F}\Omega$. Recall that $\mathrm{Aut}(\Omega)$ is a Lie group in the compact-open topology, [29]. In particular, given $x\in\mathscr{F}\Omega$, $\mathrm{Aut}(\Omega)x\subset\mathscr{F}\Omega$ is a closed, locally connected, properly embedded, smooth submanifold of $\mathscr{F}\Omega$. But the projection, $p\colon\mathscr{F}\Omega\to\Omega$ is proper, so $\mathrm{Aut}(\Omega)x\subset\Omega$ is also a closed, locally connected, properly embedded, smooth submanifold of Ω. The same argument works for any closed subgroup of $\mathrm{Aut}(\Omega)$.

(3) The assertion follow easily, in particular $\pi(\mathcal{G}x)\subset M$ is compact by compactness of M, Heine-Borel, and the local properties that $\pi(\mathcal{G}x)$ is closed and locally connected.

(4) Let $S(\mathcal{G}x)\subset\mathcal{G}'$ be the stabilizer of $\mathcal{G}x$ in $\mathcal{G}'$. Clearly $\mathcal{G}$ is a normal subgroup of S of finite index. Furthermore $\pi(\mathcal{G}x)$ compact implies that $S\cap\Gamma$ is co-compact in S (recall $\Gamma\subset\mathcal{G}'$). But $S=\mathcal{G}+(S\cap\Gamma)$, so $S/(S\cap\Gamma)=\mathcal{G}/(\mathcal{G}\cap\Gamma)$. The rest is clear.

Note that the same conclusions hold for any closed subgroup H of $\mathrm{Aut}(\Omega)$ such that $\Gamma\subset\mathcal{N}(H)$, for example the radical of $\mathcal{G}$.

Appendix: Notes on reducibility

To prove the main theorem of this paper in full generality we show that it suffices to consider the case where Ω/Γ is irreducible (in a sense made precise below). One must show that Ω has a (holomorphic) factorization, into 'irreducible' factors; $\Omega=\oplus_i\Omega_i$ such that each factor Ω_i admits

(1) a convex embedding,

(2) an automorphism group Γ_i satisfying the hypotheses of the main theorem (e.g. discrete, fixed-point free, etc.).

Note that for the purposes of this paper one can always pass to a finite index normal subgroup of Γ without loss of generality, in particular we may assume that if $\Omega=\Omega_1\times\Omega_2$, then $\Gamma\subset\mathrm{Aut}(\Omega_1)\times\mathrm{Aut}(\Omega_2)$.

Definition 9.3. Given a domain Ω and $\Gamma\subset\mathrm{Aut}(\Omega)$ we say (Ω,Γ) is *reducible* if

(1) $\Omega=\Omega_1\times\Omega_2$ is a non-trivial factorization, and

(2) $\Gamma=\Gamma_1\times\Gamma_2$ where $\Gamma\subset\mathrm{Aut}(\Omega_1)\times\mathrm{Aut}(\Omega_2)$ and $\Gamma_j=\Gamma\cap\mathrm{Aut}(\Omega_j)$.

We say (Ω,Γ) is essentially irreducible if (Ω,Γ') is not reducible for any finite index normal subgroup $\Gamma'\subset\Gamma$. (Compare [31] p. 86, Theorem 5.22.)

Since Ω is finite dimensional there is a finite index normal subgroup Γ' and a factorization of (Ω,Γ') into essentially irreducible factors.

If Γ is discrete, and if Γ_j is a factor as above then Γ_j is discrete (in $\mathrm{Aut}(\Omega_j)$). Likewise if Γ acts freely then so does Γ_j. Recall that if Ω is convex, then each Ω_i admits a convex embedding by Lemma 5.4.

In § 13 we use essential irreducibility, via the following lemma: (compare [31] p. 85, Theorem 5.19)

LEMMA 9.4. *Given a non-trivial factorization $\Omega=\Omega_1\times\Omega_2$ and Γ co-compact in Ω, such that*

(1) $\mathrm{Aut}(\Omega_1)$ *is semi-simple and has no compact factors, or* Ω_1 *is a bounded symmetric domain, and*

(2) $[\Gamma+\mathrm{Aut}(\Omega_1)]_0=\mathrm{Aut}(\Omega_1)$.

Then there is a finite index normal subgroup $\Gamma'\subset\Gamma$ *such that* $\Gamma'=\Gamma'_1\times\Gamma'_2$ *where* $\Gamma'_j=\Gamma'\cap\mathrm{Aut}(\Omega_j)$. *In particular,* (Ω,Γ) *is not essentially irreducible.*

Proof. Note $\Gamma\subset\mathcal{N}(\Gamma_1)$. By the technique of Lemma 9.2, and item 2 above, $\Gamma_1=\Gamma\cap\mathrm{Aut}(\Omega_1)$ is co-compact in $\mathrm{Aut}(\Omega_1)$.

Recall that by a technique of Bochner and Yano, if Ω is a (Kähler) manifold with negative Ricci (this is certainly the case for symmetric spaces of noncompact type) and Γ is co-compact in $\mathrm{Aut}(\Omega)$ (Γ acting by isometries), then $\mathcal{N}(\Gamma)/\Gamma$ is finite. Thus the kernel of the homomorphism of Γ to $\mathcal{N}(\Gamma_1)/\Gamma_1$ is a finite index normal subgroup, and we assume without loss of generality (by passing to a finite covering) that the homomorphism of Γ to $\mathcal{N}(\Gamma_1)/\Gamma_1$ is trivial. In particular every $\gamma\in\Gamma/\Gamma_1$ lifts to $(\gamma_1,\gamma_2)\in\mathrm{Aut}(\Omega_1)\times\mathrm{Aut}(\Omega_2)$ where $\gamma_1\in\mathscr{C}(\Gamma_1)+\Gamma_1$.

But, Ω_1 is a bounded symmetric domain, so $\mathscr{C}(\mathrm{Aut}(\Omega_1))=0$ and by the Borel density theorem [37] $\mathscr{C}(\Gamma_1)=0$, hence $\gamma_1\in\Gamma_1$.

Now letting π_2 be the projection of $\mathrm{Aut}(\Omega_1)\times\mathrm{Aut}(\Omega_2)$ to $\mathrm{Aut}(\Omega_2)$,

$$\pi_2\Gamma=(\Gamma+\mathrm{Aut}(\Omega_1))/\mathrm{Aut}(\Omega_1)=\Gamma/(\mathrm{Aut}(\Omega_1)\cap\Gamma)=\Gamma/\Gamma_1.$$

To show $\Gamma=\Gamma_1\times\Gamma_2$, with $\Gamma_j=\Gamma\cap\mathrm{Aut}(\Omega_j)$, it suffices to show $\mathrm{Aut}(\Omega_2)\cap\Gamma=\pi_2\Gamma$. But we have already shown that if $\pi_2(\gamma)=\gamma_2$, then $\exists\gamma_1\in\Gamma_1\subset\Gamma$ such that $\gamma_1\cdot\gamma=\gamma_2$.

The point is that the quotient space has a bundle structure, and some finite cover is a trivial bundle.

10. $\mathscr{G}$ is semi-simple

The main result of this section is

THEOREM 10.1. *Let* Ω *be a bounded domain and* $G\subset\mathrm{Aut}(\Omega)$ *a connected, closed subgroup. Suppose* $\Gamma\subset\mathrm{Aut}(\Omega)$ *is co-compact, and* $\Gamma\subset\mathcal{N}(G)$. *If* Γ *is discrete then* G *is semi-simple.*

Note that the hypotheses that Γ acts freely, and Ω is convex are unnecessary here. By Lemma 9.2, $\mathscr{G}$ satisfies the hypotheses of Theorem 10.1, and we may apply it in the proof of Theorem 9.1. Throughout this section *we let* H *be a connected, closed subgroup of* $\mathrm{Aut}(\Omega)$. Certain definitions and lemmas in this section are useful elsewhere, so we state them in terms of H, rather than G. (This gives them a little more generality, since G satisfies some additional hypotheses.) Lemmas stated in terms of H may be applied with $H=G$, or H equal to some subgroup of G. The reader should bear in mind that H does not represent any fixed group in the context of Theorem 10.1, and this will cause no confusion.

The technique of the proof of Theorem 10.1 is to show that the abelian radical C of G, defined below is trivial.

Definition 10.2. *Let* $\mathscr{L}H$ be the Lie algebra of H. The nilpotent radical n of $\mathscr{L}H$ is its maximal nilpotent ideal. We call the center of n the *abelian radical* of $\mathscr{L}H$, and denote it by c. c, n determine subgroups C, N (respectively) of H by the Exp map.

LEMMA 10.3.
(1) *$\mathscr{L}H$ is semi-simple iff $n=0$ iff $c=0$.*
(2) *N, C are closed subgroups of H.*
(3) $\Gamma\subset\mathscr{N}(C), \mathscr{N}(N)$.

Proof. (1) It is a standard fact that $\mathscr{L}H$ is semi-simple iff $n=0$. If $n\neq 0$ then the derived sequence $(c_{k+1}=[n, c_k])$ of n terminates, and the last step $c_{k(n)}$ is in the center, and is non-trivial.

(2) The group N associated to n is closed in H, since by definition it is maximal. Likewise, the closure of C is central in N, so C is closed in N.

(3) Since every step in the construction of C is canonical, it is preserved by automorphisms of H.

We now outline the proof of Theorem 10.1, breaking it into several steps.

(1) From Definition 10.4 to Corollary 10.6 we define a function g_H relating to H-actions and develop a Weitzenboch type formula for g_H (which we apply with the Einstein-Kähler metric on Ω).

(2) From Lemma 10.7 to Corollary 10.9 we show that for $H=C$, the abelian radical of G, $g_C\equiv 0$ iff C is trivial.

(3) It remains to produce a maximum for g_C. The point is to show g_C is Γ-invariant. From Definition 10.10 to Lemma 10.13 we develop the notion of a 'unimodular action of Γ on C', showing that (i) If $C/(C\cap\Gamma)$ is compact, then the Γ action on C is unimodular, and (ii) If the Γ action on C is unimodular then g_C is Γ-invariant.

(4) In Theorem 10.14 we exploit the hypothesis that Γ is discrete by applying the machinery of discrete subgroups of lie groups as in [31], to show $C/(C\cap\Gamma)$ is compact.

There is one hypothesis in Theorem 10.14 that requires special verification, and this is done using another Weitzenboch type formula, somewhat simpler than that for g_H.

Now we begin to develop a Weitzenboch type formula for certain group actions. The following definition presupposes the choice of an intrinsic Kähler metric on Ω. $\langle\ ,\ \rangle$ denotes the associated inner product, $\wedge$ denotes the wedge or 'exterior' product. There are actually two different exterior products, one comes from tensoring over $\mathbf{R}$, the other over $\mathbf{C}$, and both are useful to us; in Lemma 11.3 we work over $\mathbf{R}$, whereas in Definition 10.4 we work over $\mathbf{C}$. In general, if we regard vector fields $X_i(x)$ as fields in $T\Omega$, then we tensor over $\mathbf{R}$, whereas $X_i(x) \in T^{1,0}\Omega$ indicates that we tensor over $\mathbf{C}$.

Definition 10.4. Let H be a closed, connected subgroup of $\mathrm{Aut}(\Omega)$ of dimension k and let $X_i(x) \in T^{1,0}\Omega$ be complete holomorphic vector fields on Ω giving a basis for the Lie algebra $\mathscr{L}H$. We define

$$w_H(x) = \wedge_i X_i(x) \in \Lambda^k T^{1,0}\Omega \quad \text{and} \quad g_H(x) = \langle w_H(x), w_H(x) \rangle .$$

Note that if we had tensored over $\mathbf{R}$, then $w_H(x)$ would essentially be the push-forward of a volume form on H. But since we tensor over $\mathbf{C}$ it is a holomorphic tensor field. We now show that g_H satisfies a strong maximum principle, when defined in terms of the Einstein-Kähler metric [3].

Recall that on a Kähler manifold, the Laplace-Beltrami operator on functions takes the simple form: $\Delta = \frac{1}{2}\sum g^{i\bar{j}}\partial_i\partial_{\bar{j}}$ and given vector (or appropriate tensor) fields, X, Y, $\partial_i\langle X, Y\rangle = \langle D_i X, Y\rangle + \langle X, D_{\bar{i}} Y\rangle$ where D indicates covariant differentiation.

In the next lemma we will use the holomorphicity of X_i to get $D_{\bar{i}}X_i = 0 = D_{\bar{i}}w$ repeatedly.

LEMMA 10.5. *Let U be an Einstein-Kähler manifold, such that* $\mathrm{Ric}_{i\bar{j}} = -rg_{i\bar{j}}$, *let $X_i(x) \in T^{1,0}\Omega$ be holomorphic vector fields on Ω, and $w(x) = \wedge_i X_i(x) \in \Lambda^k T\Omega$. Then*

$$\frac{1}{2}\Delta|w(x)|^2 = |\nabla w|^2 + kr|w|^2. \tag{6}$$

Proof. We will sketch the proof. We refer to [22] for background.

$$\begin{aligned}\Delta|w|^2 &= g^{i\bar{j}}\partial_i\partial_{\bar{j}}\langle w, w\rangle = g^{i\bar{j}}\partial_i\langle w, D_j w\rangle = |Dw|^2 + \langle w, g^{i\bar{j}}D_{\bar{i}}D_j w\rangle \\ &= |Dw|^2 + \langle w, g^{i\bar{j}}[D_{\bar{i}}, D_j]\, w\rangle .\end{aligned}$$

But $-g^{i\bar{j}}[D_{\bar{i}}, D_j]\, w$ is a Ricci curvature term. By multilinearity of curvature operators, $\mathrm{Ric}(X\wedge Y)=(\mathrm{Ric}\,X)\wedge Y+X\wedge(\mathrm{Ric}\,Y)=-2r(X\wedge Y)$ where the last equality holds only in the Einstein-Kähler case.

If U is a bounded domain then by the work of Cheng-Yau, [3], there is a canonical Einstein-Kähler metric on U. With respect to this metric we see

COROLLARY 10.6. *If U is a bounded domain with the Cheng-Yau metric, then $g_H(x)$ is subharmonic and satisfies a very strong maximum principle: if it has a local maximum then it vanishes identically.*

We now begin the second step from the outline of the proof of Theorem 10. In a lot of cases the function g_H vanishes everywhere, hence is of no interest. One case where it is of interest is when H is abelian. Recall that a connected abelian or nilpotent Lie group has a unique maximal compact subgroup, which is a torus. It is clearly a canonical normal subgroup. The space of homomorphisms $\phi: S^1\to T$ is in $1-1$ correspondence with the fundamental group, $\pi_1(T)$, hence it is countable. Furthermore, given $\phi\in\pi_1(T)$, the fixed-point set $\phi^*\subset\Omega$ is a holomorphic subvariety. The isotropy T_x is compact, so $\{x\colon T_x\neq 0\}=\bigcup_{\phi\in\pi_1(T)}\phi^*$.

But $\Omega-\bigcup_{\phi\in\pi_1(T)}\phi^*\neq\emptyset$ by simple measure theory considerations. Hence there exists $x\in\Omega$, such that the isotropy is trivial, i.e. $T_x=\{0\}$. We conclude

LEMMA 10.7. *Let $N\subset\mathrm{Aut}(\Omega)$ be a non-trivial connected nilpotent group. Then N has no isotropy on a dense open subset S of Ω. If $X_i(x)\in T\Omega$ is a basis for $\mathscr{L}N$, then $\forall x\in S$, $X_i(x)$ are linearly independent* over $\mathbf{R}$.

LEMMA 10.8. *If $C\subset\mathrm{Aut}(\Omega)$ is abelian, and $X_i(x)\in T^{1,0}\Omega$ is a basis for $\mathscr{L}C$, C has no isotropy on a dense open subset S of Ω, and $\forall x\in S, X_i(x)$ are linearly independent* over $\mathbf{C}$.

Proof. It suffices to show that $\forall x\in\Omega, T_xCx\subset T\Omega$ is totally-real, i.e, $T_xCx\cap J(T_xCx)=0$, where J denotes the complex structure. Suppose $L\subset T_xCx$ is a complex line. Since C is abelian and acts freely, L may be regarded as a subalgebra of c, hence as a subgroup of C. Since L acts freely by holomorphic automorphisms, the orbit Lx has complex tangent space T_yLx at every $y\in Lx$, in fact Lx is a non-trivial holomorphic embedding of a complex line. But Ω is hyperbolic so this gives a contradiction.

COROLLARY 10.9. *$w_C(x)$ and $g_C(x)$ are non-zero on a dense open subset of Ω.*

We now begin the third step from the outline of the proof of Theorem 10. We want to apply the maximum principle, Corollary 10.6, to prove Theorem 10.1. It suffices to show that if C is the abelian radical of G then $g_C(x)$ is a Γ invariant function.

We exploit the fact that C is the radical of G because $G\subset\mathcal{N}(C)$, and $\Gamma\subset\mathcal{N}(G)$ implies $\Gamma\subset\mathcal{N}(C)$. In fact $\Gamma\subset\mathcal{N}(C)\Rightarrow\Gamma\subset\mathcal{N}(\Gamma\cap C)$. In Theorem 10.14 we will see that $\Gamma\cap C$ is co-compact in C.

This will verify the hypothesis of Lemma 10.12 for all $\gamma\in\Gamma$ and by Lemma 10.13 we conclude that $g_C(x)$ is Γ invariant.

From this point to Lemma 10.13, *N can represent any closed, connected subgroup of* Aut(Ω) (*including* C). (We *think* of it as a nilpotent radical for the application here, to Theorem 10, but the theory is valid in greater generality.)

Definition 10.10. (1) Given $\gamma\in\mathcal{N}(N)$, $\mathrm{Ad}_N(\gamma)\colon N\to N$ is defined by $\mathrm{Ad}_N(\gamma)(t)=\gamma^{-1}t\gamma$, in particular, $\mathrm{Ad}_N(\gamma)(e)=e$.

(2) Thus the derivative, $d\mathrm{Ad}_N(\gamma)(e)\colon \mathscr{L}N\to\mathscr{L}N$ and we define,

$$\mathrm{ad}_N(\gamma)=d(\mathrm{Ad}_N(\gamma))(e).$$

(3) The modular function of N is a homomorphism $\phi N\colon\mathcal{N}(N)\to\mathbf{R}$ defined to be $\phi_N(\gamma)=\det(\mathrm{ad}_N(\gamma))$.

Remark 10.11. (1) Given $X\in\mathscr{L}N, X(\gamma(x))=\gamma_*\,\mathrm{ad}(\gamma)X(x)$. One just differentiates $\gamma\cdot\gamma^{-1}\cdot\mathrm{Exp}(tX)\cdot\gamma\cdot x$ by t.

(2) We will use the notation $[\mathrm{ad}(\gamma)X_i](x)$ for $\Sigma_j\alpha_{ij}X_j(x)$, with holomorphic vector fields X_i, where $\mathrm{ad}(\gamma)X_i=\Sigma_j\alpha_{ij}X_j$ for the corresponding elements of the Lie algebra.

(3) Choose a left invariant volume form μ on N, and suppose that $\Gamma_N\backslash N$ is a compact quotient (of unit volume). Note that the following maps are well-defined; right multiplication, $r_h\colon\Gamma_N\backslash N\to\Gamma_N\backslash N$ for $h\in H$, left multiplication, $l_h\colon\Gamma_N\backslash N\to\Gamma_N\backslash N$ for $h\in\Gamma_N$. Clearly, for $\gamma\in\Gamma_N$,

$$\int_{\Gamma_N\backslash N}\det(d(l_{\gamma^{-1}}r_\gamma))\,d\mu=1.$$

But the integrand is a constant function, in fact $\phi_N(\gamma)=\det(d(l_{\gamma^{-1}}r_\gamma))$, so for $\gamma\in\Gamma_N$, $|\phi_N(\gamma)|=1$.

Now ϕ is a homomorphism, so by the preceeding, $|\phi_N(\gamma)|\colon\Gamma_N\backslash N\to\mathbf{R}$ is well defined, but it has compact range, so $|\phi_N(\gamma)|\equiv1$ (for all $\gamma\in N$ or $\gamma\in\mathcal{N}(\Gamma_N)$).

The last remark essentially proves the well known

LEMMA 10.12. *Let Γ_N be a co-compact subgroup of N, then $\forall\gamma\in\mathcal{N}(\Gamma_N)$, $|\phi_N(\gamma)|=1$.*

We say that γ is *unimodular* on N. Unimodularity is the key to showing g_C is Γ invariant.

LEMMA 10.13. *If $|\phi_N(\gamma)|=1$ then*

$$|w_N(\gamma(x))|^2 = |w_N(x)|^2.$$

Our use of the Kähler-Einstein metric is necessary here.

Proof. $X_i(\gamma(x))=d\gamma(x)\,[\mathrm{ad}(\gamma)X_i]\,(x)$ so $w(\gamma(x))=\Lambda_i X_i(\gamma(x))=\gamma_*\det\mathrm{ad}(\gamma)\Lambda_i X_i((x))$ but $|\phi_N(\gamma)|=1$ implies $\det^2[\mathrm{ad}(\gamma)]=1$ and γ_* is isometric, for the intrinsic metric, so

$$|w(\gamma(x))|^2 = |w(x)|^2$$

We now begin the fourth step from the outline of the proof of Theorem 10. To prove the main theorem of this section, it suffices by the lemmas above to show that $C\cap\Gamma$ is co-compact in C. By the hypotheses of Theorem 10 $G\cap\Gamma$ is co-compact in G. (From Lemma 9.2 we know that $\mathscr{G}\cap\Gamma$ is co-compact in $\mathscr{G}$.) We must appeal to some machinery from the theory of discrete subgroups of Lie groups to derive the theorem stated below. This is essentially Corollary 8.26, followed by Proposition 2.17 from the book of Raghunathan [31].

THEOREM 10.14. *Let G be a connected Lie group, $\Gamma\subset G$ a lattice. Let R be the radical of G, N the nilpotent radical, and let $S\subset G$ be a semi-simple subgroup such that $G=SR$ is a Levi-Malcev decomposition. Let σ be the action of S on R, i.e. $sr=\sigma(s,r)s$.*

[σ1] Assume that the kernel of σ has no compact factors in its identity component. Then $N/(N\cap\Gamma)$ is compact. Furthermore if C is the center of N, then $C/(C\cap\Gamma)$ is compact.

So we will finish the proof by verifying [σ1] (for $G=\mathscr{G}$). Since S is semi-simple there is unique maximal compact factor K in $\ker\sigma$ and K is semisimple. It is clear that on the level of Lie algebras $\mathscr{L}K$ is a factor of $\mathscr{L}G$, and that $\Gamma\subset\mathcal{N}(\mathscr{L}K)$. The Killing form is a canonical bi-invariant metric on $\mathscr{L}K$, so $\mathrm{Ad}(\gamma)$ acts on $\mathscr{L}K$ by isometries for all $\gamma\in\Gamma$.

Let $X_i(x)$ be an orthonormal basis for $\mathscr{L}K$. Define $f_K\colon\Omega\to\mathbf{R}$ by

$$f_K(x) = \sum |X_i(x)|^2$$

so

$$f_K(\gamma(x)) = f_K(x)$$

for all $\gamma \in \Gamma$. The proof is analogous to that in Lemma 10.13. $f_K(x)$ is precisely the energy function (in the sense of harmonic maps) of the map from K to Kx. Again we invoke the Γ invariant Cheng-Yau metric to get a very strong maximum principle for f_K.

LEMMA 10.15.

$$\Delta f_K = \sum (|\nabla X_i|^2 + r|X_i|^2).$$

The calculation is similar to Lemma 10.5, in this case we don't really need the holomorphic formalisms, we can use the riemannian structure with the fact that X_i are killing fields. This is essentially the Bochner-Yano formula.

By the strong maximum principle and co-compactness of Γ, f_K vanishes identically, thus K is trivial.

The proof of Theorem 10.1 is complete.

Theorem 10.1 may be compared to a result of Hano [10], he showed that if a unimodular group G acts transitively on a bounded domain, then G is semi-simple.

11. $\mathscr{G}$ is transitive

Definition 11.1. We use $K \overset{\mathrm{mc}}{\subset} G$ to denote 'K is a maximal compact subgroup of the Lie group G'.

The main theorem of this section is

THEOREM 11.2. *Let Ω be a convex hyperbolic domain and Γ a discrete, freely acting co-compact subgroup of* Aut(Ω). *Let G be a closed, connected, semisimple subgroup of* Aut(Ω) *such that $\Gamma \subset \mathcal{N}(G)$. Then $\forall x \in \Omega$,*

(1) $G_x \overset{\mathrm{mc}}{\subset} G$.

(2) *The orbit Gx is a holomorphic submanifold of Ω.*

(3) *There is a canonical biholomorphic map $\mathcal{I}_x\colon Gx \times G_x^* \to \Omega$.*

In particular, if Ω is essentially irreducible then G is transitive, and Ω is a bounded symmetric domain.

The proof of Theorem 11.2 hinges on an analysis of the set $\mathcal{S}$ which we now define. Given a connected, closed sub-group $G \subset \mathrm{Aut}(\Omega)$ acting on Ω let

$$\mathcal{S} = \{x \in \Omega : G_x \overset{\mathrm{mc}}{\subset} G\}. \tag{7}$$

This is the 'lowest stratum of the orbit structure'. $\mathcal{S}$ is clearly Γ invariant if $\Gamma \subset \mathcal{N}(G)$.

Note that $x \in \mathcal{S}$ satisfies item (1) of Theorem 11.2 by definition. Our strategy for the proof of Theorem 11.2 is to show that $\mathcal{S}$ satisfies many of the (hypotheses and) conclusions of Theorem 11.2 (with Ω replaced by $\mathcal{S}$). We will use these facts to compare $\mathcal{S}$ to Ω and conclude that in fact they must be the same.

Much of the analysis of $\mathcal{S}$ is based on the Cartan decomposition of a semi-simple Lie group, [12]. In the non semi-simple case (which is not relevant to this paper) the Iwasawa decomposition [14] can be used as a substitute at many points in the analysis.

The maximal compact sub-group G_x of a connected Lie group is connected (by the Iwasawa decomposition). This leads us to the following equivalent characterization of $\mathcal{S}$:

$$\mathcal{S} = \{x \in \Omega : \dim Gx \leq k\} \tag{8}$$

for an appropriate k, (note '$\leq$' could be replaced by '$=$'). (Clearly, the *real* dimension is referred to here.)

LEMMA 11.3. *$\mathcal{S}$ is a real analytic subvariety of Ω.*

Proof. Choose a basis $X_i(x) \in T\Omega$ for $\mathcal{L}G$ and let $(X_{ij}(x))$ be the matrix with rows $X_i(x)$. By equation (8)

$$\mathcal{S} = \{x \in \Omega : \mathrm{rk}(X_{ij}(x)) \leq k\} \tag{9}$$

where rk denotes 'rank over $\mathbf{R}$'. Since $X_i(x)$ are holomorphic vector fields, (real) determinants of $(k+1)\times(k+1)$ submatrices are real analytic functions. But $\mathcal{S}$ is the zero set of finitely many such functions.

The following lemma is standard material, [13]:

LEMMA 11.4. *Given Lie groups $J \subset G$, let J act on G/J. The fixed-point set J^* in G/J satisfies the following:*

(1) *$J^* = \mathcal{N}(J)/J \subset G/J$ (i.e. the $\mathcal{N}(J)$-orbit of the coset $[J]$).*

(2) *If G is compact and connected, or if J is semi-simple, then $\mathcal{N}(J) = \mathcal{C}(J) + J$.*

(3) *Furthermore, if G is semi-simple and J maximal compact, then $\mathcal{N}(J)=J$, so $J^*=[J]$, a single point.*

(Item (2) is not applied in this paper.)

By the Iwasawa decomposition, $G=K\cdot S$ differentiably, where $K\overset{\text{mc}}{\subset}G$, and S is a solvable subgroup of G such that $S=\mathbf{R}^n$ differentiably. Furthermore all maximal compact sub-groups of G are conjugate in G.

Remark 11.5. (1) $Gx=G/G_x=\mathbf{R}^n$ for all $x\in\mathcal{S}$.

(2) Given $x\in\mathcal{S}$, by the Cartan (or Iwasawa) conjugacy theorem,

$$\mathcal{S}=\bigcup_{y\in Gx} G_y^*.$$

(3)
$$\mathcal{S}=\bigcup_{y\in G_x^*} Gy.$$

(4) If G is semi-simple then by Lemma 11.4 for all $x\in\mathcal{S}$, $Gx\cap G_x^*=\{x\}$, furthermore for all $y,z\in\mathcal{S}$, $\exists!x$ such that $Gz\cap G_y^*=\{x\}$, hence the unions above (in the two preceeding items) are disjoint.

To proceed with the proof of Theorem 11.2 we will need two basic properties of the set $\mathcal{S}$;

LEMMA 11.6. (1) $\mathcal{S}\neq\varnothing$.

(2) *$\mathcal{S}$ is connected.*

Both follow from the fixed-point theory in § 12; since Ω is convex, Theorem 12.2 implies that $\mathcal{S}\neq\varnothing$, and that G_x^* is connected for $x\in\mathcal{S}$. But G is connected, so $Gx=G/G_x$ is connected, and item (2) of Remark 11.6 follows from item (2) of Remark 11.5.

Item (4) of Remark 11.5 is crucial for the next two lemmas, so for the balance of this section we assume that *G is semi-simple*.

LEMMA 11.7. *$x\in\mathcal{S}$ determines a canonical diffeomorphism $\mathcal{I}_x\colon Gx\times G_x^*\to\mathcal{S}$ (onto $\mathcal{S}$) defined by*

$$\forall\gamma\in G,\quad \gamma(x)=y\quad\Rightarrow\quad \mathcal{I}_x(y,w)=\gamma(w)$$

Proof. $\gamma_i(x)=y$, $i=1,2\Rightarrow\exists\lambda\in G_x$, such that $\gamma_2=\gamma_1\cdot\lambda$ but $\lambda(w)=w$, so $\mathcal{I}_x$ is well-defined. In fact, by Remark 11.5, item (4), $\mathcal{I}_x(y,w)=G_y^*\cap Gw$. Given $z\in\mathcal{S}$, let $y=Gx\cap G_z^*$ and $w=G_x^*\cap Gz$, then $\mathcal{I}_x(y,w)=z$.

In the case where Γ acts freely we can now apply the machinery of group cohomology to show $\mathscr{S}=\Omega$.

A good reference for the material we use in the following lemma is the book of Mosher and Tangora, [28].

LEMMA 11.8. (1) *Given contractible smooth manifolds $D_i, i=1,2$ and a proper embedding $f: D_1 \to D_2$,* if $\exists\Gamma$ *a discrete, freely acting, co-compact, properly discontinuous subgroup of diffeomorphisms of D_2, such that $f(D_1)$ is Γ invariant then f is a homeomorphism.*

(2) *Given N, M compact manifolds and an embedding $f: N \to M$ such that the induced map on the fundamental group $f_*: \pi_1(N) \to \pi_1(M)$ is an isomorphism, if both N, M are $K(\pi, 1)$ spaces (Eilenberg-MacLane) then f is a homeomorphism.*

Sylvan Cappel has pointed out that by an application of the 'Bovel trick' one can remove the hypothesis in (1) that Γ acts freely. This implies that the main theorems, Theorem 1 and Theorem 2, are true without the hypothesis that Γ acts freely.

Proof. (1) Letting $N=D_1/\Gamma$ (identify D_1 with $f(D_1)$) and $M=D_2/\Gamma$, both are $K(\pi, 1)$ spaces (with $\pi=\Gamma$). Furthermore, f induces a well defined map on the Γ-cosets, in fact an embedding, and the induced map on the fundamental group, $f_*: \pi_1(N) \to \pi_1(M)$ is just the identity map on Γ. Thus the first item of the lemma reduces to the second item, which we proceed to prove.

(2) f_* is an isomorphism and both N, M are $K(\pi, 1)$ spaces so f is a homotopy equivalence, therefore f is a homology equivalence. Therefore the homological dimensions of N, M are equal. Since M, N are compact manifolds their topological dimensions equal their homological dimensions. But an embedding into an equidimensional manifold is a homeomorphism. (The point of the proof is that the cohomology of either space is the group cohomology of Γ.)

We apply the lemma with $D_1=\mathscr{S}, D_2=\Omega$, where f is simply the inclusion map. Since we are assuming that Ω is convex, it is certainly contractible. Furthermore, in this case G_x^* is contractible by the fixed-point Theorem 12.2. Together with contractability of Gx for $x \in \mathscr{S}$ and Lemma 11.7 which says $\mathscr{S}=Gx \times G_x^*$, this implies that $\mathscr{S}$ is contractible. Note that G_x^* is a holomorphic submanifold of Ω. It is easy to see that $\mathscr{S}$ is a manifold, in fact by Remark 11.5, item (4) locally it is the product of G_x^* by a neighborhood of x in Gx. By Lemma 11.3, $\mathscr{S}$ is a smooth, locally-connected, closed submanifold of Ω, without boundary. Applying Lemma 11.8 we conclude that $\mathscr{S}=\Omega$.

LEMMA 11.9. *Given a connected semi-simple group acting by holomorphic isometries on a hermitian manifold* Ω, *define* $\mathcal{S}$ *as in equation* (7) *and* $\mathcal{I}_x$ *as in Lemma* 11.7, *suppose that* $\mathcal{S}=\Omega$. *Then* $\forall x \in \Omega$.

(1) *Gx is a holomorphic submanifold of* $\mathcal{S}$.

(2) *$\mathcal{I}_x$ is a biholomorphic map.*

Remark 11.10. Note that this lemma does not suppose that $\Gamma \subset N(G)$ for some co-compact Γ.

There is no analogous statement in riemannian geometry (replacing 'holomorphic submanifold' with 'totally-geodesic submanifold', and 'biholomorphic' with 'isometric'). There are counterexamples, without the complex structure. Even with Ω of constant curvature and $\mathcal{I}_x \subset \mathrm{Isom}(\Omega)$, Gx may not be totally geodesic and $\mathcal{I}_x$ not an isometry. In some sense this happens because in the riemannian case the group G can act on an irreducible space Ω, such that for some x, Gx is a proper totally geodesic submanifold but, (1) the normal bundle NGx to Gx is flat and (2) G acts trivially on NGx. This contrasts with the case of a (Kähler) bounded symmetric domain Ω, where (1) is impossible because the bisectional curvature on Ω is non-trivial, and this induces non-trivial curvature in NGx. On the other hand, we expect there is an analogous statement in riemannian geometry if Ω is Einstein and we suppose that $\Gamma \subset N(G)$ for some co-compact Γ.

Proof. (1) It suffices to show $T_x Gx \perp T_x G_x^*$ (since Ω is a holomorphic manifold and $T_x Gx \oplus T_x G_x^* = T_x\Omega$). In Lemma 12.1 we show that G_x^* determines a projection $p: T_x \mathcal{S} \to T_x G_x^*$ and $\ker p = (T_x G_x^*)^\perp$. But $\forall v \in T_x Gx, k \in G_x^*, k_* v \in T_x Gx$, hence $p(v) \in T_x Gx \cap T_x G_x^* = 0$ so $\ker p = T_x Gx$.

(2) We must verify that $d\mathcal{I}_x(z, y)$ is complex linear on $T_y G_x^*$ and on $T_x Gx$. The first is obvious because $\gamma \in G$ is holomorphic and $d\mathcal{I}_x(z, y)$ on $T_y G_x^*$ is just $d\gamma$. The second is more subtle; $\mathcal{I}_x(z, y) = \mathcal{I}_y(\gamma y, y) \cdot \mathcal{I}_x(x, y)$ and we apply the chain rule. The left factor is complex linear since $d\gamma$ is. The right factor, $d\mathcal{I}_x(x, y): X(x) \mapsto X(y)$ for any field $X \in \mathcal{L}G$. It follows that $d\mathcal{I}_x(x, y)$ conjugates the action of G_x on $T_x Gx$ to the action of $G_y = G_x$ on $T_x Gy$. But the complex structure of $T_x Gx, T_y Gy$ is uniquely determined (up to complex conjugation) as the central element of G_x^* satisfying $J^2 = -I$. So $d\mathcal{I}_x(x, y)$ conjugates the complex structure of $T_x Gx$ to the complex structure of $T_y Gy$. We can ignore the complex conjugate by the continuity method and connectedness of G_x^*.

By Lemma 11.9, $\Omega = Gx \times G_x^*$ is a holomorphic factorization. Now Gx is clearly a bounded symmetric domain and $\Gamma \subset \mathcal{N}(G)$, so Lemma 9.4 at the end of §9 proves the last claim of Theorem 11.2.

12. Fixed points of convex domains

The following trivial averaging trick is quite useful.

LEMMA 12.1. *Given a compact group J acting by isometries on a euclidean space V, define $\pi: V \to J^*$, by*

$$\pi(v) = \int_{j \in J} j\, v\, d\mu$$

where $d\mu$ is the bi-invariant measure (with unit volume). π is a linear projection map, it is the identity map on the linear subspace J^*, and $\ker \pi = J^{*\perp}$.

The following theorem is of general interest in the study of automorphism groups of convex domains.

THEOREM 12.2. *Given a convex hyperbolic domain Ω and a compact subgroup $J \subset \mathrm{Aut}(\Omega)$,*

(a) *J^* is non-empty and*

(b) *J^* is contractible. (In fact J^* is a retract of Ω.)*

(c) *J^* is a holomorphic submanifold of Ω, in particular it is a locally-connected, real analytic submanifold without boundary, totally-geodesic in any J-invariant metric.*

The proof resembles that of the Cartan fixed-point theorem, however we use two structures simultaneously: the Kobayashi metric which is intrinsic and the euclidean structure which is not. We exploit convexity of $K(x, r)$ in the euclidean sense, see Lemma 5.4, not the intrinsic sense.

We cannot hope to drop the convexity hypothesis on Ω in Theorem 12.2 because (we are told [8]), there are examples of finite groups, that act on pseudoconvex domains with no fixed-points. On the other hand, if we add the hypothesis that J is connected, there are generalizations.

Proof. (a) We recursively define a sequence of pointed-sets

$$(x_i, S_i), \quad x_i \in S_i \subset \Omega, \quad \forall i \geq 0.$$

(i) $S_0 = \Omega$ and S_{i+1} is defined as follows:

(ii) x_i is the center of mass of S_i.

(iii) Let $S_i(r) = S_i \bigcap_{j \in J} K(jx_i, r)$,

(iv) $r_i = \inf\{r: S_i(r) \neq \emptyset\}$,

(v) $S_{i+1} = S_i(r_i)$.

By induction, $\forall i$, S_i is compact ($i>0$), convex (since $K(x, r)$ is by Lemma 5.4), S_i is J-invariant, $\dim S_{i+1}<\dim S_i$, or it is zero. In the latter case $S_i=\{x_i\}$, in particular for $i=\dim_R(\Omega)$, $S_i=\{x_i\}$ is a fixed-point, z, so $J^*\neq\emptyset$.

(b) Contractibility follows from the same construction, roughly speaking we contract along geodesics of the Kobayashi metric. The catch is that one needs to single out a 'unique geodesic' in a continuous fashion. We do this in the next paragraph, using the (euclidean) center-of-mass, hence the contraction depends on the choice of a (convex) embedding.

Given $x, y\in J^*, \forall t\in[0,1]$ let $S_1^t=K(x, tk)\cap K(y,(1-t)k)$, where $k=k(x,y)$. Proceed as above to obtain $z^t(x,y)\in J^*$. $z^t(x,y)$ depends on t, x, y in a continuous fashion. $z^1(x,y)=y$ and $z^0(x,y)=x$, so this provides a contraction.

An examination of the proof that $J^*\neq\emptyset$ shows that we have actually constructed a retraction map from Ω to J^*.

(c) These local facts are all easily verified. One may apply Lemma 12.1 to the second fundamental form of J^* to see that J^* is totally-geodesic.

An interesting possibility for an alternative proof is to consider the holomorphic map $s\colon\Omega\to\Omega$ defined by

$$s(z)=\int_{j\in J} jz\,d\mu$$

where $d\mu$ is the bi-invariant measure (with unit volume). Iterations of s should converge to a holomorphic retraction of Ω to J^*. One hopes that in general J^* admits holomorphic convex embeddings, (note that kob=car for a holomorphic retraction of convex Ω, by work of Lempert).

13. Final steps

We recap the proofs of the main theorems.

Proof of Theorems 2 *and* 9.1. Recall $\mathcal{G}'=[\Gamma+\{\sigma_t: t\in\mathbf{R}\}]$, and $\mathcal{G}$ denotes the maximal connected subgroup of $\mathcal{G}'$. The same arguments will hold if we replace $\{\sigma_t: t\in\mathbf{R}\}$ by any connected subgroup of $\mathrm{Aut}(\Omega)$.

By Lemma 9.2, $[\Gamma+\mathcal{G}]_0=\mathcal{G}$, and Γ normalizes $\mathcal{G}$. By Theorem 10.1, $\mathcal{G}$ is semisimple. By Theorem 11.2, $\Omega=\Omega_1\times\Omega_2$ holomorphically, in particular both factors are complex manifolds, furthermore Ω_1 is non-trivial and is biholomorphic to a bounded symmetric domain.

Now we can apply Lemma 9.4, so Γ has a finite index normal subgroup Γ' such that $\Gamma'=\Gamma_1'\times\Gamma_2'$ where $\Gamma'\subset\mathrm{Aut}(\Omega_1)\times\mathrm{Aut}(\Omega_2)$ and $\Gamma_j'=\Gamma'\cap\mathrm{Aut}(\Omega_j)$.

Proof of Theorem 1. It suffices to apply Theorem 2 inductively to the factors Ω_2 and $\mathrm{Aut}(\Omega_2)$ that arise in the conclusion of that theorem. To this end we have verified that each of the hypotheses of Theorem 1 is satisfied by Ω_2: it is a complex submanifold of Ω by Theorem 11.2, and a holomorphic factor of Ω, hence it is convex by Corollary 5.5, hyperbolicity of Ω_2 is automatic. The hypotheses on Γ, (Γ1) Γ is discrete and acts freely and (Γ2) Γ is co-compact (in Ω), are discussed in the appendix to § 9 (Notes on reducibility).

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