THE MAXIMUM MODULUS AND VALENCY OF FUNCTIONS MEROMORPHIC IN THE UNIT CIRCLE.

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Introductory Abstract

1) Let E be a closed set of complex values w containing $w = 0, \infty$ and at least one other finite value. Let $p(\varrho)$ be an increasing positive function defined for $0 \leq \varrho < 1$.

We consider in this essay a function $f(z)$ meromorphic in $|z| < 1$ and such that none of the equations

$$f(z) = w,$$

where w lies in E, have more than $p(\varrho)$ roots in $|z| \leq \varrho$, $0 < \varrho < 1$. In other words the valency of $f(z)$ on the set E is at most $p(\varrho)$ in $|z| < \varrho$, $0 < \varrho < 1$. We shall also say sometimes that $f(z)$ takes no value w of E more than $p(\varrho)$ times in $|z| \leq \varrho$.

Our aim is to find bounds under this hypothesis for the maximum modulus of $f(z)$

$$M[\varrho, f(z)] = \max_{0 \leq \theta \leq 2\pi} |f(\varrho e^{i\theta})|.$$

We confine ourselves here to the case when $p(\varrho) \geq 1$. There are in this case various difficulties. We cannot use the simple theory of subordination to give the extremal functions. Further since $f(z)$ is not in general regular, we have

$$M[\varrho, f] = \infty$$

whenever the circle $|z| = \varrho$ contains a pole of $f(z)$. Lastly even if we assume that $f(z)$ is regular and takes no value more than once, we cannot from the boundedness of $f(0)$ alone deduce a bound for $M[\varrho, f]$, as is shown by the functions

$$f(z) = kz$$

for which $f(0) = 0$, while k is arbitrary. If $p(\frac{1}{2}) \leq p$, so that $f(z)$ takes no value of E more than p times in $|z| \leq \frac{1}{2}$ and $f(z)$ is regular, a bound for $M[\varrho, f]$ can be obtained in terms of

$$\mu_p = \max[1, f(0), f'(0), \ldots, f^{(p)}(0)].$$

We have preferred, however, to use the following method. Let $f(z)$ be meromorphic in $|z| < 1$ and let a_μ, $\mu = 1$ to m, b_ν, $\nu = 1$ to n, be the zeros and poles of $f(z)$ in $\left|\frac{z - \xi}{1 - \bar{\xi} z}\right| \leq \frac{1}{2}$. Then we define

$$f_*(\xi) = f(\xi)\, 2^{n-m} \frac{\prod_{\nu=1}^{n} g(\xi, b_\nu)}{\prod_{\mu=1}^{m} g(\xi, a_\mu)}$$

where

$$g(\xi, z) = \frac{z - \xi}{1 - \xi \bar{z}} \frac{|z|}{z}.$$

Thus $f_*(z)$ is obtained from $f(z)$ by dividing out the zeros and poles of $f(z)$ in the neighborhood of the point z. If $f(z)$ is regular nonzero in $|z| < 1$, we have $f_*(z) = f(z)$. The function $f_*(z)$ has a continuous modulus in $|z| < 1$. However, except near the zeros and poles $f(z)$ does not differ too much from $f(z)$. *Further we can obtain bounds for*

$$M[\varrho, f_*(z)] = \max_{0 \leq \theta \leq 2\pi} |f_*(\varrho e^{i\theta})|$$

in terms of ϱ, $|f_*(0)|$, E *and the function* $p(\varrho)$ *only.* We shall deal with $f_*(z)$ throughout and obtain bounds for $M[\varrho, f_*(z)]$. Also when $f(z)$ is regular we have

$$|f(z)| \leq |f_*(z)|$$

so that bounds for $M[\varrho, f]$ follow.

2) In Chapter I we consider the case, where E consists of 0, 1, ∞ and $p(\varrho)$ is a general increasing function such that

$$\int_0^1 p(\varrho)\, d\varrho < \infty.$$

Another way of putting this is to say that

$$N_0 = \sum_{\nu=1}^{\infty} (1 - |d_\nu|) < \infty \tag{2.1}$$

where d_ν runs over all the points such that $f(d_\nu) = 0$, 1 or ∞ in $|z| < 1$.

The basis of most of the positive results obtained in our essay is Theorem I, as stated in paragraph 20, where a bound is obtained for

$$\frac{d}{d\varrho} \log |f_*(\varrho)|$$

at $\varrho = 0$. Integration of this result yields Theorem II which is

$$\log M[\varrho, f_*(z)] < \frac{1}{1-\varrho}[(1+\varrho)\log^+ |f_*(0)| + A\varrho(\log^+ \log^+ |f_*(0)| + N_0 + 1)] \tag{2.2}$$

where N_0 is defined as in (2.1). We see that the condition $f(z) \neq 0, 1, \infty$ of Schottky's Theorem is replaced by the much more general condition (2.1). If $f(z)$ is regular, we may replace $M[\varrho, f_*(z)]$ by $M[\varrho, f(z)]$ in (2.2). Moreover it appears from a wide class of counterexamples given in Theorem IV, that the condition (2.1) is probably the weakest of its kind in order that the function $f(z)$ shall always satisfy

$$\log M[\varrho, f_*(z)] = \frac{O(1)}{1-\varrho},$$

the order of magnitude attained when $p(\varrho) \equiv 0$. Most of Chapter I is taken up by the proof of Thorem I, an inequality for

$$\left[\frac{d}{d\varrho} \log |f_*(\varrho)|\right]_{\varrho=0} = \Re \frac{f_*'(0)}{f_*(0)}.$$

The major deductions from this are stated in Theorems I to VI in paragraph 20. The deduction of Theorems II to VI from Theorem I is comparatively simple. Theorem V gives a generalization to functions $f(z)$, such that the equations

$$f(z) = 0,\ \phi(z),\ \infty$$

have at most $p(\varrho)$ roots in $|z| < \varrho$, where $\int p(\varrho)\, d\varrho < \infty$ and $\phi(z)$ is a meromorphic function in $|z| < 1$ having at most $p(\varrho)$ poles and zeros. Then we have

$$\log M[\varrho, f_*(z)] \leq \log M[\varrho, \phi(z)] + \frac{O(1)}{1-\varrho}.$$

The result follows from Theorem II by applying that result to

$$g(z) = \frac{f(z)}{\phi(z)}.$$

3) In Chapter II we take up our general problem again. In the first part, we consider the case where $p(\varrho)$ is a constant positive integer p in $0 < \varrho < 1$.[1] Suppose that $f(z)$ is regular nonzero, and that $f(z)$ takes some value w such that $|w| = r$, at most p times in $|z| < 1$. Put

$$(3.1) \qquad \phi(z) = [f(z)]^{1/p+1}.$$

Then

$$\phi(z) = w'$$

implies

$$f(z) = (w')^{p+1}.$$

Taking $(w')^{p+1} = w$ which holds for $p + 1$ distinct values w', we deduce that $\phi(z)$ defined by (3.1) satisfies

$$\phi(z) \neq w'$$

in $|z| < 1$ for some w', such that $(w')^{p+1} = r$. The same result holds if $f(z)$ takes *no* value w, such that $|w| = r$, more than $p + 1$ times in $|z| < 1$. In Theorem II, the main result of the first part of the chapter, we show that this method can be extended to the case when $f(z)$ has a finite number of zeros and poles in $|z| < 1$. The proof is based on a lemma on hyperbolic distances. This allows us to find extensions of all the positive results proved when $p = 0$ (Hayman (1), (2), (3)), to the case when $f(z)$ takes the values of E at most p times in $|z| < 1$. The method yields among other results an extension of Cartwright's[1]

[1] This was previously considered in CARTWRIGHT (1), LITTLEWOOD (1).

Theorem I. *If $f(z)$ is regular in $|z| < 1$ and takes no value more than p times, then we have*

$$M[\varrho, f] < A(p)\,\mu_p\,(1-\varrho)^{-2p}$$

where

$$\mu_p = \max\{1, |f(0)|, |f'(0)|, \ldots |f^{(p)}(0)|\}.$$

The extension is

Theorem III. *Let r_n be a sequence of real numbers such that*

$$r_0 = 0,$$
$$r_n < r_{n+1} \to \infty, \text{ as } n \to \infty$$
$$S = \sum_{n=1}^{\infty} \left(\log \frac{r_n + 1}{r_n}\right)^2 < \infty,$$

suppose also that $f(z)$ is meromorphic in $|z| < 1$ and for each r_n either $f(z)$ takes some value on the circle $|w| = r_n$ at most $p-1$ times or, $f(z)$ takes each value on the circle $|w| = r_n$ at most p times. Then we have

$$M[\varrho, f_*(z)] < A(p)\{|f_*(0)| + r_1\}\, e^{S/p+1}(1-\varrho)^{-2p}.$$

This result is an extension at once of Theorem I above and of Theorem III (Hayman (3)) from which latter it is a deduction, using Theorem II of Chapter II.

4) In the second part of Chapter II we deal with the more general problem, when $p(\varrho)$ is unbounded. The results in this case are based on Chapter I. We consider in all four problems.

(i) *What can we say when E contains the whole plane?*

(ii) *How small a set E is sufficient to have the same effect as the whole plane on the order of magnitude of* $\log M[\varrho, f_*(z)]$*?*

(iii) *What can we say if E contains some arbitrarily large values?*

(iv) *What can we say if E contains only $0, 1, \infty$ or is bounded?*

Let

$$\varrho_* = \frac{1 + 2\varrho}{2 + \varrho},$$

so that

$$\tfrac{1}{2}(1-\varrho) \le 1 - \varrho_* < \tfrac{1}{3}(1-\varrho), \quad 0 \le \varrho < 1.$$

Then we prove in Theorem VII that (iv) implies

(4.1) $$\log M[\varrho, f_*(z)] = \frac{O(1)}{1-\varrho}\int_0^{\varrho_*}[1 + p(r)]\,dr.$$

Also we prove in Theorem V that (i) gives

(4.2) $$\log M[\varrho, f_*(z)] = O\left\{\int_0^{\varrho_*}\frac{1 + p(r)}{1-r}\,dr\right\}.$$

Now if

(4.3) $$p(\varrho) = (1-\varrho)^{-a}, \quad 0 \le a < \infty,$$

then (4.1), (4.2) both give

(4.4) $$\log M[\varrho, f_*(z)] = O(1-\varrho)^{-a}$$

when $a > 1$, while (4.2) also implies (4.4) if $0 < a \le 1$. It is shown by some simple examples in paragraph 21 of Chapter II, that (4.4) gives the right order of magnitude for $\log M[\varrho, f]$ when $p(\varrho)$ is given by (4.3) and $a > 1$, and that (4.2) is still best possible if $0 \le a \le 1$ in (4.3). The inequality (4.1) is also best possible when $p(\varrho)$ is given by (4.3), $0 \le a < \infty$.

Theorem VII shows that a set E consisting of a sequence w_n satisfying

(4.5) $$1 < \left|\frac{w_{n+1}}{w_n}\right| < c,$$

and $w_n \to \infty$, is always sufficient to result in (4.2). Theorem IX shows that if $p(\varrho)$ grows as rapidly as in (4.3) with $a > 0$ we can replace (4.5) by the weaker condition

$$|w_{n+1}| < |w_n|^c.$$

Converse examples, which show that these results are all more or less best possible when $p(\varrho)$ is given by (4.3) are left to Chapter III in all but the simplest cases.

Lastly in the case of problem (iii) above, we show in Theorem VI Chapter II, that if

$$\int_0^1 p(\varrho)\,d\varrho < \infty$$

we have always

$$\overline{\lim_{\varrho\to 1}}\,(1-\varrho)\log M[\varrho, f_*(z)] = 0,$$

a result which cannot be improved, as was shown in Hayman (2), even when $p(\varrho) \equiv 0$. We prove further in Theorem X, Chapter II the more sophisticated inequalities

$$\varliminf_{\varrho \to 1} (1-\varrho)^{\frac{1+a}{3-a}} \log M[\varrho, f(z)] = 0, \quad 0 \leq a < 1;$$

$$\varliminf_{\varrho \to 1} \frac{(1-\varrho) \log M[\varrho, f]}{\log \log \frac{1}{1-\varrho}} < A, \quad a = 1;$$

which hold when $p(\varrho)$ is given by (4.3) with $0 \leq a < 1$. These are extensions of results proved when $a = 0$ in Hayman (2). They are shown to be best possible in Theorems II and III of Chapter III.

5) In Chapter III we provide converse examples to the results of Chapter II, when $p(\varrho)$ is given by (4.3). While it is easy to provide these examples in the case of problems (i) and (iv) above, the counterexamples to problems (ii) and (iii) present considerable difficulties[1] and need a good deal of preliminary general mapping Theory.

Throughout the whole essay the ideas of R. Nevanlinna have been fundamental. I have tried to indicate the most important places in the text.

An index of literature is given at the back.

I should like to express my gratitude to Miss M. L. Cartwright for suggesting the problem to me.

CHAPTER I.

Extensions of Schottky's Theorem

Notation.

1) If $z = x + iy$ is any complex number we shall write

$$x = \Re z, \qquad y = \Im z,$$
$$\bar{z} = x - iy, \quad |z| = \sqrt{(x^2 + y^2)}.$$

Throughout this chapter we shall be dealing with functions $f(z)$ meromorphic in $|z| < 1$. We suppose for the time being that $f(z)$ is meromorphic also for $|z| = 1$. We denote by

$$a_\mu = |a_\mu| e^{i\alpha_\mu}, \quad \mu = 1 \text{ to } m \tag{1.1}$$

[1] Particularly when $a = 1$ in (4.3). This case is, however, in many ways critical, and its omission would be a serious gap.

the zeros of $f(z)$ in $|z| \leq \frac{1}{2}$, arranged in order of increasing moduli and with correct multiplicity, and by

$$a_\mu, \quad \mu = m + 1 \text{ to } M \tag{1.2}$$

the zeros of $f(z)$ in $\frac{1}{2} < |z| < 1$. We write similarly

$$b_\mu = |b_\mu| e^{i\beta_\mu}, \quad \mu = 1 \text{ to } n \text{ and } n + 1 \text{ to } N \tag{1.3}$$

for the poles of $f(z)$ in $|z| \leq \frac{1}{2}$ and in $\frac{1}{2} < |z| < 1$ respectively, and

$$c_\mu = |c_\mu| e^{i\gamma_\mu}, \quad \mu = 1 \text{ to } k, \text{ and } k + 1 \text{ to } K, \tag{1.4}$$

for the points in $|z| \leq \frac{1}{2}$, $\frac{1}{2} < |z| < 1$ respectively such that $f(c_\mu) = 1$. To these we shall refer as the ones of $f(z)$.

It will be useful occasionally to consider all the zeros, poles and ones together and we accordingly write

$$d_\mu = |d_\mu| e^{i\delta_\mu}, \quad \mu = 1 \text{ to } l = m + n + k, \quad l + 1 \text{ to } L = M + N + K \tag{1.5}$$

for all the points in $|z| \leq \frac{1}{2}$, $\frac{1}{2} < |z| < 1$ respectively, such that $f(d_\mu) = 0, 1, \infty$. We also write

$$g(z, a) = \frac{a - z}{1 - \bar{a} z} \frac{|a|}{a}, \quad |z| < 1, \quad |a| < 1. \tag{1.6}$$

It will be necessary in the course of the work to use largely three integrals involving $f(z)$. We define first[1]

$$m[r, f(z)] = m[r, f] = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| \, d\theta, \quad 0 < r \leq 1. \tag{1.7}$$

Here $\log^+ x$ denotes as usual the larger of zero and $\log x$. We shall need also

$$m_0[r, f(z)] = \frac{1}{\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| (1 - \cos\theta) \, d\theta, \quad 0 < r \leq 1, \tag{1.8}$$

and

$$m_1[r, f(z)] = \max_{\frac{1}{2} r \leq \varrho \leq r} m[\varrho, f], \quad 0 < r \leq 1. \tag{1.9}$$

The expressions max, min, outside a bracket containing certain quantities denote the greatest or least respectively of these quantities, or if these do not exist, the upper and lower bounds. We denote by A any absolute constant not necessarily the same in different places and by $A(p)$ etc. constants depending on p.

[1] C. F. Nevanlinna (1) p. 6, formula 3.

2) Using the notation of (1.1) to (1.6) the result which will be the basis of this chapter, and whose proof will occupy most of it may now be stated as follows.

Theorem I. *Let $f(z)$ be meromorphic in $|z| \leq 1$ and let*

$$g(z) = 2^{n-m} f(z) \frac{\prod_{\mu=1}^{n} g(z, b_\mu)}{\prod_{\mu=1}^{m} g(z, a_\mu)}. \tag{2.1}$$

Then we have

$$\Re \frac{g'(0)}{g(0)} < 2\,|\log|g(0)|| + A\left[1 + \log^+|\log|g(0)|| + \sum_{\mu=1}^{L} |1 \mp d_\mu|^2 (1 - |d_\mu|)\right] \tag{2.2}$$

where the sign in the sum is $-$ or $+$ according as $|g(0)| \geq 1$ or $|g(0)| < 1$, respectively.

The interest of Theorem I lies in the fact that it is applicable to any meromorphic function. By mapping the unit circle onto itself, so that an arbitrary point goes into the origin, we can obtain various extension of Schottky's Theorem.

The bound obtained in (2.2) appears to be fairly sharp at least in its dependence on the d_μ, as the formula (3.4) below suggests. If we are given only the number L of the d_μ and nothing about their position we can eliminate the term $\log^+|\log|g(0)||$ as will be shown in Theorem III below. This result is, however, less useful than Theorem I. The condition that $f(z)$ is meromorphic on $|z| = 1$ can clearly be relaxed, provided that $\sum |1 \mp d_\mu|^2 (1 - |d_\mu|)$ converges.

The proof of Theorem I is rather long. It depends in the main on applications of the Poisson-Jensen formula and some other analogous formulae and owes most to the ideas of Nevanlinna.[1]

Fundamental Identities.

3) In this section we put together four fundamental identities, which we shall have occasion to use frequently at a later stage. We have firstly if, $|z| < R \leq 1$,

[1] See NEVANLINNA (1), particularly Chapter IV.

$$(3.1)\quad \log f(z) = \frac{1}{2\pi}\int_0^{2\pi} \log |f(R e^{i\theta})| \frac{R e^{i\theta} + z}{R e^{i\theta} - z} d\theta - \sum \log g\left(\frac{z}{R}, \frac{b_\nu}{R}\right) + \sum \log g\left(\frac{z}{R}, \frac{a_\mu}{R}\right) + i C.$$

This is the generalized Poisson-Jensen formula. For a proof see e g. Nevanlinna (1), Ch. I, p. 4. Putting $z = 0$ in (3.1) and taking real parts we obtain Jensen's formula

$$(3.2)\qquad \log |f(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log |f(R e^{i\theta})|\, d\theta - \sum \log^+ \frac{R}{|a_\mu|} + \sum \log^+ \frac{R}{|b_\nu|}.$$

Secondly by differentiating (3.1) w. r. t. z and then putting $z = 0$, we have

$$\frac{f'(0)}{f(0)} = \frac{1}{\pi R}\int_0^{2\pi} \log |f(R e^{i\theta})|\, e^{i\theta} d\theta - \sum \frac{R^2 - |a_\mu|^2}{R^2 a_\mu} + \sum \frac{R^2 - |b_\nu|^2}{R^2 b_\nu},$$

where the sums are taken throughout over the zeros and poles of $f(z)$ which lie in $|z| < R$. Taking real parts and multiplying by R we have

$$(3.3)\quad \Re\, R \frac{f'(0)}{f(0)} = \frac{1}{\pi}\int_0^{2\pi} \log |f(R e^{i\theta})| \cos\theta\, d\theta - \sum \frac{R^2 - |a_\mu|^2}{R |a_\mu|} \cos \alpha_\mu + \sum \frac{R^2 - |b_\nu|^2}{R |b_\nu|} \cos \beta_\nu.$$

Combining (3.2) and (3.3) we deduce

$$(3.4)\quad \Re\, R \frac{f'(0)}{f(0)} - 2 \log |f(0)| = \frac{1}{\pi}\int_0^{2\pi} \log \left|\frac{1}{f(R e^{i\theta})}\right| (1 - \cos\theta)\, d\theta - \sum \left\{\frac{R^2 - |a_\mu|^2}{R |a_\mu|} \cos \alpha_\mu - 2 \log \frac{R}{|a_\mu|}\right\} + \sum \left\{\frac{R^2 - |b_\nu|^2}{R |b_\nu|} \cos \beta_\nu - 2 \log \frac{R}{|b_\nu|}\right\}.$$

All the above formulae require the assumption $f(0) \neq 0$ or ∞. We shall assume in future that $f(0) \neq 0, 1, \infty$ and that $f'(0) \neq 0$, whenever it becomes necessary to insure the finitude of the terms of our relations.

It is now possible to outline the proof of Theorem I. We apply the formula (3.4) to the function $g(z)$. We thus obtain a bound of the right type for the left hand side of (2.2), provided that R is nearly 1, while yet

$$\frac{1}{\pi}\int_0^{2\pi} \log\left|\frac{1}{g(Re^{i\theta})}\right|(1-\cos\theta)\,d\theta$$

is not too large. Since the integral is bounded by $m_0\left[R, \frac{1}{g}\right]$ it is necessary only to obtain a sufficiently delicate bound for $m_0\left[R, \frac{1}{g(z)}\right]$. This we proceed to do, using methods similar to those employed by Nevanlinna[1] in proving his second fundamental Theorem in the Theory of Meromorphic Functions, together with certain smoothing out processes, which become necessary if the d_ν cluster too much near the origin.

4) In the next two paragraphs we prove lemma 1, which plays much the same role in a later stage of our proof as Jensen's formula (3.2) in the ordinary Nevanlinna Theory.

Lemma 1. *We have with the notation of paragraph* 1, *if* $R < 1$,

$$m_0\left[R, \frac{1}{f}\right] \leq 13\, m_1[R, f] + 13 \sum_{\nu=1}^{N} \log^+ \frac{R}{|b_\nu|}\left|1 - \frac{b_\nu}{R}\right|^2 + 7 \log \frac{1}{|f(0)|}.$$

It is significant that this bound does not depend on the zeros of $f(z)$ and on the poles only in the manner indicated.

Making use of (3.2) and (3.3) we have

$$(4.1)\quad \frac{1}{\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|(1-\cos\theta)\,d\theta - \frac{1}{\pi}\int_0^{2\pi}\log\left|f\left(\frac{1}{2}Re^{i\theta}\right)\right|(1-2\cos\theta)\,d\theta = \sum_{\mu=1}^{M}\phi\left(\frac{a_\mu}{R}\right) - \sum_{\nu=1}^{N}\phi\left(\frac{b_\nu}{R}\right),$$

where $\phi(z) = \phi(\varrho e^{i\theta})$ is given by

$$\begin{cases} \phi(z) = 2\log\frac{1}{\varrho} - \frac{1-\varrho^2}{\varrho}\cos\theta, & \frac{1}{2} \leq \varrho < 1; \quad (4.2)\\ \phi(z) = 2\log 2 - 3\varrho\cos\theta, & 0 \leq \varrho < \frac{1}{2} \quad (4.3)\\ \phi(z) = 0, & \varrho \geq 1. \quad (4.4)\end{cases}$$

To prove lemma 1 we need the following elementary inequalities whose proof we defer to the next paragraph.

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[1] Nevanlinna (1), Chapter IV, particularly p. 57—67.

Lemma 2. *We have*

(i) $$|\phi(z)| < 6 \log \frac{1}{2|z|}, \quad |z| \leq \frac{1}{3}.$$

(ii) $$-\frac{4}{3}(1-|z|)^3 < \phi(z) < 3 \log \left|\frac{1}{z}\right| |1-z|^2, \quad 0<|z|<1.$$

(iii) $$\frac{R}{6}\left(1-\frac{\varrho}{R}\right)^3 < \int_{\frac{1}{2}R}^{R} \left(1-\frac{r}{R}\right) \log^+ \frac{r}{\varrho}\, dr < \frac{R}{2}\left(1-\frac{\varrho}{R}\right)^2 \log \frac{R}{\varrho}, \quad 0<\varrho \leq R.$$

(iv) $$\log \frac{R}{2\varrho} < \left(1-\frac{\varrho}{R}\right)^2 \log \frac{R}{\varrho}, \quad 0 < \varrho \leq \frac{1}{2} R.$$

Assuming the truth of lemma 2 for the time being we deduce from (4.1) and (1.7), (1.8)

$$m_0\left[R, \frac{1}{f}\right] = m_0[R, f] + 2\frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|\frac{1}{f(\frac{1}{2} R e^{i\theta})}\right| (1-2\cos\theta)\, d\theta$$
$$+ 2\frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|f\left(\frac{1}{2} R e^{i\theta}\right)\right| (2\cos\theta - 1)\, d\theta + \sum_{\nu=1}^{N} \phi\left(\frac{b_\nu}{R}\right) - \sum_{\mu=1}^{M} \phi\left(\frac{a_\mu}{R}\right),$$

and hence

(4.5) $$m_0\left[R, \frac{1}{f}\right] \leq 2m\left[\frac{1}{2}R, f\right] + 6m\left[\frac{1}{2}R, \frac{1}{f}\right] + 4m[R,f]$$
$$+ \sum_{\nu=1}^{N} \phi\left(\frac{b_\nu}{R}\right) - \sum_{\mu=1}^{M} \phi\left(\frac{a_\mu}{R}\right),$$

since

$$-1 \leq 1 - 2\cos\theta \leq 3, \quad 0 \leq 1-\cos\theta \leq 2.$$

Also (3.2) gives

$$6m\left[\frac{1}{2}R, \frac{1}{f}\right] = 6m\left[\frac{1}{2}R, f\right] + 6\sum_{\nu=1}^{n} \log^+ \frac{R}{2|b_\nu|} - 6\sum_{\mu=1}^{m} \log^+ \frac{R}{2|a_\mu|} + 6 \log \frac{1}{|f(0)|},$$

and using lemma 2 (i) we deduce

(4.6) $$6m\left[\frac{1}{2}R, \frac{1}{f}\right] - \sum_{|a_\mu| \leq \frac{1}{3}R} \phi\left(\frac{a_\mu}{R}\right) \leq 6m\left[\frac{1}{2}R, f\right] + 6\sum_{\nu=1}^{N} \log^+ \frac{R}{2|b_\nu|} + 6\log\frac{1}{|f(0)|}.$$

Next we have from (3.2), if $\frac{1}{2}R \leq r \leq R$

(4.7) $$\sum_{R \leq |a_\mu| \leq R} \log^+ \frac{r}{|a_\mu|} \leq \sum_{\nu=1}^{N} \log^+ \frac{r}{|b_\nu|} + m[r,f] + \log\frac{1}{|f(0)|}.$$

Multiplying both sides of (4.7) by $\left(1-\frac{r}{R}\right)$ and integrating w. r. t. r we have

$$(4.8)\quad \sum_{\frac{1}{2}R\le|a_\mu|\le R}\int_{\frac{1}{2}R}^{R}\left(1-\frac{r}{R}\right)\log^+\frac{r}{|a_\mu|}dr\le\sum_{\nu=1}^{N}\int_{\frac{1}{2}R}^{R}\left(1-\frac{r}{R}\right)\log^+\frac{r}{|b_\nu|}dr$$
$$+\frac{R}{8}m_1[R,f]+\frac{R}{8}\log\frac{1}{|f(0)|}.$$

We have from (4.5) and (4.6) making use of (1.9),

$$(4.9)\quad m_0\left[R,\frac{1}{f}\right]\le 12\,m_1[R,f]+6\log\frac{1}{|f(0)|}$$
$$+\sum_{\nu=1}^{N}\left\{\phi\left(\frac{b_\nu}{R}\right)+6\log^+\frac{R}{2|b_\nu|}\right\}-\sum_{\frac{1}{2}R\le|a_\mu|\le R}\phi\left(\frac{a_\mu}{R}\right).$$

Also from lemma 2 (ii), (iii) and (4.8) we have

$$-\sum_{\frac{1}{2}R\le|a_\mu|\le R}\phi\left(\frac{a_\mu}{R}\right)\le\frac{4}{3}\sum_{\frac{1}{2}R\le|a_\mu|\le R}\left(1-\frac{|a_\mu|}{R}\right)^3$$
$$\le\sum_{\frac{1}{2}R\le|a_\mu|\le R}\frac{8}{R}\int_{\frac{1}{2}R}^{R}\left(1-\frac{r}{R}\right)\log^+\frac{r}{|a_\mu|}dr$$
$$\le\sum_{\nu=1}^{N}\frac{8}{R}\int_{\frac{1}{2}R}^{R}\left(1-\frac{r}{R}\right)\log^+\frac{r}{|b_\nu|}dr+m_1[R,f]+\log\frac{1}{|f(0)|}$$
$$\le 4\sum_{\nu=1}^{N}\left|1-\frac{b_\nu}{R}\right|^2\log^+\frac{R}{|b_\nu|}+m_1[R,f]+\log\frac{1}{|f(0)|}.$$

Combining this with (4.9) we obtain

$$m_0\left[R,\frac{1}{f}\right]\le 13\,m_1[R,f]+7\log\frac{1}{|f(0)|}$$
$$+\sum_{\nu=1}^{N}\left\{\phi\left(\frac{b_\nu}{R}\right)+6\log^+\frac{R}{2|b_\nu|}+4\left|1-\frac{b_\nu}{R}\right|^2\log^+\frac{R}{|b_\nu|}\right\}.$$

Using lemma 2, (i), (ii) and (iv), we have finally

$$m_0\left[R,\frac{1}{f}\right]\le 13\,m_1[R,f]+7\log\frac{1}{|f(0)|}+13\sum_{\nu=1}^{N}\log^+\frac{R}{|b_\nu|}\left|1-\frac{b_\nu}{R}\right|^2,$$

which proves lemma 1.

5) It remains to establish the inequalities of lemma 2. We have

$$|\phi(z)| \leq \phi\left(-\frac{1}{3}\right) = 2\log 2 + 1,\quad |z| \leq \frac{1}{3},$$

and

$$6\log\frac{1}{2|z|} \geq 6\log\frac{3}{2},\quad |z| \leq \frac{1}{3}.$$

Since also

$$6\log\frac{3}{2} - 2\log 2 - 1 = \log\frac{3^6}{2^8} - 1 > \log\frac{2.8}{e} > 0,$$

lemma 2 (i) follows.

To prove (ii) we note that

$$(5.1)\qquad |\phi(\varrho e^{i\theta})| \geq \phi(\varrho),\quad 0 < \varrho < 1.$$

Also $\phi(1) = 0$ and

$$0 \leq \phi'(\varrho) = \left(\frac{1-\varrho}{\varrho}\right)^2 \leq 4(1-\varrho)^2,\quad \frac{1}{2} \leq \varrho \leq 1.$$

On integrating we obtain

$$(5.2)\qquad \phi(\varrho) < 0,\quad \tfrac{1}{2} \leq \varrho < 1;$$

$$(5.3)\qquad \phi(\varrho) > -\tfrac{4}{3}(1-\varrho)^3,\quad \tfrac{1}{2} \leq \varrho < 1,$$

and since the left hand side of (5.3) decreases for $0 \leq \varrho \leq \frac{1}{2}$, while the right hand side increases, (5.3) holds for $0 \leq \varrho < 1$. Combining this with (5.1) we deduce the first inequality of lemma 2 (ii). To prove the second inequality, note that from (4.2), (5.2) and (5.3) we have

$$\phi(z) = \left[2\log\frac{1}{\varrho}\right](1-\cos\theta) + \phi(\varrho)\cos\theta,\quad \frac{1}{2} \leq \varrho \leq 1$$

$$< \left[2\log\frac{1}{\varrho}\right](1-\cos\theta) + \frac{4}{3}(1-\varrho)^3,$$

$$(5.4)\qquad \phi(z) < 2\left[1-\cos\theta + \frac{2}{3}(1-\varrho)^2\right]\log\frac{1}{\varrho},\quad \frac{1}{2} \leq \varrho < 1.$$

Also

$$(5.5)\qquad |1-\varrho e^{i\theta}|^2 = 1 - 2\varrho\cos\theta + \varrho^2 = (1-\varrho)^2 + 2\varrho(1-\cos\theta).$$

Combining (5.4) and (5.5), the second inequality of lemma 2 (ii) follows when $\frac{1}{2} \leq \varrho \leq 1$. When $\varrho < \frac{1}{2}$ we note that

$$\phi(\varrho e^{i\theta}) = 2 \log 2 - 3 \varrho \cos \theta < 2 \log 2 - 4 \varrho \cos \theta + \tfrac{1}{2}$$
$$< 2 - 4 \varrho \cos \theta < 2 \,| 1 - \varrho e^{i\theta} |^2,$$

by (5.5). We deduce, since $2 < 3 \log 2 \leq 3 \log \frac{1}{\varrho}$, $0 \leq \varrho \leq \frac{1}{2}$ that

$$\phi(\varrho e^{i\theta}) < 3 \log \frac{1}{\varrho} | 1 - \varrho e^{i\theta} |^2, \quad 0 < \varrho \leq \frac{1}{2}.$$

This completes the proof of lemma 2 (ii).

To prove (iii) and (iv), we may without loss in generality suppose that $R = 1$. Suppose first $\varrho \geq \frac{1}{2}$. Then

$$\int_{\frac{1}{2}}^{1} (1 - r) \log^+ \frac{r}{\varrho} dr = \int_{\varrho}^{1} (1 - r) \log \frac{r}{\varrho} dr$$
$$= \frac{1}{2} \int_{\varrho}^{1} \frac{(1 - r)^2}{r} dr > \frac{1}{2} \int_{\varrho}^{1} (1 - r)^2 dr = \frac{1}{6} (1 - \varrho)^3.$$

This proves the first half of (iii) when $\varrho \geq \frac{1}{2}$. Also when $0 < \varrho < \frac{1}{2}$

$$\frac{d}{d\varrho} \left[\int_{\frac{1}{2}}^{1} (1 - r) \log \frac{r}{\varrho} dr - \frac{1}{6} (1 - \varrho)^3 \right]$$
$$= -\frac{1}{\varrho} \int_{\frac{1}{2}}^{1} (1 - r) dr + \frac{1}{2} (1 - \varrho)^2 = \frac{1}{2} (1 - \varrho)^2 - \frac{1}{4\varrho} < \frac{1}{2} - \frac{1}{2} = 0.$$

Hence

$$\int_{\frac{1}{2}}^{1} (1 - r) \log^+ \frac{r}{\varrho} dr - \frac{1}{6} (1 - \varrho)^3$$

decreases with ϱ for $0 \leq \varrho \leq \frac{1}{2}$ and is positive when $\varrho \geq \frac{1}{2}$ and so the expression is positive for $0 < \varrho < 1$, which proves the first inequality of (iii). The second inequality of (iii) is obtained by replacing $\log^+ \frac{r}{\varrho}$ by $\log \frac{R}{\varrho}$ in the integrand, and altering the lower limit of integration to ϱ both of which can only increase the integral, since $\log^+ \frac{r}{\varrho} = 0$, $r < \varrho$.

It remains to prove (iv). We note that

$$\log \frac{1}{2x} - (1-x)^2 \log \frac{1}{x} = (2x - x^2) \log \frac{1}{x} - \log 2 = (2-x)x \log \frac{1}{x} - \log 2.$$

Since $x \log \frac{1}{x}$ has a maximum at $x = \frac{1}{e}$, and increases for $x < \frac{1}{e}$ we have

$$(2-x)x \log \frac{1}{x} - \log 2 < 2 \cdot \frac{1}{4} \log 4 - \log 2 = 0, \quad x \leq \frac{1}{4},$$

and

$$(2-x)x \log \frac{1}{x} - \log 2 < \frac{7}{4}\frac{1}{e} - \log 2 < 0, \quad x \geq \frac{1}{4}.$$

Hence lemma 2 (iv) holds for $0 < \frac{\varrho}{R} \leq \frac{1}{2}$ and the proof of that lemma and of lemma 1 is complete.

6) The next stage of the proof is very closely related to the Nevanlinna Theory.[1] The method by which Nevanlinna obtains a bound for $m\left[r, \frac{1}{f}\right]$ depending only on the d_ν and on $f(0)$, $f'(0)$ will be used. We could deduce immediately a bound for

$$m_0\left[r, \frac{1}{f}\right] \leq 4m\left[r, \frac{1}{f}\right].$$

Such a bound would, however, contain a term of the order of $\sum(1 - |d_\mu|)$ whereas we need the sharper bound involving $\sum(1 - |d_\mu|)\,|1 - d_\mu|^2$, which is smaller when the d_μ cluster near the positive real axis. This necessitates replacing the simple Jensen formula (3.2) by the more complicated lemma 1 applied to the logarithmic derivate $\frac{f'(z)}{f(z)}$ of $f(z)$, to obtain the required result.

Lemma 3. *We have with the hypotheses of lemma 1*

(i) $$m_0\left[R, \frac{1}{f}\right] \leq 17\, m_1\left[R, \frac{f'}{f-1}\right] + 4\, m_1\left[R, \frac{f'}{f}\right] + 13 \sum_{\nu=1}^{L} \log^+ \frac{R}{|d_\nu|}\left|1 - \frac{d_\nu}{R}\right|^2 + 7 \log\left|\frac{f(0)-1}{f'(0)}\right| + 4 \log 6.$$

(ii) $$m\left[R, \frac{1}{f}\right] \leq 2\, m\left[R, \frac{f'}{f-1}\right] + m\left[R, \frac{f'}{f}\right] + \sum_{\nu=1}^{L} \log^+ \frac{R}{|d_\nu|} + \log \frac{18\,|f(0)-1|}{|f'(0)|}.$$

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[1] Nevanlinna (1) p. 63—66.

Consider

$$F(z) = \frac{1}{f(z)} + \frac{1}{f(z) - 1}.$$

When $|f(z)| \leq \frac{1}{3}$ we have

$$\left|\frac{1}{f(z) - 1}\right| \leq \frac{3}{2} \leq \frac{1}{2} \frac{1}{|f(z)|},$$

and so

(6.1) $$|F(z)| \geq \frac{1}{2} \frac{1}{|f(z)|}.$$

Similarly when $|f(z) - 1| \leq \frac{1}{3}$ we have

$$|F(z)| \geq \frac{1}{2} \frac{1}{|f(z) - 1|}$$

and these two sets of points are mutually exclusive. We deduce that

(6.2) $$\log^+ F(z) \geq \log^+ \frac{1}{|f(z)|} + \log^+ \frac{1}{|f(z) - 1|} - 2 \log 3,$$

provided that either $|f| \leq \frac{1}{3}$ or $|f - 1| \leq \frac{1}{3}$ and (6.2) is trivial otherwise. We deduce at once

$$m_0\left[R, \frac{1}{f}\right] + m_0\left[R, \frac{1}{f - 1}\right] \leq m_0[R, F] + m_0[R, 2 \log 3];$$

(6.3) $$m_0\left[R, \frac{1}{f}\right] + m_0\left[R, \frac{1}{f - 1}\right] \leq m_0[R, F] + 4 \log 3.$$

We deduce also

(6.4) $$m\left[R, \frac{1}{f}\right] + m\left[R, \frac{1}{f - 1}\right] \leq m[R, F] + 2 \log 3.$$

We now write

(6.5) $$F(z) = \frac{1}{f - 1} \cdot \frac{f - 1}{f'} \left[\frac{f'}{f} + \frac{f'}{f - 1}\right].$$

It follows that

(6.6) $$m_0[R, F] \leq m_0\left[R, \frac{1}{f - 1}\right] + m_0\left[R, \frac{f - 1}{f'}\right] + m_0\left[R, \frac{f'}{f} + \frac{f'}{f - 1}\right].$$

Also since

$$\log^+ (a + b) \leq \log^+ a + \log^+ b + \log 2$$

we have

$$(6.7)\qquad \begin{aligned} m_0\left[R, \frac{f'}{f} + \frac{f'}{f-1}\right] &\le m_0\left[R, \frac{f'}{f}\right] + m_0\left[R, \frac{f'}{f-1}\right] + m_0[R, \log 2] \\ &\le 4\, m_1\left[R, \frac{f'}{f}\right] + 4\, m_1\left[R, \frac{f'}{f-1}\right] + 2 \log 2. \end{aligned}$$

The function $\dfrac{f'(z)}{f(z)-1}$ has simple poles whenever $f(z) = 1$ or ∞, i.e. at the points c_ν, b_ν.

Applying lemma 1 with $\dfrac{f'(z)}{f(z)-1}$ instead of $f(z)$, we have

$$(6.8)\qquad \begin{aligned} m_0\left[R, \frac{f-1}{f'}\right] &\le 13\, m_1\left[R, \frac{f'}{f-1}\right] + 7 \log\left|\frac{f(0)-1}{f'(0)}\right| \\ &\quad + 13 \sum_{\nu=1}^{L} \log^+ \frac{R}{|d_\nu|}\left|1 - \frac{d_\nu}{R}\right|^2. \end{aligned}$$

Combining (6.6), (6.7) and (6.8), we deduce

$$\begin{aligned} m_0[R, F] &\le m_0\left[R, \frac{1}{f-1}\right] + 17\, m_1\left[R, \frac{f'}{f-1}\right] + 4\, m_1\left[R, \frac{f'}{f}\right] \\ &\quad + 13 \sum_{\nu=1}^{L} \log^+ \frac{R}{|d_\nu|}\left|1 - \frac{d_\nu}{R}\right|^2 + 7 \log\left|\frac{f(0)-1}{f'(0)}\right| + 2 \log 2. \end{aligned}$$

Combining this with (6.3), we deduce lemma 3 (i).

Making use of (6.4), (6.5) we deduce analogously

$$m\left[R, \frac{1}{f}\right] \le 2 \log 3 + m\left[R, \frac{f-1}{f'}\right] + m\left[R, \frac{f'}{f}\right] + m\left[R, \frac{f'}{f-1}\right] + \log 2$$

and hence applying (3.2) with $\dfrac{f'(z)}{f(z)-1}$ instead of $f(z)$

$$\begin{aligned} m\left[R, \frac{1}{f}\right] &< 2\, m\left[R, \frac{f'}{f-1}\right] + m\left[R, \frac{f'}{f}\right] + \log 18 \\ &\quad + \sum_{\nu=1}^{L} \log^+ \frac{R}{|d_\nu|} + \log\left|\frac{f(0)-1}{f'(0)}\right|, \end{aligned}$$

which is lemma 3 (ii). This completes the proof of lemma 3.

7) We have obtained bounds for $m_0\left[R, \frac{1}{f}\right]$ and for $m\left[R, \frac{1}{f}\right]$ in lemma 3 which depend on the d_ν and on $f(0)$, $f'(0)$ and on the expressions $m\left[R, \frac{f'}{f}\right]$, $m\left[R, \frac{f'}{f-1}\right]$. The crux of the investigations is the result due to Borel and

Nevanlinna, according to which these latter expressions are in general small with respect to the other terms appearing in lemma 3. However, while the third and fourth terms on the right hand side of lemma 3 (i) and (ii) depend only on the behavior of $f(z)$ in $|z| \leq R$, in order to prove anything about the first two terms, we must assume that $f(z)$ is meromorphic in a larger region. For this region Nevanlinna (and his followers) have always chosen a larger concentric circle. In fact much weaker assumptions suffice in general to bound the first two terms in lemma 3 (i), e.g. the assumption that $f(z)$ is meromorphic in a larger touching circle, or more generally in any domain bounded by a finite number of analytic curves and containing all but a finite number of the points in $|z| \leq R$.

Some deductions from this will be made elsewhere. The case of the larger concentric circle is all that we need for the present.

We have first

Lemma 4. *Suppose that $f(z)$ is regular, $f(z) \neq 0$ or 1 in $|z| < R$. Then we have*

$$|f'(0)| < \frac{A}{R} |f(0)| \, [1 + |\log |f(0)||].$$

It is clearly sufficient to suppose $R = 1$. In this case lemma 4 is an immediate consequence of Theorem V, Hayman (1).

We have next

Lemma 5. *Suppose that $f(z)$ is meromorphic on $|z| = R$, except perhaps for a set of points of measure zero. Let $d_0(\theta)$ denote the radius of the largest circle centre $z_0 = Re^{i\theta}$ in which $f(z)$ is regular and not equal to 0 or 1. Then we have*

$$m\left[R, \frac{f'(z)}{f(z)}\right] \leq A + \log^+ m[R, f] + \log^+ m\left[R, \frac{1}{f}\right] + \frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{1}{d_0(\theta)} \, d\theta$$

where the integral is taken in the Lebesgue sense.

It follows from lemma 4 that

$$\left|\frac{f'(Re^{i\theta})}{f(Re^{i\theta})}\right| < A \left\{\frac{|\log |f(Re^{i\theta})|| + 1}{d_0(\theta)}\right\}$$

and hence

$$\log^+\left|\frac{f'(R e^{i\theta})}{f(R e^{i\theta})}\right| < \left[\log^+ \frac{1}{|d_0(\theta)|} + \log^+ |\log |f(R e^{i\theta})|| + A\right]$$

$$< \log^+ \frac{1}{d_0(\theta)} + \log^+ \log^+ |f| + \log^+ \log^+ \frac{1}{|f|} + A.$$

Integrating from $\theta = 0$ to $\theta = 2\pi$ we deduce

$$m\left[R, \frac{f'}{f}\right] < \frac{1}{2\pi}\int_0^{2\pi} \log^+ \frac{1}{d_0(\theta)} d\theta + \frac{1}{2\pi}\int_0^{2\pi} \log^+ \log^+ |f(R e^{i\theta})| \, d\theta$$

$$(7.1) \qquad + \frac{1}{2\pi}\int_0^{2\pi} \log^+ \log^+ \frac{1}{|f(R e^{i\theta})|} d\theta + A.$$

Now it follows from the geometric-arithmetic mean theorem that if $\phi(x)$ is a real positive function of x we have

$$\frac{1}{b-a}\int_a^b \log \phi(x) \, dx \le \log \left\{\frac{1}{b-a}\int_a^b \phi(x) \, dx\right\}.$$

Hence writing $\psi(x) = \max[1, \phi(x)]$ we have

$$\frac{1}{b-a}\int_a^b \log^+ \phi(x) \, dx = \frac{1}{b-a}\int_a^b \log \psi(x) \, dx$$

$$\le \log \left\{\frac{1}{b-a}\int_a^b \psi(x) \, dx\right\}$$

$$\le \log \left\{\frac{1}{b-a}\int_a^b (\phi(x) + 1) \, dx\right\}$$

$$\le \log^+ \left\{\frac{1}{b-a}\int_a^b \phi(x) \, dx\right\} + 1.$$

On applying this inequality to the second and third integral on the right hand side of (7.1) we obtain lemma 5.

8) Before proceeding further we need a simple lemma which will help us to deal with the last term in lemma 5. This is

Lemma 6. *Let* $0 < r < \infty$, *let* z *be any complex number and let* E_k *be the set of all* θ *such that* $|z - re^{i\theta}| < kr$, *where* $0 < k \leq 1$. *Then we have*

$$\int_{E_k} \log^+ \frac{1}{|z - re^{i\theta}|}\, d\theta < \pi k \left[\log \frac{1}{k} + \log^+ \frac{1}{r} + 1\right].$$

We may without loss in generality suppose z real and positive or zero. Then E_k consists of an interval $|\theta| < \theta_0 < \frac{\pi}{2}$, or is void. The last case is trivial. In the first case we have for θ on E_k

$$|re^{i\theta} - z| \geq r \sin \theta$$

and therefore

$$\sin \theta_0 \leq k$$

i.e.

$$\theta_0 \leq \frac{\pi k}{2}.$$

Hence

$$\int_{E_k} \log^+ \frac{1}{|z - re^{i\theta}|}\, d\theta \leq 2 \int_0^{\theta_0} \log^+ \frac{1}{r \sin \theta}\, d\theta$$

$$\leq 2 \int_0^{\theta_0} \log^+ \frac{\pi}{2r\theta}\, d\theta \leq 2 \int_0^{\frac{\pi k}{2}} \left\{\log^+ \frac{1}{r} + \log \frac{\pi}{2\theta}\right\} d\theta$$

$$= \pi k \left[\log^+ \frac{1}{r} + \log \frac{1}{k} + 1\right],$$

which proves the lemma.

We can now prove

Lemma 7. *Suppose that* $f(z)$ *is meromorphic in a domain* D *containing almost all points of* $|z| = R$. *Let* $d(\theta)$ *be the distance from* $z = Re^{i\theta}$ *to the frontier of* D, *let* $d_0(\theta)$ *be the radius of the largest circle centre* $z = Re^{i\theta}$ *in which* $f(z)$ *is regular and unequal to* 0 *or* 1 *and let* $n(\theta)$ *be the number of roots of the equations* $f(z) = 0, 1, \infty$ *at points distant at least* $\frac{1}{2} d(\theta)$ *from the frontier of* D. *Then we have*

$$\int_0^{2\pi} \log^+ \frac{1}{d_0(\theta)}\, d\theta \leq A \left\{\int_0^{2\pi} \left[\log^+ n(\theta) + \log^+ \frac{1}{d(\theta)}\right] d\theta + \log^+ \frac{1}{R} + 1\right\}.$$

Let $d_1, d_2, \ldots, d_n \ldots$ be the roots of $f(z) = 0, 1, \infty$ enumerated in the order of their distance from the frontier of D. Let $d_1(\theta)$ be the distance from $z = Re^{i\theta}$

to the nearest point d_ν. Then

$$d_0(\theta) = \min(d_1(\theta), d(\theta))$$

and so

$$(8.1) \qquad \int_0^{2\pi} \log^+ \frac{1}{d_0(\theta)} d\theta \le \int_0^{2\pi} \log^+ \frac{1}{d_1(\theta)} d\theta + \int_0^{2\pi} \log^+ \frac{1}{d(\theta)} d\theta.$$

Let E be the set of all θ for which $n(\theta) > 0$ and

$$(8.2) \qquad d_1(\theta) < \frac{1}{[n(\theta)]^2} \min \left[R, \frac{d(\theta)}{2}\right].$$

Then if θ lies in E there is a point d_ν such that $\nu \le n(\theta)$ and

$$d_0(\theta) = |R e^{i\theta} - d_\nu| < \tfrac{1}{2} d(\theta).$$

We thus deduce from (8.2) that if θ lies in E we have

$$d_1(\theta) = |R e^{i\theta} - d_\nu| < \frac{R}{\nu^2}$$

for some ν. Hence it follows from lemma 6 that

$$(8.3) \qquad \int_E \log^+ \frac{1}{d_1(\theta)} d\theta \le \sum_{\nu=1}^{\infty} \frac{\pi}{\nu^2}\left[\log \nu^2 + \log^+ \frac{1}{R} + 1\right] < A\left[1 + \log^+ \frac{1}{R}\right].$$

Again if θ is not in E we see from (8.2) that

$$(8.4) \qquad \log^+ \frac{1}{d_1(\theta)} \le 2 \log^+ n(\theta) + \log^+ \frac{1}{R} + \log^+ \frac{1}{d(\theta)} + \log 2.$$

Hence we deduce from (8.3) and (8.4) that

$$\int_0^{2\pi} \log^+ \frac{1}{d_1(\theta)} d\theta \le A\left\{\int_0^{2\pi}\left[\log^+ n(\theta) + \log^+ \frac{1}{d(\theta)}\right] d\theta + \log^+ \frac{1}{R} + 1\right\},$$

and combining this with (8.1) we have lemma 7.

9) We now combine lemmas 3, 5, 7 to prove

Lemma 8. *Suppose that*

$$f(z) = p_0 + p_1 z + \cdots, \quad p_0 \ne 0, 1, \infty, \; p_1 \ne 0,$$

is meromorphic in $|z| \le 1$. *Then we have with the notation of paragraph* 1

$$m_0\left[R, \frac{1}{f}\right] < A\left\{\sum_{\nu=1}^{L} \log^+ \frac{1}{|d_\nu|}\,|1 - d_\nu|^2 + \log^+ \log^+ |p_0| + \right.$$
$$\left. \log^+ \log^+ \frac{1}{|p_1|} + \log^+ \left|\frac{p_0 - 1}{p_1}\right| + \log \frac{1}{1 - R} + 1\right\}, \quad \tfrac{1}{2} \leq R < 1.$$

We use the notation of lemma 7 and write

$$I = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \frac{1}{d_0(\theta)}\, d\theta \tag{9.1}$$

$$B = \max\left\{m[R, f],\ m\left[R, \frac{1}{f}\right],\ m[R, f - 1],\ m\left[R, \frac{1}{f - 1}\right]\right\} \tag{9.2}$$

$$C = \max\left\{m\left[R, \frac{f'}{f}\right],\ m\left[R, \frac{f'}{f - 1}\right]\right\}. \tag{9.3}$$

The functions f, $\frac{1}{f}$, $1 - f$, and $\frac{1}{1 - f}$ all have the same points d_ν. Also if we put $\phi = f$, $\frac{1}{f}$, $1 - f$ and $\frac{1}{1 - f}$ in turn we obtain

$$\frac{\phi'}{\phi} = \frac{f'}{f},\ \frac{-f'}{f},\ \frac{-f'}{1 - f},\ \frac{f'}{1 - f};$$
$$\frac{\phi'}{\phi - 1} = \frac{f'}{f - 1},\ \frac{-f'}{1 - f} - \frac{f'}{f},\ -\frac{f'}{f},\ \frac{f'}{1 - f} + \frac{f'}{f}.$$

Thus in any case we see that

$$m\left[R, \frac{\phi'}{\phi}\right] \leq C$$
$$m\left[R, \frac{\phi'}{\phi - 1}\right] \leq m\left[R, \frac{f'}{f}\right] + m\left[R, \frac{f'}{f - 1}\right] + \log 2 \leq 2\,C + \log 2$$

and

$$\log^+ \left|\frac{\phi(0) - 1}{\phi'(0)}\right| \leq \log^+ |p_0 - 1| + \log^+ |p_0| + \log^+ \frac{1}{|p_1|}$$
$$\leq 2 \log^+ |p_0| + \log 2 + \log^+ \frac{1}{|p_1|}.$$

Thus we obtain, on applying lemma 3 (ii) with f, $\frac{1}{f}$, $1 - f$, $\frac{1}{1 - f}$ instead of $f(z)$ in turn

$$B \leq 5\,C + \sum_{\nu=1}^{L} \log^+ \frac{R}{|d_\nu|} + 2 \log^+ |p_0| + \log^+ \frac{1}{|p_1|} + A. \tag{9.4}$$

Applying lemma 5 with f, $1-f$ instead of $f(z)$ in turn, we haev

$$C \leq 2 \log^+ B + I + A. \tag{9.5}$$

Combining (9.4) and (9.5) we deduce

$$C \leq A\left[\log^+\left(\sum_{\nu=1}^{L} \log^+ \frac{R}{|d_\nu|}\right) + \log^+ \log^+ |p_0| + \log^+ \log^+ \frac{1}{|p_1|} + I + 1\right]. \tag{9.6}$$

We now use lemma 7, taking for D the domain $|z| < 1$. This gives

$$I \leq A\left[\log^+ \frac{1}{1-R} + \log^+ n(R') + 1 + \log \frac{1}{R}\right], \tag{9.7}$$

where $n(R')$ is the number of the d_ν lying in $|z| \leq R' = \frac{1}{2}(1 + R)$. Then

$$\log^+ n(R') \leq \log^+ \frac{1}{(1-R')^3} \sum_{\nu=1}^{L} (1 - |d_\nu|)^3$$

$$\leq 3 \log^+ \frac{1}{1-R'} + \sum_{\nu=1}^{L} (1 - |d_\nu|)^3,$$

$$\log^+ n(R') \leq 3 \log \frac{1}{1-R} + 3 \log 2 + \sum_{\nu=1}^{L} |1 - d_\nu|^2 \log^+ \frac{1}{|d_\nu|}. \tag{9.8}$$

Similarly

$$\log^+\left(\sum \log^+ \frac{R}{|d_\nu|}\right) \leq \log^+ \frac{1}{(1-R)^2} \sum_{\nu=1}^{L} |1 - d_\nu|^2 \log^+ \frac{1}{|d_\nu|};$$

$$\log^+\left(\sum \log^+ \frac{R}{|d_\nu|}\right) \leq 2 \log \frac{1}{1-R} + \sum_{\nu=1}^{L} |1 - d_\nu|^2 \log^+ \frac{1}{|d_\nu|}. \tag{9.9}$$

Combining (9.6) to (9.9) we deduce that

$$\begin{aligned} C \leq A\Bigg[\sum_{\nu=1}^{L} |1 - d_\nu|^2 \log^+ \frac{1}{|d_\nu|} + \log \frac{1}{1-R} + \log \frac{1}{R} \\ + \log^+ \log^+ |p_0| + \log^+ \log^+ \frac{1}{|p_1|} + 1\Bigg] \end{aligned} \tag{9.10}$$

where C is defined by (9.3) and A is an absolute constant. Substituting in (9.10) for R any number r such that $\frac{1}{2} R \leq r \leq R$ we have $r \geq \frac{1}{4}$ if $R \geq \frac{1}{2}$ so that we have

$$m_1\left[R, \frac{f'}{f}\right] + m_1\left[R, \frac{f'}{f-1}\right]$$

$$\leq A\left[\sum_{\nu=1}^{L} |1 - d_\nu|^2 \log^+ \frac{1}{|d_\nu|} + \log \frac{1}{1-R} + \log^+ \log^+ |p_0| + \log^+ \log^+ \frac{1}{|p_1|} + 1\right].$$

Combining this with lemma 3 (i), we have lemma 8.

10) We have now found a bound for $m_0\left[R, \frac{1}{f}\right]$, when $f(z)$ is meromorphic in $|z| \leq 1$, which depends on the position of the d_ν and on R in the right way, at least when the d_ν lie near $|z| = 1$. The bound has, however, the disadvantage of becoming infinite whenever $f(0) = 0, 1, \infty$ or when $f'(0) = 0$.

In order to eliminate this difficulty we introduce the function $g(z)$ of Theorem I which is not equal to 0 or ∞ in $|z| \leq \frac{1}{2}$, and so shows a more regular behaviour than $f(z)$. We shall also employ a transformation of $|z| \leq R$ onto itself, which will move the origin to a point z_0, near which the d_ν do not cluster too much, and which is so chosen that $f(z_0)$ is not much greater than $g(0)$. We shall then obtain a bound for $m_0\left[R, \frac{1}{g}\right]$ which is of the required form, unless $\frac{f'(z)}{f(z)}$ is small everywhere on the circle $|z| = |z_0|$, in which case Theorem I can be proved directly.

We use the notation of (1.1) to (1.6) and write

$$\Pi_1(z) = \prod_{\mu=1}^{m} g(z, a_\mu) \tag{10.1}$$

$$\Pi_2(z) = \prod_{\nu=1}^{n} g(z, b_\nu) \tag{10.2}$$

$$\Pi_3(z) = \prod_{\nu=1}^{n} g(z, c_\nu) \tag{10.3}$$

$$g(z) = 2^{n-m} f(z) \frac{\Pi_2(z)}{\Pi_1(z)}. \tag{10.4}$$

We note also that

$$|g(z, a)| < 1, \quad |z| < 1, \ |a| < 1; \tag{10.5}$$

so that

$$\log^+ \frac{1}{|g(z)|} \leq \log^+ \frac{2^m}{|f(z)||\Pi_2(z)|} \leq \log^+ \frac{1}{|f(z)|} + \sum_{\nu=1}^{n} \log^+ \frac{1}{|g(z, b_\nu)|} + m \log 2. \tag{10.6}$$

In order to obtain a bound for $m_0\left[R, \frac{1}{g(z)}\right]$ we must first calculate $m_0\left[R, \frac{1}{g(z, a)}\right]$. We have

Lemma 9. *If* $a = \varrho e^{i\phi}$ *then*

$$m_0\left[R, \frac{1}{g(z, a)}\right] = \begin{cases} 2\log\frac{1}{R} - \varrho\frac{1-R^2}{R}\cos\phi, & 0 < \varrho \le R < 1. \\ 2\log\frac{1}{\varrho} - R\frac{1-\varrho^2}{\varrho}\cos\phi, & 0 < R \le \varrho < 1. \end{cases}$$

This is immediate on applying the formula (3.4) and noting that since (10.5) holds we have

$$\log^+ \frac{1}{|g(z, a)|} = \log\frac{1}{|g(z, a)|}.$$

Combining lemma 9 and (10.6) we have

Lemma 10. *We have with the above notation*

$$m_0\left[R, \frac{1}{g(z)}\right] \le m_0\left[R, \frac{1}{f(z)}\right] + 3\,l, \quad R \ge \tfrac{1}{2}.$$

In fact (10.5) and (10.6) yield

$$(10.7) \qquad m_0\left[R, \frac{1}{g}\right] \le m_0\left[R, \frac{1}{f}\right] + \sum_{\nu=1}^{n} m_0\left[R, \frac{1}{g(z, b_\nu)}\right] + 2\,m\log 2$$

and we see from lemma 9 that if $|b_\nu| \le \frac{1}{2}$, $R \ge \frac{1}{2}$, we have

$$m_0\left[R, \frac{1}{g(z, b_\nu)}\right] \le 2\log\frac{1}{R} + 1 - R^2 < 2\log 2 + 1 < 3,$$

so that

$$(10.8) \qquad \sum_{\nu=1}^{n} m_0\left[R, \frac{1}{g(z, b_\nu)}\right] \leqq 3\,n.$$

Hence, combining (10.7) and (10.8) we have lemma 10.

11) We next prove a lemma which will help us to find a point near which there are not too many of the d_ν. This is

Lemma 11. *Let* $d_1, \ldots d_l$ *be* l *complex numbers such that* $|d_\nu| < 1$, $\nu = 1$ to l. *Then there exists* ϱ, $\frac{1}{8} \le \varrho \le \frac{1}{4}$ *such that*

$$\left|\prod_{\nu=1}^{l} g(z, d_\nu)\right| > A^{-l}, \quad |z| = \varrho.$$

Suppose $|a| = r$, $|z| = \varrho$. Then

$$\mu(\varrho, r) = \min_{|z|=\varrho,\ |a|=r} |g(z, a)| = \left|\frac{r-\varrho}{1-\varrho r}\right| \geq \frac{3}{4}|r-\varrho|, \quad \varrho \leq \frac{1}{4}.$$

Hence

$$\int_{\frac{1}{8}}^{\frac{1}{4}} \log \mu(\varrho, r)\, d\varrho > \frac{1}{8}\log\frac{3}{4} + \int_{\frac{1}{8}}^{\frac{1}{4}} \log|r-\varrho|\, d\varrho > -A. \tag{11.1}$$

Hence if

$$\mu[\varrho, \Pi] = \min_{|z|=\varrho} \prod_{\nu=1}^{l} |g(z, d_\nu)|$$

we have

$$\int_{\frac{1}{8}}^{\frac{1}{4}} \log \mu[\varrho, \Pi]\, d\varrho \geq \sum_{\nu=1}^{l} \int_{\frac{1}{8}}^{\frac{1}{4}} \log \mu(\varrho, |d_\nu|)\, d\varrho > -Al$$

by (11.1). It follows that there exists ϱ, $\frac{1}{8} \leq \varrho \leq \frac{1}{4}$, such that

$$\log \mu[\varrho, \Pi] > -8Al = -Al,$$

and lemma 11 follows.

To continue with the proof we shall have to distinguish two possibilities. The first is essentially that $f'(z)$ is small everywhere on the circle $|z| = \varrho$ which is constructed in lemma 11. In this case we can give a direct proof of the truth of Theorem I. This is the aim of lemma 12. If the hypotheses of the lemma are not satisfied we can proceed with the main course of the argument, obtain a bound for $m_0\left[R, \frac{1}{g(z)}\right]$ and hence prove Theorem I.

12) *Partial Proof of Theorem I.*

We have

Lemma 12. *Let* ϱ, $\frac{1}{8} \leq \varrho \leq \frac{1}{4}$ *be such that*

$$\prod_{\nu=1}^{l} |g(z, d_\nu)| > A^{-l}, \quad |z| = \varrho.$$

Suppose also that $|g(0)| \geq 1$ *and that at each point of* $|z| = \varrho$

$$\left|\frac{f'(z)}{f(z)}\right| \leq 2. \tag{12.1}$$

Then we have

$$\left|\frac{g'(0)}{g(0)}\right| \leq A(1 + l).$$

Thus Theorem I holds under the hypotheses of lemma 12.

Suppose that (12.1) holds when $z = e^{i\theta}$ for $\theta_1 \leq \theta \leq \theta_2$. Then we have

$$\log |f(\varrho e^{i\theta_2})| - \log |f(\varrho e^{i\theta_1})| \leq \int_{\theta_1}^{\theta_2} \left| \frac{f'(\varrho e^{i\theta})}{f(\varrho e^{i\theta})} \right| \varrho \, d\theta$$

$$\leq 2\varrho(\theta_2 - \theta_1) \leq \tfrac{1}{2}(\theta_2 - \theta_1) \leq \pi,$$

since $\varrho \leq \frac{1}{4}$. We deduce that if $0 \leq \theta_1 \leq \theta_2 \leq 2\pi$ and (12.1) holds whenever $|z| = \varrho$, then we have

$$\left| \log |f(\varrho e^{i\theta_2})| - \log |f(\varrho e^{i\theta_1})| \right| < \pi, \quad 0 \leq \theta_1 < \theta_2 \leq 2\pi. \tag{12.2}$$

Now $g(z)$ is regular nonzero in $|z| < \frac{1}{2}$, and so its maximum modulus increases and its minimum modulus decreases in $|z| < \frac{1}{2}$. It follows that there is a point $z_1 = \varrho e^{i\theta_1}$ such that

$$|g(z_1)| = |g(0)|. \tag{12.3}$$

Next we see from (10.4) and the hypothesis of lemma 12 that

$$A^{-l} |f(z)| \leq |g(z)| \leq A^l |f(z)|, \quad |z| = \varrho. \tag{12.4}$$

Hence if $z_2 = \varrho e^{i\theta_2}$ is any point on $|z| = \varrho$, it follows from (12.2) to (12.4) that

$$\left| \log |g(z_2)| - \log |g(0)| \right| < Al + \pi. \tag{12.5}$$

We deduce from (12.5) that $\log g(z)$ which is regular in $|z| \leq \varrho$ satisfies there

$$\Re \log g(z) > \log |g(0)| - A(1 + l).$$

Hence $\log g(z)$ is subordinate to

$$\psi(z) = \log g(0) + A(1 + l) \frac{2z}{\varrho - z}$$

in $|z| < \varrho$ so that

$$\left| \frac{d}{dz} \log g(z) \right|_{z=0} = \left| \frac{g'(0)}{g(0)} \right| \leq |\psi'(0)| = \frac{2A(1 + l)}{\varrho} \leq A(1 + l), \tag{12.6}$$

which proves the inequality of lemma 12. Also if $|d_\nu| \leq \frac{1}{2}$ we have

$$(1 - |d_\nu|) \, |1 - d_\nu|^2 \geq \tfrac{1}{8}$$

so that

$$l \leq 8 \sum_{\nu=1}^{l} (1 - |d_\nu|) \, |1 - d_\nu|^2 \leq 8 \sum_{\nu=1}^{L} (1 - |d_\nu|) \, |1 - d_\nu|^2.$$

Thus (12.6) implies Theorem I and the proof of lemma 12 is complete.

13) We consider now the case where $f'(z)$ is not small on the whole circle $|z| = \varrho$. We have in this case.

Lemma 13. *Suppose that* ϱ *is the number constructed in lemma 11 and suppose that*

$$\max_{|z|=\varrho} \left| \frac{f'(z)}{f(z)} \right| \geq 2.$$

Then there exists a point z_0 *with the following properties.*

(i) $$\tfrac{1}{8} \leq |z_0| \leq \tfrac{1}{4}$$

(ii) $$\prod_{\nu=1}^{l} |g(z_0, d_\nu)| > A^{-l}$$

(iii) $$|\log|f(z_0)| - \log|g(0)|| < A(1 + l)$$

(iv) $$\left| \frac{f'(z_0)}{f(z_0)} \right| \geq 2.$$

Since $g(z)$ is regular nonzero in $|z| < \frac{1}{2}$, there exists a point $z_1 = \varrho e^{i\theta_1}$ such that

(13.1) $$|g(z_1)| = |g(0)|.$$

Let θ_0 be the smallest number not less than θ_1 and such that for $z_0 = \varrho e^{i\theta_0}$

(13.2) $$\left| \frac{f'(z_0)}{f(z_0)} \right| \geq 2.$$

By hypothesis θ_0 exists. Also

(13.3) $$\left| \frac{f'(\varrho e^{i\theta})}{f(\varrho e^{i\theta})} \right| \leq 2, \quad \theta_1 < \theta < \theta_0 < \theta_1 + 2\pi,$$

and hence

(13.4) $$|\log|f(z_0)| - \log|f(z_1)|| \leq \pi.$$

Again as (12.4) still holds we have

$$|\log|f(z_1)| - \log|g(z_1)|| = |\log|f(z_1)| - \log|g(0)|| \leq A l,$$

and combining this with (13.1) and (13.4) we deduce

(13.5) $$|\log|f(z_0)| - \log|g(0)|| \leq A(1 + l).$$

Then (13.2) and (13.5) show that z_0 satisfies the conditions (iii) and (iv) of lemma 13. Also (i) and (ii) are satisfied, by lemma 11. Thus the proof of lemma 13 is complete.

We shall consider now the transformation

$$(13.6) \qquad w = l(z) = \frac{R^2[(R-z_0)z + z_0(R-\bar{z}_0)]}{\bar{z}_0(R-z_0)z + R^2(R-\bar{z}_0)}, \quad R \geq \tfrac{1}{2},$$

which sends $|z| \leq R$ onto $|w| \leq R$ and $z = 0$ onto $w = z_0$.

We consider instead of $f(z)$ the function

$$\psi(z) = f[l(z)],$$

and deduce from lemma 8 applied to $\psi(z)$ instead of $f(z)$ and from lemma 13 that $m_0\left[R, \frac{1}{\psi}\right]$ has an upper bound of the required form. From this follows a bound for $m_0\left[R, \frac{1}{g}\right]$ and the proof of Theorem I.

14) In this paragraph we investigate the function $l(z)$ of (13.6). We have

Lemma 14. *The transformation $w = l(z)$ of (13.6) is the unique bilinear transformation of $|z| \leq R$ onto $|w| \leq R$, such that $l(0) = z_0$ and $l(R) = R$. The inverse transformation is given by*

$$(14.1) \qquad z = \lambda(w) = \frac{R^2(w - z_0)}{R^2 - \bar{z}_0 w} \frac{R - \bar{z}_0}{R - z_0}.$$

Let $R_0 = R + \frac{1}{8}(1 - R)$. Then we have

$$(14.2) \qquad \tfrac{1}{6} < |l'(z)| < 6, \quad |z| \leq R_0.$$

Also if $|z_i| \leq R_0$, $i = 1, 2$ and $l(z_i) = w_i$ we have

$$(14.3) \qquad \tfrac{1}{6}|z_1 - z_2| \leq |w_1 - w_2| \leq 6|z_1 - z_2|.$$

The statements of lemma 14 up to (14.1) are evident by inspection. Consider now

$$|l'(z)| = \frac{R^2(R^2 - |z_0|^2)}{\left|R^2 + \bar{z}_0 \frac{R - z_0}{R - \bar{z}_0} z\right|^2}.$$

Since $|z_0| \leq \frac{1}{4}$, $R \geq \frac{1}{2}$, we have

$$(14.4) \qquad \frac{R^2 \cdot \frac{3}{4} R^2}{|R^2 + \frac{1}{4}|z||^2} \leq |l'(z)| \leq \frac{R^4}{|R^2 - \frac{1}{4}|z||^2}, \quad |z| < 1.$$

Also if $|z| \leq R_0 = R + \frac{1}{8}(1 - R) < 1$, then $R^2 + \frac{1}{4}|z| < 2R^2$ since $R \geq \frac{1}{2}$, so that (14.4) gives

(14.5) $$|l'(z)| > \frac{\frac{3}{4}R^4}{4R^4} = \frac{3}{16} > \frac{1}{6}.$$

Again if $|z| \leq R_0 = \frac{1}{8} + \frac{7}{8}R$ we have

$$\frac{R^4}{\left|R^2 - \frac{1}{4}|z|\right|^2} \leq \frac{R^4}{\left|R^2 - \frac{1}{4}R_0\right|^2} = \frac{1}{\left(1 - \frac{7}{32R} - \frac{1}{32R^2}\right)^2} \leq \frac{1}{\left(1 - \frac{7}{16} - \frac{1}{8}\right)^2}$$

since $R \geq \frac{1}{2}$, i.e.

(14.6) $$|l'(z)| \leq \left(\frac{16}{7}\right)^2 < 6.$$

Combining (14.5) and (14.6) we have (14.2).

If z_1, z_2 lie in $|z| \leq R_0$, so does the line segment joining z_1, z_2 and so we have from (14.2)

$$|w_2 - w_1| = |l(z_2) - l(z_1)| \leq \int_{z_1}^{z_2} |l'(z)|\,|dz| \leq 6|z_2 - z_1|$$

where the integral is taken along the straight line joining z_1, z_2. This proves the second inequality of (14.3).

Conversely $l(z)$ maps the circle $|z| \leq R_0$ onto another circle, C say, and w_1, w_2 lie in C. Hence so does the segment joining w_1, w_2 and integrating along this segment we have

$$|z_1 - z_2| = |\lambda(w_1) - \lambda(w_2)| \leq \int_{w_1}^{w_2} |\lambda'(w)|\,|dw| \leq 6|w_1 - w_2|$$

since

$$|\lambda'(w)| = \left|\frac{1}{l'(z)}\right| \leq 6,$$

when w lies in C. This completes the proof of (14.3) and so of lemma 14.

15) Consider now

(15.1) $$\psi(z) = f[l(z)].$$

It follows from lemma 14 that $|l(z)| \leq R$ for $|z| \leq R$. Suppose next that

$$z = re^{i\theta}, R \leq r \leq R_0 = \tfrac{1}{8} + \tfrac{7}{8}R.$$

Then by (14.3)

$$|l(re^{i\theta}) - l(Re^{i\theta})| \leq 6(r - R)$$

so that

$$|l(re^{i\theta})| \leq R + 6(r - R) \leq R + 6(R_0 - R) = R + \tfrac{3}{4}(1 - R)$$

i.e.

(15.2) $$|l(re^{i\theta})| < R_1$$

where

$$R_1 = 1 - \tfrac{1}{4}(1 - R) < 1.$$

Thus $\psi(z)$ is meromorphic for $|z| \le R_0$. Consider next $m_0\left[R, \frac{1}{\psi(z)}\right]$. By applying lemma 8 with $\frac{R}{R_0}$ instead of R and $\psi(R_0 z)$ instead of $f(z)$ we see that

(15.3) $$m_0\left[R, \frac{1}{\psi(z)}\right] = m_0\left[\frac{R}{R_0}, \frac{1}{\psi(R_0 z)}\right] \le A\left\{\Sigma_{d'} + \log^+ \log^+ \frac{1}{R_0|\psi'(0)|} + \log^+\log^+|\psi(0)| + \log^+\left|\frac{\psi(0) - 1}{R_0\psi'(0)}\right| + \log\frac{R_0}{R_0 - R} + 1\right\}$$

where

(15.4) $$\Sigma_{d'} = \Sigma \log^+\left|\frac{R_0}{d'_\nu}\right| \left|1 - \frac{d'_\nu}{R_0}\right|^2$$

and the sum is taken over all points d'_ν in $|z| \le R_0$, such that $\psi(d'_\nu) = 0, 1$, or ∞.

We consider the terms on the right hand side of (15.3) in turn. We have first.

Lemma 15. *Let $\Sigma_{d'}$ be as defined in* (15.4). *Then we have*

$$\Sigma_{d'} < A \sum_{\nu=1}^{L} |1 - d_\nu|^2 (1 - |d_\nu|).$$

Suppose that $\psi(d') = 0, 1$ or ∞. Then it follows from (15.1) that $l(d') = d_\nu$, where $f(d_\nu) = 0, 1$ or ∞. In this case we write $d' = d'_\nu$ and thus obtain an ordering of the points d'_ν.

Suppose first that $\nu \le l$ so that $|d_\nu| \le \frac{1}{2}$. Then

$$|l(d'_\nu) - l(0)| \le 6|d'_\nu|$$

by (14.3) and so, since $l(0) = z_0$, we have

$$|d'_\nu| \ge \tfrac{1}{6}|d_\nu - z_0|,$$

and hence

(15.5) $$\left|1 - \frac{d'_\nu}{R_0}\right|^2 \log^+\left|\frac{R_0}{d'_\nu}\right| < A \log\left|\frac{1}{d'_\nu}\right| < A \log\left|\frac{6}{d_\nu - z_0}\right|.$$

Also

$$|g(z_0, d_\nu)| = \left|\frac{z_0 - d_\nu}{1 - \bar{z}_0 d_\nu}\right| < A|z_0 - d_\nu|, \quad |d_\nu| < 1,$$

so that (15.5) gives

$$\left|1-\frac{d'_\nu}{R_0}\right|^2 \log^+\left|\frac{R_0}{d'_\nu}\right| < A \log\left|\frac{A}{g(z_0, d_\nu)}\right|$$

if $\nu \leq l$. Thus

$$(15.6) \qquad \begin{aligned} \sum_{\nu=1}^{l}\left|1-\frac{d'_\nu}{R_0}\right|^2 \log^+\left|\frac{R_0}{d'_\nu}\right| &< A\sum_{\nu=1}^{l}\left[\log\left|\frac{1}{g(z_0, d_\nu)}\right| + 1\right] \\ &= A\left[\log\frac{1}{|\Pi_1(z_0)\,\Pi_2(z_0)\,\Pi_3(z_0)|} + l\right] \end{aligned}$$

where the $\Pi_i(z)$ are as defined in (10.1) to (10.3). Combining (15.6) and lemma 13 (ii), we see that

$$(15.7) \qquad \sum_{\nu=1}^{l}\left|1-\frac{d'_\nu}{R_0}\right| \log^+\left|\frac{R_0}{d'_\nu}\right| < A\,l < A\sum_{\nu=1}^{l}(1-|d_\nu|)\,|1-d_\nu|^2.$$

Suppose next that $\frac{1}{2} \leq |d_\nu| \leq R$. Then (14.3) yields

$$|d'_\nu| \geq \frac{1}{6}|d_\nu - z_0| \geq \frac{1}{24}$$

so that

$$(15.8) \qquad \begin{aligned} \log^+\left|\frac{R_0}{d'_\nu}\right| &< A(R_0 - |d'_\nu|) = A[(R_0-R) + R - |d'_\nu|] \\ &< A[1 - R + R - |d'_\nu|]. \end{aligned}$$

Again if $d_\nu = r e^{i\theta}$ (14.3) yields

$$|\lambda(d_\nu) - \lambda(R e^{i\theta})| \leq 6\,|d_\nu - R e^{i\theta}| = 6(R - |d_\nu|).$$

Again since $\lambda(d_\nu) = d'_\nu$, $|\lambda(R e^{i\theta})| = R$ we have

$$(R - |d'_\nu|) \leq 6(R - |d_\nu|)$$

so that (15.8) gives

$$(15.9) \qquad \log^+\left|\frac{R_0}{d'_\nu}\right| < A[1 - R + R - |d_\nu|] = A[1 - |d_\nu|], \quad \tfrac{1}{2} \leq |d_\nu| \leq R.$$

Suppose next $R \leq |d'_\nu| \leq R_0$. Then $1 - |d_\nu| > \frac{1}{2}(1-R)$ by (15.2) so that

$$(15.10) \qquad \log\left|\frac{R_0}{d'_\nu}\right| \leq \log\frac{R_0}{R} < A(R_0 - R) = A(1-R) < A(1 - |d_\nu|)$$

and combining (15.9), (15.10) we have

$$(15.11) \qquad \log^+\left|\frac{R_0}{d'_\nu}\right| \leq A(1 - |d_\nu|), \quad \nu > l.$$

Consider lastly $\left|1 - \frac{d'_\nu}{R_0}\right|^2$, $\nu > l$. We have

$$\left|1 - \frac{d'_\nu}{R_0}\right| < A\,|R_0 - d'_\nu| < A\,[|R_0 - R| + |R - d'_\nu|]$$
$$< A\,[|R_0 - R| + |l(R) - l(d'_\nu)|],$$

making use of (14.3). Since $l(R) = R$, $l(d'_\nu) = d_\nu$, we deduce

(15.12) $$\left|1 - \frac{d'_\nu}{R_0}\right| < A\,(R_0 - R) + |R - d_\nu| < A\,[1 - R + |1 - d_\nu|].$$

Also (15.2) yields

$$|1 - d_\nu| > 1 - R_1 = \tfrac{1}{4}(1 - R).$$

Thus (15.12) gives

(15.13) $$\left|1 - \frac{d'_\nu}{R_0}\right|^2 < A\,|1 - d_\nu|^2, \quad |d'_\nu| \leq R_0.$$

Combining (15.11) and (15.13) we obtain

(15.14) $$\sum_{\nu=l+1}^{L} \left|1 - \frac{d'_\nu}{R_0}\right|^2 \log^+\left|\frac{R_0}{d'_\nu}\right| < A \sum_{\nu=l+1}^{L} (1 - |d_\nu|)\,|1 - d_\nu|^2.$$

Now lemma 15 follows from (15.7) and (15.14).

16) The other terms on the right hand side of (15.3) are easier to deal with. We have

$$\psi(0) = f(z_0)$$
$$\psi'(0) = f'(z_0)\,l'(0).$$

Also by (14.2)

$$A < |l'(0)| < A'.$$

Hence we have

$$\log^+\log^+\left|\frac{1}{R_0\,\psi'(0)}\right| < \log^+\log^+\left|\frac{1}{f'(z_0)}\right| + A$$
$$< \log^+\left[\log^+\left|\frac{f(z_0)}{f'(z_0)}\right| + \log^+\left|\frac{1}{f(z_0)}\right|\right] + A$$
$$< \log^+\log^+\left|\frac{1}{g(0)}\right| + A\,(1 + l)$$

making use of lemma (iii) and (iv). Thus

(16.1) $$\log^+\log^+\left|\frac{1}{R_0\,\psi'(0)}\right| < \log^+\log^+\left|\frac{1}{g(0)}\right| + A\,(1 + l).$$

Next

(16.2)
$$\log^+\log^+|\psi(0)| = \log^+\log^+|f(z_0)| < \log^+[\log^+|g(0)| + A(1+l)]$$
$$< \log^+\log^+|g(0)| + l + A$$

making use of lemma 13 (iii). Again

$$\log^+\left|\frac{\psi(0)-1}{R_0\,\psi'(0)}\right| = \log^+\left|\frac{f(z_0)-1}{R_0\,|l'(0)|\,f(z_0)}\right| < \log^+\left|\frac{f(z_0)-1}{f(z_0)}\right| + \log^+\left|\frac{f(z_0)}{f'(z_0)}\right| + A$$
$$< \log^+\left|\frac{1}{f(z_0)}\right| + \log^+\left|\frac{f(z_0)}{f'(z_0)}\right| + A.$$

We deduce that

(16.3)
$$\log^+\left|\frac{\psi(0)-1}{R_0\,\psi'(0)}\right| < \log^+\left|\frac{1}{g(0)}\right| + A(1+l) + A$$

making use of lemma 13 (iii) and (iv). Lastly we have

(16.4)
$$\log\frac{R_0}{R_0-R} < \log\frac{1}{R_0-R} = \log\frac{8}{1-R} < \log\frac{1}{1-R} + A.$$

Making use of the inequalities (16.1) to (16.4) and lemma 15 for the terms on the right hand side of (15.3) we have finally

Lemma 16. *If* $|g(0)| \geq 1$ *and* $\psi(z)$ *is defined by* (15.1) *then*

$$m_0\left[R, \frac{1}{\psi(z)}\right] < A\left\{\sum_{\nu=1}^{L}|1-d_\nu|^2(1-|d_\nu|) + \log^+\log|g(0)| + \log\frac{1}{1-R} + 1\right\}.$$

Proof of Theorem I.

17) Having obtained lemma 16 it remains to deduce a bound for $m_0\left[R, \frac{1}{g(z)}\right]$ and to apply (3.4). We may assume without loss in generality that $|g(0)| \geq 1$. For if $|g(0)| \leq 1$, we apply our result to $\frac{1}{f(-z)}$ instead of $f(z)$. This changes the points d_ν to $-d_\nu$ and $g(z)$ becomes $\frac{1}{g(-z)}$. Also

$$\frac{g'(z)}{g(z)} = \frac{1}{g(-z)}\,\frac{d}{dz}\,\frac{1}{g(-z)}.$$

Thus when we have proved Theorem I for $|g(0)| \geq 1$ the result for $|g(0)| \leq 1$ follows.

Further we have proved Theorem I if the hypotheses of lemma 12 hold and $|g(0)| \geq 1$, so that we may assume further that these hypotheses are not satisfied so that $f(z)$ satisfies the conditions of lemma 13.

Suppose now that in (13.6)

$$w = R e^{i\phi} = l(R e^{i\theta}).$$

Then (14.2) gives

$$\frac{1}{6} \leq \left|\frac{d\phi}{d\theta}\right| \leq 6. \tag{17.1}$$

Since $\theta = 0$ corresponds to $\phi = 0$ we deduce

$$\left|\frac{\theta}{6}\right| \leq |\phi| \leq 6|\theta|. \tag{17.2}$$

Now

$$\begin{aligned} m_0\left[R, \frac{1}{\psi}\right] &= \frac{1}{\pi}\int_0^{2\pi} \log^+ \left|\frac{1}{f[l(R e^{i\theta})]}\right| (1 - \cos\theta)\, d\theta \\ &= \frac{1}{\pi}\int_0^{2\pi} \log^+ \left|\frac{1}{f(R e^{i\phi})}\right| (1 - \cos\theta)\, d\theta. \end{aligned} \tag{17.3}$$

Also (17.1) and (17.2) give

$$1 - \cos\phi < A(1 - \cos\theta),$$

$$|d\phi| < A\,|d\theta|,$$

so that from (17.3)

$$\begin{aligned} m_0\left[R, \frac{1}{f}\right] &= \frac{1}{\pi}\int_0^{2\pi} \log^+ \left|\frac{1}{f(R e^{i\phi})}\right| (1 - \cos\phi)\, d\phi \\ &< A\int_0^{2\pi} \log^+ \left|\frac{1}{f(R e^{i\phi})}\right| (1 - \cos\theta)\, d\theta \\ &= A\, m_0\left[R, \frac{1}{\psi}\right]. \end{aligned} \tag{17.4}$$

Again from lemma 10

$$m_0\left[R, \frac{1}{g}\right] < m_0\left[R, \frac{1}{f}\right] + 3l < m_0\left[R, \frac{1}{f}\right] + A\sum_{\nu=1}^{l}(1 - |d_\nu|)\,|1 - d_\nu|^2,$$

and combining this with (17.4) and lemma 16 we have

$$(17.5)\quad m_0\left[R, \frac{1}{g}\right] < A\left[1 + \log\frac{1}{1-R} + \log^+\log^+|g(0)| + \sum_{\nu=1}^{L}(1-|d_\nu|)|1-d_\nu|^2\right].$$

We have from (4.2) and lemma 2 (ii) if $z = \varrho e^{i\theta}$,

$$\left|2\log\frac{1}{\varrho} - \frac{1-\varrho^2}{\varrho}\cos\theta\right| < A|1-z|^2 \log\left|\frac{1}{z}\right|, \ \tfrac{1}{2} \le \varrho < 1.$$

Writing $\frac{\varrho}{R}$ for ϱ in this we obtain

$$(17.6)\qquad \left|2\log\frac{R}{\varrho} - \frac{R^2-\varrho^2}{\varrho R}\cos\theta\right| < A\left|1-\frac{z}{R}\right|^2 \log\left|\frac{R}{z}\right|, \ \tfrac{1}{2} \le \varrho \le R < 1.$$

We now apply (3.4) with $g(z)$ instead of $f(z)$ and obtain, making use of (17.6),

$$(17.7)\qquad \begin{aligned} \Re R\frac{g'(0)}{g(0)} - 2\log|g(0)| < m_0\left[R, \frac{1}{g}\right] &+ A\sum_{\mu=m+1}^{M}\left|1-\frac{a_\mu}{R}\right|^2 \log^+\left|\frac{R}{a_\mu}\right| \\ &+ A\sum_{\nu=n+1}^{N}\left|1-\frac{b_\nu}{R}\right|^2 \log^+\left|\frac{R}{b_\nu}\right|, \end{aligned}$$

since the zeros and poles of $g(z)$ are the zeros and poles of $f(z)$ which lie in $\frac{1}{2} \le z < 1$. Thus we have

$$\log^+\left|\frac{R}{a_\mu}\right| < A(R-|a_\mu|) < A(1-|a_\mu|)$$

$$\left|1-\frac{a_\mu}{R}\right| < A|R-a_\mu| < A[(1-R)+|1-a_\mu|] < A|1-a_\mu|$$

since $|a_\mu| \le R$, and so

$$\log^+\left|\frac{R}{a_\mu}\right|\left|1-\frac{a_\mu}{R}\right|^2 < A(1-|a_\mu|)|1-a_\mu|^2.$$

Combining this with (17.5) and (17.7) we obtain

$$\begin{aligned} \Re R\frac{g'(0)}{g(0)} - 2\log|g(0)| < A\Bigg[1 + \log\frac{1}{1-R} &+ \log^+\log|g(0)| \\ &+ \sum_{\nu=1}^{L}|1-d_\nu|^2(1-|d_\nu|)\Bigg], \end{aligned}$$

or

$$(17.8) \qquad \Re\frac{g'(0)}{g(0)} - \frac{2}{R}\log g(0) < A\left[1 + \log\frac{1}{1-R} + \log^+\log|g(0)| + \sum_{\nu=1}^{L}|1-d_\nu|^2(1-|d_\nu|)\right].$$

Now

$$\frac{2}{R} = 2 + \frac{2(1-R)}{R} < 2 + 4(1-R), \quad R \geq \tfrac{1}{2}.$$

We choose

$$(17.9) \qquad R = \max\left[\tfrac{1}{2},\ 1 - \frac{1}{\log|g(0)|}\right]$$

so that

$$(1-R)\log|g(0)| \leq 1.$$

Then (17.8) and (17.9) give

$$\Re\frac{g'(0)}{g(0)} < 2\log|g(0)| + A\left[1 + \log^+\log|g(0)| + \sum_{\nu=1}^{L}|1-d_\nu|^2(1-|d_\nu|)\right],$$

which proves Theorem I.

Applications of Theorem I.

18) Having proved the fundamental Theorem I we shall devote the rest of the chapter to some applications of this result. These follow relatively easily. Our aim is to obtain upper bounds for the maximum modulus of a function $f(z)$, regular or more generally meromorphic in $|z| < 1$, given the roots of the equations $f(z) = 0, 1, \infty$, or more generally $f(z) = \phi_1(z), \phi_2(z), \infty$, where $\phi_1(z)$, $\phi_2(z)$ are assigned meromorphic functions. The feature which distinguishes our investigations from previous work, e.g. that deducible from the ordinarily Nevanlinna Theory is that we obtain results of the type

$$(18.1) \qquad \log M(\varrho, f) = \frac{O(1)}{1-\varrho},$$

even when the equations $f(z) = 0, 1, \infty$ may have infinitely many roots in $|z| < 1$, provided that the total number $n(r)$ of these roots in $|z| \leq r < 1$ satisfies

$$(18.2) \qquad \int_0^1 n(r)\,dr < \infty.$$

Moreover the condition (18.2) seems to be the weakest condition of its kind which still implies (18.1). (See Theorem IV.) Even in the case of a finite number of roots our bounds appear to be sharper than those previously obtained.

Naturally we cannot obtain (18.1) generally when $f(z)$ is meromorphic, since $M(\varrho, f) = \infty$, if $f(z)$ has a pole on the circle $|z| = \varrho$. We circumvent this difficulty by introducing a function $f_*(z)$, the star function of $f(z)$, which behaves locally as the function $g(z)$ of Theorem I behaves at the origin. If $f(z)$ has no zeros or poles in a small circle surrounding the point z we shall have

$$f(z) = f_*(z),$$

and if $f(z)$ has no poles in $|z| < 1$ we shall have

$$|f_*(z)| \geq |f(z)|, \quad |z| < 1.$$

The function $f_*(z)$ has a continuous non-zero modulus in $|z| < 1$; and it is regular except on certain circles. Hence $\log |f_*(z)|$ is continuous on each radius $\arg z = \theta = \text{const}$, and differentiable except at an isolated set of points. Making use of Theorem I, we can obtain for $\log |f_*(z)|$ a differential inequality, whose integration will yield our main result, Theorem II.

Notation.

19) We shall consider in the rest of this chapter a function $f(z)$, meromorphic in $|z| < 1$ and denote as in (1.1) to (1.5) by a_μ, b_μ, c_μ the zeros, poles and ones of $f(z)$ in $|z| < 1$ and by d_μ the totality of these points. We no longer assume that the set of d_μ is finite. We assume, however, that

$$N_0 = \sum_{\mu=1}^{\infty} (1 - |d_\mu|) < \infty. \tag{19.1}$$

Let $n(r, f)$ denote the number of poles of $f(z)$ in $|z| \leq r$, so that $n(r, 1/(f - w))$ denotes the number of roots of $f(z) = w$ in $|z| \leq r < 1$. Then (19.1) may also be written as

$$N_0 = \int_0^1 \left\{ n(r, f) + n\left(r, \frac{1}{f}\right) + n\left(r, \frac{1}{f - 1}\right) \right\} dr < \infty. \tag{19.2}$$

We also define a function $f_*(z)$ as follows. Let a_{μ_1}, $a_{\mu_2}, \ldots a_{\mu_p}$, $b_{\nu_1}, \ldots b_{\nu_q}$ be the zeros and poles respectively of $f(z)$ in the region

$$(19.3) \qquad |g(z, \xi)| = \left|\frac{z-\xi}{1-\bar{z}\xi}\right| \leq \tfrac{1}{2}.$$

Then we write

$$(19.4) \qquad f_*(\xi) = 2^{q-p} \frac{\prod_{j=1}^{q} g(\xi, b_{\nu_j})}{\prod_{i=1}^{p} g(\xi, a_{\mu_i})} f(\xi).$$

If $f(\xi) = 0$ or ∞, $f_*(\xi)$ is defined by continuity. We see that $f_*(\xi)$ is regular nonzero, except on the circles $|g(\xi, a_\mu)| = \frac{1}{2}$, $|g(\xi, b_\nu)| = \frac{1}{2}$, and on these $|f_*(\xi)|$ is continuous. Moreover if $f(z)$ has no zeros or poles in the region (19.3) we have

$$(19.5) \qquad f_*(\xi) = f(\xi)$$

and if $f(z)$ has no poles in the region (19.3), we have

$$(19.6) \qquad |f_*(\xi)| \geq |f(\xi)|.$$

In particular (19.6) and (19.5) respectively hold throughout $|\xi| < 1$ if $f(\xi)$ is regular, or regular nonzero throughout $|\xi| < 1$.

Lastly we define if $\phi(z)$ is any function of z in $|z| < 1$

$$M[r, \phi(z)] = \max_{0 \leq \theta \leq 2\pi} |\phi(r e^{i\theta})|.$$

We use this notation both for meromorphic functions, in which case we have $M[r, \phi] = +\infty$ whenever $\phi(z)$ has a pole on $|z| = r$ or for discontinuous functions such as $f_*(z)$. Here max denotes the upper bound.

We shall in the sequel be dealing frequently with the derivative of $f_*(z)$ at a point $z = \xi$. When doing this we assume that this derivative is defined at $z = \xi$, so that $|g(\xi, b_\nu)| \neq \frac{1}{2}$, $|g(\xi, a_\mu)| \neq \frac{1}{2}$ for any pole b_ν or zero a_μ. We write this derivative as $f_*'(\xi)$ which is not to be confused with $f'_*(\xi)$, the star function of the derivative of $f(z)$ at $z = \xi$, which latter, however, will not be used in this paper.

20) With the notation defined above Theorem I may be rewritten as follows

Theorem I. *Let $f(z)$ be meromorphic in $|z| = |x + iy| < 1$ and let the roots d_μ of $f(z) = 0, 1, \infty$ satisfy* (19.1). *Then we have*

$$\left\{\frac{d}{dx}\log|f_*(x)|\right\}_{x=0} \le 2\left\{|\log|f_*(0)|| + A[1+\log^+|\log|f_*(0)||] \right.$$
$$\left. + \sum_{\mu=1}^{\infty}|1 \pm d_\mu|^2(1-|d_\mu|)\right\}$$

where the sign in the sum is + or — according as $|f_*(0)| \le 1$ *or* $|f_*(0)| \ge 1$.

If $f(z)$ is meromorphic in $|z| \le 1$ this result is obtained at once from Theorem I of paragraph 2. In the general case we apply that result to $f\left(\frac{z}{r}\right)$ with $r > 1$ and make $r \to 1$.

By integrating Theorem I we can prove

Theorem II. *With the hypotheses of Theorem I we have for* $0 < r < 1$,

$$\log M[r, f_*(z)] < \frac{1}{1-r}\{(1+r)\log^+|f_*(0)| + Ar[\log^+\log^+|f_*(0)| + N_0 + 1]\},$$

where N_0 *is defined as in* (19.1).

This result could not be obtained by integrating the inequality of Theorem I, if the sum in that Theorem was replaced by N_0. Thus to obtain Theorem II it is not sufficient to obtain a bound for $|f_*'(0)|$. Both Theorems I and II contain the term $\log^+|\log|f_*(0)||$. If there are only a finite number L of d_μ we can eliminate this term, by replacing the terms depending on d_μ in Theorems I and II by L. Whether the term $\log^+|\log|f_*(0)||$ can be eliminated in Theorems I and II without otherwise weakening those results remains an open question. We have

Theorem III. *Suppose that the equations* $f(z) = 0, 1, \infty$ *have at most* L *roots in* $|z| < 1$. *Then we have*

(i) $$|f_*'(0)| < |f_*(0)|\,[2\log|f_*(0)| + A(1+L)]$$

(ii) $$M[r, f_*(z)] < \exp\left\{\frac{1+r}{1-r}\log^+|f_*(0)| + \frac{Ar}{1-r}(1+L)\right\}.$$

Both parts of this result are best possible even in the case $L = 0$, except for the constant A.[1]

We also prove the following converse Theorem to Theorem II.

[1] C. F. HAYMAN (1), Theorems V and VI.

Theorem IV. *Let*

$$\phi(w) = e^{bw} \prod_{\nu=1}^{\infty} \left(1 + \frac{w}{r_\nu}\right) e^{-\frac{w}{r_\nu}}$$

be an integral function of genus 0 *or* 1, *which is real for real* w, *has real negative zeros and satisfies* $\phi(0) = 1$. *Let*

$$f(z) = \left\{\phi\left(\frac{1+z}{1-z}\right)\right\}^{-1}.$$

Then $f(z)$ *is regular nonzero in* $|z| < 1$ *and we have for any* a

$$\log^+ |f(r)| \geq \frac{1+r}{1-r}\left\{\log |f(0) - a| + \int_0^r n(t, a) \frac{dt}{t} - \log^+ |a| - \log 2\right\}, 0 < r < 1, \tag{20.1}$$

where $n(t, a)$ *denotes the number of zeros of* $f(z) - a$ *in* $|z| \leq t$. *Further if* $\sum \frac{1}{r_\nu}$ *converges* $\phi(w)$ *has genus* 0 *and* $\lim_{r\to 1} (1 - r) \log M(r, f)$ *exists finitely, so that* $\int_0^1 n(t, a)\, dt$ *converges for every* a. *If* $\sum \frac{1}{r_\nu}$ *diverges* $\phi(w)$ *has genus* 1,

$$\lim_{r \to 1} (1 - r) \log^+ f(r) = \infty$$

and $\int_0^1 n(t, a)\, dt$ *diverges for every* a *other than* $a = 0$.

Theorem IV shows that for the wide variety of functions $f(z)$ introduced in that Theorem, the condition that $\int_0^1 n(t, a)\, dt$ converges for every a with one exception is necessary in order that $(1 - r) \log M[r, f]$ shall be bounded. Thus we cannot hope to weaken the conditions

$$\int_0^1 n(t, a)\, dt < \infty, \quad a = 0, 1, \infty$$

to obtain an inequality similar to that of Theorem II.

It is not difficult to deduce from Theorem II a generalization in which the equations $f(z) = 0$ or 1 are replaced by $f(z) = \phi_1(z)$ or $\phi_2(z)$, where the $\phi_i(z)$ are meromorphic functions of z.

The case when $\phi_1(z) \equiv 0$ is particularly simple and we confine ourselves to it. We can deduce a result in the more general case by considering $f(z) - \phi_1(z)$ which is equal to 0, $\phi_2(z) - \phi_1(z)$ when $f(z) = \phi_1(z)$, $\phi_2(z)$ respectively. We have

Theorem V. *Suppose that $f(z)$, $\phi(z)$ are meromorphic in $|z| < 1$ and also that*

$$N_1 = \int_0^1 \left\{ n(t, f) + n\left(t, \frac{1}{f}\right) + n\left(t, \frac{1}{f-\phi}\right) + n\left(t, \frac{1}{\phi}\right) + n(t, \phi) \right\} dt < \infty.$$

Then we have for $0 < r < 1$

$$\log M[r, f_*(z)] < \log M\left[r, \phi_*(z)\right] + \frac{1}{1-r}[(1+r)\log^+ \mu + A r(\log^+ \log^+ \mu + N + 1)]$$

where

$$\mu = \left|\frac{f_*(0)}{\phi_*(0)}\right|.$$

The results of Theorems I, II, III, and V have all been stated in terms of star-functions. This is justified by their simplicity in this form whether $f(z)$ is regular or merely meromorphic. It is not difficult to deduce results for $M[r, f]$ when those for $M[r, f_*]$ are known. The following Theorem enables us to do this.

Theorem VI. *If $f(z)$ is regular in $|z| < 1$ we have*

$$M[r, f_*(z)] \geq M[r, f(z)], \quad 0 \leq r < 1. \tag{20.2}$$

If $f(z)$ is meromorphic in $|z| < 1$, then given ϱ, $0 < \varrho < 1$, we can find r, such that $\varrho \leq r < \frac{1}{2}(1 + \varrho)$ and such that when $|z| = r$ we have

$$\log \left|\frac{f(z)}{f_*(z)}\right| < A\, n\left(\frac{3+\varrho}{4}\right), \tag{20.3}$$

where $n\left(\frac{3+\varrho}{4}\right)$ denotes the number of poles of $f(z)$ in $|z| \leq \frac{3+\varrho}{4}$.

Corollary. *We may replace $M[r, f_*(z)]$ by $M[r, f]$ in Theorems II, III, (ii) and V for all r if $f(z)$ is regular in $|z| < 1$ and for some r in every range $\varrho \leq r < \frac{1}{2}(1 + \varrho)$ otherwise.*

Proof of Theorem II.

21) Having stated our results we proceed to prove them in turn. Theorem I has already been proved at the end of paragraph 17. The other results follow fairly simply.

Before we can prove Theorem II, we need two lemmas.

Lemma 17. *Let $\lambda(r)$ be a real continuous function of r for $0 \le r < 1$ and suppose that $\lambda'(r)$ exists at all but a finite number of points in every range $0 \le r \le k < 1$. Suppose further that we have*

$$(1 - r^2)\lambda'(r) \le 2\lambda(r) + C\log^+\lambda(r) + \mu(r) \tag{21.1}$$

whenever $\lambda(r) \ge 0$, where C is a positive constant and $\mu(r)$ is positive and continuous for $0 \le r < 1$. Then we have

$$\lambda(r) < \frac{1}{1-r}\left[(1+r)\lambda + 2\,Cr(\log^+\lambda + C + 3) + 2(1+C)\int_0^r \mu(t)\,dt\right], \quad 0 \le r < 1,$$

where $\lambda = \max\{0, \lambda(0)\}$.

Suppose that $0 < R < 1$ and that $\lambda(R) > 0$. Let α be the smallest non-negative number such that $\lambda(r) > 0$ for $\alpha < r \le R$. Then we must have either $\alpha = 0$ or $\lambda(\alpha) = 0$, and (21.1) holds for $\alpha < r \le R$, except perhaps at a finite number of points. Thus

$$(1 - r^2)\lambda'(r) < (2 + C)\lambda(r) + \mu(r), \quad \alpha \le r < R$$

except at a finite number of points, where $\lambda(r)$ is continuous. Thus

$$\left[\left(\frac{1-r}{1+r}\right)^{1+\frac{C}{2}}\lambda(r)\right]_\alpha^R < \int_\alpha^R \frac{(1-r)^{C/2}}{(1+r)^{2+C/2}}\mu(r)\,dr < \int_0^R \mu(r)\,dr = M. \tag{21.2}$$

say. Since we have either $\lambda(\alpha) = 0$ or $\alpha = 0$, we have $\lambda(\alpha) \le \lambda$ and we deduce that

$$\lambda(R) < (\lambda + M)\left(\frac{1+R}{1-R}\right)^{1+C/2}, \quad 0 < R < 1. \tag{21.3}$$

Substituting from this expression for $\log^+\lambda(r)$ in (21.1) we have for $r \le R$

$$(1 - r^2)\lambda'(r) \le 2\lambda(r) + C\left\{\log^+\lambda + \log^+ M + \log 2 + \left(1 + \frac{C}{2}\right)\log\left(\frac{1+r}{1-r}\right)\right\} + \mu(r),$$

whenever (21.1) holds. This is an inequality of the same type as (21.1) with $C = 0$ and $\mu(r)$ replaced by

$$\nu(r) = C\left\{\log^+ \lambda + \log^+ M + \log 2 + \left(1 + \frac{C}{2}\right)\log\frac{1+r}{1-r}\right\} + \mu(r).$$

We deduce from (21.3) that we have

$$\lambda(R) < \frac{1+R}{1-R}\left(\lambda + \int_0^R \nu(r)\,dr\right). \tag{21.4}$$

Now

$$\int_0^R \nu(r)\,dr = CR\,[\log^+\lambda + \log^+ M + \log 2] + C\left(1+\frac{C}{2}\right)\int_0^R \log\left(\frac{1+r}{1-r}\right)dr + M,$$

$$\int_0^R \nu(r)\,dr < (1+C)\,M + CR\,[\log^+\lambda + C + 3], \tag{21.5}$$

since

$$\int_0^R \log\frac{1+r}{1-r}\,dr = (1-R)\log(1-R) + (1+R)\log(1+R) < 2R,$$

and M is defined in (21.2). This proves the lemma on combining (21.4) and (21.5).

We have next

Lemma 18. *If $\mu(r)$ is defined for $0 \leq r < 1$ by*

$$\mu(r) = \left|1 - \frac{z-r}{1-rz}\right|^2\left(1 - \left|\frac{z-r}{1-rz}\right|\right)$$

where z is complex and $|z| < 1$, then we have

$$\int_0^r \mu(t)\,dt < Ar(1-|z|), \quad 0 < r < 1.$$

We have

$$\mu(r) = \left|\frac{1-rz-z+r}{1-rz}\right|^2\left(1-\left|\frac{z-r}{1-rz}\right|\right) = \frac{(1+r)^2\,|1-z|^2}{|1-rz|^2}\left(1-\left|\frac{z-r}{1-zr}\right|\right)$$

$$< \frac{4\,|1-z|^2}{|1-rz|^2}\left(1-\left|\frac{z-r}{1-rz}\right|^2\right) = \frac{4\,|1-z|^2(1-|z|^2)(1-r^2)}{|1-rz|^4}.$$

Thus

$$\int_0^r \mu(t)\,dt < 8\,|1-z|^2(1-|z|^2)\int_0^r \frac{(1-t)\,dt}{|1-tz|^4}. \tag{21.6}$$

Suppose first $r \leq \frac{1}{2}$. Then (21.6) yields

$$\int_0^r \mu(t)\,dt < 8 \cdot 16\,|1 - z|^2 (1 - |z|^2) \int_0^r dt < A\,r\,(1 - |z|). \tag{21.7}$$

So that the lemma holds in this case. Suppose next $r > \frac{1}{2}$. Then we have

$$\int_0^r \frac{1 - t}{|1 - tz|^4}\,dt < \int_0^{r_1} \frac{(1 - t)\,dt}{|1 - tz|^4} + \int_{r_1}^1 \frac{(1 - t)}{|1 - tz|^4} = I_1 + I_2 \tag{21.8}$$

where $r_1 = \max\{0, 1 - |1 - z|\}$. Also

$$I_1 < \int_0^{r_1} \frac{1 - t}{(1 - t)^4}\,dt \leq \frac{1}{2}\,\frac{1}{(1 - r_1)^2} \leq \frac{2}{|1 - z|^2}. \tag{21.9}$$

Since $|1 - tz| > \frac{1}{2}|1 - z|$, $0 < t < 1$ we have

$$I_2 = \int_{r_1}^1 \frac{(1 - t)\,dt}{|1 - zt|^4} < \frac{16}{|1 - z|^4} \int_{r_1}^1 (1 - t)\,dt = \frac{8\,(1 - r_1)^2}{|1 - z|^4} \leq \frac{8}{|1 - z|^2}. \tag{21.10}$$

Combining (21.6), (21.8), (21.9), (21.10), we have if $r \geq \frac{1}{2}$

$$\int_0^r \mu(t)\,dt < A\,(1 - |z|^2) < A\,r\,(1 - |z|).$$

Combining this with (21.7), we have lemma 18.

22) We can now prove Theorem II. Write

$$\lambda(r) = \log^+ |f_*(r)|, \quad 0 \leq r < 1. \tag{22.1}$$

It is sufficient to prove that

$$\lambda(r) \leq \frac{1}{1 - r}\left[(1 + r)\,\lambda(0) + A\,r\,(1 + \log^+ |\lambda(0)| + N_0)\right] \tag{22.2}$$

using the notation of Theorem II. For the same upper bound then holds for $\log^+ |f_*(re^{i\theta})|$, $0 \leq \theta \leq 2\pi$, as we can prove by writing $f(ze^{i\theta})$ instead of $f(z)$.

We apply Theorem I to the function

$$g(z) = f\left(\frac{r + z}{1 + rz}\right), \quad 0 \leq r < 1.$$

Then $g(z) = 0, 1, \infty$ at the points $z = d'_\nu$, where

$$d'_\nu = \frac{d_\nu - r}{1 - r\, d_\nu} \tag{22.3}$$

and d_ν are the points such that $f(d_\nu) = 0, 1, \infty$. We deduce from Theorem I that if $|g_*(0)| \geq 1$,

$$\left[\frac{d}{dx} \log^+ |g_*(x)|\right]_{x=0} \leq 2 \log^+ |g_*(0)| + A\left[1 + \log^+ \log^+ |g_*(0)| + \sum_{\nu=1}^{\infty} |1 - d'_\nu|^2 (1 - |d'_\nu|)\right]. \tag{22.4}$$

Now the Green's Functions w.r.t. $|z| < 1$, $\log |g(z_1, z_2)|$ are invariant under a conformal mapping of $|z| < 1$ onto itself and we deduce from (19.4) that

$$\log |g_*(z)| = \log \left| f_*\left(\frac{r + z}{1 + rz}\right)\right| \tag{22.5}$$

$$\Re \frac{g'_*(0)}{g_*(0)} = \Re \frac{f'_*(r)}{f_*(r)} (1 - r^2). \tag{22.6}$$

Combining (22.1), (22.4), (22.5), (22.6), we deduce

$$(1 - r^2)\lambda'(r) \leq 2\lambda(r) + A\left[1 + \log^+ \lambda(r) + \sum_{\nu=1}^{\infty} |1 - d'_\nu|^2 (1 - |d_\nu|)\right],\ 0 < r < 1, \tag{22.7}$$

where d'_ν is defined as in (22.3). This inequality is similar to that of lemma 17 with

$$\mu(r) = A\left[\sum_{\nu=1}^{\infty} \left|1 - \frac{d_\nu - r}{1 - r d_\nu}\right|^2 \left(1 - \left|\frac{d_\nu - r}{1 - r d_\nu}\right|\right) + 1\right] \tag{22.8}$$

and $C = A$. Hence that lemma yields for $0 < r < 1$

$$\lambda(r) < \frac{1}{1 - r}\left[(1 + r)\lambda(0) + A r[1 + \log^+ \lambda(0)] + A\int_0^r \mu(t)\, dt\right]. \tag{22.9}$$

Also it follows from lemma 18 that

$$\begin{aligned}\int_0^r \mu(t)\, dt &= A r + \sum_{\nu=1}^{\infty} \int_0^r \left|1 - \frac{d_\nu - t}{1 - t d_\nu}\right|^2 \left(1 - \left|\frac{d_\nu - t}{1 - t d_\nu}\right|\right) dt \\ &< A r + A r \sum_{\nu=1}^{\infty} (1 - |d_\nu|) \\ &= A r (1 + N_0).\end{aligned}$$

Combining this with (22.9), we have (22.2), which proves Theorem II.

Proof of Theorem III.

23) Theorem III can be deduced from Theorem II and lies less deep than Theorem II. Suppose that a_μ, $\mu = 1$ to m, $\mu = m + 1$ to M are the zeros and b_ν, $\nu = 1$ to n, $\nu = n + 1$ to N the poles of $f(z)$ lying in $|z| \leq \frac{1}{2}$ and $\frac{1}{2} < |z| < 1$ respectively under the hypotheses of Theorem III. We have

$$f_*(0) = f(0)\, 2^{n-m} \frac{\prod_{\nu=1}^{n} g(0, b_\nu)}{\prod_{\mu=1}^{m} g(0, a_\mu)}.$$

We write

$$g(z) = f(z) \frac{\prod_{\nu=1}^{N} g(z, b_\nu)}{\prod_{\mu=1}^{M} g(z, a_\mu)}. \tag{23.1}$$

Then we have

$$A^{-(M+N)} \leq \left|\frac{f_*(0)}{g(0)}\right| \leq A^{M+N} \tag{23.2}$$

and also

$$\left|\frac{f_*'(0)}{f_*(0)} - \frac{g'(0)}{g(0)}\right| \leq \sum_{\mu=m+1}^{M} \left|\frac{g'(0, a_\mu)}{g(0, a_\mu)}\right| + \sum_{\nu=n+1}^{N} \left|\frac{g'(0, b_\nu)}{g(0, b_\nu)}\right|. \tag{23.3}$$

Since also by hypothesis $M + N \leq L$ we deduce from (23.2) that we have

$$A^{-L} \leq \left|\frac{f(z)}{g(z)}\right| \leq A^{L} \tag{23.4}$$

in the first instance when $z = 0$, and hence for $|z| < 1$ by mapping $|z| < 1$ onto itself conformally. Also we have from (23.3)

$$\left|\frac{f_*'(0)}{f_*(0)} - \frac{g'(0)}{g(0)}\right| \leq A\,L. \tag{23.5}$$

It follows that it is sufficient to prove Theorem III (i), with $g(z)$ instead of $f_*(z)$. Suppose first that $|g(0)| \leq 1$. Then (23.4) gives

$$|f_*(0)| \leq A^{L}$$

and so Theorem II gives

$$\log^+ |f_*(r e^{i\theta})| \leq \frac{A(1 + L)}{1 - r}.$$

Combining this with (23.4) we deduce that

$$\log |g(re^{i\theta})| \leq \frac{C}{1-r}, \qquad 0 < r < 1, \quad 0 \leq \theta \leq 2\pi \tag{23.6}$$

where

$$C = A(1 + L). \tag{23.7}$$

It follows, by mapping $|z| < 1$ onto itself conformally so that $z = 0$, $z = re^{i\theta}$ are interchanged, that if we have

$$\log^+ |g(0)| \geq \frac{C}{1-r} \tag{23.8}$$

then

$$\log |g(re^{i\theta})| \geq 0, \qquad 0 \leq \theta \leq 2\pi.$$

It follows that if (23.8) holds we have

$$|g(z)| > 1, \qquad |z| < r,$$

and hence $g(z)$ is subordinate in $|z| < r$ to

$$\psi(z) = \exp\left\{\frac{r+z}{r-z}\log|g(0)| + i \arg g(0)\right\}.$$

We deduce that if (23.8) holds we have

$$|g'(0)| \leq |\psi'(0)| = \frac{2}{r}|g(0)| \log |g(0)|. \tag{23.9}$$

Suppose now that $\log |g(0)| \geq 2C$ and let r be defined by

$$(1-r)\log|g(0)| = C. \tag{23.10}$$

Thus $r \geq \frac{1}{2}$. Then we have (23.9) and so

$$\left|\frac{g'(0)}{g(0)}\right| \leq \frac{2}{r}\log|g(0)| = 2\log|g(0)| + \frac{2(1-r)}{r}\log|g(0)|$$
$$\leq 2\log|g(0)| + 4C$$

by (23.10). Combining this with (23.4), (23.5), (23.7), we see that Theorem III (i) holds when

$$\log |f_*(0)| > 2C + AL = A(1 + L),$$

and by writing $[f(z)]^{-1}$ instead of $f(z)$ we see that this inequality also holds when

$$\log |f_*(0)| < -A(1 + L).$$

Finally the result follows from Theorem I if

$$|\log|f_*(0)|| < A(1 + L).$$

Thus Theorem III (i) is always true.

The inequality of Theorem III (ii) now follows from (i). On writing $f\left(\frac{z+r}{1+rz}\right)$ instead of $f(z)$ in (i) we have

$$(1 - r^2)|f_*'(r)| < 2|f_*(r)|\{|\log|f_*(r)|| + A(1 + L)\}, \quad 0 < r < 1.$$

Integrating this we have

$$|f_*(r)| < \mu^{\frac{1+r}{1-r}} A^{\frac{r(1+L)}{1-r}}, \quad 0 < r < 1$$

where

$$\mu = \max[1, |f_*(0)|].$$

A similar result holds with $f_*(re^{i\theta})$ instead of $f_*(r)$. This proves (ii) and completes the Proof of Theorem III.

Proof of Theorem IV.

24) To prove Theorem IV, we need two further lemmas. In lemma 19 we show that the inequality (20.1) is satisfied for a certain class of functions $f(z)$ and in lemma 20, we show that the functions $f(z)$ of Theorem IV belong to this class. The remaining part of Theorem IV then follows.

Lemma 19. *Suppose that* $\psi(w) = \psi(u + iv)$ *is regular for* $u \geq 0$, *real for* $v = 0$, *and that*

(i) $$|\psi(u + iv)| \leq \psi(u) \qquad u \geq 0$$

(ii) $$\frac{\log^+ \psi(u)}{u} \quad \text{increases with } u, \; 0 < u < \infty.$$

Let

$$f(z) = \psi\left(\frac{1+z}{1-z}\right).$$

Then $f(z)$ *satisfies the inequality* (20.1).

Suppose that the hypotheses of lemma 19 hold. Then we have

$$(24.1) \qquad \frac{1}{2\pi}\int_0^{2\pi} \log^+|f(re^{i\theta})|\,d\theta \leq \frac{1-r}{1+r}\log^+ f(r), \qquad 0 < r < 1.$$

For write

$$\frac{1 + r e^{i\theta}}{1 - r e^{i\theta}} = u(r, \theta) + i\, v(r, \theta). \tag{24.2}$$

Then it follows from lemma (19), (i) and (ii) that

$$\begin{aligned}\log^+ |f(r e^{i\theta})| &= \log^+ |\psi[u(r, \theta) + i\, v(r, \theta)]| \\ &\leq \frac{u(r, \theta)}{u(r, 0)} \log^+ |\psi[u(r, 0)]| \\ &= \frac{u(r, \theta)}{u(r, 0)} \log^+ f(r)\end{aligned}$$

and hence

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(r e^{i\theta})|\, d\theta &\leq \frac{\log^+ |f(r)|}{u(r, 0)} \frac{1}{2\pi}\int_0^{2\pi} u(r, \theta)\, d\theta \\ &= \frac{u(0, 0)}{u(r, 0)} \log^+ |f(r)|\end{aligned}$$

since $u(r, \theta)$ is harmonic in $|z| < 1$. This proves (24.1), making use of (24.2).

We now apply Jensen's formula (3.2) to $f(z) - a$ and obtain

$$\frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(r e^{i\theta}) - a|\, d\theta \geq \log |f(0) - a| + \int_0^r n(t, a)\frac{dt}{t}.$$

Also, since $\log^+ |f - a| \leq \log |f| + \log^+ |a| + \log 2$ we deduce

$$\log |f(0) - a| + \int_0^r n(t, a)\frac{dt}{t} \leq \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(r e^{i\theta})|\, d\theta + \log^+ |a| + \log 2$$

for all finite a. Combining this with (24.1), we have lemma 19.

We prove next

Lemma 20. *Let*

$$\psi(w) = [\phi(w)]^{-1}$$

where $\phi(w)$ is the integral function of Theorem IV. Then $\psi(w)$ satisfies the hypotheses of lemma 19.

We have

$$\psi(w) = e^{-bw} \prod_{\nu=1}^{\infty} \left\{ e^{\frac{w}{r_\nu}} \Big/ \left(1 + \frac{w}{r_\nu}\right)\right\}. \tag{24.3}$$

In order to show that $\psi(w)$ satisfies the hypotheses of lemma 19, it is sufficient to show that these are satisfied by each of the factors

$$e^{-bw}, \quad e^{\frac{w}{r_\nu}}\Big/\left(1 + \frac{w}{r_\nu}\right).$$

Both (i) and (ii) are trivial for $\psi_0(w) = e^{bw}$ since when b is real

$$\left|e^{b(u+iv)}\right| = e^{bu}$$

and

$$\frac{\log^+ e^{bu}}{u} = \max(0, b) = \text{const.}$$

Consider next

$$\psi_1(w) = e^{\frac{w}{r}}\Big/\left(1 + \frac{w}{r}\right). \tag{24.4}$$

We have

$$\left|\frac{\psi_1(u+iv)}{\psi_1(u)}\right| = \frac{u+r}{\sqrt{(u+r)^2 + v^2}} \leq 1$$

which proves (i). Also

$$\frac{\log^+ \psi_1(u)}{u} = \frac{\log \psi_1(u)}{u} = \frac{1}{r}\left[1 - \frac{r}{u}\log\left(1 + \frac{u}{r}\right)\right]$$

is an increasing function of u/r for fixed r. For putting $\frac{u}{r} = x$, we have

$$\frac{d}{dx}\left[1 - \frac{1}{x}\log(1+x)\right] = \frac{-1}{x(1+x)} + \frac{1}{x^2}\int_1^{1+x}\frac{dt}{t} > \frac{-1}{x(1+x)} + \frac{1}{x(1+x)} = 0, \qquad x > 0.$$

Thus $\psi_1(w)$ defined by (24.4) satisfies (i) and (ii) of lemma 19 and hence so does $\psi(w)$ defined by (24.3). This proves lemma 20 and we deduce the truth of (20.1) in Theorem IV. Suppose next that

$$S = \sum_{\nu=1}^{\infty}\frac{1}{r_\nu} < \infty.$$

Then we may write

$$\phi(w) = e^{(b-s)w}\prod_{\nu=1}^{\infty}\left(1 + \frac{w}{r_\nu}\right)$$

so that $\phi(w)$ has genus zero. Also if $w = u + iv$,

$$\left|1 + \frac{w}{r}\right| > 1, \qquad u \geq 0,$$

so that

$$|\phi(w)| \geq e^{-(b-s)w} \qquad u \geq 0$$

and hence

$$|\psi(u+iv)| = |[\phi(u+iv)]^{-1}| < e^{(s-b)u}.$$

Since

$$\frac{\log^+ \psi(u)}{u}$$

increases with u, we deduce that

$$U = \lim_{u\to\infty} \frac{\log^+ \psi(u)}{u}$$

exists finitely and hence so does

$$\lim_{r\to 1} (1-r)\log f(r) = \lim_{r\to 1} (1-r)\,\psi\left(\frac{1+r}{1-r}\right) = 2\,U.$$

Also it follows from (20.1) that $\int_0^1 n(t, a)\,dt$ converges for every a.

Suppose next that $\sum \frac{1}{r_\nu} = \infty$. Then $\phi(w)$ has genus 1. Also

$$1 + \frac{u}{r} < e^{\frac{u}{r}}$$

so that

$$\psi(u) = e^{-bu} \prod_{\nu=1}^{\infty} e^{\frac{u}{r_\nu}} \Big/ \left(1 + \frac{u}{r_\nu}\right) \geq e^{-bu} \prod_{\nu=1}^{N} e^{\frac{u}{r_\nu}} \Big/ \left(1 + \frac{u}{r_\nu}\right)$$

for every finite N. Hence

$$\varliminf_{u\to\infty} \frac{\log \psi(u)}{u} \geq -b + \sum_{\nu=1}^{N} \frac{1}{r_\nu} - \lim_{u\to\infty} \sum_{\nu=1}^{N} \frac{\log\left(1+\frac{u}{r_\nu}\right)}{u}$$

$$= \sum_{\nu=1}^{N} \frac{1}{r_\nu} - b.$$

Since $\sum \frac{1}{r_\nu}$ diverges we deduce that

$$\varliminf_{u\to\infty} \frac{\log \psi(u)}{u} = \lim_{r\to 1} \frac{1-r}{1+r} \log f(r) = \infty. \tag{24.5}$$

It follows that $\int_0^1 n(t, a)\,dt$ diverges for every finite $a \neq 0$. For suppose contrary to this that

$$N_0 = \int_0^1 n(t, a)\,dt < \infty, \qquad a \neq 0.$$

In this case we write

$$f_1(z) = \frac{f(z)}{a}.$$

Then $f_1(z) \neq 0, \infty$ in $|z| < 1$ and the number of roots of $f_1(z) = 1$ in $|z| \leq t$ is $n(t, a)$. Hence Theorem II yields

$$\log M[r, f_1(z)] = \log M[r, f_{1*}(z)] < \frac{A}{1-r}[N_0 + \log^+ |f_1(0)| + 1],$$

$$0 < r < 1,$$

and so also

$$\overline{\lim_{r \to 1}}\,(1-r) \log |f(r)| < \infty.$$

which contradicts (24.5). This completes the proof of Theorem IV.

Proof of Theorem V.

25) Theorem V is an almost immediate consequence of Theorem II. We write

$$g(z) = \frac{f(z)}{\phi(z)} \tag{25.1}$$

and see that if $g(z) = 0$, 1 or ∞, we have either $f(z) = 0$ or ∞ or $\phi(z) = 0$ or ∞ or $f(z) = \phi(z)$. Thus we have

$$N_2 = \int_0^1 \left\{n(t, g) + n\left(t, \frac{1}{g}\right) + n\left(t, \frac{1}{g-1}\right)\right\} dt \leq N_1$$

where N_1 is defined as in Theorem V. It follows that we have

$$\log M[r, g_*(z)] < \frac{1}{1-r}\{(1+r)\log^+ |g_*(0)| + Ar[\log^+ \log^+ |g_*(0)| + N_1 + 1]\} \tag{25.2}$$

on applying Theorem II to $g(z)$ instead of $f(z)$.

It follows from (25.1) and (19.4) that

$$g_*(z) = \frac{f_*(z)}{\phi_*(z)}$$

and hence we have

$$\log M[r, f_*(z)] \leq \log M[r, \phi_*(z)] + \log M[r, g_*(z)].$$

Combining this with (25.2) we have Theorem V.

Proof of Theorem VI.

26) It follows from (19.3) and (19.4) that if $f(z)$ has poles b_ν and zeros a_μ in $|z| < 1$, then we have

$$\log\left|\frac{f(z)}{f_*(z)}\right| = \sum \log^+\left|\frac{1 - z\bar{b}_\nu}{2(z - b_\nu)}\right| - \sum \log^+\left|\frac{1 - z\bar{a}_\mu}{2(z - a_\mu)}\right|$$

and from this (20.2) follows at once. We deduce further that

$$\log\left|\frac{f(z)}{f_*(z)}\right| \le \sum \log^+\left|\frac{1 - |z|\,|b_\nu|}{2(|z| - |b_\nu|)}\right|$$

and hence

$$\log^+ M\left[t, \frac{f(z)}{f_*(z)}\right] \le \sum \log^+\left|\frac{1 - t|b_\nu|}{2(t - |b_\nu|)}\right|. \tag{26.1}$$

Thus if $0 < \varrho < \varrho' < 1$, we have

$$\int_\varrho^{\varrho'} \log^+ M\left[t, \frac{f}{f_*}\right] dt \le \sum \int_\varrho^{\varrho'} \log^+\left|\frac{1 - t|b_\nu|}{2(t - |b_\nu|)}\right| dt. \tag{26.2}$$

Now

$$\int_\varrho^{\varrho'} \log^+\left|\frac{1 - t|b_\nu|}{2(t - |b_\nu|)}\right| dt \le \int_{x=-1/2}^{1/2} \log^+ \frac{1}{2x}\, dt$$

where

$$x = \frac{t - |b_\nu|}{1 - t|b_\nu|}, \qquad t = \frac{x + |b_\nu|}{1 + x|b_\nu|},$$

so that

$$dt = \frac{1 - |b_\nu|^2}{(1 + x|b_\nu|)^2}\, dx \le 8(1 - |b_\nu|)\, dx.$$

Thus we have always

$$\int_\varrho^{\varrho'} \log^+\left|\frac{1 - t|b_\nu|}{2(t - |b_\nu|)}\right| dt \le A(1 - |b_\nu|) \int_{-1/2}^{1/2} \log\left|\frac{1}{2x}\right| dx \tag{26.3}$$

$$= A(1 - |b_\nu|).$$

Also the integral on the left hand side of (26.3) vanishes if

$$\frac{\varrho - |b_\nu|}{1 - \varrho|b_\nu|} \ge \frac{1}{2}$$

i.e. if

$$\varrho \ge \frac{1 + 2|b_\nu|}{2 + |b_\nu|}$$

which is true in particular unless

$$1-\varrho \geq \tfrac{1}{3}(1-|b_\nu|).$$

Hence the integral on the left hand side of (26.3) either vanishes or satisfies

$$\int_{\varrho}^{\varrho'} \log^+ \left|\frac{1-t|b_\nu|}{2(t-|b_\nu|)}\right| dt < A(1-\varrho) \tag{26.4}$$

so that (26.4) holds generally. Again the integral in (26.4) vanishes if

$$\frac{|b_\nu|-\varrho'}{1-\varrho'|b_\nu|} \geq \frac{1}{2}$$

i.e. if

$$|b_\nu| \geq \frac{1+2\varrho'}{2+\varrho'}$$

which is true if

$$1-|b_\nu| < \tfrac{1}{3}(1-\varrho').$$

Chose now ϱ' so that

$$(1-\varrho') = \tfrac{3}{4}(1-\varrho). \tag{26.5}$$

Then it follows that the integral in (26.4) vanishes if

$$1-|b_\nu| < \tfrac{1}{4}(1-\varrho).$$

Thus the total number of b_ν for which the integral in (26.4) does not vanish cannot exceed the number of b_ν in $|z| \leq 1-\frac{1}{4}(1-\varrho)$ i.e $n\left(\frac{3+\varrho}{4}\right)$. Using this fact and (26.4) we have

$$\sum \int_{\varrho}^{\varrho'} \log^+ \left|\frac{1-t|b_\nu|}{2(t-|b_\nu|)}\right| dt < A(1-\varrho)\, n\left(\frac{3+\varrho}{4}\right).$$

It follows that there exists r, such that $\varrho < r < \varrho'$ and

$$\sum \log^+ \left|\frac{1-r|b_\nu|}{2(r-|b_\nu|)}\right| < A\frac{(1-\varrho)}{\varrho'-\varrho}\, n\left(\frac{3+\varrho}{4}\right) < A\, n\left(\frac{3+\varrho}{4}\right), \tag{26.6}$$

using (26.5). Also it follows from (26.5) that $\varrho' < \frac{1}{2}(1+\varrho)$ so that $\varrho < r < \frac{1}{2}(1+\varrho)$. Combining (26.1), (26.6), we have (20.3).

It remains to prove the corollary to Theorem VI. We may suppose that the quantity r in (20.3) satisfies $r \geq \frac{1}{4}$. This is trivial if $\varrho \geq \frac{1}{4}$. If $0 \leq \varrho < \frac{1}{4}$ (26.5) shows that r exists such that (20.3) holds and also $\frac{1}{4} \leq r \leq \varrho'$, where

$$1 - \varrho' = \frac{3}{4}\left(1 - \frac{1}{4}\right) = \frac{9}{16}$$

so that $\varrho' < \frac{1}{2}$. Thus we have in this case

$$\varrho < \tfrac{1}{4} \le r < \tfrac{1}{2} \le \tfrac{1}{2}(1 + \varrho).$$

Let $n(t)$ be the number of poles of $f(z)$ in $|z| \le t$. Then $n(t)$ is an increasing function of t. Hence we have

$$\int_{\frac{3+\varrho}{4}}^{1} n(t)\,dt \ge \left[1 - \frac{3+\varrho}{4}\right] n\left(\frac{3+\varrho}{4}\right) = \frac{1-\varrho}{4} \cdot n\left(\frac{3+\varrho}{4}\right)$$

and so

$$n\left(\frac{3+\varrho}{4}\right) \le \frac{A}{1-r} \int_{0}^{1} n(t)\,dt. \tag{26.7}$$

Thus we have

$$n\left(\frac{3+\varrho}{4}\right) \le \frac{A}{1-r} N_0$$

where N_0 is defined as in (19.2). Combining this with (20.3) and Theorem II, we deduce that if r satisfies (20.3) we have

$$\log M[r, f(z)] \le \frac{1}{1-r}\{(1+r)\log^+|f_*(0)| + A r [\log^+\log^+|f_*(0)| + N_0 + 1]\} + \frac{A N_0}{1-r}$$

$$\le \frac{1}{1-r}\{(1+r)\log^+|f_*(0)| + A r [\log^+\log^+|f_*(0)| + N_0 + 1]\}$$

since by hypothesis $r \ge \frac{1}{4} = A$. This proves the part of the corollary which involves Theorem II. Similarly the part involving Theorem III (ii) follows on noting that with the hypotheses of that Theorem we have

$$n\left(\frac{3+\varrho}{4}\right) \le L \le \frac{L}{1-r}.$$

Finally in the case of Theorem V we have in the notation of that Theorem

$$n\left(\frac{3+\varrho}{4}\right) \le \frac{A}{1-r} N_1.$$

This completes the proof of the corollary.

CHAPTER II.

The Main Problem.

Statement of Problem.

1) In the last chapter we obtained bounds for the maximum modulus of a function which has a restricted number of zeros, poles and ones. In this chapter we consider the more general problem of a meromorphic function $f(z)$ which takes none of an assigned set E of complex values w more than $p(\varrho)$ times in $|z| < \varrho < 1$. Clearly $p(\varrho)$ may be taken to be increasing. This is, however, the only assumption we make on $p(\varrho)$.

It is easy to see that we may without loss in generality suppose E to be closed. For let w_0 be a limit point of E and let z_0 be a point in $|z| < 1$, such that

$$f(z_0) = w_0. \tag{1.1}$$

Then the values taken by $f(z)$ in any neighborhood of z_0 contain w_0 as an interior point. It follows that if the equation (1.1) has exactly $p(\varrho)$ roots in $|z| < \varrho$, then $f(z)$ takes all values of E sufficiently near to w_0 at least $p(\varrho)$ times in $|z| < \varrho$. Hence if the equation $f(z) = w$ has at most $p(\varrho)$ roots in $|z| < \varrho$, whenever w lies in E the same is true of the equation (1.1) and so we may suppose E closed.

If E is unbounded it follows from this that we may assume that E contains $w = \infty$. We shall make this assumption also when E is bounded. In addition we shall have to suppose that E contains at least two finite numbers, one of which we may without loss in generality take to be $w = 0$.

Thus we assume altogether that $p(\varrho)$ is nondecreasing and that E contains 0 and ∞ and is closed.

We shall continue to use throughout this chapter the function $f_*(z)$ defined in paragraph 19 of the previous chapter. We shall obtain bounds for the maximum modulus $M[\varrho, f_*(z)]$ of $f_*(z)$. Bounds for $M[\varrho, f]$ can be deduced by means of Theorem VI of chapter I. The use of $f_*(z)$ has two advantages. In the first instance it allows us to study meromorphic functions as easily as regular functions. Secondly we shall be able to obtain our bounds in a very simple form depending only on $p(\varrho)$, ϱ, $|f_*(0)|$ and E.

Part I.

The Case when $p(\varrho)$ is Constant.

2) We have already studied the case when $p(\varrho) \equiv 0$, $0 < \varrho < 1$ in some detail in previous work (Hayman (1), (2), (3)). In addition the case when $p(\varrho) \equiv p$, a positive integer has received some attention. Ostrowsky[1], Milloux[2] and others have studied functions with only a finite number of zeros, poles and ones. Further there are the results of Littlewood[3] and Cartwright[4] about functions regular in the unit circle and taking none of a sequence of values w_n such that

$$|w_n| \to \infty$$

more than p times. Littlewood showed that if w_{n+1}/w_n is bounded then

$$\log M[\varrho, f] = O\left\{\log \frac{1}{1-\varrho}\right\}.$$

Miss Cartwright[4] showed that if in addition

$$\left|\frac{w_{n+1}}{w_n}\right| \to 1$$

then we have

$$M[\varrho, f] = O(1-\varrho)^{-2(p+1)-\varepsilon}$$

for every $\varepsilon > 0$. Here the index $2(p+1)$ is best possible as is shown by the functions

$$f(z) = \left(\frac{1+z}{1-z}\right)^{2(p+1)}$$

which take no real negative value more than p times in $|z| < 1$.

Cartwright[4] proved also the following result

Theorem I. *Suppose that* $f(z) = a_0 + a_1 z + \cdots$ *is regular in* $|z| < 1$ *and takes no value more than* p *times, where* p *is a positive integer. Then we have*

$$M[\varrho, f] < A(p)\mu(1-\varrho)^{-2p}$$

where

$$\mu = \max[1, |a_0|, |a_1|, \ldots |a_p|].$$

This result was generalized by Spencer[5] to functions which take values on the average p times in $|z| < 1$ and now p need not be an integer. We cannot even state Spencer's many beautiful results without going into his rather intricate definition of mean valency, which lies outside our scope. We must refer the reader to Spencer's papers.

[1] Ostrowsky (1). [2] Milloux (1). [3] Littlewood (2), p. 228.
[4] Cartwright (1). [5] Spencer (1), and references there given.

3) We do not aim here to prove explicitly all the above results. We shall, however, prove a general Theorem by which the study of functions taking values p times can be reduced to the case when $p = 0$, which has already been studied. This result is Theorem II. Having proved this we shall give an application in Theorem III, which will include Cartwright's Theorem I as a special case. The second half of the chapter will be occupied with the case when $p(\varrho)$ is unbounded. We reserve all except the simplest counterexamples to the next chapter.

The functions

$$f(z) = \left(\frac{1+z}{1-z}\right)^{2(p+1)}$$

take no value more than $p+1$ times and no real negative value more than p times in $|z| < 1$, and their rate of growth is extremal under these conditions. By taking the $(p+1)$th root we obtain functions which take no real negative value in $|z| < 1$. This simple process lies at the basis of our result. The most serious difficulty lies in the zeros and poles, which would yield singularities when taking the $(p+1)$th root. This difficulty is not insuperable, however, as we shall see.

Theorem II. *Suppose that $f(z)$ is meromorphic in $|z| < 1$ and has at most q zeros and q poles. Suppose that p is a positive integer or zero and that $f(z)$ satisfies one or more conditions of one of the following two types.*

(i) *$f(z)$ takes no value on the circle $|w| = r$ more than $p+1$ times in $|z| < 1$;*

(ii) *$f(z)$ takes some value on the circle $|w| = r$ at most p times in $|z| < 1$;*

where the numbers r are real and positive.

Then given ϱ, $0 < \varrho < 1$, there exists ϱ' such that

$$1 - \varrho \leq A^q (1 - \varrho') \tag{3.1}$$

and a function $\phi(z)$ regular nonzero in $|z| < 1$ and such that

$$|\phi(0)| \leq \{A^q |f_*(0)|\}^{1/(p+1)} \tag{3.2}$$

$$\log M[\varrho', \phi(z)] \geq \frac{1}{p+1}\{\log M[\varrho, f_*(z)] - qA\}. \tag{3.3}$$

Further if r is a number for which (i) *or* (ii) *holds then $\phi(z)$ never takes some value w^1, such that $|w^1|^{p+1} = r$, in $|z| < 1$.*

Thus we can use the known bounds for $M[\varrho, \phi]$ to obtain bounds for $M[\varrho, f_*]$, and as we shall see in paragraph 10, the bounds obtainable in this way are fairly sharp.

4) The proof of Theorem II consists of two parts. The first, which is almost trivial, is to prove the result when $q = 0$. The second rather more intricate part consists in eliminating the zeros and poles of $f(z)$. This depends on conformal mapping and hyperbolic distances[1], which were introduced earlier. We have first

Lemma 1. *Theorem II holds when* $q = 0$, *with* $\varrho' = \varrho$, $\phi(z) = [f(z)]^{1/(p+1)}$.

Suppose $q = 0$ and let $f(z)$ be the function of Theorem II. We write

$$\phi(z) = [f(z)]^{1/(p+1)} \tag{4.1}$$

where the principal branch of $\phi(z)$ is taken at the origin. We write $\varrho' = \varrho$. Then (3.1) to (3.3) are clearly satisfied.

Suppose next that $f(z)$ satisfies a condition of type (ii), so that the equation

$$f(z) = w \tag{4.2}$$

has at most p roots in $|z| < 1$ for some w such that $|w| = r$. Then if

$$(w^1)^{p+1} = w \tag{4.3}$$

we can only have

$$\phi(z) = w^1 \tag{4.4}$$

when (4.2) holds. Since there are $p + 1$ different values w^1 satisfying (4.3) it follows that for at least one of them the equation (4.4) can have no solution in $|z| < 1$. This proves the Theorem except for the case when $f(z)$ satisfies one or more hypotheses of the type (i).

To complete the proof of lemma 1, we shall show that if $q = 0$ and $f(z)$ satisfies a hypothesis (i) then $f(z)$ also satisfies the corresponding hypothesis (ii). Let

$$g(z) = \log f(z)$$

and let $\eta_1 < \eta < \eta_2$ be the largest interval such that the equation

$$g(z) = \log r + i\,\eta, \qquad \eta_1 < \eta < \eta_2$$

has roots in $|z| < 1$. If the interval does not exist, $f(z) \neq r$ and our result is proved. The interval cannot be infinite since otherwise the equation

$$f(z) = r$$

would have infinitely many roots in $|z| < 1$. Then the equation

$$f(z) = r\,e^{i\eta_1}$$

has at most p roots in $|z| < 1$.

[1] Hayman (1).

For if not let $z_1, z_2, \ldots, z_{p+1}$ be roots of this equation (possibly coincident in the case of multiple roots) so that

$$(4.5) \qquad f(z_j) = r e^{i\eta_1}, \qquad j = 1 \text{ to } p+1.$$

Then we have

$$g(z_j) = \log r + i(\eta_1 + 2 m_j \pi)$$

where $m_j \neq 0$, since $g(z) \neq \log r + i\eta_1$ by hypothesis. Hence if ε is sufficiently small we can find z_j' near z_j, such that

$$g(z_j') = \log r + i\eta_1 + 2(m_j + \varepsilon)\pi i$$

and by hypothesis we can also find z_0 such that

$$g(z_0) = \log r + i\eta_1 + 2\pi i \varepsilon.$$

Hence the equation

$$f(z) = r e^{i\eta_1 + 2\pi i \varepsilon}$$

has $p+2$ distinct roots in $|z| < 1$ if ε is small enough, contrary to hypothesis. Thus the assumption that (4.5) holds is incorrect, so that the equation

$$f(z) = r e^{i\eta_1}$$

has at most p roots in $|z| < 1$. Thus when $q = 0$ the hypothesis (i) of Theorem II yields (ii). This completes the proof of lemma 1.

We note incidentally that the argument breaks down when $q > p$. In fact the function z^{p+1} takes every value in $|w| < 1$ exactly $p+1$ times in $|z| < 1$.

5) We now approach the task of eliminating the zeros and poles. To do this we proceed to construct a simply-connected domain lying in $|z| < 1$, and containing neither poles nor zeros of $f(z)$. We then consider the function $f[\lambda(z)]$, where $\lambda(z)$ maps $|z| < 1$ onto this domain and have a function satisfying the hypotheses of Theorem II, with $q = 0$.

We first deal with the zeros and poles in a manner analogous to that used in paragraps 11 and 12 of chapter I. We have

Lemma 2. *Suppose that $f(z)$ is meromorphic in $|z| < 1$ and has at most q zeros and q poles in $|z| < 1$. Then if $|z_0| < 1$, there exists z' such that*

$$\text{(i)} \qquad \left| \frac{z' - z_0}{1 - \bar{z}_0 z'} \right| \leq \frac{1}{4}.$$

(ii) *If* z_ν, $\nu \geq 1$ *is a pole or zero of* $f(z)$ *we have*

$$|g(z_\nu, z')| = \left|\frac{z_\nu - z'}{1 - \bar{z}_\nu z'}\right| \geq A^{-q}.$$

(iii) *We have*

$$A^{-q}|f_*(z_0)| \leq |f(z')| \leq A^q |f_*(z_0)|.$$

To prove lemma 2 we may suppose without loss in generality that $z_0 = 0$. For, if not, we can consider $f\left(\frac{z - z_0}{1 - \bar{z}_0 z}\right)$ instead of $f(z)$. Let a_μ, $\mu = 1$ to m be the zeros and b_ν, $\nu = 1$ to n the poles of $f(z)$ in $|z| \leq \frac{1}{2}$, and let

$$g(z) = f(z)\, 2^{n-m} \frac{\prod_{\nu=1}^{n} g(z, b_\nu)}{\prod_{\mu=1}^{m} g(z, a_\mu)}. \tag{5.1}$$

We have $m \leq q$, $n \leq q$ by hypothesis and so it follows from lemma 11 of chapter I, that we can find ϱ, $\frac{1}{8} \leq \varrho \leq \frac{1}{4}$ such that

$$\left|\prod_{\mu=1}^{m} g(z, a_\mu) \prod_{\nu=1}^{n} g(z, b_\nu)\right| > A^q, \quad |z| = \varrho. \tag{5.2}$$

Since $g(z)$ is regular nonzero in $|z| < \frac{1}{2}$, its maximum modulus increases and its minimum modulus decreases. Hence we can find z' such that

$$|z'| = \varrho \tag{5.3}$$

and

$$g(z') = |g(0)| = |f_*(0)|. \tag{5.4}$$

It follows from (5.3) and $\varrho \leq \frac{1}{4}$ and $z_0 = 0$, that z' satisfies (i). It follows from (5.1), (5.2) and (5.3) that

$$A^q |g(z')| < |f(z')| < A^q |g(z')|,$$

which combined with (5.4) yields (iii). Also (ii) follows from (5.2) when $|z_\nu| \leq \frac{1}{2}$, and (ii) is trivial for $|z_\nu| \geq \frac{1}{2}$ since then

$$|g(z_\nu, z)| \geq \frac{\frac{1}{2} - \frac{1}{4}}{1 - \frac{1}{2}} = \frac{1}{2}.$$

This completes the proof of lemma 2.

6) The crux in the proof of Theorem II is lemma 3, which is best expressed in terms of hyperbolic distances. We recall the definitions given in Hayman (1), which we denote by S. T., particularly (3.1) and (3.3). With this notation we have

Lemma 3. *Let D be a simply connected domain in the z plane and let z', z'', $z_1, z_2, \ldots z_q$ be $q + 2$ points of D such that*

$$d[z', z_i; D] \geq \delta, \quad i = 1 \text{ to } q, \qquad d[z'', z_i; D] \geq \delta, \quad i = 1 \text{ to } q, \tag{6.1}$$

where $\delta > 0$. Then there exists a simply connected domain D', containing z', z'' but none of the z_i, and contained in D such that

$$d[z', z''; D'] \leq d[z', z''; D] + A\left(q + \log^+ \frac{1}{\delta}\right). \tag{6.2}$$

To prove lemma 3 we remind the reader of the following properties of hyperbolic distances. They obey the triangle relation (S. T. lemma 5). They are left invariant by 1 : 1 conformal mappings (S. T. lemma 6). They increase with a contracting domain. In other words if $D_1 < D_2$, we have

$$d[z_1, z_2; D_1] \geq d[z_1, z_2; D_2].$$

This is implicit in S. T. lemma 6, since we may take the function $w = f(z) = z$ in that lemma, which maps D_1 into D_2. Also if D is the circle $|z| < 1$, then we have

$$d[0, z'; D] = \tfrac{1}{2} \log \frac{1 + |z'|}{1 - |z'|}.$$

This follows from S. T. (3.4). It also follows more generally that if D is the circle $|z - z_0| < R$, we have

$$d[z_0, z'; D] = \tfrac{1}{2} \log \frac{R + |z' - z_0|}{R - |z' - z_0|}.$$

It follows from the invariance of hyperbolic distances that we may suppose without loss in generality, that the domain D of lemma 3 is the circle $|z| < 1$. We suppose first that $q = 1$. By a conformal mapping of $|z| < 1$ onto itself, we then map z_1 onto the point $+ iy_1$ on the positive imaginary axis and z', z'' onto the real axis. This is clearly always possible. We now take for D' the domain obtained from D by cutting along the positive imaginary axis from iy_1 to i.

Hyperbolic distances in the new domain could be worked out explicitly, but it is easier to approximate, making use of the methods of S. T. Suppose first $0 < z' < \frac{1}{2}$. Then z' is contained in the circle C, namely $|z - \frac{1}{2}| < \frac{1}{2}$, itself contained in D'. Thus

$$d[z', \tfrac{1}{2}; D'] < d[z', \tfrac{1}{2}, C] = \tfrac{1}{2} \log \left(\frac{1 - z'}{z'}\right). \tag{6.3}$$

Now we have by hypothesis

$$d[z', iy; D] \geq \delta.$$

Hence we have

$$d[z', 0; D] + d[0, iy_1; D] \geq \delta$$

so that either

$$d[z', 0; D] = \tfrac{1}{2} \log \left(\frac{1 + z'}{1 - z'}\right) > \frac{\delta}{2} \tag{6.4}$$

or

$$d[0, iy_1; D] = \tfrac{1}{2} \log \frac{1 + y_1}{1 - y_1} > \frac{\delta}{2}. \tag{6.5}$$

If (6.4) holds we have at once from (6.4) and (6.3)

$$d[z', \tfrac{1}{2}; D'] < A\left[1 + \log^+ \frac{1}{\delta}\right]. \tag{6.6}$$

If (6.5) holds and $|z'| \geq \frac{1}{2} y_1$, we see again from this, (6.5) and (6.3) that (6.6) holds. If $0 < z' < \frac{1}{2} y_1$ we note that z' is contained in the circle C', $|z| < y_1$, which is contained in D', so that

$$\begin{aligned} d[z', \tfrac{1}{2} y_1; D'] &< d[z', \tfrac{1}{2} y_1; C'] \\ &< d[z', 0; C'] + d[0, \tfrac{1}{2} y_1; C'] \\ &= \tfrac{1}{2} \log \frac{y_1 + z'}{y_1 - z'} + \tfrac{1}{2} \log \frac{y_1 + \frac{1}{2} y_1}{y_1 - \frac{1}{2} y_1} < \log 3, \end{aligned}$$

since $z' < \frac{1}{2} y_1$. Thus in this case

$$\begin{aligned} d[z', \tfrac{1}{2}; D'] &< d[z', \tfrac{1}{2} y_1; D'] + d[\tfrac{1}{2} y_1, \tfrac{1}{2}; D'] \\ &< A + \tfrac{1}{2} \log^+ \frac{1}{y_1} \end{aligned}$$

from (6.3), and since (6.5) holds we deduce again (6.6). Thus (6.6) holds for $0 \leq z' \leq \frac{1}{2}$.

Suppose now that $\frac{1}{2} \leq z' \leq z'' < 1$. Then z', z'' are both contained in the circle C, $|z - z'| < 1 - z'$, which is contained in D', so that

$$d[z', z''; D'] < d[z', z''; C] = \tfrac{1}{2}\log\left(\frac{1 - z' + z'' - z'}{1 - z' - (z'' - z')}\right)$$

$$< \tfrac{1}{2}\log\frac{1 - z'}{1 - z''} + \tfrac{1}{2}\log 2,$$

$$(6.7) \qquad d[z', z''; D'] < \tfrac{1}{2}\log\frac{1 - z'}{1 - z''} + A\left(1 + \log^{+}\frac{1}{\delta}\right).$$

Again if $0 < z' < \frac{1}{2} \leq z'' < 1$ we have

$$d[z', z''; D'] < d[z', \tfrac{1}{2}; D'] + d[\tfrac{1}{2}, z''; D']$$

$$< A\left[1 + \log^{+}\frac{1}{\delta}\right] + \tfrac{1}{2}\log\frac{1}{1 - z''} + A\left(1 + \log^{+}\frac{1}{\delta}\right),$$

on using (6.6) and (6.7) with $\frac{1}{2}$ instead of z'. Thus (6.7) holds also if $0 < z' \leq \leq \frac{1}{2} < z''$. Suppose next that $0 < z' < z'' < \frac{1}{2}$. Then we have

$$d[z', z''; D'] < d[z', \tfrac{1}{2}; D'] + d[z'', \tfrac{1}{2}; D']$$

$$< A\left[1 + \log^{+}\frac{1}{\delta}\right],$$

using (6.6) in turn with z', z'' instead of z'. Thus (6.7) holds whenever $0 \leq z' < < z'' < 1$. Also in this case

$$d[z', z''; D] = d[z', z''; |z| < 1] = \tfrac{1}{2}\log\left\{\frac{1 + z''}{1 - z''}\Big/\frac{1 + z'}{1 - z'}\right\} > \tfrac{1}{2}\log\frac{1 - z'}{1 - z''}.$$

Thus (6.7) yields if $0 \leq z' < z'' < 1$,

$$(6.8) \qquad d[z', z''; D'] < d[z', z''; D] + A\left(1 + \log^{+}\frac{1}{\delta}\right).$$

Clearly (6.8) also holds if $-1 < z' < z'' \leq 0$. Suppose lastly $-1 < z' < 0 < z'' < 1$. We note that D' is always contained in the domain D'' given by

$$\frac{-3\pi}{2} < \arg z < \frac{\pi}{2}, \quad |z| < 1,$$

which contains $z = -\frac{1}{2}, \frac{1}{2}$. Thus

$$d[-\tfrac{1}{2}, \tfrac{1}{2}; D'] < d[-\tfrac{1}{2}, \tfrac{1}{2}; D''] = A.$$

Hence if $-1 < z' < 0 < z'' < 1$ we have, using (6.8)

$$d[z', z''; D'] < d[z', -\tfrac{1}{2}; D'] + d[-\tfrac{1}{2}, \tfrac{1}{2}; D'] + d[\tfrac{1}{2}, z''; D'],$$

$$(6.9) \qquad d[z', z''; D'] < d[z', -\tfrac{1}{2}, D] + d[\tfrac{1}{2}, z''; D] + A\left(1 + \log^{+}\frac{1}{\delta}\right).$$

Again since $z' < 0 < z''$, we have

$$\begin{aligned} d[z', z''; D] &= d[z', 0; D] + d[0, z''; D] \\ &> d[z', -\tfrac{1}{2}; D] - d[0, -\tfrac{1}{2}; D] + d[\tfrac{1}{2}, z''; D] - d[0, \tfrac{1}{2}; D] \\ &= d[z', -\tfrac{1}{2}; D] + d[\tfrac{1}{2}, z''; D] - A. \end{aligned}$$

Combining this with (6.9), we see that (6.8) holds in this case also. This completes the proof of lemma 3, when $q = 1$.

7) It remains to prove lemma 3 when $q > 1$. Let D be the domain of lemma 3. Suppose that the z_i are so numbered that

$$d[z_i, z'; D] \geq d[z_1, z'; D], \quad i \geq 1, \tag{7.1}$$

$$d[z_i, z''; D] \geq d[z_2, z''; D] \quad i \geq 2. \tag{7.2}$$

Let D_1 be a simply connected domain, containing z', z'' but not z_1, contained in D, and such that

$$d[z', z''; D_1] < d[z', z''; D] + A\left(1 + \log^+ \frac{1}{\delta}\right).$$

Since we have proved lemma 3, when $q = 1$, D_1 exists. Since D_1 is contained in D we have further

$$\begin{aligned} d[z', z_i; D_1] &\geq d[z', z_i; D] \geq \delta, \quad i = 2 \text{ to } q \\ d[z'', z_i; D_1] &\geq d[z'', z_i; D] \geq \delta, \quad i = 2 \text{ to } q. \end{aligned}$$

Thus we can similarly construct a simply connected domain D_2 contained in D_1, not containing z_2, and such that

$$d[z', z''; D_2] < d[z', z''; D_1] + A\left(1 + \log^+ \frac{1}{\delta}\right)$$

$$d[z', z''; D_2] < d[z', z''; D] + A\left(1 + \log^+ \frac{1}{\delta}\right). \tag{7.3}$$

We may suppose without loss in generality that D is $|z| < 1$ and that $z' = 0$. Let

$$z = \psi(w) = a_1 w + a_2 w^2 + \cdots$$

map $|w| < 1$ onto D_2 so that $\psi(0) = z' = 0$. Since D_2 does not contain z_1 it follows from the theory of schlicht functions[1], that

[1] LITTLEWOOD (2) p. 207, Theorems 242, 243.

$$|\psi(w)| \leq \frac{|a_1||w|}{(1-|w|)^2} \leq \frac{4|z_1||w|}{(1-|w|)^2}.$$

Also it follows from (7.1) that $|z_i| \geq |z_1|$, $i \geq 2$, so that if $\psi(w_i) = z_i$, $i > 2$, we have

$$\frac{4|z_1||w_i|}{(1-|w_i|)^2} \geq |z_1|,$$

$$(1-|w_i|)^2 \leq 4|w_i|,$$

$$|w_i| \geq A > 0.$$

It follows that we have

$$d[z', z_i; D_2] = d[0, w_i; |w| < 1] > A, \quad i > 2.$$

Similarly we have

$$d[z'', z_i; D_2] > A, \ i > 2,$$

since D_2 does not contain z_2 and (7.2) holds. We can thus repeat our construction $q-2$ times more with A instead of δ, and finally obtain a domain $D' = D_q$, which contains none of the points $z_1, z_2, \ldots z_q$. Also if D_3, D_4, etc. are domains containing none of the z_i, for $i \leq 3$, $i \leq 4$, etc., we deduce that

$$\begin{aligned} d[z', z''; D_q] &\leq d[z', z''; D_{q-1}] + A \leq \cdots \\ &\leq d[z', z''; D_2] + A(q-2) \\ &\leq d[z', z''; D] + A\left[q + \log^+ \frac{1}{\delta}\right], \end{aligned}$$

making use of (7.3). This completes the proof of lemma 3.

8) We can now combine lemmas 2 and 3 to give

Lemma 4. *Let z_0, $|z_0| < 1$, be given and suppose that $f(z)$ is meromorphic in $|z| < 1$ and has at most q zeros and q poles. Then we can find a function $\lambda(z)$, which is schlicht in $|z| < 1$ and maps $|z| < 1$ into itself, such that $f[\lambda(z)]$ is regular non zero in $|z| < 1$ and*

(i) $$|f[\lambda(0)]| < A^q |f_*(0)|.$$

(ii) *There exists a points $z_{(1)}$, such that*

$$|f[\lambda(z_{(1)})]| > A^{-q} |f_*(z_0)|$$

where

(iii) $$1 - |z_{(1)}| > A^{-q}(1 - |z_0|).$$

We may suppose $q \geq 1$, since otherwise lemma 4 is trivial taking $\lambda(z) = z$. Let z' be chosen to satisfy the conclusions of lemma 2. Let z'' be chosen so as to satisfy the conclusions of lemma 2 for z', with $z_0 = 0$. It follows that if z_ν, $\nu > 0$, is a pole or zero of $f(z)$ we have

$$d[z', z_\nu; |z| < 1] = \tfrac{1}{2} \log \frac{1 + |g(z', z_\nu)|}{1 - |g(z', z_\nu)|} > A^q$$

$$d[z'', z_\nu; |z| < 1] > A^q.$$

It now follows from this and lemma 3, that we can find a domain D' contained in $|z| < 1$ and containing none of the poles or zeros of $f(z)$, of which there are at most $2q$ in all, such that

$$\begin{aligned} d[z', z''; D'] &< d[z', z''; |z| < 1] + Aq \\ &< d[z', 0; |z| < 1] + d[0, z_0; |z| < 1] + d(z_0, z'', |z| < 1] + Aq \\ &< \tfrac{1}{2} \log \frac{1 + |z_0|}{1 - |z_0|} + Aq + A \end{aligned}$$

$$(8.1) \qquad d[z', z''; D'] < \tfrac{1}{2} \log \frac{1 + |z_0|}{1 - |z_0|} + Aq$$

making use of lemma 2 (i) and $q \geq 1$.

Let $\lambda(z)$ map $|z| < 1$ onto D' so that $\lambda(0) = z''$, $\lambda(z_{(1)}) = z'$. Then the conclusions (i) to (iii) of lemma 4 hold. For $\lambda(z)$ is schlicht in $|z| < 1$. Also D' contains no zeros or poles of $f(z)$ so that $f[\lambda(z)]$ is regular, nonzero. Further

$$|f[\lambda(0)]| = |f(z'')| < A^q |f_*(0)|,$$

by lemma 2 (iii) with 0, z'' instead of z_0, z'. This proves lemma 4 (i). Again

$$|f[\lambda(z_{(1)})]| = |f(z')| > A^{-q} |f_*(z_0)|$$

by lemma 2 (iii), which proves lemma 4 (ii). Finally

$$d[0, z_{(1)};\ |z| < 1] = d[z', z''; D'] < \tfrac{1}{2} \log \frac{1 + |z_0|}{1 - |z_0|} + Aq$$

by (8.1), i.e.

$$\tfrac{1}{2} \log \frac{1 + |z_{(1)}|}{1 - |z_{(1)}|} < \tfrac{1}{2} \log \frac{1 + |z_0|}{1 - |z_0|} + Aq,$$

which yields lemma 4 (iii). This completes the proof of lemma 4.

Proof of Theorem II.

9) We can now prove Theorem II. Let $f(z)$ be the function of that Theorem and let z_0 be so chosen that $|z_0| = \varrho$ and

$$M[\varrho, f_*(z)] = |f_*(z_0)|. \tag{9.1}$$

Let $\lambda(z)$ be defined as in lemma 4 and let

$$\phi(z) = f[\lambda(z)]^{1/(p+1)}. \tag{9.2}$$

Let $z_{(1)}$ be defined as in lemma 4 and let $|z_{(1)}| = \varrho'$. Then (3.1) follows from lemma 4 (iii). Further

$$|\phi(0)| = |f[\lambda(0)]|^{1/(p+1)} < \{A^q |f_*(0)|\}^{1/(p+1)}$$

by lemma 4 (i). This proves (3.2). Again

$$\log M[\varrho', \phi(z)] \geq \log|\phi(z_{(1)})| = \frac{1}{p+1} \log|f[\lambda(z_{(1)})]|$$

$$> \frac{1}{p+1}[\log|f_*(z_0)| - A\,q]$$

making use of lemma 4 (ii), whence (9.1) gives

$$\log M[\varrho', \phi] > \frac{1}{p+1}[\log M[\varrho, f_*(z)] - A\,q]$$

which proves (1.3). Again the function $f[\lambda(z)]$ takes no value more often than $f(z)$ in $|z| < 1$, since $\lambda(z)$ is schlicht. Thus $f[\lambda(z)]$ satisfies the hypotheses for $f(z)$ of lemma 1. Hence it follows from that lemma that if $\phi(z)$ is defined by (9.2) and $f(z)$ satisfies a hypothesis (i) or (ii) of Theorem II, then there exists a number w', such that $|w'|^{p+1} = r$ and such that $\phi(z) \neq w'$, in $|z| < 1$. This completes the proof of Theorem II.

An Application of Theorem II.

10) As has already been stated, Theorem II can be used to prove extensions to most of the theorems, which have been proved earlier[1], concerning upper bounds for the rate of growth of functions omitting certain values, to the case when the functions take certain values at most p times. We give one application

[1] Hayman (1), (2), (3).

of this, which will contain as a special case a slightly different form of Cartwright's Theorem I. This is

Theorem III. *Let r_n be a strictly increasing sequence of real numbers tending to infinity and satisfying*

$$r_0 = 0 \tag{10.1}$$

$$s = \sum_{n=1}^{\infty} \left(\log \frac{r_{n+1}}{r_n}\right)^2 < \infty. \tag{10.2}$$

Suppose also that $f(z)$ is meromorphic in $|z| < 1$ and that for each n either

(i) *$f(z)$ takes some value w_n, such that $|w_n| = r_n$ at most p times in $|z| < 1$, or*

(ii) *$f(z)$ takes no value w_n, such that $|w_n| = r_n$ more than $p + 1$ times in $|z| < 1$,*

where p is a positive integer or zero. Then we have

$$M[\varrho, f_*(z)] < A(p)\, e^{S/(p+1)} (r_1 + |f_*(0)|)(1 - \varrho)^{-2(p+1)}, \quad 0 < \varrho < 1. \tag{10.3}$$

Of course if $f(z)$ is regular we can replace $M[\varrho, f_*]$ by the smaller $M[\varrho, f]$ in (10.3). To prove Theorem III suppose first that $p = 0$ and that (i) holds for every n. Suppose further that

$$|f(0)| < r_1 \tag{10.4}$$

$$w_1 = -1. \tag{10.5}$$

Since $f(z)$ never takes a sequence of values tending to infinity, $f(z)$ is regular in $|z| < 1$. Also it follows from this and (10.1), that $f(0) = f_*(0)$, and hence it follows from Hayman (3), Theorem III, that (10.3) holds in this case.

Again if (10.4) is satisfied but not necessarily (10.5), (10.3) still follows on applying the result with $\dfrac{f(z)}{-w_1}$ instead of $f(z)$. Suppose now that (10.4) is false. Let n_0 be the greatest integer such that

$$r_{n_0} \leq |f_*(0)|,$$

and put

$$\psi(z) = \frac{f(z)}{-w_{n_0+1}}. \tag{10.6}$$

Then $\psi(z)$ satisfies the hypotheses for $f(z)$ of Theorem III, with

$$s_{n_0+1} = \sum_{n_0+1}^{\infty} \left(\log \frac{r_{n+1}}{r_n}\right)^2$$

instead of s, and also (10.4) and (10.5), if we relabel the sequence r_n so that r_{n_0+k}, $k > 0$ becomes r_k. We deduce that (10.3) holds for $\psi(z)$, so that

$$\log M[\varrho, \psi(z)] < s_{n_0+1} + 2 \log \frac{1}{1-\varrho} + A.$$

Then (10.6) gives

$$\begin{aligned}\log M[\varrho, f] &\le \log r_{n_0+1} + s_{n_0+1} + 2\log\frac{1}{1-\varrho} + A \\ &= \log r_{n_0} + \log\frac{r_{n_0+1}}{r_{n_0}} + s_{n_0+1} + 2\log\frac{1}{1-\varrho} + A \\ &\le \log|f(0)| + \sum_{n=n_0}^{\infty}\left(\log\frac{r_{n+1}}{r_n}\right)^2 + 2\log\frac{1}{1-\varrho} + A.\end{aligned}$$

Thus (10.3) holds in this case also. It follows that Theorem III holds generally when $p = 0$ and the hypothesis (i) is satisfied for each n.

Suppose next that $p > 0$. We apply Theorem II. Let $\phi(z)$ be the function of that theorem. Then for each n, $\phi(z)$ never takes in $|z| < 1$ some value w_n', such that

$$|w_n'|^{p+1} = r_n.$$

Hence

$$s' = \sum_{n=1}^{\infty}\left(\log\left|\frac{w_{n+1}'}{w_n'}\right|\right)^2 = \frac{1}{(p+1)^2}\sum_{n=1}^{\infty}\left(\log\frac{r_{n+1}}{r_n}\right)^2 = \frac{s}{(p+1)^2}.$$

Also $|w_1'| = r_1^{1/(p+1)}$. Thus it follows from our previous argument that

$$M[\varrho', \phi(z)] < A\, e^{s/(p+1)^2}(|\phi(0)| + r_1^{1/(p+1)})(1-\varrho')^{-2}. \tag{10.7}$$

Now $f(z)$ has at most $p+1$ poles or zeros so that in Theorem II $q = p+1$. It then follows from Theorem II, (3.1) that given ϱ, $0 < \varrho < 1$, we may choose ϱ' with

$$\frac{1}{1-\varrho'} < \frac{A^{p+1}}{1-\varrho} \tag{10.8}$$

such that from (3.3)

$$M[\varrho, f_*(z)] < \{A\, M[\varrho', \phi(z)]\}^{p+1} \tag{10.9}$$

and finally by (3.2)

$$|\phi(0)| \le A\,|f_*(0)|^{1/(p+1)}. \tag{10.10}$$

Combining (10.7) to (10.10) we deduce that for $0 < \varrho < 1$

$$\begin{aligned}M[\varrho, f_*(z)] &\le [A^{p+1} e^{s/(p+1)^2}\{|f_*(0)|^{1/(p+1)} + r_1^{1/(p+1)}\}(1-\varrho)^{-2}]^{p+1} \\ &\le A^{(p+1)^2} e^{s/(p+1)}(|f_*(0)| + r_1)(1-\varrho)^{-2(p+1)},\end{aligned}$$

which proves (10.3), i.e. Theorem III.

Suppose now that $f(z)$ takes no value more than p times in $|z| < 1$. Then we can apply Theorem III, with $r_1 = |f_*(0)|$, and $s = \varepsilon$ where ε is arbitrarily small. We deduce that

$$M[\varrho, f_*] \leq A(p) |f_*(0)| (1 - \varrho)^{-2p}.$$

This is the conclusion of Theorem I due to Cartwright with

$$|f_*(0)| = 2^{n-m} \left| f(0) \frac{b_1 b_2 \ldots b_n}{a_1 a_2 \ldots a_m} \right|$$

replacing μ, where $a_1 \ldots a_m$ are the zeros and $b_1 \ldots b_n$ the poles of $f(z)$ in $|z| \leq \frac{1}{2}$. It is not difficult to deduce the exact bound in the form of Theorem I.

The chief disadvantage of the method lies in the constant $A(p)$ of Theorem III. This takes the form $A^{(p+1)^2}$, while Cartwright proved her original result with $A(p) = A^p$. This makes it impracticable in general to extend the method of Theorem II to the case of functions having infinitely many zeros and poles. To consider this case we shall adopt different methods, based on the results of Chapter I.

Part II.

The Case when $p(\varrho)$ is Unbounded.

11) The problem of this chapter when $p(\varrho)$ is unbounded does not seem to have received so much study to date. The most important work in this case is probably Nevanlinna's Theory of meromorphic functions. Nevanlinna's results involve bounds for the characteristic of a function meromorphic in a circle in terms of its valency on a finite number of values. For a more detailed study of this problem which lies outside the scope of this essay, we refer the reader to Nevanlinna's books.[1] The methods are not easily applied to give best possible results in our problem. In fact the strongest results obtainable on the Nevanlinna Theory are

$$T[r, f] = O\left\{\log \frac{1}{1 - r}\right\}.$$

From this we can only deduce

$$\log M[r, f_*(z)] = O\left\{\frac{1}{1 - r} \log \frac{1}{1 - r}\right\}$$

[1] NEVANLINNA (1) and (2).

so that the results

$$\log M[r, f_*] = \frac{O(1)}{1-r} \tag{11.1}$$

of chapter I are not immediate deductions from Nevanlinna's Theory. Rather surprisingly the Nevanlinna Theory can be adapted to yield results of the type (11.1) but we must consider this as outside our scope. We shall base our method on the results of Chapter I.

To do this we consider $f(z)$ in relation to the roots of the equations $f(z) = 0, w, \infty$, where w is the value of E nearest in modulus to $f_*(z)$. In this way the general theory reduces essentially to the case where E contains only 3 values which we may take to be $0, 1, \infty$, and it is this case which we consider first of all.

This problem has been considered in Chapter I. We must, however, extend the results of the Chapter to the case when $\int p(\varrho)\, d\varrho$ diverges. In order to do this we shall have to obtain results in terms of the roots of the equations $f(z) = 0, 1, \infty$ in $|z| \leq R$ where $R < 1$.

If $z = \varrho$, $f_*(\varrho)$ is defined as in Chapter I, 19 in terms of the poles and zeros of $f(z)$ in the circle

$$\left|\frac{z-\varrho}{1-\varrho z}\right| \leq \tfrac{1}{2}. \tag{11.2}$$

If (11.2) holds we have

$$|z| \leq \frac{1+2\varrho}{2+\varrho}$$

and conversely if z is real and satisfies $\varrho < z < \dfrac{1+2\varrho}{2+\varrho}$ we have (11.2). Thus in order to obtain results involving $f_*(\varrho)$ we shall certainly have to consider the behavior of $f(z)$ in a region containing (11.2) and the smallest circle centre the origin, which contains (11.2) has radius

$$\varrho_* = \frac{1+2\varrho}{2+\varrho}. \tag{11.3}$$

Throughout this work we shall consequently, in obtaining bounds for $|f_*(z)|$ in the circle $|z| \leq \varrho$, have to consider the number of roots of equations $f(z) = 0, w, \infty$, where w lies in E, in the circle $|z| < \varrho_*$, where ϱ_* is given in terms of ϱ by (11.3). We note that

$$\tfrac{1}{2}(1-\varrho) \leq 1-\varrho_* \leq \tfrac{1}{3}(1-\varrho),$$

so that if $p(\varrho)$ does not grow more rapidly than a power of $\frac{1}{1-\varrho}$ we shall have

$$p(\varrho_*) = O\{p(\varrho)\}.$$

After this preamble we can state our first and fundamental result as follows

Theorem IV. *Suppose that $f(z)$ is meromorphic in $|z| < 1$ and that each of the equations $f(z) = 0, w_0, \infty$ have at most $p(\varrho)$ roots in $|z| < \varrho$ for $0 < \varrho < 1$. Let*

$$\lambda(\varrho) = \log M[\varrho, f_*(z)] = \max_{0 \leq \theta \leq 2\pi} \log |f_*(\varrho e^{i\theta})|, \quad 0 \leq \varrho < 1. \tag{11.4}$$

Suppose that $0 \leq \varrho < \varrho' < 1$ and that ϱ_*, ϱ'_* are given by (11.3). Then

(i) *If $\lambda(\varrho') \geq \log |w_0|$, we have*

$$\lambda(\varrho) > \log |w_0| - \frac{A(1-\varrho)}{(1-\varrho')}[1 + p(\varrho'_*)].$$

(ii) *If $\lambda = \max\{0, \lambda(\varrho) - \log|w_0|\}$, we have*

$$\lambda(\varrho') < \lambda + \log |w_0| + \frac{A}{1-\varrho'}\left[(\varrho' - \varrho)\lambda + \int_{\varrho_*}^{\varrho'_*} [1 + p(t)]\, dt\right].$$

By repeated application of Theorem IV we shall obtain all the results of this chapter, which will turn out to give the correct order of growth in most cases. Roughly speaking the more rapidly $p(\varrho)$ grows, the less effect the form of E has. A sufficiently slowly growing sequence will always have nearly the same effect as the whole plane, and if $p(\varrho)$ grows as rapidly as $(1-\varrho)^{-a}$ with $a > 1$, a set E consisting of 2 finite values and ∞ will have much the same effect as the whole plane.

12) We need two subsidiary lemmas for the proof of Theorem IV. We have first

Lemma 5. *Let $0 \leq \varrho < 1$, $\varrho_* \leq R < 1$, and let*

$$g_R(\varrho, z) = \frac{R(z-\varrho)}{R^2 - \varrho \bar{z}}$$

$$g(\varrho, z) = g_1(\varrho, z) = \frac{z-\varrho}{1-\varrho\bar{z}}$$

$$\lambda(\varrho, z) = \log^+ \left|\frac{1}{2g(\varrho, z)}\right| - \log^+ \left|\frac{1}{2g_R(\varrho, z)}\right|.$$

Then we have

(i) *If* $|g(\varrho, z)| \geq \frac{1}{2}$ *then* $\lambda(\varrho, z) = 0$

(ii) *If* $|g(\varrho, z)| \leq \frac{1}{2}$ *we have*

$$0 \leq \lambda(\varrho, z) \leq A\left[\tfrac{1}{2} - |g(\varrho, z)|\right]$$

(iii)
$$\left|\frac{\partial}{\partial \varrho} \lambda(\varrho, z)\right| \leq \frac{A}{1 - \varrho}$$

whenever the differential coefficient exists.

Suppose that $|z| \geq R$. Then we have

$$|g_R(\varrho, z)| \geq 1$$

and also

$$(12.1) \qquad |g(\varrho, z)| = \left|\frac{z - \varrho}{1 - \varrho \bar{z}}\right| \geq \frac{R - \varrho}{1 - \varrho R} \geq \tfrac{1}{2}.$$

Thus in this case $\lambda(\varrho, z) = 0$ so that lemma 5 (i) holds.

Next when $|z| = R$, we have

$$|g_R(\varrho, z)| = 1,$$

and hence using (12.1) we deduce

$$(12.2) \qquad \tfrac{1}{2} \leq \left|\frac{g(\varrho, z)}{g_R(\varrho, z)}\right| \leq 1, \quad |z| \leq R.$$

The inequalities (12.2) which were proved for $|z| = R$ hold also when $|z| < R$ since the function

$$\frac{g(\varrho, z)}{g_R(\varrho, z)} \text{ is equal in modulus to } \frac{R^2 - \varrho z}{R(1 - \varrho z)},$$

which is regular nonzero in $|z| \leq R$, and attains its maximum and minimum modulus in this region on the boundary. Then lemma 5 (i) follows from (12.2).

We also deduce from (12.2) that if $|g(\varrho, z)| \leq \frac{1}{4}$ we have

$$(12.3) \qquad \lambda(\varrho, z) \leq \log 2,$$

and also we have clearly if $\frac{1}{4} \leq |g(\varrho, z)| \leq \frac{1}{2}$,

$$(12.4) \qquad \lambda(\varrho, z) \leq \log\left|\frac{1}{2g(\varrho, z)}\right|.$$

Combining (12.3), (12.4) we have lemma 5 (ii).

The inequality of lemma 5 (iii) is trivial by (i) if $|g(\varrho, z)| \geq \frac{1}{2}$. Suppose next that

$$|g(\varrho, z)| < \tfrac{1}{2}, \quad |g_R(\varrho, z)| > \tfrac{1}{2}.$$

In this case it follows from (12.2) that

$$|g(\varrho, z)| = \left|\frac{z-\varrho}{1-\varrho z}\right| \geq \frac{1}{4},$$

so that

(12.5) $$|z-\varrho| > A(1-\varrho).$$

Also

$$\left|\frac{\partial}{\partial \varrho}\lambda(\varrho, z)\right| = \left|\frac{1}{z-\varrho} - \frac{\bar{z}}{1-\varrho\bar{z}}\right| \leq \frac{A}{1-\varrho} + \frac{A}{1-\varrho} \leq \frac{A}{1-\varrho}$$

by (12.5). Suppose lastly that

$$g(\varrho, z) < \tfrac{1}{2}, \quad g_R(\varrho, z) < \tfrac{1}{2}.$$

Then

$$\left|\frac{\partial}{\partial \varrho}\lambda(\varrho, z)\right| = \left|\frac{\bar{z}}{R^2-\varrho\bar{z}} - \frac{\bar{z}}{1-\varrho\bar{z}}\right| \leq \frac{A}{1-\varrho}.$$

This completes the proof of lemma 5 (iii) and so of lemma 5.

13) We have next

Lemma 6. *Let* $0 \leq \varrho < 1$, $\varrho_* \leq R < 1$, *and suppose that* $f(z)$ *is meromorphic in* $|z| < 1$, *and that none of the equations* $f(z) = 0, 1, \infty$ *have more than* $p(R)$ *roots in* $|z| < R$. *Then we have*

$$\left|\frac{f_*'(\varrho)}{f_*(\varrho)}\right| < \frac{2R}{R^2-\varrho^2}\{|\log|f_*(\varrho)|| + A[p(R)+1]\}.$$

Put

(13.1) $$z = l(w) = \frac{R(\varrho + Rw)}{R+\varrho w},$$

$$w = \frac{R(z-\varrho)}{R^2-\varrho z},$$

(13.2) $$\phi(w) = f(z) = f[l(w)].$$

The function $z = l(w)$ maps $|w| < 1$ onto $|z| < R$ so that the equations $\phi(w) = 0, 1, \infty$ each have at most $p(R)$ roots in $|w| < 1$. Thus applying Theorem III (i) of Chapter I to $\phi(w)$ we have

$$(13.3)\qquad \left|\frac{\phi_*'(0)}{\phi_*(0)}\right| < 2\{|\log|\phi_*(0)|| + A(p(R)+1)\}.$$

Let a_μ, $\mu = 1$ to M be the zeros and b_ν, $\nu = 1$ to N the poles of $f(z)$ in $|z| < R$ and let a'_μ, b'_ν be the corresponding zeros and poles of $\phi(w)$. Thus we have

$$(13.4)\qquad l(a'_\mu) = a_\mu, \quad \mu = 1 \text{ to } M$$

$$(13.5)\qquad l(b'_\nu) = b_\nu, \quad \nu = 1 \text{ to } N.$$

Then if z, w are real and non-negative we have from Chapter I, (19.4)

$$(13.6)\qquad \log|f_*(z)| = \log|f(z)| + \sum_{\mu=1}^{M} \log^+\left|\frac{\frac{1}{2}}{g(z, a_\mu)}\right| - \sum_{\nu=1}^{N} \log^+\left|\frac{\frac{1}{2}}{g(z, b_\nu)}\right|.$$

$$(13.7)\qquad \log|\phi_*(w)| = \log|\phi(w)| + \sum_{\mu=1}^{M} \log^+\left|\frac{\frac{1}{2}}{g(w, a'_\mu)}\right| - \sum_{\nu=1}^{N} \log^+\left|\frac{\frac{1}{2}}{g(w, b'_\nu)}\right|.$$

Suppose now that z, w are connected by (13.1). Then (13.4), (13.5) give easily

$$|g(w, a'_\mu)| = \left|\frac{w - a'_\mu}{1 - w\bar{a}'_\mu}\right| = \left|\frac{w - a'_\mu}{1 - w a'_\mu}\right| = \left|\frac{\dfrac{R(z-\varrho)}{R^2 - \varrho z} - \dfrac{R(a_\mu - \varrho)}{R^2 - \varrho a_\mu}}{1 - \dfrac{R(z-\varrho)}{R^2 - \varrho z}\cdot\dfrac{R(a'_\mu - \varrho)}{(R^2 - \varrho a_\mu)}}\right|$$
$$= \left|\frac{R(z - a_\mu)}{R^2 - a_\mu z}\right| = |g_R(z, a_\mu)|$$

with the notation of lemma 5. Making use of this and (13.2), (13.6), (13.7) we deduce that

$$(13.8)\qquad \log|f_*(z)| - \log|\phi_*(w)| = \sum_{\mu=1}^{M} \lambda(z, a_\mu) - \sum_{\nu=1}^{N} \lambda(z, b_\nu).$$

Putting $z = \varrho$, $w = 0$ we have at once from (13.8) and lemma 5 (ii)

$$(13.9)\qquad |\log|\phi_*(0)| - \log|f_*(\varrho)|| < A(M+N) < A\,p(R).$$

Differentiating (13.8) with respect to z and then putting $z = \varrho$ we have, using (13.1)

$$\frac{f_*'(\varrho)}{f_*(\varrho)} - \frac{R}{R^2 - \varrho^2}\,\frac{\phi_*'(0)}{\phi_*(0)} = \left[\sum_{\mu=1}^{M} \frac{\partial}{\partial z}\lambda(z, a_\mu) - \sum_{\nu=1}^{N} \frac{\partial}{\partial z}\lambda(z, b_\nu)\right]_{z=\varrho}.$$

Making use of lemma 5 (iii) we deduce

$$(13.10)\qquad \left|\frac{f_*'(\varrho)}{f_*(\varrho)} - \frac{R}{R^2 - \varrho^2}\,\frac{\phi_*'(0)}{\phi_*(0)}\right| < \frac{A}{1-\varrho}(M+N) \leq \frac{A}{1-\varrho}\,p(R) \leq \frac{AR}{R^2 - \varrho^2}\,p(R).$$

Using (13.3), (13.9) and (13.10) we have lemma 6.

Proof of Theorem IV.

14) The remainder of the proof of Theorem IV is not easily split into lemmas. In this paragraph we prove Theorem IV (i). We shall suppose throughout the proof of Theorem IV that $w_0 = 1$. We may clearly do this without loss in generality, since otherwise we can consider $\frac{f(z)}{w_0}$ instead of $f(z)$. By applying Theorem IV to this function, the complete result will follow.

Suppose now that

(14.1) $$\log |f_*(\varrho')| = 0$$

and that

(14.2) $$\log |f_*(r)| < 0, \quad \varrho \leq r < \varrho'.$$

Let

(14.3) $$R = \varrho'_* = \frac{1 + 2\varrho'}{2 + \varrho'}.$$

Then it follows from Lemma 6 and (14.2) that

$$\left|\frac{f'_*(r)}{f_*(r)}\right| < \frac{2R}{R^2 - r^2}\{-\log |f_*(r)| + A[p(R) + 1]\}, \quad \varrho < r < \varrho'.$$

This may be written as

$$\frac{d}{dr}\frac{R + r}{R - r}[-\log |f_*(r)| + A[p(R) + 1]] > 0, \quad \varrho < r < r'.$$

This yields combined with (14.1)

$$\frac{R + \varrho'}{R - \varrho'}[A(1 + p(R))] > \frac{R + \varrho}{R - \varrho}\{A[1 + p(R)] - \log |f_*(\varrho)|\},$$

i.e.

$$\frac{R + \varrho}{R - \varrho}\log |f_*(\varrho)| > \left\{\frac{R + \varrho}{R - \varrho} - \frac{R + \varrho'}{R - \varrho'}\right\}A[1 + p(R)],$$

or

$$\log |f_*(\varrho)| > \frac{-2AR(\varrho' - \varrho)}{(R + \varrho)(R - \varrho')}[p(R) + 1].$$

Making use of (14.3) we deduce that

(14.4) $$\log |f_*(\varrho)| > \frac{-A(\varrho' - \varrho)}{1 - \varrho'}\{p(\varrho'_*) + 1\}.$$

Suppose next that

(14.5) $$\log |f_*(\varrho')| \geq 0$$

and that

$$\log |f_*(\varrho)| < 0.$$

Let ϱ_1 be the least number greater than ϱ such that

$$\log |f_*(\varrho_1)| = 0.$$

Then we have

$$\log |f_*(r)| < 0, \quad \varrho < r < \varrho_1$$

and hence by what precedes (14.4) holds with ϱ_1 replacing ϱ'. Since $\varrho' \geq \varrho_1$, (14.4) holds a fortiori with ϱ'. Also (14.4) is trivial if $|f_*(\varrho)| > 1$. Thus (14.4) holds whenever $0 \leq \varrho < \varrho' < 1$ and (14.5) holds.

Suppose now that with the notation of (11.4),

$$\lambda(\varrho') \geq 0.$$

Then we can find $\theta = \theta(\varrho')$ such that

$$\log |f_*(\varrho' e^{i\theta}| \geq 0.$$

Then our previous argument leading to (14.4), when applied to $f(z e^{i\theta})$ gives

$$\log |f_*(\varrho e^{i\theta})| > \frac{-A(\varrho' - \varrho)}{1 - \varrho'} \{1 + p(\varrho'_*)\}.$$

This proves Theorem IV (i) since

$$\lambda(\varrho) \geq \log |f_*(\varrho e^{i\theta})|.$$

15) It remains to prove Theorem IV (ii). Suppose first that

$$\log |f_*(\varrho)| \geq 0 \tag{15.1}$$

and that

$$\log |f_*(\varrho)| \leq \log |f_*(r)|, \quad \varrho \leq r \leq \varrho'. \tag{15.2}$$

Suppose also that

$$\varrho' \leq \varrho_* = \frac{1 + 2\varrho}{2 + \varrho}. \tag{15.3}$$

In this case we take

$$R = r_* = \frac{1 + 2r}{2 + r}, \quad \varrho \leq r \leq \varrho', \tag{15.4}$$

and use Lemma 6 with r instead of ϱ. This gives

$$\left|\frac{f_*'(r)}{f_*(r)}\right| < \frac{2R}{R^2 - r^2}\{\log|f_*(r)| + A[1 + p(R)]\}$$

$$= \frac{2(2 + 5r + 2r^2)}{(1 + 4r + r^2)(1 - r^2)}\{\log|f_*(r)| + A[1 + p(R)]\};$$

$$\left|\frac{f_*'(r)}{f_*(r)}\right| \le \frac{4}{1 - r^2}\{\log|f_*(r)| + A[1 + p(R)]\}.$$

This may be written as

$$\frac{d}{dr}\left(\frac{1 - r}{1 + r}\right)^2 \log|f_*(r)| \le \frac{4(1 - r)}{(1 + r)^3}[1 + p(R)] \le A(1 - \varrho)[1 + p(R)]$$

since $\varrho \le r \le \varrho'$ and (15.3) holds. Thus we have on integrating

$$\left[\left(\frac{1 - r}{1 + r}\right)^2 \log|f_*(r)|\right]_{\varrho}^{\varrho'} \le A(1 - \varrho)\int_{\varrho}^{\varrho'}[1 + p(R)]\,dr$$

$$= A(1 - \varrho)\int_{\varrho_*}^{\varrho_*'}[1 + p(R)]\frac{3\,dR}{(2 - R)^2}$$

$$\le A(1 - \varrho)\int_{\varrho_*}^{\varrho_*'}[1 + p(R)\,dR,$$

making use of (15.4), i.e.

$$\log|f_*(\varrho')| < \left(\frac{1 + \varrho'}{1 - \varrho'}\right)^2\left\{\left(\frac{1 - \varrho}{1 + \varrho}\right)^2 \log|f_*(\varrho)| + A(1 - \varrho)\int_{\varrho_*}^{\varrho_*'}[1 + p(R)]\,dR\right\}$$

$$< \left[1 + \frac{2(\varrho' - \varrho)}{(1 + \varrho)(1 - \varrho')}\right]^2 \log|f_*(\varrho)| + \frac{A(1 - \varrho)}{(1 - \varrho')^2}\int_{\varrho_*}^{\varrho_*'}[1 + p(R)]\,dR,$$

and since (15.3) holds so that $(1 - \varrho) < A(1 - \varrho')$, we deduce

(15.5) $$\log|f_*(\varrho')| < \left[1 + \frac{A(\varrho' - \varrho)}{1 - \varrho'}\right]\log|f_*(\varrho)| + \frac{A}{1 - \varrho'}\int_{\varrho_*}^{\varrho_*'}[1 + p(R)]\,dR.$$

16) Suppose next that (15.3) is false, so that

(16.1) $$\frac{1 + 2\varrho}{2 + \varrho} < \varrho' < 1.$$

We still suppose that (15.1), (15.2) hold. We put

(16.2) $$R = \varrho_*' = \frac{1 + 2\varrho'}{2 + \varrho'}$$

and

(16.3) $$z = l(w) = \frac{R(\varrho + Rw)}{R + \varrho w}.$$

(16.4) $$\phi(w) = f(z) = f[l(w)].$$

The function $l(w)$ maps $|w| < 1$ onto $|z| < R$. Also Theorem II of Chapter I yields

(16.5) $$\log|\phi_*(w)| \le \frac{A}{1 - |w|}[\log^+|\phi_*(0)| + \sum(1 - |d'_\nu|) + 1],$$

where d'_ν are the points such that $\phi(d'_\nu) = 0, 1, \infty$ in $|w| < 1$. We have from (16.3), (16.4)

$$d'_\nu = \frac{R(d_\nu - \varrho)}{R^2 - \varrho d_\nu},$$

where d_ν are the points such that $f(d_\nu) = 0, 1, \infty$ and $|d_\nu| < R$. Hence

$$1 - |d'_\nu|^2 = 1 - d'_\nu \bar{d}'_\nu = \frac{(R^2 - \varrho \bar{d}_\nu)(R^2 - \varrho d_\nu) - R^2(d_\nu - \varrho)(\bar{d}_\nu - \varrho)}{(R^2 - \varrho d_\nu)(R^2 - \varrho \bar{d}_\nu)}$$

$$= \frac{(R^2 - \varrho^2)(R^2 - |d_\nu|^2)}{|R^2 - \varrho d_\nu|^2} < \frac{(R^2 - \varrho^2)(R^2 - |d_\nu|^2)}{R^2(R^2 - \varrho^2)} < \frac{A(R - |d_\nu|)}{1 - \varrho},$$

since (16.1), (16.2) hold. Also $1 - |d'_\nu| \le 1$ so that we deduce

(16.6) $$\sum(1 - |d'_\nu|) < \frac{A}{1 - \varrho}\sum \min\{R - |d_\nu|, 1 - \varrho\}.$$

Now from (16.1), (16.2) we deduce

$$R > \tfrac{1}{2}(1 + \varrho_*)$$

so that

$$\min\{R - |d_\nu|, 1 - \varrho\} < A \min\{R - |d_\nu|, R - \varrho_*\}$$

and so (16.6) yields with (16.2)

(16.7) $$\sum(1 - |d'_\nu|) < \frac{A}{1 - \varrho}\int_{\varrho_*}^{\varrho'_*} p_0(r)\,dr$$

where $p_0(r)$ denotes the number of d_ν such that $|d_\nu| \le r$.[1]

17) To complete the proof, we apply the argument which leads to lemma 6. The formulae (16.3), (16.4) are the same as (13.1), (13.2) and hence if w, z are related as in (16.3) and are both real and positive, we have again as in (13.8)

(17.1) $$\log|f_*(z)| - \log|\phi_*(w)| = \sum_{\mu=1}^{M}\lambda(z, a_\mu) - \sum_{\nu=1}^{N}\lambda(z, b_\nu).$$

[1] We have clearly $p_0(r) \le 3p(r)$, in the notation of Theorem IV.

Here $\lambda(z, a_\mu)$ is the function of lemma 5 with $\varrho = z$, $z = a_\mu$, and b_ν, a_μ are the poles, zeros of $f(z)$ in $|z| < R$.

When $z = \varrho$, it follows from lemma 5 (i) that $\lambda(z, a_\mu) = 0$, if

$$\left|\frac{a_\mu - \varrho}{1 - \varrho a_\mu}\right| \geq \frac{1}{2},$$

and so a fortiori if

$$\frac{|a_\mu| - \varrho}{1 - \varrho |a_\mu|} \geq \frac{1}{2}$$

i.e., if $|a_\mu| \geq \varrho_*$. Also if $|a_\mu| \leq \varrho_*$, lemma 5 (ii) yields

$$|\lambda(\varrho, a_\mu)| \leq A.$$

Similar results hold for $\lambda(\varrho, b_\nu)$, and hence (17.1) yields

$$\text{(17.2)} \qquad \big|\log|f_*(\varrho)| - \log|\phi_*(0)|\big| < A\, p_0(\varrho_*)$$

where $p_0(\varrho_*)$ is the number of roots of $f(z) = 0, 1, \infty$ in $|z| < \varrho_*$. Again let w' be the number corresponding to $z = \varrho'$ by (16.3). We have from (17.1)

$$\text{(17.3)} \qquad \log|f_*(\varrho')| - \log|\phi_*(w')| = \sum_{\mu=1}^{M} \lambda(\varrho', a_\mu) - \sum_{\nu=1}^{N} \lambda(\varrho', b_\nu).$$

Now it follows from lemma 5 (i) and (ii) that either $\lambda(\varrho', a_\mu) = 0$, or

$$|\lambda(\varrho', a_\mu)| \leq A\left[\frac{1}{2} - \left|\frac{a_\mu - \varrho'}{1 - \varrho' a_\mu}\right|\right] \leq A\left[\frac{1}{2} - \frac{|a_\mu| - \varrho'}{1 - \varrho' |a_\mu|}\right]$$
$$= \frac{A}{2}\left\{\frac{1 + 2\varrho' - |a_\mu|(2 + \varrho')}{1 - \varrho' |a_\mu|}\right\} \leq \frac{A}{1 - \varrho'}\left[\frac{1 + 2\varrho'}{2 + \varrho'} - |a_\mu|\right].$$

Thus using (16.2), we have if $\lambda(\varrho', a_\mu) \neq 0$

$$\text{(17.4)} \qquad |\lambda(\varrho', a_\mu)| \leq \frac{A}{1 - \varrho'}(R - |a_\mu|).$$

Again we have from lemma 5 (ii)

$$|\lambda(\varrho', a_\mu)| \leq A$$

so that (17.4) yields

$$|\lambda(\varrho', a_\mu)| \leq \frac{A}{1 - \varrho'} \min\{1 - \varrho', R - |a_\mu|\},$$

and since $R - \varrho' > A(1 - \varrho')$ we deduce that

$$\lambda(\varrho', a_\mu) \leq \frac{A}{1 - \varrho'} \min[R - \varrho', R - |a_\mu|].$$

A similar result holds for the poles b_ν. Thus (17.3) yields

$$|\log|f_*(\varrho')| - \log|\phi_*(w')|| < \frac{A}{1-\varrho'}\{\sum \min[R-|a_\mu|, R-\varrho'] + \sum \min[R-b_\nu, R-\varrho']\}$$

$$(17.5) \quad |\log|f_*(\varrho')| - \log|\phi_*(w')|| \leq \frac{A}{1-\varrho'}\int_{\varrho'}^{R} p_0(r)\,dr,$$

where $p_0(r)$ has the same meaning as in (16.7).

Putting $w = w'$ in (16.5), we have from (16.7), (17.2) and (17.5)

$$(17.6) \quad \log|f_*(\varrho')| < \frac{A}{1-w'}\left[\log|f_*(\varrho)| + A(1 + p_0(\varrho_*)) + \frac{A}{1-\varrho}\int_{\varrho_*}^{\varrho'_*} p_0(r)\,dr\right] + \frac{A}{1-\varrho'}\int_{\varrho'}^{R} p_0(r)\,dr.$$

Also w' corresponds to ϱ' by (16.3) so that

$$\varrho' = \frac{R(\varrho + Rw')}{R + \varrho w'}$$

and hence

$$w' = \frac{R(\varrho' - \varrho)}{R^2 - \varrho\varrho'}$$

so that

$$(17.7) \qquad 1 - w' = \frac{(R+\varrho)(R-\varrho')}{R^2 - \varrho\varrho'} > \frac{A(1-\varrho')}{1-\varrho}.$$

Also

$$(17.8) \qquad p_0(\varrho_*) \leq \frac{1}{R-\varrho_*}\int_{\varrho_*}^{R} p_0(r)\,dr < \frac{A}{1-\varrho}\int_{\varrho_*}^{\varrho'_*} p_0(r)\,dr$$

by (16.1) and (16.2). Using (16.1), (16.2), (17.6), (17.7), (17.8) we deduce that

$$(17.9) \qquad \log|f_*(\varrho')| < \frac{A(1-\varrho)}{1-\varrho'}\left\{\log|f_*(\varrho)| + \frac{A}{1-\varrho}\int_{\varrho_*}^{\varrho'_*} p_0(r)\,dr + A\right\}.$$

We deduce from (16.1) that $(1-\varrho) < A(\varrho' - \varrho)$, so that it follows from (17.9) that (15.5) still holds if (16.1) is true. Thus (15.5) holds whenever (15.1) and (15.2) are satisfied.

18) Suppose now that $w_0 = 1$ in Theorem IV so that

$$\lambda = \max\{0, \lambda(\varrho)\}$$

where $\lambda(\varrho)$ is defined as in (11.4). We have proved Theorem IV (i). To prove (ii) it is clearly sufficient to show that

$$\log|f_*(\varrho')| \leq \left\{1 + \frac{A(\varrho'-\varrho)}{1-\varrho'}\right\}\lambda + \frac{A}{1-\varrho'}\int_{\varrho_*}^{\varrho'_*}[1+p(r)]\,dr. \tag{18.1}$$

For then the same upper bound clearly holds for $\log|f_*(\varrho' e^{i\theta})|$ for every θ, on applying (18.1) to $f(z e^{i\theta})$, instead of $f(z)$. Now (18.1) is trivial unless

$$\log|f_*(\varrho')| > \lambda. \tag{18.2}$$

If (18.2) holds let ϱ_1 be the greatest number less than ϱ' such that

$$\log|f_*(\varrho_1)| = \lambda. \tag{18.3}$$

Then it follows from (18.2) and the definition of λ that ϱ_1 exists and $\varrho_1 \geq \varrho$. Also

$$\log|f_*(r)| \geq \log|f_*(\varrho_1)|, \quad \varrho_1 < r \leq \varrho'.$$

Thus (15.1), (15.2) are satisfied with ϱ_1 instead of ϱ and λ instead of $\log|f_*(\varrho)|$. Then (18.1) follows with ϱ_1 instead of ϱ from (15.5). Since $\varrho_1 \geq \varrho$, it follows that (18.1) holds a fortiori with ϱ, so that Theorem IV (ii) always holds. This completes the proof of Theorem IV.

Applications of Theorem IV.

19) We can solve most of the problems of the type we consider in this chapter by repeated application of Theorem IV. General theorems are rather cumbersome. We shall prefer to give some particular applications. Our aim is to obtain the right order of magnitude for the growth of $\lambda(\varrho) = \log M[\varrho, f_*(z)]$. This is less than we can achieve in favorable circumstances by the methods of Theorem II. For instance we obtained in Theorem III the right order of magnitude of $M[\varrho, f_*(z)]$. On the other hand, the method of Theorem IV, dealing as it does with a general increasing function $p(\varrho)$, has of course much wider scope than that of Theorem II.

We recall the statement of the problem in paragraph 1. We shall restrict ourselves to results of the following types.

(i) *The case when E includes the whole plane.*

(ii) *How small a set E is sufficient to have the same effect as the whole plane for a given function $p(\varrho)$, i.e. such that the order of growth of $\lambda(\varrho)$ is the same as if E occupied the whole plane?*

(iii) *What can we prove if E is merely unbounded?*

(iv) *What can we prove if E contains only a finite set of values?*

Some of these results will be proved for general $p(\varrho)$, in some cases we shall restrict ourselves to the functions $p(\varrho) = (1-\varrho)^{-a}$. Converse theorems, except in the simplest cases will be left to Chapter III. They will show that at any rate when $p(\varrho) = (1-\varrho)^{-a}$, $0 \le a < \infty$, all the orders of magnitude obtained are best possible.

The result in case (iv) is obtained in Theorem IV (ii). It appears that the three values $0, w_0, \infty$ have much the same effect as any bounded set E. Writing $\varrho = 0$, ϱ for ϱ' in Theorem IV (ii) and putting $w_0 = 1$, we obtain

Theorem V. *Suppose that $f(z)$ is meromorphic in $|z| < 1$ and that none of the equations $f(z) = 0, 1, \infty$ have more than $p(\varrho)$ roots in $|z| \le \varrho < 1$. Then if $\lambda(\varrho)$ is defined as in (11.4), and ϱ_* as in (11.3), whe have*

$$\lambda(\varrho) < \frac{A}{1-\varrho}\left[\log^+|f_*(0)| + \int_0^{\varrho*}[1 + p(r)]\,dr\right], \quad 0 < \varrho < 1.$$

Consider next the case (iii) above. We have

Theorem VI.[1] *Suppose that E is an unbounded set containing zero, and that none of the equations $f(z) = w$, where w lies in E, have more than $p(\varrho)$ roots in $|z| \le \varrho$. Then if*

(19.1) $$\int_0^1 p(\varrho)\,d\varrho < \infty$$

we have

(19.2) $$\overline{\lim_{\varrho \to 1}}\,(1-\varrho)\lambda(\varrho) = 0.$$

Theorem VI shows that if (19.1) holds, we can sharpen the result of Theorem V by merely assuming that E is unbounded. It does not appear that this is possible in general if (19.1) is false.[2] We shall, however, later prove some results in this case also, which hold for some ϱ arbitrarily near 1. (Theorem X.) These results lie rather deeper than Theorem VI and will be proved only in the case

$$p(\varrho) = (1-\varrho)^{-a}.$$

[1] This result was proved when $p(\varrho) = 0$ in HAYMAN (2), Theorem V. Even in this case (19.2 is best possible as is shown in Theorem VI of that paper.

[2] See Theorem III of Chapter III, which proves this if $p(\varrho) = 1/(1-\varrho)$.

We come now to our fundamental result in the problem (i) and (ii) stated above. This is

Theorem VII. *Suppose that E contains zero and a sequence of complex numbers w_n satisfying*

$$|w_n| \leq |w_{n+1}| \leq k|w_n|, \quad n = 1, 2, \ldots, \tag{19.3}$$

where k is a constant greater than one, and

$$|w_n| \to \infty, \quad as \quad n \to \infty. \tag{19.4}$$

Then if none of the equations $f(z) = w$, for w in E, have more than $p(r)$ roots in $|z| \leq r < 1$, we have

$$\lambda(\varrho) \leq \log \mu + A \int_{\frac{1}{2}}^{\varrho^*} [1 + \log k + p(t)] \frac{dt}{1-t}, \quad 0 < \varrho < 1,$$

where $\mu = \max [|f_*(0)|, |w_1|]$.

Corollary. *If E consists of the whole plane, so that $f(z)$ takes no value more than $p(\varrho)$ times in $|z| \leq \varrho$, we have*

$$\lambda(\varrho) \leq \log |f_*(0)| + A \int_{\frac{1}{2}}^{\varrho^*} [1 + p(t)] \frac{dt}{1-t}.$$

We shall see that the corollary to Theorem VII gives in general the right answer for our problem (i) above. Also when $p(\varrho)$ is bounded the conditions (19.3), (19.4) cannot be essentially weakened.[1] On the other hand, we shall see in Theorem IX, that if $p(\varrho)$ grows as rapidly as $(1-\varrho)^{-a}$ with $a > 0$, we can relax (19.3) to $|w_{n+1}| < |w_n|^k$, and this condition cannot be further weakened.[2]

We shall first prove Theorems VI and VII and then give some simple converse examples to the corollary to Theorem VII. Theorem V has been proved already in Theorem IV (ii).

Proof of Theorems VI and VII.

20) To prove Theorem VI, take ϱ so nearly 1 that

$$\int_{\varrho}^{1} [1 + p(r)] \, dr < \varepsilon. \tag{20.1}$$

[1] See Theorem IV of Chapter III.
[2] See Theorem V of Chapter III.

Then take a value w_0 in E, such that

$$\log |w_0| > \lambda(\varrho).$$

Since E is unbounded $f(z)$ has at most $p(r)$ poles in $|z| \leq r < 1$, and so we can apply Theorem IV (ii).

We obtain

$$\lambda(\varrho') \leq \log |w_0| + \frac{A}{1-\varrho'} \int_{\varrho}^{1} [1 + p(r)]\, dr, \quad \varrho' \geq \varrho.$$

We deduce that

$$\overline{\lim_{\varrho' \to 1}} (1 - \varrho')\lambda(\varrho') \leq A\,\varepsilon,$$

and since ε is arbitrary Theorem VI follows.

To prove Theorem VII, note that Theorem IV (ii) yields on making $\varrho' \to \varrho$,

$$\frac{d}{d\varrho}\lambda(\varrho) \leq \frac{A}{1-\varrho}[\lambda(\varrho) - \log |w_0| + p(\varrho_*) + 1] \tag{20.2}$$

provided that $\lambda(\varrho) \geq \log |w_0|$. It follows from (19.3) and (19.4), that if $\lambda(\varrho) \geq \log |w_1|$, we can always find n such that

$$\log |w_n| \leq \lambda(\varrho) \leq \log |w_n| + \log k. \tag{20.3}$$

By applying (20.2) with $w_0 = w_n$, we deduce that

$$\frac{d}{d\varrho}\lambda(\varrho) \leq \frac{A}{1-\varrho}[\log k + p(\varrho_*) + 1] \tag{20.4}$$

provided that

$$\lambda(\varrho) \geq \log |w_1|.$$

We note that Theorem VII is trivial if $\lambda(\varrho) \leq \log |w_1|$. Suppose then that $\lambda(\varrho) > \log |w_1|$, and that ϱ_0 is the smallest positive number such that

$$\lambda(t) \geq \log |w_1|, \quad \varrho_0 \leq t \leq \varrho.$$

Then (20.4) yields on writing t instead of ϱ and integrating from $t = \varrho_0$ to ϱ

$$\lambda(\varrho) - \lambda(\varrho_0) \leq A \int_{\varrho_0}^{\varrho} (p(t_*) + \log k + 1)\frac{dt}{1-t} \leq A \int_{\frac{1}{2}}^{\varrho_*} [p(t) + \log k + 1]\frac{dt}{1-t}. \tag{20.5}$$

Also we have either $\lambda(\varrho_0) = \log |w_1|$ or $\varrho_0 = 0$ so that

$$\lambda(\varrho_0) \leq \max\{\log |w_1|, \log |f_*(0)|\} = \log \mu.$$

Thus Theorem VII follows from (20.5).

To prove the corollary we take $w_n = f_*(0)\, e^{n-1}$, and apply the main Theorem. We deduce that

$$\lambda(\varrho) \leq \log |f_*(0)| + A \int_{\frac{1}{2}}^{\varrho_*} [1 + 1 + p(t)] \frac{dt}{1-t},$$

and the corollary follows.

Some Simple Counterexamples to Theorem VII.

21) In this paragraph we study some simple examples to show that the order of growth obtained in the corollary to Theorem VII is the correct one, at any rate when

(21.1) $$p(\varrho) = b(1-\varrho)^{-a}, \quad 0 \leq a < \infty.$$

In fact if $a = 0$ in the above, the corollary to Theorem VII yields

$$\lambda(\varrho) \leq \log |f_*(0)| + A(1+b) \log \frac{1}{2(1-\varrho_*)}$$

$$\leq \log |f_*(0)| + A(1+b) \log \frac{3}{2(1-\varrho)}.$$

On the other hand, the function

(21.2) $$f(z) = \left(\frac{1+z}{1-z}\right)^{2b}$$

is regular nonzero in $|z| < 1$ and takes no value more than $b + 1$ times if b is any real number, while at the same time

$$\lambda(\varrho) = \log M[\varrho, f] > 2b \log \frac{1}{1-\varrho}.$$

Thus the order of growth given by Theorem VII, corollary for $\lambda(\varrho)$ is correct, when $p(\varrho)$ is constant.

On the other hand, when $a > 0$ and $p(\varrho)$ is given by (21.1) we have

$$\int_0^{\varrho_*} p(t) \frac{dt}{1-t} = b \int_0^{\varrho_*} \frac{dt}{(1-t)^{a+1}} = \frac{O(1)}{(1-\varrho_*)^a} = \frac{O(1)}{(1-\varrho)^a} = O[p(\varrho)].$$

Thus in this case Theorem VII corollary gives

(21.3) $$\lambda(\varrho) = O\{p(\varrho)\}.$$

Let

$$f(z) = u + iv = \left(\frac{1+z}{1-z}\right)^a, \quad 0 < a < \infty$$

be the function of (21.2), which takes no value more than $\frac{a}{2}+1$ times. Also we have

$$|v| < \left(\frac{1+\varrho}{1-\varrho}\right)^a < 2^a(1-\varrho)^{-a}. \tag{21.4}$$

Hence if we put

$$\phi(z) = e^{f(z)}$$

the equation $\phi(z) = w \neq 0$ has a root only if

$$f(z) = \log w \pm 2n\pi i. \tag{21.5}$$

For given w, n the equation (21.5) has at most $\frac{a}{2}+1$ roots in $|z|<1$ and it follows from (21.4) that the equation (21.5) only has roots in $|z|<\varrho$ for $2^{a+1}(1-\varrho)^{-a}$ different values of n. Thus for given w (21.5) has at most $\left(\frac{a}{2}+1\right)2^{a+1}(1-\varrho)^{-a}$ different roots in $|z|<1$, while clearly

$$\lambda(\varrho) = \log M[\varrho, f] \geq (1-\varrho)^{-a}, \quad 0 < a < \infty.$$

Thus we see that (21.3) cannot be sharpened so that the corollary to Theorem VII gives the right order for $\lambda(\varrho)$, when $p(\varrho)$ is given by (21.1), and $a = 0$ or $0 < a < \infty$.

22) We proceed to obtain some general conditions under which the inequality (21.3) holds. We have seen in the preceding paragraph, that we cannot hope to prove more than this, even if $f(z)$ takes no value more than $p(\varrho)$ times in $|z|<\varrho$. Also as stated in paragraph 11, without making more assumptions on $p(\varrho)$, we can only prove the slightly weaker inequality $\lambda(\varrho) = O\{p(\varrho_*)\}$. If $p(\varrho)$ does not grow too rapidly, this implies (21.3).

We need first two lemmas, which give bounds for the integrals occurring in Theorems V and VII.

Lemma 7. *Suppose that $p(t)$ is a positive increasing function, defined for $0 \leq t < 1$, and satisfying*

$$p[\tfrac{1}{2}(1+t)] \geq k\,p(t), \quad 0 \leq t < 1, \tag{22.1}$$

where k is a constant greater than 2. Then we have

$$\frac{1}{1-\varrho}\int_{\frac{1}{2}}^{\varrho} p(t)\,dt \leq \frac{8k}{k-2}(\varrho-\tfrac{1}{2})\,p(\varrho), \quad \tfrac{1}{2} < \varrho < 1. \tag{22.2}$$

Suppose first $\varrho \leq \frac{3}{4}$. Then

$$\int_{\frac{1}{2}}^{\varrho} p(t)\,dt \leq (\varrho - \tfrac{1}{2})\,p(\varrho) < 4(1-\varrho)(\varrho - \tfrac{1}{2})\,p(\varrho) < \frac{8k}{k-2}(1-\varrho)(\varrho-\tfrac{1}{2})\,p(\varrho).$$

Thus (22.2) holds in this case. If $\frac{3}{4} \leq \varrho < 1$, we define inductively

$$\varrho_0 = \varrho,$$

(22.3)
$$\varrho_n = 2\varrho_{n-1} - 1, \quad n \geq 1,$$

and we have

$$\int_{\varrho_1}^{\varrho_0} p(t)\,dt < 2(\varrho_0 - \varrho_1)\,p(\varrho_0) = 2(1-\varrho)\,p(\varrho),$$

$$\int_{\varrho_2}^{\varrho_1} p(t)\,dt < 2(\varrho_1 - \varrho_2)\,p(\varrho_1) = 2\cdot 2(1-\varrho)\,p(\varrho_1)$$

$$< 2\cdot\frac{2}{k}(1-\varrho)\,p(\varrho),$$

by (22.1). Continuing we deduce that if $\varrho_{n+1} \geq 0$

(22.4)
$$\int_{\varrho_{n+1}}^{\varrho_n} p(t)\,dt < 2\left(\frac{2}{k}\right)^n (1-\varrho)\,p(\varrho).$$

Let n_0 be the largest integer for which ϱ_{n_0}, defined as in (22.3) is positive. Then $\varrho_{n_0} \leq \frac{1}{2}$, and so

$$\int_{\frac{1}{2}}^{\varrho} p(t)\,dt \leq \int_{\varrho_{n_0}}^{\varrho} p(t)\,dt = \sum_{n=1}^{n_0} \int_{\varrho_n}^{\varrho_{n-1}} p(t)\,dt$$

$$< 2(1-\varrho)\,p(\varrho)\left[1 + \frac{2}{k} + \left(\frac{2}{k}\right)^2 + \cdots\right] < \frac{2k}{k-2}(1-\varrho)\,p(\varrho)$$

$$< \frac{8k}{k-2}(\varrho - \tfrac{1}{2})(1-\varrho)\,p(\varrho)$$

since $\varrho \geq \frac{3}{4}$. This completes the proof of lemma 7.

We deduce

Lemma 8. *Suppose that $p(t)$ is a positive increasing function of t for $0 \leq t < 1$ and satisfies (22.1) with $k > 1$. Then we have*

$$\int_{\frac{1}{2}}^{\varrho} p(t)\frac{dt}{1-t} < \frac{8k}{k-1}(\varrho - \tfrac{1}{2})\,p(\varrho).$$

We apply lemma 7 with $p(t)/(1-t)$, which increases since $p(t)$ does, instead of $p(t)$. We have

$$\frac{p[\frac{1}{2}(1+t)]}{1-\frac{1}{2}(1+t)} = \frac{2p[\frac{1}{2}(1+t)]}{1-t} \geq 2k\frac{p(t)}{1-t}$$

since (22.1) holds. Also by hypothesis $2k > 2$. Thus we have from lemma 7

$$\frac{1}{1-\varrho}\int_{\frac{1}{2}}^{\varrho} p(t)\frac{dt}{1-t} < \frac{8\cdot 2k}{2k-2}(\varrho - \tfrac{1}{2})\frac{p(\varrho)}{1-\varrho},$$

which proves lemma 8.

Using lemma 7 and Theorem V we can now prove

Theorem VIII. *Suppose that $p(\varrho)$ is an increasing function of ϱ for $0 < \varrho < 1$, satisfying* (22.1) *with $k > 2$. Then if $f(z)$ is meromorphic in $|z| < 1$ and none of the equations $f(z) = 0, 1, \infty$ have more than $p(\varrho)$ roots in $|z| \leq \varrho$ for $0 < \varrho < 1$, we have*

$$\log M[\varrho, f_*(z)] < \log^+|f_*(0)| + A\varrho\left[\frac{1+\log^+|f_*(0)|}{1-\varrho} + \frac{k}{k-2}p(\varrho_*)\right]$$

where

$$\varrho_* = \frac{1+2\varrho}{2+\varrho}.$$

In fact Theorem V yields

$$\log M[\varrho, f_*(z)] < \log^+|f_*(0)| + \frac{A}{1-\varrho}\left\{\varrho(1+\log^+|f_*(0)|) + \int_{\frac{1}{2}}^{\varrho_*} p(t)\,dt\right\},$$

$$< \log^+|f_*(0)| + \frac{A\varrho}{1-\varrho}(1+\log^+|f_*(0)|) + \frac{Ak(\varrho_* - \frac{1}{2})}{k-2}p(\varrho_*)$$

by lemma 7. Also $\varrho_* - \frac{1}{2} < A\varrho$, and hence Theorem VIII follows.

If

(22.5) $$p(\varrho) = b(1-\varrho)^{-a}, \quad a > 1,$$

(22.1) holds with $k = 2^a > 2$. Thus Theorem VIII yields

$$\lambda(\varrho) = \log M[\varrho, f_*(z)] = O\{p(\varrho_*)\} = O\{p(\varrho)\}.$$

This order of growth cannot be sharpened, even if we assume that $f(z)$ takes no value more than $p(\varrho)$ times in $|z| < \varrho$. Thus to revert to our original problem (ii) of paragraph 19, we see that if $p(\varrho)$ grows as rapidly as in (22.5), any set E containing more than 2 values has roughly the same effect on the order of growth of $\lambda(\varrho)$.

This agrees with Nevanlinna's theory, who showed that if $f(z)$ takes any value as frequently as this, $f(z)$ takes all values with at most two exceptions roughly equally often.[1] There is not, however, entire agreement between the two theories, since Nevanlinna's theory still holds, at least in amended form when $p(\varrho) = 1/(1-\varrho)$, while Theorem VIII breaks down, as we shall see in Theorem I, Chapter III.

23) In the general case when $p(\varrho)$ grows less rapidly, the situation is not so simple. In fact if $M > 1$

$$f(z) = M \exp \frac{1+z}{1-z},$$

takes in $|z| < 1$ no value w such that $|w| < M$, while at the same time

$$\log M[\varrho, f] > \frac{1+\varrho}{1-\varrho}.$$

Thus in order to prove stronger results we shall have to assume that E is unbounded at least. Theorem V shows that if E contains a sequence satisfying

$$(23.1) \qquad |w_{n+1}| < k\,|w_n| \to \infty, \quad \text{as } n \to \infty$$

and $p(\varrho) = (1-\varrho)^{-a}$, then if $f(z)$ takes no value of E more than $p(\varrho)$ times in $|z| \leq \varrho$ we have

$$\log M[\varrho, f] = O(1-\varrho)^{-a}, \quad a > 0.$$

We shall show that if $p(\varrho)$ grows as rapidly as this, we can replace the condition (23.1) by the weaker condition

$$(23.2) \qquad |w_{n+1}| < |w_n|^c, \quad n = 1, 2, \ldots, \quad c > 1,$$

and yet obtain

$$\log M[\varrho, f_*] = O[p(\varrho_*)]$$

if $f(z)$ takes no value w_n more than $p(\varrho)$ times. We have

Theorem IX. *Suppose that E is a set of values containing zero and a sequence w_n satisfying* (23.2) *and*

$$(23.3) \qquad w_n \to \infty, \quad \text{as} \quad n \to \infty.$$

Suppose also that $p(\varrho)$ is an increasing function of ϱ satisfying (22.1) *with some $k > 1$ and $p(0) \geq 1$.*

[1] NEVANLINNA (1) Chapter IV.

Then if $f(z)$ is meromorphic in $|z| < 1$ and takes no value of E more than $p(\varrho)$ times in $|z| \leq \varrho$, $0 < \varrho < 1$, we have

$$\lambda(\varrho) = \log M[\varrho, f_*(z)] \leq A(c, k)[1 + \log^+|f_*(0)| + \log|w_1|]\, p(\varrho_*).$$

On the other hand, it will be seen in Chapter III, Theorem V, that if E does not contain a sequence satisfying (23.2) and (23.3), then $f(z)$ exists taking no value of E more than $(1-\varrho)^{-a}$ times, $0 < a \leq 1$ in $|z| < 1$, while yet $\overline{\lim\limits_{\varrho \to 1}}\, (1-\varrho)^a \log M[\varrho, f_*(z)] = \infty$.

Thus the condition on E to contain a sequence satisfying (23.2) and (23.3) is the necessary and sufficient condition that E shall have the same effect as the whole plane when $p(\varrho) = (1-\varrho)^{-a}$, $0 < a \leq 1$. Thus Theorems VII, VIII and IX solve problem (ii) of paragraph 19, when $p(\varrho) = (1-\varrho)^{-a}$, $0 \leq a < \infty$.

24) To prove Theorem IX we first extract from the sequence w_n a certain subsequence. Let n_2 be the smallest integer such that

$$|w_{n_2}| > |w_1|^k \tag{24.1}$$

where k is the constant of (22.1) and Theorem IX. Then

$$w_{n_2 - 1} \leq |w_1|^k$$

and hence by (23.3) we have

$$|w_{n_2}| < |w_1|^{kc}. \tag{24.2}$$

Thus (24.1) and (24.2) give

$$|w_1|^k \leq |w_{n_2}| \leq |w_1|^{kc}.$$

We can similarly find n_3 such that

$$|w_{n_2}|^k \leq |w_{n_3}| \leq |w_{n_2}|^{kc}$$

and finally a sequenae n_p, such that

$$w_{n_1} = w_1$$

$$|w_{n_p}|^k \leq |w_{n_{p+1}}| \leq |w_{n_p}|^{kc}, \quad p > 1.$$

We now ignore all the w_n expect those for which $n = n_p$ and this remaining sequence we relabel simply w_p. We thus obtain a sequence w_n, none of which $f(z)$ takes more than $p(\varrho)$ times in $|z| < \varrho$, and which satisfies

$$|w_n|^k < |w_{n+1}| \leq |w_n|^{kc}, \quad n \geq 1. \tag{24.3}$$

Also w_1 is the same in the old and the new notation.

Let ϱ_n be the least number such that

$$M[\varrho_n, f_*(z)] = |w_n|.$$

If $|f_*(0)| > |w_n|$, we put $\varrho_n = 0$. If $|f_*(z)| < |w_n|$ for $|z| < 1$ we put $\varrho_n = 1$. In any case we have on applying Theorem IV (ii), with $\varrho = \varrho_n$, $\varrho' = \varrho_{n+1}$,

$$\log\left|\frac{w_{n+1}}{w_n}\right| < \frac{A}{1-\varrho_{n+1}} \int_{t_n}^{t_{n+1}} [1 + p(t)]\, dt, \tag{24.4}$$

provided that $\varrho_n > 0$, where

$$t_n = \varrho_{n*} = \frac{1 + 2\varrho_n}{2 + \varrho_n}, \quad 0 < n < \infty. \tag{24.5}$$

Now we have $p(t) \geq p(0) \geq 1$ by hypothesis, so that

$$\frac{A}{1-\varrho_{n+1}} \int_{t_n}^{t_{n+1}} [1 + p(t)]\, dt \leq \frac{2A}{1-\varrho_{n+1}} \int_{t_n}^{t_{n+1}} p(t)\, dt$$

$$\leq \frac{A}{1-t_{n+1}} \int_{t_n}^{t_{n+1}} p(t)\, dt$$

since

$$1 - t_n = \frac{1-\varrho_n}{2+\varrho_n} \leq \frac{1-\varrho_n}{2}.$$

Thus (24.4) yields

$$\log\left|\frac{w_{n+1}}{w_n}\right| < \frac{A}{1-t_{n+1}} \int_{t_n}^{t_{n+1}} p(t)\, dt. \tag{24.6}$$

Suppose now that $\varrho_n > 0$, and that

$$\log|w_n| < B\, p(t_n), \tag{24.7}$$

where B is a constant to be determined. Let p be an integer, such that

$$k^{p-1} > c. \tag{24.8}$$

Then if

$$1 - t_{n+1} < 2^{-p}(1 - t_n) \tag{24.9}$$

we deduce by successive application of (22.1) that

$$p(t_{n+1}) \geq k^p p(t_n) \geq k c\, p(t_n)$$

by (24.8). Also by (24.3) we have

$$\log |w_{n+1}| \leq k c \log |w_n| \leq k c B p(t_n)$$

using (24.7), and hence (24.7) and (24.9) give

$$\log |w_{n+1}| < B p(t_{n+1}). \tag{24.10}$$

Suppose next that (24.9) is false. Then (24.6) gives

$$\log \left|\frac{w_{n+1}}{w_n}\right| < \frac{A\, 2^p}{1 - t_n} \int_{t_n}^{t_{n+1}} p(t)\, dt \leq A\, 2^p \int_{t_n}^{t_{n+1}} \frac{p(t)\, dt}{1-t}$$

$$< A(c, k)\, p(t_{n+1})$$

making use of lemma 8. Using the first inequality of (24.3), which gives

$$\log \left|\frac{w_{n+1}}{w_n}\right| \geq \left(1 - \frac{1}{k}\right) \log |w_{n+1}|$$

we deduce that in this case also

$$\log |w_{n+1}| \leq A(c, k)\, p(t_{n+1}). \tag{24.11}$$

Thus if (24.7) holds, we have either (24.10) or (24.11). Choosing

$$B \geq A(c, k), \tag{24.12}$$

we see that (24.7) implies (24.10), provided that (24.12) holds. Let n_0 be the smallest integer such that

$$|w_{n_0}| > |f_*(0)|.$$

Then (24.7) holds with $n \leq n_0$, and $B = \log |w_{n_0}|$. Also if $|f_*(0)| > |w_1|$ it follows from (24.3), that

$$\log |w_{n_0}| < k c \log |f_*(0)|.$$

Thus we have in any case

$$\log |w_{n_0}| \leq \log |w_1| + k c \log^+ |f_*(0)|.$$

It follows from this and (24.12), that (24.7) holds for all n with

$$B = \log |w_1| + k c \log^+ |f_*(0)| + A(c, k). \tag{24.13}$$

Suppose now that $\varrho > 0$, and let n be the largest integer (if any), such that

$$|w_n| < M[\varrho, f_*]. \tag{24.14}$$

If no such integer exists, Theorem IX is trivial. Otherwise we have from (24.3) and (24.7)

$$\log M[\varrho, f_*(z)] < k c \log |w_n| < k c B p(t_n) < k c B p(\varrho_*)$$

since

$$\varrho_n < \varrho$$

by (24.14) and so $t_n \leq \varrho_*$ by (24.5). Making use of (24.13), Theorem IX follows.

Some Further Results when E is Unbounded.

25). We have considered in the last few sections what kind of sets E have the effect of the whole plane for different functions $p(\varrho)$, on the rate of growth of $\lambda(\varrho)$. Theorem VIII shows that if $p(\varrho)$ satisfies (22.1) with $k > 2$, any set E containing at least 2 finite values has this property, while Theorem IX shows that if (22.1) is satisfied with $k > 1$, then it is sufficient to assume that E contains a sequence satisfying (23.2). These results may be considered as generalizations of Theorem VII.

We conclude the chapter by proving certain results on the assumption that E is unbounded only. These results will take the form of showing that $M[\varrho, f_*]$ satisfies a certain inequality for some values of ϱ arbitrarily near 1. Also we can in this case replace $M[\varrho, f_*]$ by $M[\varrho, f]$, even if $f(z)$ is meromorphic. Since the form of the best possible inequalities is rather intricate and cannot easily be stated in the simple form of e.g. Theorems V and VII, we confine ourselves to the case

(25.1) $$p(\varrho) = c(1-\varrho)^{-a}, \quad 0 \leq a < \infty, \ c > 0.$$

We note that in this case Theorem VI gives

(25.2) $$\log^+ M[\varrho, f_*(z)] = o(1-\varrho)^{-1}, \quad 0 \leq a < 1,$$

while Theorem V gives

(25.3) $$\log M[\varrho, f_*(z)] = O\left\{\frac{1}{1-\varrho} \log \frac{1}{1-\varrho}\right\}, \quad a = 1$$

(25.4) $$\log M[\varrho, f_*(z)] = O(1-\varrho)^{-a}, \qquad a > 1.$$

These inequalities are the strongest which hold for *all* ϱ sufficiently near 1.[1] Also the functions

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[1] See Chapter III, Theorems II and III, section 17, for converse examples to (25.2), (25.3), (25.5), (25.6). (25.5) and its converse were proved for $a = 0$ in HAYMAN (2).

$$f(z) = \exp\left(\frac{1+z}{1-z}\right)^a$$

show that we cannot hope to strengthen (25.4) at all, even if $f(z)$ takes no value more than $p(\varrho)$ times in $|z| < 1$. We can, however, sharpen (25.2) and (25.3) for some values of ϱ arbitrarily near 1. We have in fact

Theorem X. *Suppose that E is an unbounded set of complex values and that $f(z)$ is meromorphic in $|z| < 1$ and takes no value of E more than $p(\varrho)$ times in $|z| < 1$, where $p(\varrho)$ is given by (25.1). Then we have*

$$\varliminf_{\varrho \to 1} (1-\varrho)^{\frac{1+a}{3-a}} \log M[\varrho, f] \leq 0, \quad 0 \leq a < 1. \tag{25.5}$$

$$\varliminf_{\varrho \to 1} \frac{\log M(\varrho, f)}{p(\varrho) \log\log \frac{1}{1-\varrho}} < A, \quad a = 1. \tag{25.6}$$

$$\varliminf_{\varrho \to 1} \frac{\log M[\varrho, f]}{p(\varrho)} < A(a), \quad a > 1. \tag{25.7}$$

26). To prove Theorem X we may suppose without loss in generality that E contains zero. For if not, we may consider $f(z) - w_0$ instead of $f(z)$, where w_0 is a value of E. This will not alter any of the inequalities of Theorem X. Also if w is a value of E, $f(z) - w_0$ does not take the value $w - w_0$ more than $p(\varrho)$ times in $|z| < \varrho$, and $w - w_0$ is unbounded if w is.

We next write

$$\lambda_1(\varrho) = \max_{0 \leq r \leq \varrho} \log M[r, f_*(z)] = \max_{0 \leq r \leq \varrho} \lambda(r). \tag{26.1}$$

Then it is sufficient to prove Theorem X with $\lambda_1(\varrho)$ replacing $\log M[\varrho, f]$. In fact it follows from Theorem VI of Chapter I, that given ϱ, $0 < \varrho < 1$, we can find r such that $\varrho \leq r \leq \frac{1}{2}(1+\varrho)$ and

$$\log M[r, f] \leq \log M[r, f_*(z)] + A\, n\left[\frac{3+\varrho}{4}\right] \tag{26.2}$$

where $n\left[\frac{3+\varrho}{4}\right]$ denotes the number of poles of $f(z)$ in $|z| \leq \frac{3+\varrho}{4}$. Also since $f(z)$ takes no value of the unbounded set E more than $p(\varrho)$ times in $|z| < \varrho$, $f(z)$ has also at most $p(\varrho)$ poles, so that with the notation of (26.1) we deduce from (26.2) and the hypotheses of Theorem X, that given ϱ, $0 < \varrho < 1$, we can find r, $\varrho \leq r \leq \frac{1}{2}(1+\varrho)$ such that

$$\log M[r, f] \leq \lambda_1[\tfrac{1}{2}(1+\varrho)] + A(a)\,c\,(1-\varrho)^{-a}. \tag{26.3}$$

Hence if $a < 1$ and ϱ is so chosen that

$$\lambda_1[\tfrac{1}{2}(1+\varrho)] < \varepsilon(1-\varrho)^{-\frac{1+a}{3-a}}$$

we have

$$\log M[r, f] \leq \varepsilon(1-\varrho)^{-\frac{1+a}{3-a}} + A(a)\,c\,(1-\varrho)^{-a}$$

$$\leq \varepsilon(1-r)^{-\frac{1+a}{3-a}} + A(a)\,c\,(1-r)^{-a}$$

and since $a < \dfrac{1+a}{3-a}$, $0 < a < 1$, (26.3) holds for some values of r arbitrarily near 1, and ε is arbitrary we deduce that

$$\varliminf_{r\to 1}(1-r)^{\frac{1+a}{3-a}} \log M[r, f] \leq 0,$$

i.e., (25.5).

Suppose next that $a = 1$. Again (26.3) holds for some r in every range $\varrho \leq r \leq \frac{1}{2}(1+\varrho)$. Choose ϱ so that

$$\lambda_1[\tfrac{1}{2}(1+\varrho)] \leq A\,c\,\frac{1}{1-\varrho}\log\log\frac{1}{1-\varrho}\cdot$$

Then (26.3) gives

$$\log M[r, f] \leq A\,c\,\frac{1}{1-\varrho}\left[\log\log\frac{1}{1-\varrho} + A\right]$$

$$\leq A\,c\,\frac{1}{1-r}\left[\log\log\frac{1}{1-r} + A\right]\cdot$$

Since this holds for some values of r arbitrarily near 1, (25.6) follows.

Lastly we can prove (25.7) direct. It follows from Theorem IV (ii), applied with $\varrho = 0$, that in this case

$$\lambda_1(\varrho') \leq \frac{O(1)}{1-\varrho'} + \frac{A}{1-\varrho'}\int_0^{\varrho'_*} c\,(1-t)^{-a}\,dt$$

$$\leq \frac{A(a)\,c}{(1-\varrho')^a} + \frac{O(1)}{1-\varrho'}, \quad 0 < \varrho' < 1,$$

taking for w any value of E. Hence it follows from (26.3), that we have for some r in every range $\varrho' \leq r \leq \frac{1}{2}(1+\varrho')$

$$\log M[r, f] < \frac{A(a)\,c}{(1-r)^a} + \frac{O(1)}{1-r}\cdot$$

Since $a > 1$, (25.7) follows on taking r sufficiently near 1.

Thus to complete the proof of Theorem X, we may suppose that E contains zero and we need only prove that

(26.4) $$\overline{\lim_{\varrho \to 1}} (1-\varrho)^{\frac{1+a}{3-a}} \lambda_1(\varrho) \leq 0, \quad a < 1$$

and

(26.5) $$\overline{\lim_{\varrho \to 1}} \frac{\lambda_1(\varrho)}{p(\varrho) \log \log (1/(1-\varrho))} < A, \quad a = 1,$$

where $\lambda_1(\varrho)$ is defined as in (26.1).

27) We prove first (26.4). The function $\lambda_1(\varrho)$ is non-decreasing and clearly (26.4) is trivial if $\lambda_1(\varrho)$ is bounded. Thus we may suppose that

$$\lambda_1(\varrho) \to \infty, \text{ as } \varrho \to 1.$$

It follows from Theorem VI that given $\varepsilon > 0$, we can find ϱ_0 such that

(27.1) $$(1-\varrho)\lambda_1(\varrho) < \varepsilon, \quad \varrho \geq \varrho_0,$$

if $a < 1$. Choose w in E such that

(27.2) $$\log |w| > \lambda_1(\varrho_0)$$

and let ϱ_1 be the smallest number such that

$$\lambda(\varrho_1) = \log M[\varrho_1, f_*] = \log |w|.$$

Thus

(27.3) $$\lambda(\varrho_1) = \lambda_1(\varrho_1) = \log |w|.$$

We then apply Theorem IV (i) with ϱ_1 instead of ϱ' and we deduce that if $0 < \varrho < \varrho_1$ then

$$\lambda(\varrho_1) - \lambda(\varrho) < \frac{A(1-\varrho)}{(1-\varrho_1)}[1 + p(\varrho_{1*})],$$

and since from (27.3), $\lambda_1(\varrho_1) = \lambda(\varrho_1)$, we have a fortiori,

(27.4) $$\lambda_1(\varrho_1) < \lambda_1(\varrho) + \frac{A_1 c(1-\varrho)}{(1-\varrho_1)^{1+a}}$$

provided ϱ_1 is so near 1 that

$$p(\varrho_{1*}) = c(1-\varrho_{1*})^{-a} \geq 1,$$

which we may assume by taking w large enough. Suppose now further that

$$\frac{\varepsilon}{(1-\varrho_0)} \leq \frac{A_1 c(1-\varrho_0)}{(1-\varrho_1)^{1+a}}$$

where ϱ_0 is the number of (27.2). This again can be achieved by taking w sufficiently large so that ϱ_1 is sufficiently near 1. We then choose ϱ in (27.4) to satisfy

$$\frac{\varepsilon}{1-\varrho} = \frac{A_1 c (1-\varrho)}{(1-\varrho_1)^{1+a}} \tag{27.5}$$

and it follows that $\varrho \geq \varrho_0$, so that (27.1) holds. Then (27.4), (27.5) give

$$\lambda_1(\varrho_1) \leq \frac{2\varepsilon}{1-\varrho} = 2(A_1 c \varepsilon)^{\frac{1}{2}} (1-\varrho_1)^{-(1+a)/2}. \tag{27.6}$$

We next apply Theorem IV (ii) with ϱ_1 instead of ϱ. We may take $\lambda_1(\varrho')$ instead of $\lambda(\varrho')$ in that theorem, since $\lambda(\varrho) \leq \log|w_1|$, $\varrho \leq \varrho_1$ and the right hand side of Theorem IV (ii) increases with ϱ'. Then Theorem IV (ii) gives

$$\lambda_1(\varrho') - \log|w| \leq \frac{A}{1-\varrho'} \int_{\varrho_{1*}}^{\varrho'_*} \left[1 + \frac{c}{(1-t)^a}\right] dt$$
$$\leq \frac{A(a)\, c\, (1-\varrho_1)^{1-a}}{1-\varrho'}.$$

Then using (27.3), (27.6) we deduce that

$$\lambda_1(\varrho') \leq (A_1 c \varepsilon)^{\frac{1}{2}} (1-\varrho_1)^{-(1+a)/2} + \frac{A(a)\, c\, (1-\varrho_1)^{1-a}}{1-\varrho'}. \tag{27.7}$$

We may suppose ϱ_1 so near 1 that the second term on the right hand side of (27.7) is less than the first when $\varrho' = \varrho_1$. We then choose ϱ' so that the terms are equal, i.e.,

$$(A_1 c \varepsilon)^{\frac{1}{2}} (1-\varrho_1)^{-(1+a)/2} = A(a)\, c \frac{(1-\varrho_1)^{1-a}}{1-\varrho'}$$
$$= \left\{(A_1 c \varepsilon)^{\frac{1\times 2}{2(1+a)}} (A(a)\, c)^{1/(1-a)} (1-\varrho')^{-1/(1-a)}\right\}^{1/\left(\frac{2}{1+a}+\frac{1}{1-a}\right)}$$
$$= K \varepsilon^{K'} (1-\varrho')^{-(1+a)/(3-a)},$$

where K, K' are positive constants independent of ε. Then (27.7) gives

$$\lambda_1(\varrho') \leq 2 K \varepsilon^{K'} (1-\varrho')^{-\frac{1+a}{3-a}}$$

and this holds for values of ϱ' arbitrarily near 1. Also ε is arbitrary. Thus (26.4) holds.

28) To complete the proof of Theorem X it remains to prove (26.5). It follows from Theorem IV (ii) that

$$\lambda_1(\varrho) \leq \frac{O(1)}{1-\varrho} + \frac{A}{1-\varrho} \int_0^{\varrho_*} \frac{c\,dr}{1-r},$$

$$\leq \frac{O(1)}{1-\varrho} + \frac{A\,c}{1-\varrho} \log \frac{1}{1-\varrho_*},$$

i.e.,

(28.1) $$\lambda_1(\varrho) \leq \frac{A\,c}{1-\varrho} \log \frac{1}{1-\varrho}, \quad \varrho \geq \varrho_0.$$

Choose now w in E so large that

(28.2) $$\log |w| > \lambda_1(\varrho_0),$$

and let ϱ be the smallest number such that

(28.3) $$\lambda_1(\varrho) = \log |w|.$$

We apply Theorem IV (ii) and deduce that

(28.4) $$\lambda_1(\varrho') < \lambda_1(\varrho) + \frac{A}{1-\varrho'} \int_{\varrho_*}^{\varrho'_*} \left(1 + \frac{c}{1-r}\right) dr$$

$$\leq \lambda_1(\varrho) + \frac{A\,c}{1-\varrho'} \left[1 + \log \frac{1-\varrho}{1-\varrho'}\right]$$

if $\dfrac{c}{1-\varrho} \geq 1$, which we assume.

We suppose, as we may, that w is so large that $1-\varrho < \dfrac{1}{e}$ and we then choose ϱ' so that

(28.5) $$\frac{1-\varrho}{1-\varrho'} = \log \frac{1}{1-\varrho}.$$

Then (28.3) and (28.4) give

$$\lambda_1(\varrho') \leq \lambda_1(\varrho) + \frac{A\,c}{1-\varrho'} \left[1 + \log\log \frac{1}{1-\varrho}\right]$$

$$\leq \frac{A\,c}{1-\varrho} \log \frac{1}{1-\varrho} + \frac{A}{1-\varrho'} \left[1 + c \log\log \frac{1}{1-\varrho}\right]$$

on making use of (28.1) and (28.2). Using (28.5) and the fact that $\varrho' \geq \varrho$, we deduce that

$$\lambda_1(\varrho') \leq \frac{A c}{1-\varrho'}\left[\log\log\frac{1}{1-\varrho'} + O(1)\right]$$

and since this holds for values of ϱ' arbitrarily near 1, we deduce (26.5). This completes the proof of Theorem X.

We may remark here that the above argument clearly also yields the inequality

$$\lambda_1(\varrho') = O\left[p(\varrho')\log\log\frac{1}{1-\varrho'}\right]$$

for some values of ϱ' arbitrarily near 1 if $p(\varrho)$ is a function of the form

$$p(\varrho) = \frac{c}{1-\varrho}\left(\log\frac{1}{1-\varrho}\right)^{a}$$

and $f(z)$ takes no value of an unbounded set E more than $p(\varrho)$ times in $|z| \leq \varrho$. Thus the convergence or divergence of $\int p(\varrho)\,d\varrho$ does not seem to have the same fundamental difference in effect that appears in Theorem VI. Nevertheless the rate of growth

$$p(\varrho) = \frac{c}{1-\varrho}$$

is nearly critical in the sense that if $p(\varrho)$ grows as rapidly as $(1-\varrho)^{-a}$, with $a > 1$ the results are fundamentally different in character from those when $p(\varrho)$ grows like $(1-\varrho)^{-a}$ with $a < 1$. In the critical case $a = 1$, converse theorems appear to be most difficult to construct. We shall show, however, in Chapter III that all theorems which apply to the case $p(\varrho) = (1-\varrho)^{-a}$ are best possible. These results will appear in the next issue.