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# A compactification of Hénon mappings in $\mathbf{C}^2$ as dynamical systems

by

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## 1. Introduction

Let $H: \mathbf{C}^2 \to \mathbf{C}^2$ be a Hénon mapping

$$H\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} p(x) - ay \\ x \end{bmatrix}, \quad a \neq 0,$$

where $p$ is a polynomial of degree $d \geqslant 2$, which without loss of generality we may take to be monic.

In [HO1], it was shown that there is a topology on $\mathbf{C}^2 \sqcup S^3$ homeomorphic to a 4-ball such that the Hénon mapping extends continuously. That paper used a delicate analysis of some asymptotic expansions, for instance, to understand the structure of forward images of lines near infinity. The computations were quite difficult, and it is not clear how to generalize them to other rational maps.

In this paper we present an alternative approach, involving blow-ups rather than asymptotics. We apply it here to Hénon mappings and their compositions, and in doing so prove a result suggested by Milnor, involving embeddings of solenoids in $S^3$ which are topologically different from those obtained from Hénon mappings. But the method should work quite generally, and help to understand the dynamics of rational maps $f\colon \mathbf{P}^2 \rightsquigarrow \mathbf{P}^2$ with points of indeterminacy. In the papers [V] and [HHV], the method is applied to some other families of rational maps, and in [HP2] it is applied to Newton's method in several variables. The critical points which appear in that setting require new tools; the discussion below concerns only birational mappings.

The approach consists of three steps, which we describe below.

*Resolving points of indeterminacy.* The general theory asserts that a "birational map" $f\colon \mathbf{P}^2 \rightsquigarrow \mathbf{P}^2$ is defined except at finitely many points, and that after a finite number of blow-ups at these points, the map becomes well-defined [Sh2, IV.3, Theorem 3]. Let us denote by $\widetilde{X}_f$ the space obtained after these blow-ups, and by $\tilde{f}\colon \widetilde{X}_f \to \mathbf{P}^2$ the lifted morphism. For Hénon mappings, this is done in §2.

*The complex sequence space.* The mapping $\tilde{f}$ cannot be considered a dynamical system, since the domain and range are different. One way to obtain a dynamical system is to blow up the inverse images of the points we just blew up to construct $\widetilde{X}_f$, then their inverse images, etc. On the projective limit, we finally obtain a dynamical system $f'_\infty\colon X'_\infty \to X'_\infty$. This is not yet the space we want: we started with a birational map and should end up with an automorphism. We can repeat the procedure with $(f'_\infty)^{-1}\colon X'_\infty \rightsquigarrow X'_\infty$, which again may have points of indeterminacy. Blow these up, and then their inverse images, etc., and take a second projective limit. This finally yields a compact space $X_\infty$ and an automorphism $f_\infty\colon X_\infty \to X_\infty$.

The notation to keep track of successive blow-ups grows exponentially and soon becomes intractable. It is much easier to describe the projective limit in terms of sequence spaces. We learned of this construction from [Fr2]; something analogous was constructed by Hirzebruch [Hirz] when resolving the cusps of Hilbert modular surfaces, and was also considered by Inoue, Dloussky and Oeljeklaus [Dl], [DO].

The space $X_\infty$ will usually have a big subset $X^*_\infty$ which is an algebraic variety, but $X_\infty$ will have some points which are extremely singular, in the sense that every neighborhood has infinite-dimensional homology.

We will construct $X_\infty$ for Hénon mappings in §3; its topology (homology, etc.) is studied in §4. A surprise arises when we compute $H_2(X^*_\infty)$: the manifold $X^*_\infty$ contains projective lines constructed during the blow-ups, which define elements in the homology group, but these lines quite unexpectedly do not generate that group (Theorem 4.13).

There is a natural way to complete the homology $H_2(X^*_\infty, \mathbf{C})$ to a Hilbert space, so that $f_\infty$ induces an operator, and we plan in a future publication to show that the spectral invariants of this operator interact with the dynamics.

*The real-oriented blow-up.* In order to "resolve" the terrible singularities of $X_\infty$, we will use a further *real* blow-up, in which we consider complex surfaces as 4-dimensional real algebraic spaces, and divisors as real surfaces in them. One way of thinking of this blow-up of a surface $X$ along a divisor $D$ is to take an open tubular neighborhood $W$ of $D$ in $X$, together with a projection $\pi: \overline{W} \to D$. Excise $W$, to form $X' = X - W$. If $W$ is chosen properly, $X'$ will be a real 4-dimensional manifold with boundary $\partial X' = \partial \overline{W}$. The interior of $X'$ will be homeomorphic to $X - D$. Then $X'$ is some sort of blow-up of $X$ along $D$, with $\pi: \partial X' \to D$ the exceptional divisor. This construction is topologically correct, but non-canonical; the real-oriented blow-up is a way of making it canonical.

We can pass to the projective limit with these real-oriented blow-ups, constructing a space $\mathcal{B}^+_{\mathbf{R}}(X_\infty)$ which is topologically much simpler and better behaved than the original compactification, being a 4-dimensional manifold with boundary. The real interest is in the inner structures of the boundary 3-manifold, where we find solenoids (in this paper and in [V]), tori with irrational foliations (in [HHV]), etc.

The definition and first properties of these blow-ups are given in §5, along with the methods needed to construct them. Theorems 5.7 and 5.8 are the principal tools for understanding real-oriented blow-ups, and we expect that they will be useful for many examples besides the ones explored in this paper.

In §6, we show that the classical Hopf fibration is an example of a real-oriented blow-up, in two different ways, and we construct the real-oriented blow-up for Hénon maps. This is quite an exciting space, and we further explore its structure in §7, using toroidal decompositions. These results allow us to prove (Theorem 7.7) that extensions of the Hénon maps to their sphere at infinity are not all conjugate, even when they have the same degree; the conjugacy class of the extension remembers the argument of the Jacobian of the Hénon mapping.

In §8, we construct the real-oriented blow-up for compositions of Hénon maps, obtaining a 3-sphere with two embedded solenoids $\Sigma^+$ and $\Sigma^-$, but such that the incompressible tori in $S^3 - (\Sigma^+ \cup \Sigma^-)$ are different from the incompressible tori we obtain from just one Hénon mapping. This difference was first conjectured by Milnor [Mi4].

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## 2. Making Hénon mappings well-defined

Consider the Hénon mapping

$$H\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p(x)-ay \\ x \end{pmatrix} \tag{2.1}$$

with $a\neq 0$, which we will consider as a birational mapping $\mathbf{P}^2 \rightsquigarrow \mathbf{P}^2$ given in homogeneous coordinates as

$$H\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \tilde{p}(x,z)-ayz^{d-1} \\ xz^{d-1} \\ z^d \end{bmatrix}, \tag{2.2}$$

where $\tilde{p}(x,z)=z^d p(x/z)=x^d+\ldots$ is a homogeneous polynomial of degree $d$ in the two variables $x$ and $z$.

LEMMA 2.1. (a) *The mapping $H$ has a unique point of indeterminacy at*

$$\mathbf{p}=\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

*and collapses the line at infinity $l_\infty$ to the point*

$$\mathbf{q}=\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

(b) *The mapping $H^{-1}$ has a unique point of indeterminacy at $\mathbf{q}$, and collapses $l_\infty$ to the point $\mathbf{p}$.*

*Proof.* A point of indeterminacy of a mapping written in homogeneous coordinates without common factors is a point where all coordinate functions vanish. In order for this to happen, we must have $z=0$ of course, and the only remaining term is then $x^d$, so that at a point of indeterminacy, we also have $x=0$. Thus $\mathbf{p}$ is the unique point of indeterminacy of $H$. Clearly any other point of $l_\infty$ is mapped to $\mathbf{q}$. Part (b) is similar. □

*Blow-ups of points on surfaces.* The blow-up of an algebraic variety along a subvariety is a standard construction of algebraic geometry [Har]. For the next three sections, we will need only the most elementary case of this construction: the blow-up of a point on a smooth surface, which we review here largely to set the notation.

Let $X$ be a complex surface, and $\mathbf{x}\in X$ a smooth point. Then $X$ blown up at $\mathbf{x}$ is the surface $\widetilde{X}_\mathbf{x}$ defined as follows. Choose a neighborhood $U$ of $\mathbf{x}$ and local coordinates $u_1, u_2$ centered at $\mathbf{x}$. First consider $\widetilde{U}_\mathbf{x}\subset U\times\mathbf{P}^1$ to be the surface of equation

$$\left\{\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}\right) \,\middle|\, u_1U_2=u_2U_1\right\},$$

where

$$\begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$$

represents a point in $\mathbf{P}^1$ written in homogeneous coordinates. Note that $\widetilde{U}_{\mathbf{x}}$ is a smooth surface, and that the projection $\pi: \widetilde{U}_{\mathbf{x}} \to U$ onto the first coordinate is an isomorphism except above $\mathbf{x}$. Therefore, we can define $\widetilde{X}_{\mathbf{x}}$ to be the quotient of the disjoint union $X - \{\mathbf{x}\} \sqcup \widetilde{U}_{\mathbf{x}}$ by the equivalence relation which identifies

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in U - \{\mathbf{x}\} \quad \text{to} \quad \left( \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \right) = \pi^{-1}(\mathbf{u}) \in \widetilde{U}_{\mathbf{x}}.$$

A standard result [Har] asserts that the surface $\widetilde{X}_{\mathbf{x}}$ does not depend on the choice of the neighborhood $U$ or on the local coordinates $u_1, u_2$. The fiber $E = \pi^{-1}(\mathbf{x}) \subset \widetilde{X}_{\mathbf{x}}$, called the *exceptional divisor*, is canonically the projective line associated to the tangent plane $T_{\mathbf{x}}X$.

The space $\widetilde{U}_{\mathbf{x}}$ cannot be covered by a single coordinate patch, since it contains a projective line, but it is covered by two coordinate patches $\widetilde{U}'_{\mathbf{x}}$ and $\widetilde{U}''_{\mathbf{x}}$ where $u_1 \neq 0$ and $u_2 \neq 0$ respectively, which admit local coordinates $u_1$, $u_2/u_1$ and $u_2$, $u_1/u_2$ respectively.

We will frequently describe this situation by the phrase "blow up $\mathbf{x}$, setting $u_1 = u_2 v$". This means "blow up $\mathbf{x}$, and in $\widetilde{X}_{\mathbf{x}}$, consider the coordinate patch parametrized by $u_2$ and $v = u_1/u_2$".

If $C \subset X$ is a curve, the *proper transform* of $C$ is the curve $C' \subset \widetilde{X}_{\mathbf{x}}$ which is the closure of $\pi^{-1}(C) - E$.

*Self-intersections*. It is often necessary to know the self-intersection numbers of lines obtained when we make successive blow-ups. In this case, where we are blowing up a surface at a smooth point $\mathbf{x}$, the rules for computing these numbers are simple [Sh2, IV.3, Theorem 2 and its corollaries]:

- The exceptional divisor has self-intersection $-1$;
- The proper transform $C'$ of any smooth curve $C$ passing through $\mathbf{x}$ has its self-intersection decreased by 1.

*Making H well-defined*. We will now go through a sequence of $2d-1$ blow-ups, required to make the Hénon mapping (2.2) well-defined. The results are summarized at the end in Theorem 2.3, which we cannot state without the terminology which we create during the construction. We will denote by $H_1, \ldots, H_{2d-1}$ the extension of the Hénon mapping to the successive blow-ups.

To focus on the point of indeterminacy, we will begin work in the coordinates $u = x/y$, $v = 1/y$, so that the point of indeterminacy $\mathbf{p}$ is the point $u = v = 0$. The Hénon mapping,

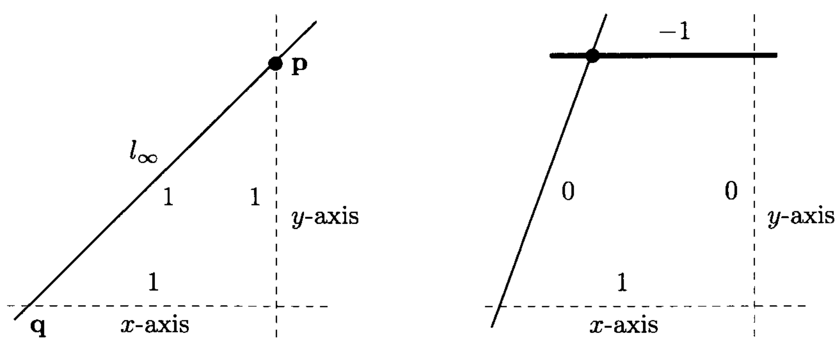



Fig. 1. Left: the original configuration of the axes (dotted) and the line at infinity in $\mathbf{P}^2$. Right: the configuration after the first blow-up. The last exceptional divisor is denoted by a thick line; the heavy dots are the points of indeterminacy of $H$ and $H_1$. The numbers labeling irreducible components are self-intersection numbers.

using the affine coordinates $u, v$ in the domain and homogeneous coordinates in the range, is written

$$H\colon \begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{bmatrix} \tilde{p}(u,v) - av^{d-1} \\ uv^{d-1} \\ v^d \end{bmatrix}.$$

At a point of indeterminacy all three homogeneous coordinates vanish; we knew that this happens only at $\mathbf{p}$, but it is clear again from this formula that it happens only at $u=v=0$.

*The first blow-up.* Blow up $\mathbf{P}^2$ at the point $u=v=0$, using the chart $v=uX_1$; i.e., use $u$ and $X_1=v/u$ as coordinates on the blown-up surface, and discard $v$ as a coordinate.

The extension of the Hénon mapping, using the affine coordinates $u, X_1$ in the domain and homogeneous coordinates in the range, is written

$$H_1\colon \begin{pmatrix} u \\ X_1 \end{pmatrix} \mapsto \begin{bmatrix} \tilde{p}(u, X_1 u) - a(X_1 u)^{d-1} \\ u^d X_1^{d-1} \\ u^d X_1^d \end{bmatrix} = \begin{bmatrix} uq(X_1) - aX_1^{d-1} \\ uX_1^{d-1} \\ uX_1^d \end{bmatrix}, \tag{2.3}$$

where we have set $\tilde{p}(1,X)=q(X)$, so that $q$ is a polynomial of degree $d$ whose constant term is 1.

Again, the only point where all three homogeneous coordinates vanish is $u=X_1=0$. We invite the reader to check that the one point of the blow-up not covered by the chart $u, X_1$ is not a point of indeterminacy. Figure 1 shows the self-intersection numbers: originally $l_\infty$ has self-intersection number 1; after one blow-up its proper transform acquires self-intersection number 0, and the exceptional divisor has self-intersection number $-1$.

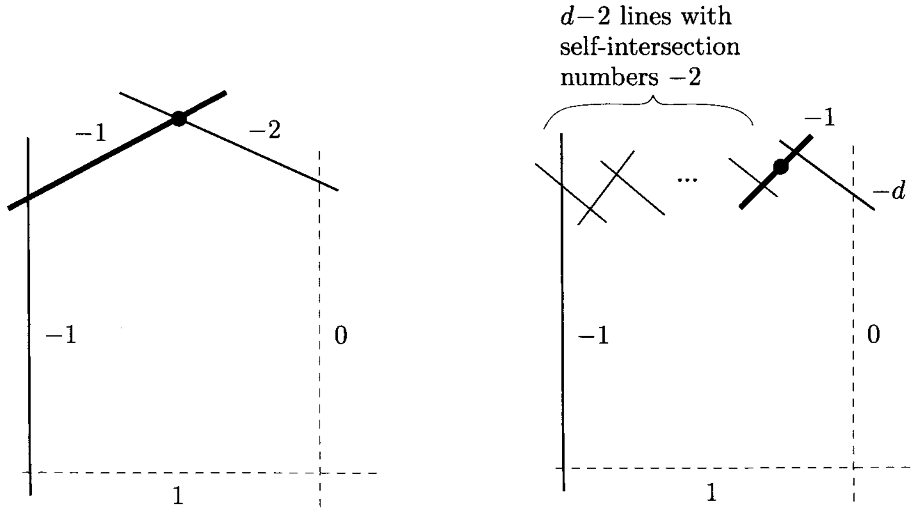



Fig. 2. The configuration after the second blow-up (left) and after the $d$th blow-up (right). The heavy dots are the points of indeterminacy of $H_2$ and $H_d$; note that the black dot on the right is an ordinary point; all the earlier ones except the very first were double points. Again, the numbers labeling components are the self-intersection numbers.

*Several more blow-ups.* We will now make a sequence of $d-1$ further blow-ups, setting

$$u = X_1 X_2, \quad X_2 = X_1 X_3, \quad \ldots, \quad X_{d-1} = X_1 X_d.$$

In other words, blow up the point $u = X_1 = 0$, focusing on the coordinate patch where $u$ and $X_2 = u/X_1$ are coordinates. Then blow up the point in that coordinate patch where $X_1 = X_2 = 0$, and focus on the coordinate patch of that surface where $X_1$ and $X_3 = X_1/X_2$ are coordinates, etc.

The Hénon mapping in these coordinates is given by the formula

$$H_k\colon \begin{pmatrix} X_1 \\ X_k \end{pmatrix} \mapsto \begin{bmatrix} X_1 X_k q(X_1) - aX_1^{d-k+1} \\ X_1^d X_k \\ X_1^{d+1} X_k \end{bmatrix} = \begin{bmatrix} X_k q(X_1) - aX_1^{d-k} \\ X_1^{d-1} X_k \\ X_1^d X_k \end{bmatrix}. \tag{2.4}$$

We see that when $k<d$, the unique point of indeterminacy is $X_1 = X_k = 0$, but when $k=d$, the unique point of indeterminacy is $X_1 = 0$, $X_d = a$. At each step, there is one point $X_k = \infty$ which is not in the domain of our chart; we leave it to the reader to check that this is never a point of indeterminacy.

Again the self-intersection numbers are as indicated in Figure 2. At each step we are blowing up the intersection of the last exceptional divisor with the proper transform of the first exceptional divisor. Therefore, after this sequence of blow-ups, the last exceptional divisor (now next-to-last) acquires self-intersection $-2$, and the first exceptional divisor has its self-intersection number decreased by 1, going from $-1$ to $-d$.

*The blow-ups that depend on the coefficients of $p$.* The next $d-1$ blow-ups have rather more unpleasant formulas, because each occurs at a smooth point of the last exceptional divisor, and we need to specify this point. To lighten the notation, we will define the polynomials $q_0, \ldots, q_{d-1}$ by induction:

$$q_0(X)=q(X) \quad \text{and} \quad q_{k+1}(X)=\frac{q_k(X)-q_k(0)}{X}, \quad k=0, \ldots, d-2;$$

and the numbers $Q_0, \ldots, Q_{d-1}$ (they are really coordinates of points on exceptional divisors) by

$$Q_0=1 \quad \text{and} \quad Q_{k+1}=-\sum_{j=0}^{k} Q_j q_{k-j+1}(0).$$

Set $Y_0=X_d$, and make the successive blow-ups

$$Y_k-aQ_k=X_1Y_{k+1}, \quad k=0, \ldots, d-2. \tag{2.5}$$

*Remark.* This means that $Y_{k+1}$ is the slope of a line through the point $X_1=0$, $Y_k=aQ_k$. In that sense the number $aQ_k$ is really the coordinate on the line parametrized by $Y_k$ of the next point at which to blow up.

LEMMA 2.2. (a) *In the coordinates $X_1, Y_k$, the Hénon mapping is given by the formula*

$$H_{d+k}\colon \begin{pmatrix} X_1 \\ Y_k \end{pmatrix} \mapsto \begin{bmatrix} X_1Y_kq_1(X_1)+a\sum_{j=0}^{k-1} Q_jq_{k-j}(X_1)+Y_k \\ X_1^{d-k-1}\left(X_1^kY_k+a\sum_{j=0}^{k-1} Q_jX_1^j\right) \\ X_1^{d-k}\left(X_1^kY_k+a\sum_{j=0}^{k-1} Q_jX_1^j\right) \end{bmatrix}, \quad k=1, \ldots, d-1. \tag{2.6}$$

(b) *The mapping $H_{d+k}$ has the unique point of indeterminacy $X_1=0$, $Y_k=aQ_k$, for $k=1, \ldots, d-2$.*

(c) *The mapping $H_{2d-1}$ has no point of indeterminacy, and maps the last exceptional divisor to $l_\infty \subset \mathbf{P}^2$ by an isomorphism.*

*Proof.* This is an easy induction: all the work was in finding the formula. To start the induction, use formula (2.4) to compute the extension of the Hénon mapping in the chart $X_d-a=X_1Y_1$:

$$H_{d+1}\colon \begin{pmatrix} X_1 \\ Y_1 \end{pmatrix} \mapsto \begin{bmatrix} (a+X_1Y_1)(q(X_1)-1)-X_1Y_1 \\ X_1^{d-1}(a+X_1Y_1) \\ X_1^d(a+X_1Y_1) \end{bmatrix} = \begin{bmatrix} X_1Y_1q_1(X_1)+aq_1(X_1)+Y_1 \\ X_1^{d-2}(X_1Y_1+a) \\ X_1^{d-1}(X_1Y_1+a) \end{bmatrix}, \tag{2.7}$$

where we have used $q(X_1)-1=X_1q_1(X_1)$, and factored out $X_1$. Observe that formula (2.7) is exactly the case $k=1$ of formula (2.6).

Now suppose that Lemma 2.2 is true for $k$, and substitute $Y_k=X_1Y_{k+1}+aQ_k$ from formula (2.5). For the first coordinate of $H_{d+k+1}$ we find

$$\begin{aligned}
X_1(X_1Y_{k+1}+aQ_k)q_1(X_1)&+a\sum_{j=0}^{k-1}Q_jq_{k-j}(X_1)+X_1Y_{k+1}+aQ_k\\
&=X_1^2Y_{k+1}q_1(X_1)+aX_1Q_kq_1(X_1)+a\sum_{j=0}^{k-1}Q_j(q_{k-j}(X_1)-q_{k-j}(0))+X_1Y_{k+1}\\
&=X_1^2Y_{k+1}q_1(X_1)+aX_1Q_kq_1(X_1)+aX_1\sum_{j=0}^{k-1}Q_jq_{k-j+1}(X_1)+X_1Y_{k+1}\\
&=X_1\Big(X_1Y_{k+1}q_1(X_1)+a\sum_{j=0}^{k}Q_jq_{k-j+1}(X_1)+Y_{k+1}\Big).
\end{aligned}$$

The second and third coordinates are similar. In particular, we can factor out $X_1$, until $k=d-1$. This proves part (a).

At each step, any points of indeterminacy must be on the last exceptional divisor of equation $X_1=0$. But if we substitute $X_1=0$ in the first coordinate, we find $Y_k=aQ_k$, so that indeed there is only one point of indeterminacy, proving (b).

The restriction to the last exceptional divisor of the mapping $H_{2d-1}$ is given by

$$Y_{d-1}\mapsto\begin{bmatrix}Y_{d-1}-aQ_{d-1}\\ a\\ 0\end{bmatrix};$$

since $a\neq0$, the map is well-defined. Since $Y_{d-1}$ appears in the first coordinate with degree 1, this last exceptional divisor maps by an isomorphism to the line at infinity. □

Figure 3 shows the self-intersection numbers. We are now always blowing up an ordinary point of the last exceptional divisor, so this last exceptional divisor (now next-to-last) acquires self-intersection $-2$. At the end, the last exceptional divisor keeps self-intersection $-1$.

To summarize, we have proved the following result. Denote by $\widetilde{X}_H$ the space obtained from $\mathbf{P}^2$ by the sequence of $2d-1$ blow-ups described above.

THEOREM 2.3. *The Hénon map $H:\mathbf{C}^2\to\mathbf{C}^2$ extends to a morphism $H_{2d-1}=\widetilde{H}:\widetilde{X}_H\to\mathbf{P}^2$, and maps $\widetilde{D}=\widetilde{X}_H-\mathbf{C}^2$, the divisor at infinity, to $l_\infty$, mapping all of $\widetilde{D}$ to the point $\mathbf{q}$ except the last exceptional divisor, which is mapped to $l_\infty$ by an isomorphism.*

*Terminology.* To state the next result, we need to name the irreducible components of $\widetilde{D}$. Let us label $A'$ the proper transform of the line at infinity, $B'$ the proper transform

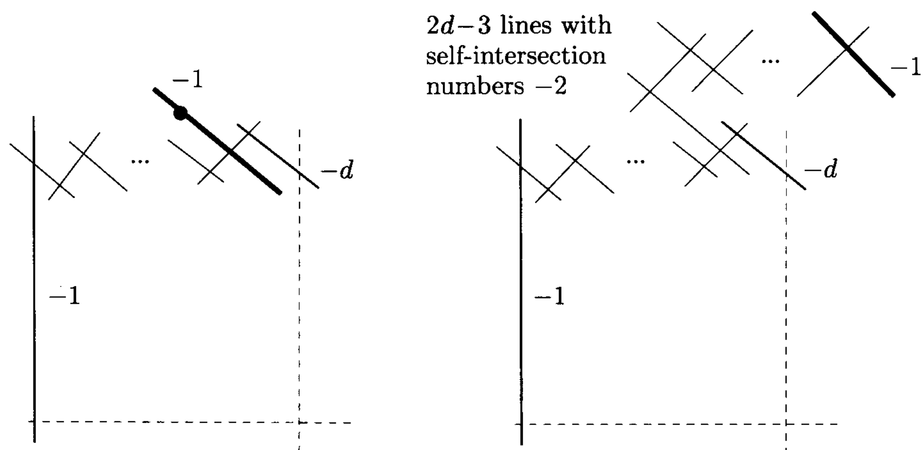



Fig. 3. Left: the configuration after the $(d+1)$st blow-up. Right: after all $2d-1$ blow-ups.

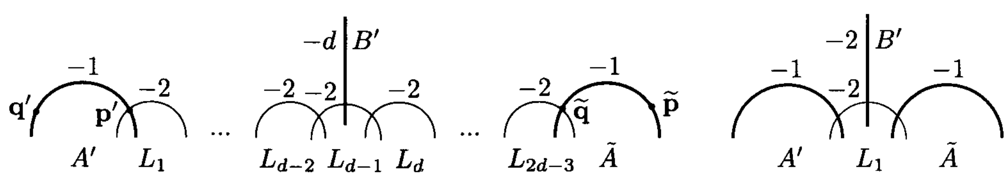



Fig. 4. The divisor $\widetilde{D}$. The numbers labeling the components are the self-intersection numbers. On the right, the simpler drawing for degree 2.

of the first exceptional divisor, then in order of creation $L_1, L_2, \ldots, L_{2d-3}$, and finally $\tilde{A}$ the components of the divisor $\widetilde{D}$. The line $\tilde{A}$, i.e., the last exceptional divisor, will play a special role.

Since $A'$ is the proper transform of $l_\infty$, the projection $A' \to l_\infty$ is an isomorphism, and we can define $\mathbf{p}', \mathbf{q}'$ as the points of $A'$ that correspond to $\mathbf{p}, \mathbf{q}$. Note that $\{\mathbf{p}'\} = L_1 \cap A'$.

The points $\widetilde{\mathbf{p}} = \widetilde{H}^{-1}(\mathbf{p})$, $\widetilde{\mathbf{q}} = \widetilde{H}^{-1}(\mathbf{q})$ play a parallel role; note that $\{\widetilde{\mathbf{q}}\} = L_{2d-3} \cap \tilde{A}$. This terminology is illustrated in Figure 4, which represents $\widetilde{D}$, i.e., Figure 3 (right), redrawn in a more symmetrical way.

*The rational mapping $H^\sharp$.* In the next section, we will want to consider $\widetilde{H}$ as a birational map $H^\sharp : \widetilde{X}_H \rightsquigarrow \widetilde{X}_H$.

THEOREM 2.4. *The rational map $H^\sharp : \widetilde{X}_H \rightsquigarrow \widetilde{X}_H$ is defined at all points except $\widetilde{\mathbf{p}}$. It collapses $\widetilde{D} - \tilde{A}$ to $\mathbf{q}'$ and maps $\tilde{A} - \widetilde{\mathbf{p}}$ to $A' - \mathbf{p}'$ by an isomorphism.*

*The rational map $(H^\sharp)^{-1}$ is defined at all points except $\mathbf{q}'$. It collapses $\widetilde{D} - A'$ to $\widetilde{\mathbf{p}}$ and maps $A' - \mathbf{q}'$ to $\tilde{A} - \widetilde{\mathbf{q}}$ by an isomorphism.*

Theorem 2.4 is illustrated in Figure 5.

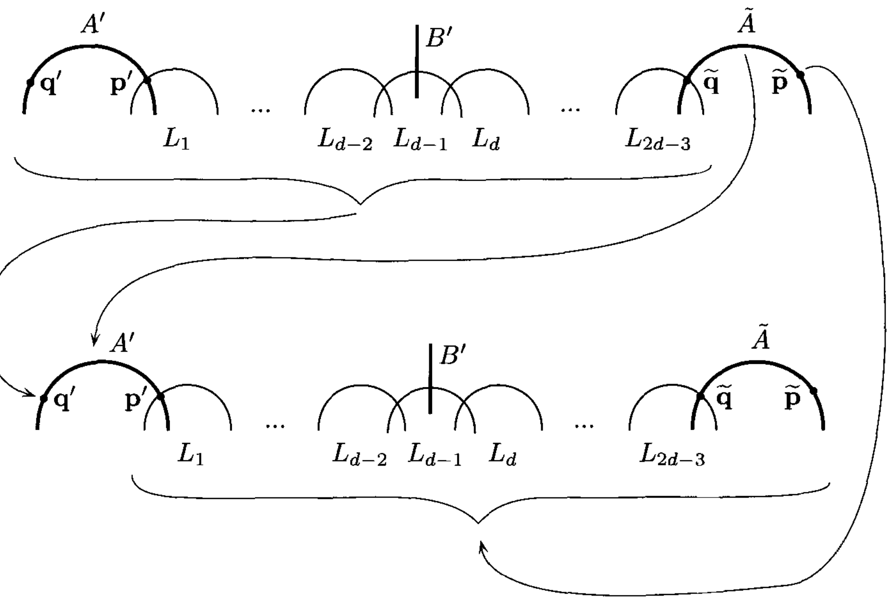



Fig. 5. The "mapping" $H^\sharp$ acting on the divisor $\widetilde{D}$. It is not a true mapping because it has a point of indeterminacy at $\widetilde{\mathbf{p}}$.

*Proof.* This is really a corollary of Theorem 2.3. Clearly, $H^\sharp$ is well-defined on $\widetilde{X}_H - \widetilde{H}^{-1}(\mathbf{p})$, i.e., on $\widetilde{X}_H - \{\widetilde{\mathbf{p}}\}$, and coincides with $\widetilde{H}$ there.

Further, $\widetilde{H}$ is an isomorphism from a neighborhood of $\widetilde{\mathbf{p}}$ to a neighborhood of $\mathbf{p}$. So if we perform any sequence of blow-ups at $\mathbf{p}$, $\widetilde{H}$ will become undetermined at $\widetilde{\mathbf{p}}$. The statements about the inverse map are similar. □

## 3. Closures of graphs and sequence spaces

We now have a well-defined map $\widetilde{H}: \widetilde{X}_H \to \mathbf{P}^2$, but that does not solve our problem of compactifying $H: \mathbf{C}^2 \to \mathbf{C}^2$ as a dynamical system. We cannot consider $\widetilde{H}$ as a dynamical system, since the domain and the range are different. Neither does $H^\sharp$ solve our problem, since it still has a point of indeterminacy. In this section we show how to perform infinitely many blow-ups so that in the projective limit we do get a dynamical system. We will construct this infinite blow-up as a sequence space, as this simplifies the presentation and proof (this description was inspired by Friedland [Fr2], who considered the analog in $(\mathbf{P}^2)^{\mathbf{Z}}$). To make this construction, we need to analyze the graph of $H^\sharp$.

Let $X, Y$ be compact smooth algebraic surfaces, and $f: X \rightsquigarrow Y$ be a birational transformation. Let us suppose that $f$ is undefined at $\mathbf{p}_1, \ldots, \mathbf{p}_n$, and that $f^{-1}$ is undefined at $\mathbf{q}_1, \ldots, \mathbf{q}_m$. Let $\Gamma_f \subset (X - \{\mathbf{p}_1, \ldots, \mathbf{p}_n\}) \times Y$ be the graph of $f$, and let $\overline{\Gamma}_f \subset X \times Y$ be its closure.

LEMMA 3.1. *The space $\bar{\Gamma}_f$ is a smooth manifold, except perhaps at points $(\mathbf{x},\mathbf{y})\in\bar{\Gamma}_f$ such that*

$$\mathbf{x}\in\{\mathbf{p}_1,\dots,\mathbf{p}_n\} \quad \textit{and} \quad \mathbf{y}\in\{\mathbf{q}_1,\dots,\mathbf{q}_m\}.$$

*Proof.* Clearly the projection $\mathrm{pr}_1\colon\bar{\Gamma}_f\to X$ onto the first coordinate is locally an isomorphism near $(\mathbf{x},\mathbf{y})$ if $\mathbf{x}\notin\{\mathbf{p}_1,\dots,\mathbf{p}_n\}$, and $\mathrm{pr}_2\colon\bar{\Gamma}_f\to Y$ is locally an isomorphism near $(\mathbf{x},\mathbf{y})$ unless $\mathbf{y}\in\{\mathbf{q}_1,\dots,\mathbf{q}_m\}$. □

*Example* 3.2. Points $(\mathbf{p}_i,\mathbf{q}_j)\in\bar{\Gamma}_f$ can genuinely be quite singular. For instance, if $X=Y=\mathbf{P}^2$ and $f=H$ is a Hénon mapping, then $f$ (resp. $f^{-1}$) has a unique point of indeterminacy $\mathbf{p}$ (resp. $\mathbf{q}$) (see Lemma 2.1). The pair $(\mathbf{p},\mathbf{q})$ is in $\bar{\Gamma}_H$, and near $(\mathbf{p},\mathbf{q})$ we can find equations of $\bar{\Gamma}_H$ as follows.

In local coordinates

$$u=x/y \quad \text{and} \quad v=1/y \quad \text{near } \mathbf{p},$$
$$s=y/x \quad \text{and} \quad t=1/x \quad \text{near } \mathbf{q},$$

the space $\bar{\Gamma}_H$ is given by the two equations

$$v^d=t(\tilde{p}(u,v)-av^{d-1}) \quad \text{and} \quad ut=sv;$$

it is quite singular indeed at the origin; one way to understand §2 is as a resolution of this singularity, as Proposition 3.3 shows.

Let $H$ be a Hénon mapping, $\widetilde{X}_H$ be the blow-up on which $\widetilde{H}\colon\widetilde{X}_H\to\mathbf{P}^2$ is well-defined, and $H^\sharp\colon\widetilde{X}_H\rightsquigarrow\widetilde{X}_H$ be $\widetilde{H}$ viewed as a rational mapping from $\widetilde{X}_H$ to itself.

PROPOSITION 3.3. *The closure $\bar{\Gamma}_{H^\sharp}\subset\widetilde{X}_H\times\widetilde{X}_H$ is a smooth submanifold.*

*Proof.* The mapping $H^\sharp\colon\widetilde{X}_H\rightsquigarrow\widetilde{X}_H$ is birational, and as we saw in Theorem 2.4, it has a unique point of indeterminacy at $\widetilde{\mathbf{p}}=\widetilde{H}^{-1}(\mathbf{p})$, and the inverse birational mapping $(H^\sharp)^{-1}$ also has a unique point of indeterminacy $\mathbf{q}'$. But the point $(\widetilde{\mathbf{p}},\mathbf{q}')$ is *not* in $\bar{\Gamma}_{H^\sharp}$, so $\bar{\Gamma}_{H^\sharp}$ is a smooth (compact) manifold by Lemma 3.1. □

There is another description of $\bar{\Gamma}_{H^\sharp}$, which we will need in a moment.

PROPOSITION 3.4. *The space $\bar{\Gamma}_{H^\sharp}$, together with the projections $\mathrm{pr}_1$ and $\mathrm{pr}_2$ onto the first and second factor respectively, make the diagram*

$$\begin{tikzcd}
\bar{\Gamma}_{H^\sharp} \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & \widetilde{X}_H \arrow[d, "\pi"] \\
\widetilde{X}_H \arrow[r, "\widetilde{H}"] & \mathbf{P}^2
\end{tikzcd}$$

*a fibered product in the category of analytic spaces* [DD].

*Proof.* The diagram

$$\begin{tikzcd} \widetilde{X}_H \times \widetilde{X}_H \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & \widetilde{X}_H \arrow[d, "\pi"] \\ \widetilde{X}_H \arrow[r, "\widetilde{H}"] & \mathbf{P}^2 \end{tikzcd}$$

commutes on the graph $\Gamma_H$, and also commutes on a closed set, so it commutes on $\bar{\Gamma}_{H^\sharp}$.

Since all the spaces involved are manifolds, it is enough to prove that the diagram is a fibered product in the category of analytic manifolds, i.e., set-theoretically. Since $\pi(\mathbf{y})=\widetilde{H}(\mathbf{x})$ on $\Gamma_H$ this is still true on the closure $\bar{\Gamma}_{H^\sharp}$. □

It is time to construct one of our main actors. The space $X_\infty$, constructed below, is a compact space which contains $\mathbf{C}^2$ as a dense open subset, and such that $H:\mathbf{C}^2\to\mathbf{C}^2$ extends to $H_\infty: X_\infty\to X_\infty$. The locus $D_\infty=X_\infty-\mathbf{C}^2$ is an infinite divisor at infinity, the geometry of which encodes the behavior of $H$ at infinity.

*Definition* 3.5. Let $X_\infty\subset(\widetilde{X}_H)^{\mathbf{Z}}$ be the set of sequences $\underline{\mathbf{x}}=(\dots,\mathbf{x}_{-1},\mathbf{x}_0,\mathbf{x}_1,\dots)$ such that successive pairs belong to $\bar{\Gamma}_{H^\sharp}\subset\widetilde{X}_H\times\widetilde{X}_H$ above.

Let $H_\infty: X_\infty\to X_\infty$ be the shift map

$$(H_\infty(\underline{\mathbf{x}}))_k=\mathbf{x}_{k+1},\quad k\in\mathbf{Z},$$

where $\underline{\mathbf{x}}=(\dots,\mathbf{x}_{-1},\mathbf{x}_0,\mathbf{x}_1,\dots)$ is a point of $X_\infty$.

Clearly $X_\infty$ is a compact space, since it is a closed subset of a product of compact sets, and $H_\infty$ is a homeomorphism $X_\infty\to X_\infty$. We will see below why $H_\infty$ can be understood as an extension of $H$.

PROPOSITION 3.6. (a) *Each point of $X_\infty$ is of one of three types*:

(1) *sequences with all entries in* $\mathbf{C}^2$;

(2) *sequences of the form* $(\dots,\widetilde{\mathbf{p}},\widetilde{\mathbf{p}},\mathbf{a},\mathbf{b},\mathbf{q}',\mathbf{q}',\dots)$ *with* $\mathbf{a}\in\widetilde{D}$, $\mathbf{a}\neq\widetilde{\mathbf{p}}$;

(3) *the two sequences* $\mathbf{p}^\infty=(\dots,\widetilde{\mathbf{p}},\widetilde{\mathbf{p}},\dots)$ *and* $\mathbf{q}^\infty=(\dots,\mathbf{q}',\mathbf{q}',\dots)$.

(b) *The sequences of type* (1) *are dense in* $X_\infty$.

*Proof.* If a sequence has any entry in $\mathbf{C}^2$, it is the full orbit of that point, forwards and backwards. Otherwise, all entries are in the divisor $\widetilde{D}=\widetilde{X}_H-\mathbf{C}^2$. If these entries are not all $\widetilde{\mathbf{p}}$, or all $\mathbf{q}'$, then there is a first entry $\mathbf{a}$ that is not $\widetilde{\mathbf{p}}$; it must be preceded by all $\widetilde{\mathbf{p}}$'s. It is followed by the orbit of $\mathbf{a}$, which is well-defined. Note that $\mathbf{b}=\widetilde{H}(\mathbf{a})$ may be $\mathbf{q}'$ (this will happen unless $\mathbf{a}\in\tilde{A}$), and all the successive terms must be $\mathbf{q}'$. This proves (a).

For part (b), we must show that a sequence $\underline{\mathbf{x}}$ of type 2 or 3 can be approximated by an orbit, i.e., that for any $\varepsilon>0$ and any integer $N$, there is a point $\mathbf{y}\in\mathbf{C}^2$ such that

$d(H^n(\mathbf{y})-\mathbf{x}_n)<\varepsilon$ when $|n|<N$. If $\underline{\mathbf{x}}$ is of type 2, we may assume that $\mathbf{x}_0\neq\widetilde{\mathbf{p}},\mathbf{q}'$. Then all iterates of $\widetilde{H}$ and of $\widetilde{H}^{-1}$ are defined and continuous in a neighborhood of $\mathbf{x}_0$, so any point in this neighborhood and close to $\mathbf{x}_0$ will have a long stretch of forward-and-backwards orbits close to $\underline{\mathbf{x}}$; but every neighborhood of $\mathbf{x}_0$ contains points of $\mathbf{C}^2$, which is dense in $\widetilde{X}$.

Similarly, the orbit of a point with $|x|$ very large and $y=0$ will approximate $\mathbf{q}^\infty$, and a point with $|y|$ very large and $x=0$ will approximate $\mathbf{q}^\infty$. □

*Example* 3.7. The fact that $\mathbf{C}^2$ is dense in $X_\infty$ is not quite so obvious as one might think, and there are examples of birational maps where it is not. For instance, consider the mapping

$$f:\begin{pmatrix}x\\y\end{pmatrix}\mapsto\begin{pmatrix}xy\\x\end{pmatrix},$$

a priori well-defined on $(\mathbf{C}^*)^2$. Denote $\mathbf{p}$ and $\mathbf{q}$ the points at infinity on the $x$-axis and $y$-axis respectively. Then when $r$ is in the line at infinity, the pair $(r,\mathbf{q})$ belong to $\bar{\Gamma}_F\subset\mathbf{P}^2\times\mathbf{P}^2$, as does the pair $(\mathbf{q},r)$. Thus the sequence space contains points like

$$(\dots,\mathbf{q},r,\mathbf{q},\mathbf{q},r,\mathbf{q},r,\mathbf{q},\mathbf{q},\dots)$$

with symbols $\mathbf{q}$ and $r$ in any order. Such sequences cannot be approximated by orbits in $(\mathbf{C}^*)^2$; they also form subsets which have dimension equal to the number of appearances of $r\neq\mathbf{q}$, which may be infinite. Such sequence spaces are a little scary, as well as pathological, and irrelevant to the original dynamical system. In our case, if we had not blown up $\mathbf{P}^2$, then $\mathbf{C}^2$ would still have been dense in the sequence space, but it would have had bad singularities.

PROPOSITION 3.8. *The space* $X^*_\infty=X_\infty-\{\mathbf{p}^\infty,\mathbf{q}^\infty\}$ *is an algebraic manifold. More precisely,*

(1) *the projection* $\pi_0$ *onto the* 0*-th coordinate induces an isomorphism of the space of orbits of the first type to* $\mathbf{C}^2$;

(2) *if* $\underline{\mathbf{x}}=(\dots,\widetilde{\mathbf{p}},\widetilde{\mathbf{p}},\mathbf{a},\mathbf{b},\mathbf{q}',\mathbf{q}',\dots)$ *is a point of the second type, and* $\mathbf{a}$ *appears in the* $k$*-th position, then the projection* $\pi_k$ *onto the* $k$*-th position induces a homeomorphism of a neighborhood of* $\underline{\mathbf{x}}$ *onto* $\widetilde{X}_H-\{\widetilde{\mathbf{p}},\mathbf{q}'\}$.

*Proof.* The first part is clear. For the second, if a point $\underline{\mathbf{y}}$ satisfies $\mathbf{y}_k\neq\widetilde{\mathbf{p}},\mathbf{q}'$, then the entire forward-and-backwards orbit of $\mathbf{y}_k$ is defined: forwards it will never land on $\widetilde{\mathbf{p}}$, and backwards it will never land on $\mathbf{q}'$.

Let us call $\phi_k:\widetilde{X}_H-\{\widetilde{\mathbf{p}},\mathbf{q}'\}\to X_\infty$ the map which maps $\mathbf{x}$ to the unique sequence $\underline{\mathbf{x}}\in X_\infty$ with $\mathbf{x}_k=\mathbf{x}$. The change of coordinate map $\phi_l^{-1}\circ\phi_k$ is then simply $H^{l-k}$ on $\mathbf{C}^2$.

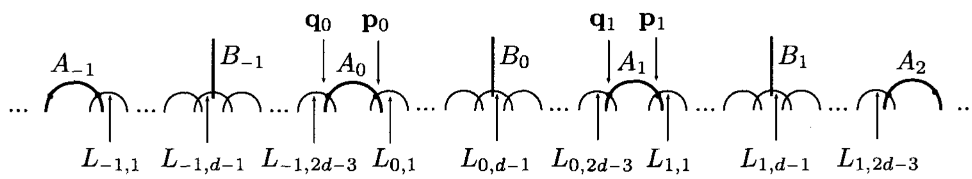



Fig. 6. The divisor $D_\infty$.

This shows that the coordinate changes are algebraic on the intersections of coordinate neighborhoods, except for one detail. The set $\mathbf{C}^2 \subset X_\infty$ is exactly the intersection of the images of $\phi_k$ and $\phi_l$ when $|l-k| \geqslant 2$, but when $l=k+1$, the intersection then contains the sequences with entries $(..., \tilde{\mathbf{p}}, \tilde{\mathbf{p}}, \mathbf{a}, \mathbf{b}, \mathbf{q}', \mathbf{q}', ...)$ with $\mathbf{a} \in \tilde{A} - \{\tilde{\mathbf{p}}\}$ and $\mathbf{b} \in A' - \{\mathbf{q}'\}$. In this case also the change of coordinates is given by $H^\sharp$ and is still algebraic. □

*Remark.* Contrary to what usually happens in algebraic geometry (and is sometimes required in the definition of an algebraic manifold), $X_\infty^*$ is not quasi-compact for the Zariski topology, i.e., it cannot be covered by a finite number of affine algebraic manifolds.

We can now see why $H_\infty$ is an extension of $H$. On the subset isomorphic to $\mathbf{C}^2$ formed of sequences in $\mathbf{C}^2$, with $\pi_0$ the isomorphism, we have

$$\pi_0(H_\infty(\underline{\mathbf{x}})) = H(\pi_0(\underline{\mathbf{x}}));$$

i.e., on that subset, $\pi_0$ conjugates $H_\infty$ to $H$.

*Notation.* We will systematically identify $\mathbf{C}^2$ with $\phi_0(\mathbf{C}^2) \subset X_\infty$. With this identification, $H_\infty$ does extend $H$ continuously, and algebraically in $X_\infty^*$. Moreover, we will set $D_\infty = X_\infty - \mathbf{C}^2$. Figure 6 shows $D_\infty$.

The lines denoted by $A_i$, $i \in \mathbf{Z}$, are formed of those sequences whose $i$th entry is in $A'$; each such line connects the sequences whose $i$th entry is in $L_1$ (denoted by $L_{i,1}$) with those whose $(i-1)$st entry is in $L_{2d-3}$. In particular, the points $\mathbf{q}_0, \mathbf{p}_0, \mathbf{q}_1, \mathbf{p}_1 \in X_\infty$ correspond to the sequences

$$\begin{array}{lllllll}
\mathbf{q}_0 = (..., & \tilde{\mathbf{p}}, & \tilde{\mathbf{q}}, & \mathbf{q}', & \mathbf{q}', & \mathbf{q}', & ...), \\
..., & -2, & -1, & 0, & 1, & 2, & ...
\end{array}$$

$$\begin{array}{lllllll}
\mathbf{p}_0 = (..., & \tilde{\mathbf{p}}, & \tilde{\mathbf{p}}, & \mathbf{p}', & \mathbf{q}', & \mathbf{q}', & ...), \\
..., & -2, & -1, & 0, & 1, & 2, & ...
\end{array}$$

$$\begin{array}{lllllll}
\mathbf{q}_1 = (..., & \tilde{\mathbf{p}}, & \tilde{\mathbf{p}}, & \tilde{\mathbf{q}}, & \mathbf{q}', & \mathbf{q}', & ...), \\
..., & -2, & -1, & 0, & 1, & 2, & ...
\end{array}$$

$$\begin{array}{lllllll}
\mathbf{p}_1 = (..., & \tilde{\mathbf{p}}, & \tilde{\mathbf{p}}, & \tilde{\mathbf{p}}, & \mathbf{p}', & \mathbf{q}', & ...). \\
..., & -2, & -1, & 0, & 1, & 2, & ...
\end{array}$$

## 4. The homology of $X_\infty^*$

In this section we will study the homology groups $H_i(X_\infty^*)$, more particularly $H_2(X_\infty^*)$ and the quadratic form on it coming from the intersection product.

Although nasty spaces (solenoids, etc.) are lurking around every corner, here we will compute only the homology groups of manifolds, being careful to exclude the nasty parts. All homology theories coincide for such spaces, and we may use singular homology, for instance. Unless stated otherwise, we use integer coefficients; at the end we will use complex coefficients. Using $d$-torsion coefficients would give quite different results, which can easily be derived using the universal coefficient theorem.

*Inductive limits.* It is fairly easy to represent $X_\infty^*$ as an increasing union of subsets whose homology can be computed. Since inductive limits and homology commute, it is enough to understand these subsets.

First some terminology. If $G$ is an Abelian group, then $G^{\mathbf{N}}$ is the product of infinitely many copies of $G$, indexed by $\mathbf{N}$, i.e., the set of all sequences $(g_1, g_2, \dots)$ with $g_i \in G$. The group $G^{(\mathbf{N})} \subset G^{\mathbf{N}}$ is the set of sequences with only finitely many non-zero terms; in the category of Abelian groups, this is the *sum* of copies of $G$ indexed by $\mathbf{N}$; it is also easy to show that it is the inductive limit of the diagram

$$G \to G^2 \to G^3 \to \dots$$

where the map

$$G^k \to G^{k+1} \quad \text{is} \quad (g_1, \dots, g_k) \mapsto (g_1, \dots, g_k, 0).$$

The inductive limit we will encounter is not quite elementary. We will start with an example, which has many features in common with our direct limit of homology groups.

*Example* 4.1. Consider the inductive system

$$\mathbf{Z} \xrightarrow{f_1} \mathbf{Z}^2 \xrightarrow{f_2} \mathbf{Z}^3 \xrightarrow{f_3} \dots$$

where $f_n \colon \mathbf{Z}^n \to \mathbf{Z}^{n+1}$ is defined by

$$f_n(\mathbf{e}_{n,i}) = \begin{cases} \mathbf{e}_{n+1,i} & \text{if } i < n, \\ \mathbf{e}_{n+1,n} + \mathbf{e}_{n+1,n+1} & \text{if } i = n, \end{cases}$$

using the standard basis $\mathbf{e}_{n,1}, \dots, \mathbf{e}_{n,n}$ of $\mathbf{Z}^n$.

It certainly seems as if the inductive limit of this system should be the group $\mathbf{Z}^{(\mathbf{N})}$ of sequences of integers which are eventually 0. But this is not true: the inductive limit is bigger. It is isomorphic to $\mathbf{Z}^{(\mathbf{N})}$, but not in the obvious way.

Let $(v_m, v_{m+1}, ...)$ represent an element of $\varinjlim_n(\mathbf{Z}^n, f_n)$, with $v_m \in \mathbf{Z}^m$. Then for any $j$, the coordinate $(v_m)_j$ is constant as soon as $m > j$. This defines a map

$$\varinjlim_n (\mathbf{Z}^n, f_n) \to \mathbf{Z}^{\mathbf{N}}$$

which is easily seen to be injective.

PROPOSITION 4.2. *The image of* $\varinjlim_n(\mathbf{Z}^n, f_n)$ *in* $\mathbf{Z}^{\mathbf{N}}$ *is not reduced to* $\mathbf{Z}^{(\mathbf{N})}$, *it consists of those sequences* $(a_j)_{j\in\mathbf{N}}$ *that are eventually constant.*

*Proof.* Any element of the inductive limit has a representative $v_m \in \mathbf{Z}^m$ for some $m$. The $m$th entry of $v_m$ will be replicated as both the $m$th and $(m+1)$st entry of $v_{m+1}$, and then as the last three entries of $v_{m+2}$, etc. Clearly the image in $\mathbf{Z}^{\mathbf{N}}$ will be constant from the $m$th term on. □

Thus there is an exact sequence

$$0 \to \mathbf{Z}^{(\mathbf{N})} \to \varinjlim_n (\mathbf{Z}^n, f_n) \to \mathbf{Z} \to 0$$

where the third arrow associates to an eventually constant sequence the value of that constant.

We see that there is an extra generator to the inductive limit, which one may take to be the constant sequence of 1's in $\mathbf{Z}^{\mathbf{N}}$.

*Example* 4.3. Now let us elaborate our example a little. Modify $f_n: \mathbf{Z}^n \to \mathbf{Z}^{n+1}$ so that $f_n(\mathbf{e}_{n,n}) = \mathbf{e}_{n+1,n} + d\mathbf{e}_{n+1,n+1}$ for some integer $d \geqslant 1$.

Most of the computation above still holds, except that a sequence

$$\underline{v} = (v_1, v_2, ...) \in \mathbf{Z}^{\mathbf{N}}$$

belongs to the inductive limit if and only if it is eventually geometric with ratio $d$. We will denote $\mathbf{Z}[1/d]$ the rational numbers with only powers of $d$ in the denominator, i.e., the subring of $\mathbf{Q}$ generated by $\mathbf{Z}$ and $1/d$. If we set $\underline{v}^+ = (1, d, d^2, ...)$, then $\underline{v}$ belongs to the inductive limit if and only if there exists $a \in \mathbf{Z}[1/d]$ such that $\underline{v} - a\underline{v}^+$ has only finitely many non-zero entries. In other words, there is an exact sequence

$$0 \to \mathbf{Z}^{(\mathbf{N})} \to \varinjlim_n (\mathbf{Z}^n, f_n) \to \mathbf{Z}[1/d] \to 0.$$

Note that our inductive limit is still a free Abelian group, since a countable subgroup of $\mathbf{Z}^{\mathbf{N}}$ is free Abelian [Gri, Theorem 138]. In our case, the elements

$$(1, d, d^2, d^3, ...), \quad (0, 1, d, d^2, ...), \quad (0, 0, 1, d, ...), \quad ...$$

form a basis. On the other hand, $\mathbf{Z}[1/d]$ is not free (it is divisible by $d$).

*The homology of blow-ups.* Before attacking the homology of $X_\infty^*$, we will remind the reader of some facts about the homology of algebraic surfaces.

PROPOSITION 4.4. *If $X$ is a smooth algebraic surface (or more generally a 4-dimensional topological manifold), and $Z \subset X$ is a finite subset, then the inclusion $X - Z \hookrightarrow X$ induces an isomorphism on 1- and 2-dimensional homology.*

The proof comes from considering the long exact sequence of the pair $(X, X-Z)$, which gives in part

$$\dots \to H_3(X, X-Z) \to H_2(X-Z) \to H_2(X) \to H_2(X, X-Z) \to \dots .$$

For more details, see [HPV, pp. 25–26].

PROPOSITION 4.5. *If $X$ is a smooth algebraic surface (or more generally an orientable 4-dimensional topological manifold), and $Z \subset X$ is a finite subset, then the inclusion $X - Z \hookrightarrow X$ induces an exact sequence*

$$0 \to H_4(X) \to \mathbf{Z}^Z \to H_3(X-Z) \to H_3(X) \to 0.$$

*In particular, if $X$ is compact and $Z$ is a single point, then the inclusion induces an isomorphism $H_3(X-Z) \to H_3(X)$.*

The proof comes from considering the same exact sequence as above; we omit it.

We will now see that if you blow up a point of a surface, you increase the 2-dimensional homology by the class of the exceptional divisor.

Let $X$ be a surface, and $\mathbf{z}$ a smooth point. Let $\pi: \widetilde{X}_{\mathbf{z}} \to X$ be the canonical projection, and $E = \pi^{-1}(\mathbf{z})$ be the exceptional divisor.

Consider the homomorphism

$$i: H_2(X) \to H_2(\widetilde{X}_{\mathbf{z}}) \tag{4.1}$$

given by the composition

$$H_2(X) \to H_2(X - \{\mathbf{z}\}) \to H_2(\widetilde{X}_{\mathbf{z}});$$

i.e., first the inverse of the isomorphism $H_2(X - \{\mathbf{z}\}) \to H_2(X)$ in Proposition 4.4, followed by the map induced by inclusion.

PROPOSITION 4.6. *The map*

$$H_2(X) \oplus \mathbf{Z} \to H_2(\widetilde{X}_{\mathbf{z}})$$

*given by $(\alpha, m) \mapsto i(\alpha) + m[E]$ is an isomorphism.*

*Proof.* Apply the Mayer–Vietoris exact sequence to $\widetilde{U}_{\mathbf{z}}$ and $X - \{\mathbf{z}\}$, where $U$ is an open neighborhood of $\mathbf{z}$ in $X$ homeomorphic to a 4-ball. We omit the details. □

The same Mayer–Vietoris exact sequence, together with Proposition 4.5, will also show the following result.

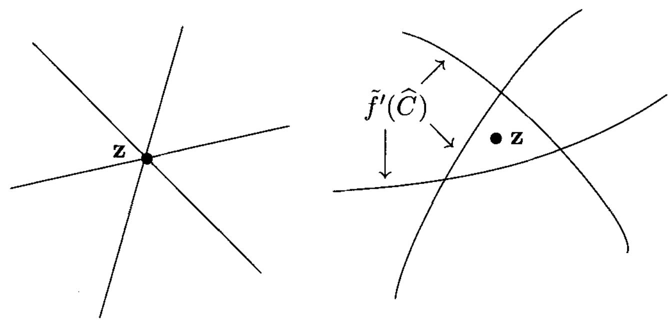



Fig. 7. Left: a curve with 3 smooth branches through $\mathbf{z}$. Right: a deformation which avoids $\mathbf{z}$.

PROPOSITION 4.7. *If $X$ is compact, the canonical projection induces isomorphisms $H_1(\widetilde{X}_\mathbf{z}) \to H_1(X)$ and $H_3(\widetilde{X}_\mathbf{z}) \to H_3(X)$.*

The next proposition will be the key to most of our computations.

PROPOSITION 4.8. *Consider the composition*

$$H_2(X) \to H_2(X - \{\mathbf{z}\}) \to H_2(\widetilde{X}_\mathbf{z}).$$

*Let $C$ be a curve in $X$ with $m$ smooth branches through $\mathbf{z}$. Then the image of $[C]$ in $H_2(\widetilde{X}_\mathbf{z})$ is $[C']+m[E]$, where $C'$ is the proper transform of $C$ in $\widetilde{X}_\mathbf{z}$.*

*Proof.* Let $\widehat{C}$ be the normalization of $C$, which is in particular a smooth 2-dimensional differentiable manifold, and let $f: \widehat{C} \to C$ be the normalizing map. The mapping $f$ can be deformed (differentiably, but perhaps not analytically) to a map $f': \widehat{C} \to X$ that avoids $\mathbf{z}$; so $f'$ lifts to a map $\tilde{f}': \widehat{C} \to \widetilde{X}_\mathbf{z}$, as illustrated by Figure 7.

Then $[\tilde{f}'(\widehat{C})]$ is the image of $[C]$ in $H_2(\widetilde{X}_\mathbf{z})$. The homology class $[\tilde{f}'(\widehat{C})]$ is of the form $[C']+n[E]$ for some $n$: indeed, $f'$ can be chosen so that $[\tilde{f}'(\widehat{C})]$ is contained in a small neighborhood of $C' \cup E$, which will retract onto $C' \cup E$, and hence whose 2-dimensional homology is generated by $[C']$ and $[E]$. We discover what $n$ is by observing that the intersection number $[\tilde{f}'(\widehat{C})] \cdot [E]$ vanishes, since the corresponding cycles are disjoint. So

$$0 = [\tilde{f}'(\widehat{C})] \cdot [E] = ([C']+n[E]) \cdot [E] = m - n,$$

since each branch of $C$ through $\mathbf{z}$ contributes 1 to $[C'] \cdot [E]$. $\square$

*The finite approximations to $X_\infty$.* Now let us consider the set

$$X_{[N,M]} \subset \prod_{i=N}^{M} \widetilde{X}_H, \quad N \leqslant M,$$

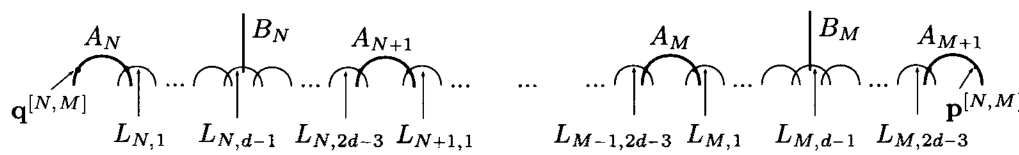



Fig. 8. The divisor $D_{[N,M]}$. The last line of one block coincides with the first of the next.

of finite sequences $(\mathbf{x}_N, \mathbf{x}_{N+1}, \ldots, \mathbf{x}_M)$ with pairs of successive points in $\bar{\Gamma}_{H^\sharp}$, and the subset $D_{[N,M]} \subset X_{[N,M]}$ with all coordinates in $\widetilde{D}$.

The set $D_{[N,M]}$ contains the point $\mathbf{p}^{[N,M]}$ all of whose coordinates are $\widetilde{\mathbf{p}}$, and the point $\mathbf{q}^{[N,M]}$ all of whose coordinates are $\mathbf{q}'$. Let us set

$$X^*_{[N,M]} = X_{[N,M]} - \{\mathbf{p}^{[N,M]}, \mathbf{q}^{[N,M]}\} \quad \text{and} \quad D^*_{[N,M]} = X^*_{[N,M]} \cap D_{[N,M]}.$$

PROPOSITION 4.9. *If $-\infty \leqslant N' \leqslant N \leqslant M \leqslant M' \leqslant \infty$, then the natural projection $X_{[N',M']} \to X_{[N,M]}$ has an inverse $X^*_{[N,M]} \to X_{[N',M']}$ defined on $X^*_{[N,M]}$.*

*Proof.* Any point of $X^*_{[N,M]}$ has a well-defined backwards orbit, since its $N$th coordinate is not $\mathbf{q}'$, and it has a well-defined forwards orbit, since its $M$th coordinate is not $\widetilde{\mathbf{p}}$. These orbits define an inclusion of $X^*_{[N,M]}$ into $X_\infty$. □

The point of this proposition is that we can compute the homology of $X^*_{[N,M]}$. If $V$ is an algebraic variety, let $\operatorname{Irr}(V)$ denote the set of irreducible components of $V$.

PROPOSITION 4.10. (a) *The space $X_{[N,M]}$ is a smooth algebraic surface, and $D_{[N,M]}$ is a divisor in $X_{[N,M]}$.*

(b) *The divisor $D_{[N,M]}$ consists of $M+1-N$ ordered blocks, each consisting of $2d$ projective lines, with the last line of one block coinciding with the first of the next, as in Figure* 8.

*Proof.* Part (a) is more or less obvious, except perhaps for the points $\mathbf{p}^{[N,M]}, \mathbf{q}^{[N,M]}$. The projection onto the $M$th coordinate gives an isomorphism of a neighborhood of $\mathbf{p}^{[N,M]}$ onto a neighborhood of $\widetilde{\mathbf{p}}$, and the projection onto the $N$th coordinate works for $\mathbf{q}^{[N,M]}$.

A point of $D_{[N,M]}$ is a sequence of points at infinity in $\widetilde{X}_H$. It consists of either

- all $\widetilde{\mathbf{p}}$ or all $\mathbf{q}'$, or
- a certain number of $\widetilde{\mathbf{p}}$'s (perhaps none), then a first element different from $\widetilde{\mathbf{p}}$, then something (perhaps $\mathbf{q}'$), then all $\mathbf{q}'$'s.

Let us denote by $D_k$ the $k$th block

$$D_k = \{\underline{\mathbf{x}} \in D_{[N,M]} \mid \mathbf{x}_k \in (\widetilde{D} - \tilde{A}) \cup \{\widetilde{\mathbf{q}}\} \text{ and } \mathbf{x}_{k-1} \neq \mathbf{q}'\},$$


Fig. 9. The self-intersections of the components of $D_{[N,M]}$.

for $N \leqslant k \leqslant M$ (if $k=N$, the condition $x_{k-1} \neq \mathbf{q}'$ is void). This set is parametrized by $\mathbf{x}_k \in (\widetilde{D}-\tilde{A}) \cup \{\widetilde{\mathbf{q}}\}$. Every point of $D_{[N,M]}$ belongs to precisely one $D_k$, except for:

- The points whose $M$th coordinate belongs to $\tilde{A}-\{\widetilde{\mathbf{q}}\}$; these form a projective line denoted $A_{M+1}$.
- The points $\mathbf{q}_k$, $k=N+1, ..., M$, whose $k$th coordinate is $\mathbf{q}'$ and whose $(k-1)$st coordinate is $\widetilde{\mathbf{q}}$. The point $\mathbf{q}_k$ is simultaneously the left-most point of $D_k$ and the right-most point of $D_{k-1}$.
- The point $\mathbf{q}_{M+1} = A_{M+1} \cap D_M$. □

All lines have the same self-intersection numbers as the corresponding lines in $\widetilde{D}$, except for the connecting lines, i.e., the lines $A_k$, $k=N+1, ..., M$, where $\mathbf{x}_k \in A'$ and $x_{k-1} \neq \mathbf{q}'$. These have self-intersection $-3$, as indicated in Figure 9. This is proved in the proof of Proposition 4.11.

PROPOSITION 4.11. (a) *The map that associates to each irreducible component of $D_{[N,M]}$ the* 2*-dimensional homology class which it carries induces an isomorphism*

$$\mathbf{Z}^{\mathrm{Irr}(D_{[N,M]})} \to H_2(X_{[N,M]}) \quad \textit{when } -\infty < N \leqslant M < \infty.$$

(b) *The inclusion $X^*_{[N,M]} \to X_{[N,M]}$ induces an isomorphism on* 2*-dimensional homology.*

Note that this proposition represents the second homology group of $X^*_\infty$ as

$$\varinjlim_{N} \mathbf{Z}^{\mathrm{Irr}(D_{[-N,N]})}, \tag{4.2}$$

since an increasing union of open sets is an inductive limit in the category of topological spaces, and homology commutes with inductive limits [Sp, Chapter IV, 1.7]. This is very similar to Example 4.3 above, and we will need to look carefully at the inclusions.

*Proof.* (a) With a slightly different definition of $X_{[N,M]}$, this follows from Proposition 4.6. We need to know that $X_{[N,M]}$ is obtained from the projective plane $\mathbf{P}^2$ by a sequence of blow-ups, corresponding naturally to the irreducible components of $D_{[N,M]}$.

First notice that we may assume that $N=0$: clearly shifting the indices gives an isomorphism $X_{[N,M]} \to X_{[0,M-N]}$.

Next observe that $X_{[0,0]} = \widetilde{X}_H$, which as we saw is obtained from $\mathbf{P}^2$ by a sequence of blow-ups, each of which creates one component of $\widetilde{D} = D_{[0,0]}$ other than $A' = A_0$. The component $A'$ is the proper transform of $l_\infty$ which was there to begin with and which generated the homology $H_2(\mathbf{P}^2)$. So the theorem is true when $M=0$.

If $M=1$, notice that $X_{[0,1]} = \Gamma_{H^\sharp}$, so the diagram

$$\begin{tikzcd} X_{[0,1]} \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & X_{[0,0]} \arrow[d, "\pi"] \\ X_{[0,0]} \arrow[r, "\widetilde{H}"] & \mathbf{P}^2 \end{tikzcd}$$

is a fibered product by Proposition 3.4. But the bottom mapping $\widetilde{H}$ is an isomorphism on a neighborhood of $\widetilde{\mathbf{p}}$, mapping $\widetilde{\mathbf{p}}$ to $\mathbf{p} = [0:1:0]$. Thus the inverse image by $\mathrm{pr}_1$ of this neighborhood maps under $\mathrm{pr}_1$ to its image just as the inverse image of the neighborhood of $\mathbf{p}$ maps under $\pi$.

This same argument shows that the component $A_1$ of $D_{[0,1]}$ has self-intersection 3. Indeed, the line $A_1 \subset D_{[0,0]}$ has self-intersection $-1$, and the first two blow-ups required to build $X_{[0,1]}$ are blow-ups of points of $A_1$.

To show that $X_{[0,2]}$ is constructed from $X_{[0,1]}$ by a sequence of blow-ups, etc., apply the same argument, using the diagram

$$\begin{tikzcd} X_{[0,2]} \arrow[r] \arrow[d] & X_{[1,2]} \arrow[d] \\ X_{[0,1]} \arrow[r] & \widetilde{X}_H. \end{tikzcd}$$

As above, we see that on $A_2$, which has self-intersection $-1$ in $X_{[0,1]}$, a point was blown up twice, so $A_2$ has self-intersection $-3$ in $X_{[0,2]}$. By induction, $A_k$ will have self-intersection $-1$ in $X_{[0,k+1]}$ and self-intersection $-3$ in $X_{[0,k+2]}$.

(b) This follows immediately from Propositions 4.4 and 4.10. □

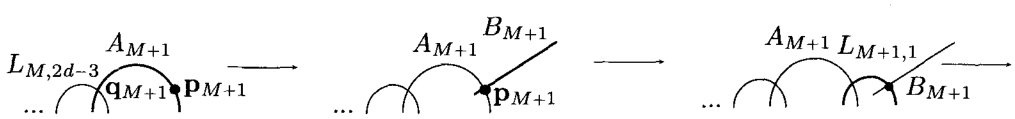
Next, we need to compute the homomorphism $H_2(X_{[-N,N]}) \to H_2(X_{[-(N+1),N+1]})$ induced by the composition of the isomorphism

$$H_2(X_{[-N,N]}) \to H_2(X^*_{[-N,N]})$$

and the mapping

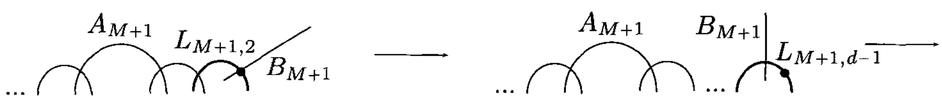
$$H_2(X^*_{[-N,N]}) \to H_2(X_{[-(N+1),N+1]})$$

induced by the inclusion.



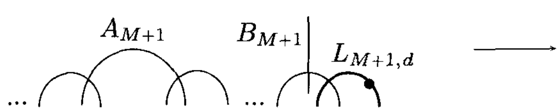

$$[A_{M+1}] \longmapsto [A_{M+1}]+[B_{M+1}] \longmapsto [A_{M+1}]+[B_{M+1}]+2[L_{M+1,1}]$$

Fig. 10. The first two blow-ups performed on $A_{M+1} \subset X_{[N,M]}$. Note that $A_{M+1}$ and $B_{M+1}$ each contribute 1 to the coefficient of the exceptional divisor $L_{M+1,1}$.



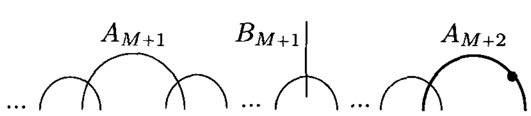

$$[A_{M+1}]+[B_{M+1}]+2[L_{M+1,1}]+3[L_{M+1,2}] \longmapsto [A_{M+1}]+[B_{M+1}]+2[L_{M+1,1}] +3[L_{M+1,2}]+\ldots+d[L_{M+1,d-1}]$$

Fig. 11. Left: the next blow-up. Right: the configuration after $d$ blow-ups. For the figure on the left, $B_{M+1}$ contributes 1 and $2L_{M+1,1}$ contributes 2 to the coefficient of the exceptional divisor $L_{M+1,2}$.




$$\longmapsto [A_{M+1}]+[B_{M+1}]+2[L_{M+1,1}] +3[L_{M+1,2}]+\ldots+d[L_{M+1,d-1}]+d[L_{M+1,d}] \longmapsto$$

Fig. 12. The configuration after $d+1$ blow-ups. Here, $dL_{M+1,d-1}$ contributes $d$ to the coefficient of the exceptional divisor $L_{M+1,d}$. It is the only contribution since this time we are blowing up an ordinary point.




$$\longmapsto [A_{M+1}]+[B_{M+1}]+2[L_{M+1,1}]+3[L_{M+1,2}] +\ldots+d([L_{M+1,d-1}]+[L_{M+1,d}]+\ldots+[L_{M+1,2d-3}]+[A_{M+1}])$$

Fig. 13. The configuration after all the blow-ups required to pass from $X_{[N,M]}$ to $X_{[N,M+1]}$ have been made. We have blown up ordinary points on lines with weight $d$, so the new exceptional divisor always has weight $d$.

PROPOSITION 4.12. *The homomorphism*

$$i_N: H_2(X_{[N,M]}) \to H_2(X_{[N-1,M+1]})$$

*described above is given by the following formula:*

$$\begin{aligned}
i_N[C] &= [C] \quad \text{if } C \neq A_{M+1}, A_N,\\
i_N[A_{M+1}] &= [A_{M+1}]+[B_{M+1}]+2[L_{M+1,1}]+3[L_{M+1,2}]+\dots\\
&\quad +d([L_{M+1,d-1}]+[L_{M+1,d}]+\dots+[L_{M+1,2d-3}]+[A_{M+2}]),\\
i_N[A_N] &= [A_N]+[B_{N-1}]+2[L_{N-1,2d-3}]+3[L_{N-1,2d-4}]+\dots\\
&\quad +d([L_{N-1,d-1}]+[L_{N-1,d-2}]+\dots+[L_{N-1,1}]+[A_{N-1}]).
\end{aligned}$$

*Proof.* This is straightforward, using Proposition 4.8. Figures 10 through 13 should explain exactly the sequence of blow-ups. □

THEOREM 4.13. (a) *The inductive limit*

$$\varinjlim_N H_2(X_{[-N,N]}) = H_2(X^*_\infty)$$

*embeds naturally in* $\mathbf{Z}^{\mathrm{Irr}(D^*_\infty)}$.

(b) *If* $\underline{v} \in \mathbf{Z}^{\mathrm{Irr}(D^*_\infty)}$ *is an element of* $H_2(X^*_\infty)$, *then the limits*

$$\nu^+(\underline{v}) = \lim_{n\to\infty} \frac{\underline{v}(A_n)}{d^n} \quad \text{and} \quad \nu^-(\underline{v}) = \lim_{n\to\infty} \frac{\underline{v}(A_{-n})}{d^n}$$

*both exist and lie in* $\mathbf{Z}[1/d]$. *Actually, the sequences are eventually constant.*

(c) *The following sequence is exact:*

$$0 \to \mathbf{Z}^{(\mathrm{Irr}(D^*_\infty))} \to H_2(X^*_\infty) \xrightarrow{(\nu^+,\nu^-)} Z[1/d]\oplus Z[1/d] \to 0. \tag{4.3}$$

*Proof.* (a) Any element $\underline{v}$ of the inductive limit is the image of some

$$\underline{v}_N \in H_2(X_{[-N,N]}) = \mathbf{Z}^{\mathrm{Irr}(D_{[-N,N]})}$$

for all sufficiently large $N$, and the coefficient $\underline{v}_N(L)$ of any irreducible component $L \in \mathrm{Irr}(D_{[-N,N]})$ is then the same as the coefficient $\underline{v}_{N'}(L)$ for all $N' \geqslant N$ by Proposition 4.12. This proves (a).

(b) The element $\underline{v}$ of the limit is determined by the corresponding element of $v_N \in H_2(X_{[-N,N]})$. In particular, $v_N$ assigns some integer weights $\alpha$ to $[A_{-N}]$ and $\beta$ to $[A_{N+1}]$. Then, again by Proposition 4.12, we see that

$$\underline{v}(A_{-N-1}) = d\alpha, \quad \underline{v}(A_{-N-2}) = d^2\alpha, \quad \dots,$$
$$\underline{v}(A_{N+2}) = d\beta, \quad \underline{v}(A_{N+3}) = d^2\beta, \quad \dots .$$

In particular, the sequences defining $\nu^-$ and $\nu^+$ are constant after $N$, so the limits exist. This proves (b).

(c) For any element $\underline{v} \in \mathbf{Z}^{(\mathrm{Irr}(D^*_\infty))}$, there exists $N$ such that $\underline{v}$ has coefficient 0 for all irreducible components $L \in \mathrm{Irr}(D^*_\infty)$ which do not belong to $D_{[-N,N]}$. Then $\underline{v}$ is in the image of $H_2(X_{[-(N+1),N+1]})$, and we see that $\mathbf{Z}^{(\mathrm{Irr}(D^*_\infty))}$ is included in $H_2(X^*_\infty)$. Clearly it is the kernel of the mapping $(\nu^-, \nu^+)$, which is surjective. □

THEOREM 4.14. *The Hénon mapping $H_\infty: X^*_\infty \to X^*_\infty$ induces a commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{Z}^{(\mathrm{Irr}(D^*_\infty))} \arrow[r] \arrow[d, "\alpha"'] & H_2(X^*_\infty) \arrow[r, "(\nu^-,\nu^+)"] \arrow[d, "H_2(H_\infty)"'] & \mathbf{Z}[1/d] \oplus \mathbf{Z}[1/d] \arrow[r] \arrow[d, "\beta"'] & 0 \\
0 \arrow[r] & \mathbf{Z}^{(\mathrm{Irr}(D^*_\infty))} \arrow[r] & H_2(X^*_\infty) \arrow[r] & \mathbf{Z}[1/d] \oplus \mathbf{Z}[1/d] \arrow[r] & 0
\end{tikzcd}$$

*where $\alpha$ is the shift*

$$\alpha([A_k]) = [A_{k-1}], \quad \alpha([B_k]) = [B_{k-1}], \quad \alpha([L_{k,i}]) = [L_{k-1,i}],$$

*and $\beta$ is the mapping $\beta(a,b) = (a/d, bd)$.*

*Proof.* The action of $H$ on the homology is induced by shifting (to the left) by one block in $\mathbf{Z}^{\mathrm{Irr}(D^*_\infty)}$. Clearly, this induces the same shift on $\mathbf{Z}^{(\mathrm{Irr}(D^*_\infty))}$, and the statement about $\alpha$ is true. To see that $\beta$ is correct, consider a homology class $\underline{v} \in \mathbf{Z}^{\mathrm{Irr}(D^*_\infty)}$ in the image of $H_2(X_{[-N,N]})$. It will satisfy

$$v_{A_{N+1}} = b, \quad v_{A_{N+2}} = db, \quad v_{A_{N+3}} = d^2 b, \quad \dots,$$

for some $b \in \mathbf{Z}$, and $\nu^+(\underline{v}) = b/d^N$. The sequence $(H_2(H_\infty))(\underline{v})$ is the same sequence shifted, so that

$$(H_2(H_\infty))(\underline{v})_{A_{N+1}} = db, \quad (H_2(H_\infty))(\underline{v})_{A_{N+2}} = d^2 b, \quad (H_2(H_\infty))(\underline{v})_{A_{N+3}} = d^3 b, \quad \dots$$

and

$$\nu^-(H_2(H_\infty))(\underline{v}) = \frac{db}{d^N} = d\nu^-(\underline{v}).$$

The computation for $\nu^+$ is similar. □

One way of understanding the exact sequence (4.3) is as part of the homology exact sequence of the pair $D^*_\infty \subset X^*_\infty$.

PROPOSITION 4.15. (a) *There exists a unique isomorphism*

$$\mathbf{Z}[1/d]\oplus\mathbf{Z}[1/d]\to H_2(X^*_\infty, D^*_\infty)$$

*which makes the following diagram commute*:

$$\begin{array}{ccc} H_2(X^*_\infty) & \xrightarrow{(\nu^-,\nu^+)} & \mathbf{Z}[1/d]\oplus\mathbf{Z}[1/d] \\ {\scriptstyle \mathrm{Id}}\downarrow & & \downarrow \\ H_2(X^*_\infty) & \longrightarrow & H_2(X^*_\infty, D^*_\infty). \end{array}$$

(b) *Both* $H_3(X^*_\infty)$ *and* $H_3(X^*_\infty, D^*_\infty)$ *are isomorphic to* $\mathbf{Z}$, *and the canonical map*

$$H_3(X^*_\infty)\to H_3(X^*_\infty, D^*_\infty)$$

*is an isomorphism.*

(c) *Both* $H_1(X^*_\infty)$ *and* $H_1(X^*_\infty, D^*_\infty)$ *are zero.*

*Remark.* We will see in §7 (in the proof of Theorem 7.6) that the homology group $H_2(X^*_\infty, D^*_\infty)$ can also be understood as $H_1(S^3-(\Sigma^+\cup\Sigma^-))$, where $\Sigma^+$ and $\Sigma^-$ are solenoids embedded in a 3-sphere obtained by an appropriate real-oriented blow-up. A classical result of algebraic topology says that for the standard $d$-adic solenoid $\Sigma_d$ embedded in the 3-sphere in the standard way, $H_1(S^3-\Sigma_d)=\mathbf{Z}[1/d]$. This explains why these bizarre groups appear in this complex-analytic setting, by making precise the sentence "at the ends of $D^*_\infty$ there are two solenoids".

*Proof.* The exact sequence of the pair $D^*_\infty\subset X^*_\infty$ reads in part

$$H_2(D^*_\infty)\to H_2(X^*_\infty)\to H_2(X^*_\infty, D^*_\infty)\to H_1(D^*_\infty).$$

The first term is $\mathbf{Z}^{(\mathrm{Irr}(D^*_\infty))}$, and the last term vanishes, since $D^*_\infty$ is a union of 2-spheres identified at points, with the quotient topology from the disjoint union. This, together with Theorem 4.13 (c), proves (a).

For (b), another part of the long exact sequence reads

$$H_3(D^*_\infty)\to H_3(X^*_\infty)\to H_3(X^*_\infty, D^*_\infty)\to H_2(D^*_\infty)\to H_2(X^*_\infty).$$

Since the last map is injective and $H_3(D^*_\infty)=0$, the canonical map

$$H_3(X^*_\infty)\to H_3(X^*_\infty, D^*_\infty)$$

is an isomorphism.

To see what it is an isomorphism between, notice that $H_3(\mathbf{P}^2)=0$. It then follows from Proposition 4.7 that $H_3(X_{[N,M]})=0$ for all $N\leqslant M$. Next, the inclusion

$$X^*_{[N,M]}\in X_{[N,M]}$$

induces (still by Proposition 4.7 for the final 0) an exact sequence

$$\begin{array}{ccccccc} 0\to H_4(X_{[N,M]}) & \to & H_4(X_{[N,M]},X^*_{[N,M]}) & \to & H_3(X^*_{[N,M]}) & \to & H_3(X_{[N,M]}). \\ \| & & \| & & & & \| \\ \mathbf{Z} & & \mathbf{Z}^2 & & & & 0 \end{array}$$

Thus $H_3(X^*_{[N,M]})$ is canonically the quotient of $\mathbf{Z}^2$ by the image of $\mathbf{Z}$ under the diagonal map, i.e., it is isomorphic to $\mathbf{Z}$.

The argument for (c) is similar but easier. □

*The intersection form on the homology.* The homology space $H_2(X^*_\infty)$ carries a quadratic form coming from intersection. In order to describe it, we need to define homology classes which are not in the kernel of $\nu^\pm$. Call $v^-,v^+\in H_2(X^*_\infty)$ the images of $[A']$ and $[\tilde{A}]$ in $H_2(X_{[0,0]})=H_2(\widetilde{X})$ under the inclusions

$$H_2(X_{[0,0]})\to H_2(X_{[-1,1]})\to H_2(X_{[-2,2]})\to \dots .$$

We can also describe $v^\pm$ as elements of $\mathbf{Z}^{\operatorname{Irr} D^*_\infty}$; in Figure 14, the top block indicates the irreducible components of $D^*_\infty$ for which we are giving weights; the next two blocks describe the weights assigned by $v^+$ and $v^-$.

The middle block describing $v^+$ has all zeroes above, and each succeeding line below is the previous multiplied by $d$; the bottom block describing $v^-$ has all zeroes below, and above each preceding line is the next multiplied by $d$.

Then $\nu^+(v^+)=\nu^-(v^-)=1$, so that the values of the quadratic form on $\mathbf{Z}^{(\operatorname{Irr}(D^*_\infty))}$ and on $v^\pm$ determine the quadratic form completely.

PROPOSITION 4.16. *On $\mathbf{Z}^{(\operatorname{Irr}(D^*_\infty))}$, the quadratic form is determined by the self-intersections and mutual intersections of the irreducible components of $D^*_\infty$.*

*The classes $v^+$ and $v^-$ satisfy the rules*

$$\begin{gathered} v^+\cdot v^+=v^-\cdot v^-=-1,\quad v^+\cdot v^-=0,\\ v^+\cdot[L_{0,2d-3}]=v^-\cdot[L_{0,1}]=1,\quad v^+\cdot[A_1]=v^-\cdot[A_0]=-1, \end{gathered}\tag{4.4}$$

*with all other intersections* 0.

*Proof.* The statement about $\mathbf{Z}^{(\operatorname{Irr}(D^*_\infty))}$ should be clear.

$$
\begin{array}{cccccccc}
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
A_{-1} & L_{-1,1} & \dots & L_{-1,d-2} & \overset{B_{-1}}{L_{-1,d-1}} & L_{-1,d} & \dots & L_{-1,2d-3} \\
A_0 & L_{0,1} & \dots & L_{0,d-2} & \overset{B_0}{L_{0,d-1}} & L_{0,d} & \dots & L_{0,2d-3} \\
A_1 & L_{1,1} & \dots & L_{1,d-2} & \overset{B_1}{L_{1,d-1}} & L_{1,d} & \dots & L_{1,2d-3} \\
A_2 & L_{2,1} & \dots & L_{2,d-2} & \overset{B_2}{L_{2,d-1}} & L_{2,d} & \dots & L_{2,2d-3} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots
\end{array}
$$

$$
v^+ = \begin{array}{cccccccc}
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \dots & 0 & \overset{0}{0} & 0 & \dots & 0 \\
0 & 0 & \dots & 0 & \overset{0}{0} & 0 & \dots & 0 \\
1 & 2 & \dots & d-1 & \overset{1}{d} & d & \dots & d \\
d & 2d & \dots & d(d-1) & \overset{d}{d^2} & d^2 & \dots & d^2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots
\end{array}
$$

$$
v^- = \begin{array}{cccccccc}
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
d & d & \dots & d & \overset{1}{d} & d-1 & \dots & 2 \\
1 & 0 & \dots & 0 & \overset{0}{0} & 0 & \dots & 0 \\
0 & 0 & \dots & 0 & \overset{0}{0} & 0 & \dots & 0 \\
0 & 0 & \dots & 0 & \overset{0}{0} & 0 & \dots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots
\end{array}
$$

Fig. 14

For the other classes, one way to do it is to construct a differentiable surface $C^+ \subset \widetilde{X}_H$ (not an algebraic curve) which represents $\tilde{A}$, and which avoids $\widetilde{\mathbf{q}}$ and $\mathbf{q}'$. Note that $C^+$ cannot be algebraic (or analytic): the self-intersection of $\tilde{A}$ is $-1$, so it is *rigid* as an algebraic curve. The curve $C^+$ is then contained in $X_\infty^*$ and represents $v^+$. But a neighborhood of the curve $C^+$ is also contained in $X_\infty^*$, so $v^+$ only intersects those curves of $D_\infty$ that $C^+$ intersects. Thus $v^+ \cdot v^+ = C^+ \cdot [A_0] = C^+ \cdot C^+ = -1$ and $v^+ \cdot [L_{0,2d-3}] = C^+ \cdot [L_{0,2d-3}] = 1$.

Similarly, construct a differentiable surface $C^- \subset \widetilde{X}_H$ which is a deformation of $A'$; clearly we can take $C^+ \cap C^- = \varnothing$. □

Of course the quadratic form is invariant under the action of $H_\infty$, since $H_\infty$ is a homeomorphism of $X^*_\infty$. This certainly is not obvious from the formulas. Let us check one case. Take $d=2$. Since $H_\infty$ induces the shift, we see that

$$(H_\infty)_*(v^+) = 2v^+ + [A_0] + [B_0] + 2[L_{0,1}].$$

The intersection product gives, as it should,

$$\begin{aligned}((H_\infty)_*(v^+))^2 &= 4(v^+)^2 + (A_0)^2 + (B_0)^2 + 4(L_{0,1})^2 + 4B_0 \cdot L_{0,1} + 4A_0 \cdot L_{0,1} + 8v^+ \cdot L_{0,1} \\ &= -4-3-2-8+4+4+8 = -1.\end{aligned}$$

This quadratic form on $H_2(X^*_\infty)$ is of course neither positive nor negative definite. For instance $\Delta$, the closure of the diagonal of $\mathbf{C}^2$ in $X^*_\infty$, has self-intersection $+1$, whereas all the irreducible components of $D^*_\infty$ have negative self-intersection. The following proposition says that the form is mainly negative.

THEOREM 4.17. *The intersection form is negative definite on* $\mathbf{Z}^{(\mathrm{Irr}(D^*_\infty))}$.

*Proof.* We will give two proofs, one conceptual and one computational. Each proves a stronger (but not the same stronger) result.

*First proof.* An element $v \in \mathbf{Z}^{(\mathrm{Irr}(D^*_\infty))}$ comes from an element of $H_2(X_{[N,M]})$ which assigns coefficient 0 to the first and the last irreducible component of the divisor $D_{[N,M]}$. Its self-intersection in $X_{[N,M]}$ and in $X^*_\infty$ coincide. We will in fact prove that if $v$ has coefficient 0 with respect to one of these components, then $v \cdot v < 0$ unless $v=0$.

Indeed, the complement of the last exceptional divisor in $D_{[N,M]}$ can be blown down to a point, so by [Gra], the intersection matrix of this divisor is negative definite, and hence $v \cdot v$ is negative if $v \neq 0$.

*Second proof.* Let us call $a_n, b_n, x_{i,j}$ the coefficients of $A_n, B_n, L_{i,j}$ respectively. Thus we are considering the quadratic form

$$\begin{aligned}&\ldots - 3a_n^2 + 2a_n x_{n,1} - 2x_{n,1}^2 + 2x_{n,1}x_{n,2} + \ldots - 2x_{n,d-2}^2 + 2x_{n,d-2}x_{n,d-1} \\ &- db_n^2 + 2b_n x_{n,d-1} - 2x_{n,d-1}^2 + 2x_{n,d-1}x_{n,d} + \ldots + 2x_{n,2d-3}a_{n+1} - 3a_{n+1}^2 + \ldots\ .\end{aligned}$$

It is clearly enough to show that the quadratic form obtained by allocating half the coefficient $a_n$ to the next term and half to the previous term is negative definite, i.e., that the quadratic term in $2d$ variables

$$\begin{aligned}&-\tfrac{3}{2}a_0^2+2a_0x_1-2x_1^2+2x_1x_2+\ldots-2x_{d-2}^2+2x_{d-2}x_{d-1}\\&-db^2+2bx_{d-1}-2x_{d-1}^2+2x_{d-1}x_d+\ldots+2x_{2d-3}a_1-\tfrac{3}{2}a_1^2\end{aligned}$$

is negative definite. This is something like working in one block at a time.

If we isolate the terms containing $b$ and complete squares, this quadratic form can be written

$$\begin{aligned}&-\left(db^2-2bx_{d-1}+\frac{1}{d}x_{d-1}^2\right)-\frac{3}{2}a_0^2+2a_0x_1-2x_1^2+2x_1x_2+\ldots-2x_{d-2}^2+2x_{d-2}x_{d-1}\\&-\left(2-\frac{1}{d}\right)x_{d-1}^2+2x_{d-1}x_d-2x_d^2+\ldots+2x_{2d-3}a_1-\frac{3}{2}a_1^2.\end{aligned}$$

If we complete squares from both ends, we can write this as

$$\begin{aligned}&-\left(db^2-2bx_{d-1}+\frac{1}{d}x_{d-1}^2\right)\\&-\left(\frac{3}{2}a_0^2-2a_0x_1+\frac{2}{3}x_1^2\right)-\left(\frac{3}{2}a_1^2-2a_1x_{2d-3}+\frac{2}{3}x_{2d-3}^2\right)\\&-\left(\frac{4}{3}x_1^2-2x_1x_2+\frac{3}{4}x_2^2\right)-\left(\frac{4}{3}x_{2d-3}^2-2x_{2d-3}x_{2d-2}+\frac{3}{4}x_{2d-2}^2\right)\\&-\ldots\\&-\left(\frac{d+1}{d}x_{d-2}^2-2x_{d-2}x_{d-1}+\frac{d}{d+1}x_{d-1}^2\right)-\left(\frac{d+1}{d}x_d^2-2x_dx_{d-1}+\frac{d}{d+1}x_{d-1}^2\right)\\&-\frac{d-1}{d(d+1)}x_{d-1}^2.\end{aligned}$$

□

It works, with a tiny bit to spare, so we actually get a slightly stronger result:

PROPOSITION 4.18. *There exists $K$ depending only on $d$ such that*

$$\frac{1}{K}\sum_{i\in \mathrm{Irr}(D_\infty)} v_i^2\leqslant -v\cdot v\leqslant K\sum_{i\in \mathrm{Irr}(D_\infty)} v_i^2$$

*for any* $v\in \mathbf{Z}^{(\mathrm{Irr}(D_\infty^*))}$.

Thus we can complete $\mathbf{C}^{(\mathrm{Irr}(D_\infty^*))}$ with respect to the intersection inner product, to get a Hilbert space, which we denote $\widehat{H}_2^0(X_\infty^*;\mathbf{C})$. By Proposition 4.18, this intersection product norm is equivalent to the $l_2$-norm on the space of sequences.

The exact sequence

$$0\to \mathbf{Z}^{(\mathrm{Irr}(D_\infty^*))}\to H_2(X_\infty^*)\to \mathbf{Z}[1/d]\oplus\mathbf{Z}[1/d]\to 0$$

gives, tensoring with $\mathbf{C}$,

$$0 \to \mathbf{C}^{(\mathrm{Irr}(D^*_\infty))} \to H_2(X^*_\infty;\mathbf{C}) \to \mathbf{C}\oplus\mathbf{C} \to 0,$$

so it is natural to complete the entire homology, i.e., to set

$$\widehat{H}_2(X^*_\infty;\mathbf{C}) = \widehat{H}^0_2(X^*_\infty;\mathbf{C})\oplus\mathbf{C}v^+\oplus\mathbf{C}v^-.$$

On this completed homology space (unlike homology with infinite chains, which is the dual of cohomology with compact supports), the pseudo-inner product given by the intersection is still defined (e.g. by the formulas (4.4)).

This completed homology $\widehat{H}_2(X^*_\infty;\mathbf{C})$ is contained in $\mathbf{C}^{\mathrm{Irr}(D_\infty)}$, but if $v=(v_i)$, $i\in \mathrm{Irr}(D_\infty)$, the quadratic form is not given by $v\cdot v=\sum v_i^2$. This is not even true for $v\in \widehat{H}^0_2(X^*_\infty;\mathbf{C})$, though in that case (only) the series is convergent. But the series is divergent for $v=v^+$ and $v=v^-$.

Neither are the subspaces $\mathbf{C}v^+$ and $\mathbf{C}v^-$ orthogonal to $H^0_2(X^*_\infty)$; formulas (4.4) say that each of these subspaces are orthogonal to all but one standard basis vector. So the pseudo-inner products $v\cdot v^\pm$ are well-defined even if $v\in\widehat{H}^0_2(X^*_\infty;\mathbf{C})$ is a series with infinitely many non-zero terms.

Clearly the subspace $\mathbf{C}^{(\mathrm{Irr}(D^*_\infty))}\subset H_2(X^*_\infty;\mathbf{C})$ is invariant under the Hénon mapping $H$, which is simply a shift in $D_\infty$, so it induces a unitary operator on the Hilbert space $\widehat{H}^0_2(X^*_\infty;\mathbf{C})$. This unitary operator has only continuous spectrum, on the unit circle, and with spectral density $2d-1$. There are in addition two eigenvectors of

$$(H_\infty)_*\colon \widehat{H}_2(X^*_\infty;\mathbf{C}) \to \widehat{H}_2(X^*_\infty;\mathbf{C}),$$

one with eigenvalue $d$ and one with eigenvalue $1/d$. One way of defining them is as

$$w^+ = \lim_{n\to\infty}\frac{1}{d^n}(H_\infty)^n_*(v^+) \quad\text{and}\quad w^- = \lim_{n\to\infty}\frac{1}{d^n}(H_\infty)^{-n}_*(v^-).$$

These do belong to the completed homology (but not to the homology), since $w^+$ is $v^+$ on the positive part of $D^*_\infty$, and decreases like a geometric series on the negative part.

These homology classes are already well-known in the theory: they are the homology classes of the currents $\mu^-$ and $\mu^+$, as defined by [BS1].

## 5. Real-oriented blow-ups

In this section we will define a way of "resolving" the non-algebraic singularities $\mathbf{p}^\infty, \mathbf{q}^\infty$. The idea is to "cut $X_\infty$ along $D_\infty$". In topology, "cutting a manifold $M$ along $Z\subset M$"

is a standard construction, at least when $Z$ is a submanifold: it means to remove the interior of an appropriately chosen tubular neighborhood of $Z$, leaving a manifold with boundary.

This correctly describes the topology of our *real-oriented blow-ups*, but lacks the naturality we need for mappings to extend canonically, so we will use a different approach, closely related to "blowing up $X$ along $Z$".

*Remark.* We will discuss the real-oriented blow-up in the real-analytic category. Most of the discussion of real-oriented blow-ups would work just as well in the real-algebraic or the $C^\infty$-category; in fact, we will work entirely with algebraic varieties, but it is convenient to be able to restrict to "ordinary" open sets, rather than Zariski-open sets. In any case, the definition (especially the endpoint modification below) should probably be viewed as preliminary. It works when $Z$ is a divisor with normal crossings, but it does not create the right object (something like the complement of a tubular neighborhood) when the singularities of $Z$ are too nasty. A full discussion of real-oriented blow-ups will require a separate paper.

*The oriented blow-up $\widehat{X}_{\mathbf{f}}^{+}$: a preliminary definition.* Suppose that $X$ is a real-analytic manifold, and that $Z\subset X$ is an analytic subset, perhaps with singularities. Suppose that $U\subset X$ is a coordinate patch in which the ideal $\mathcal{I}(Z)$ of real-analytic functions on $U$ vanishing on $Z$ is generated by $m$ functions $f_1, ..., f_m: U \to \mathbf{R}^m$. In other words, if we set

$$\mathbf{f} = \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix},$$

then $Z\cap U=\mathbf{f}^{-1}(0)$. We need to say things in terms of ideals rather than in terms of equations defining the set because, for instance, the origin in $\mathbf{R}^2$ is defined by the equation $x^2+y^2=0$, but the one function $x^2+y^2$ generates a much smaller ideal than $\mathcal{I}(Z)$, which is generated by $x$ and $y$.

We can then define $\widehat{U}_{\mathbf{f}}^{+}\subset U\times S^{m-1}$ to be the closure of the set

$$\{(x,P)\in U\times S^{m-1} \mid \mathbf{f}(x)\neq 0 \text{ and } \mathbf{f}(x)/|\mathbf{f}(x)|=P\}.$$

Denote by $p: \widehat{U}_{\mathbf{f}}^{+}\to U$ the canonical projection, and set $\widehat{Z}^{+}=p^{-1}(Z)$.

By contrast, the blow-up is the closure of the set

$$\widehat{U}_{\mathbf{f}}=\{(x,l)\in U\times \mathbf{P}_{\mathbf{R}}^{m-1} \mid \mathbf{f}(x)\neq 0 \text{ and } \mathbf{f}(x)\in l\};$$

and the exceptional divisor is $p^{-1}(Z)$. The main difference, indicated by the $^+$, is that we are now allowing for the orientability of the line $l\in\mathbf{P}_{\mathbf{R}}^{m-1}$.

We will sketch in Proposition 5.2 the fact that $\widehat{U}_{\mathbf{f}}^{+}$ depends only on $Z$ and not on the chosen generators of $\mathcal{I}(Z)$, and what to do when $Z$ is not defined by global equations.

Let us see that this does correspond to "cutting a manifold $X$ along $Z$" when $Z$ is a smooth hypersurface, defined locally by $f(x)=0$, where $f: X \to \mathbf{R}$ is a real-analytic submersion. Recall that $S^0=\{-1,1\}\subset\mathbf{R}$, and $\widehat{X}_Z\subset X\times\{-1,1\}$ is $X-Z$, with $Z\times\{1\}$ attached to the part where $f$ is positive and $Z\times\{-1\}$ attached to the part where $f$ is negative.

When $Z$ is the origin of $\mathbf{R}^n$, it also gives what we want: $\mathbf{R}^n$ with the origin replaced by $S^{n-1}$, and naturally parametrized by "polar coordinates" $(r,p)\in[0,\infty)\times S^{n-1}$, but with no identifications when $r=0$.

*The real-oriented blow-up.* Already we run into trouble when $Z$ is the union of the axes in $\mathbf{R}^2$. The ideal $\mathcal{I}(Z)$ is generated by the single function $xy$, so the space $(\widehat{\mathbf{R}^2})_Z\subset\mathbf{R}^2\times S^0$ is

$$(Q_1\cup Q_3)\times\{1\}\cup(Q_2\cup Q_4)\times\{-1\},$$

where $Q_i$ denotes the closed $i$th quadrant.

This is not quite what we want: the plane cut along $Z$, i.e., the disjoint union of the four closed quadrants. We will replace the space $\widehat{X}_{\mathbf{f}}^{+}$ by a space which maps to $\widehat{X}_{\mathbf{f}}^{+}$, and whose points above $x\in\widehat{X}_{\mathbf{f}}^{+}$ are the ends of $\widehat{X}_{\mathbf{f}}^{+}-\widehat{Z}^{+}$ at $x$. This does not change the points with a basis of connected neighborhoods, but it does separate the first quadrant from the third and the second from the fourth, as the points of contact of these quadrants each correspond to two ends.

More formally, let $X$ be any topological space, and $Y\subset X$ a closed subset. We will define the endpoint modification $E(X,Y)$ to be the space $X$ where every point $y\in Y$ has been replaced by the set of ends of $X-Y$ at $y$, i.e., by the points of the projective limit

$$\varprojlim \pi_0(V-Y),$$

where $\pi_0$ is the functor which associates to a space its set of connected components, and the projective limit is taken over all open neighborhoods $V\subset X$ of $y$. The space $E(X,Y)$ comes with a natural topology, which we will leave to the reader to define (see [DD, Vol. II, pp. 197–198]), and there is a canonical map $p: E(X,Y)\to X$. We will denote by $E(Y)\subset E(X,Y)$ the subset $p^{-1}(Y)$.

This construction can lead to pretty wild things when $Y$ is complicated, but when $X$ is a finite simplicial complex and $Y$ is a subcomplex, which will always be the case in this paper, then $E(X,Y)$ is a finite simplicial complex, and there are finitely many inverse images of every $y\in Y$.

*Definition* 5.1. Let $X$ be a real-analytic manifold, and $Z \subset X$ an analytic subset such that $\mathcal{I}(Z)$ is generated by $f_1, \ldots, f_m$. The real-oriented blow-up $\mathcal{B}^+(X, \mathbf{f})$ is the endpoint modification $E(\widehat{X}^+_{\mathbf{f}}, \widehat{Z}^+)$, with the exceptional divisor $\mathcal{B}^+(Z) \subset \mathcal{B}^+(X, Z)$ being $E(\widehat{Z}^+) \subset E(\widehat{X}^+_{\mathbf{f}}, \widehat{Z}^+)$.

This actually defines the oriented blow-up $\widehat{U}^+_{\mathbf{f}}$ only locally, and further it appears that $\widehat{U}^+_{\mathbf{f}}$ depends not only on $Z \subset U$, but also on the chosen generators for $\mathcal{I}(Z)$. The following proposition deals with these problems.

PROPOSITION 5.2. (a) *Let $U$ be a real-analytic manifold, and $Z \subset U$ an analytic subset. Suppose that $f_1, \ldots, f_m$ and $g_1, \ldots, g_k$ are two sets of generators of $\mathcal{I}(Z)$. Then the identity of $U - Z$ extends uniquely to an isomorphism $\widehat{U}^+_{\mathbf{f}} \to \widehat{U}^+_{\mathbf{g}}$ of semi-analytic spaces.*

*Thus we can write $\widehat{U}^+_Z$.*

(b) *The identity of $U - Z$ extends uniquely to an isomorphism*

$$E(\widehat{X}^+_{\mathbf{f}}, \widehat{Z}^+) \to E(\widehat{X}^+_{\mathbf{g}}, \widehat{Z}^+)$$

*of semi-analytic spaces.*

*Thus we can write $\mathcal{B}^+(U, Z)$.*

(c) *Let $U, V$ be real-analytic manifolds, and $F: V \to U$ a real-analytic map. Suppose that $Z \subset U$ is an analytic subset, and that $F$ is a local isomorphism at all points of $F^{-1}(Z)$. Then the restriction of $F$ to $V - F^{-1}(Z)$ extends to a mapping*

$$\mathcal{B}^+(F): \mathcal{B}^+(V, F^{-1}(Z)) \to \mathcal{B}^+(U, Z)$$

*which is a local isomorphism at all points of $\mathcal{B}^+(F^{-1}Z)$, so that the diagram*

$$\begin{tikzcd}
\mathcal{B}^+(V, F^{-1}(Z)) \arrow[r, "\mathcal{B}^+(F)"] \arrow[d, "p_1"'] & \mathcal{B}^+(U, Z) \arrow[d, "p_2"'] \\
V \arrow[r, "F"] & U
\end{tikzcd}$$

*commutes, where the $p_i$ are the canonical projections.*

Part (c) of Proposition 5.2 allows us to construct oriented blow-ups globally. If $Z \subset X$ is an analytic subspace of a manifold, we can cover $X$ by open subsets $U_i$, and construct $\mathcal{B}^+(U_i, Z)$ for each $i$. Part (c) evidently applies to the inclusions $U_i \cap U_j \subset U_i$, and allows us to glue the spaces $\mathcal{B}^+(U_i, Z)$ canonically.

*Sketch of proof.* The blow-up has a universal property: replacing a subspace by a divisor. The main part of the proof is showing that the space $\widehat{X}_Z$ has this property: [Har] proves this in the algebraic setting, and the same proof works in the analytic setting, and is Theorem II.7 of [HPV]. This implies that the blow-up is independent of the chosen generators. It is easy to see that this result goes over to the oriented context [HPV, Proposition VI.4], allowing us to define a space $\widehat{X}_Z^+$ as required in part (a).

The endpoint modification is a functor, giving us part (b).

Part (c) follows from the fact that the universal property defining blow-ups is evidently local on the set being blown-up. □

*The central example.* The main example we will want to consider is the real-oriented blow-up of $X=\mathbf{C}^2$ along $Z=\mathbf{C}\times\{0\}\cup\{0\}\times\mathbf{C}$, both viewed as real-analytic spaces; i.e., $\mathbf{C}^2=\mathbf{R}^4$, parametrized by $x_1,y_1,x_2,y_2$ (where $z_1=x_1+iy_1$, $z_2=x_2+iy_2$), and $Z$ is the union of the $(x_1,y_1)$-coordinate plane and the $(x_2,y_2)$-coordinate plane.

This is quite complicated to work out directly from the equations: $Z$ requires four equations to define it, and the equations for $\widehat{X}_Z$ are quite messy (this computation is done in [HPV, pp. 45–46]). It seems simpler to do the computation in its natural setting, where tensor products hide the complications of the explicit formulas.

THEOREM 5.3. *Let $X_1,X_2$ be smooth manifolds, and $Y_1\subset X_1$, $Y_2\subset X_2$ be smooth submanifolds. Then the identity of $(X_1-Y_1)\times(X_2-Y_2)$ extends to a unique isomorphism*

$$\mathcal{B}^+(X_1,Y_1)\times\mathcal{B}^+(X_2,Y_2)\to\mathcal{B}^+(X_1\times X_2,(X_1\times Y_2)\cup(Y_1\times X_2)).$$

*Proof.* The uniqueness is clear, since $(X_1-Y_1)\times(X_2-Y_2)$ is dense in both spaces. Thus it is enough to prove the statement locally, and we may assume that $X_1,X_2$ are vector spaces, and that $Y_i\subset X_i$ are vector subspaces. Choose complementary subspaces $E_i$ so that $X_i=Y_i\oplus E_i$; then

$$X_i=Y_i\times E_i \quad \text{and} \quad Y_i=Y_i\times\{0\}.$$

The following lemma, whose proof is immediate and left to the reader, allows us to ignore the $Y_i$ and focus on the $E_i$.

LEMMA 5.4. *If $Y\subset X$ are an analytic manifold and an analytic subset, and $Z$ is an analytic manifold, then*

$$\mathcal{B}^+(X\times Z,Y\times Z)=\mathcal{B}^+(X,Y)\times Z.$$

A straightforward application of Lemma 5.4, setting $Y_1\times Y_2=Z$, shows that it is enough to prove that

$$\mathcal{B}^+(E_1\times E_2,E_1\times\{0\}\cup\{0\}\times E_2)=\mathcal{B}^+(E_1,\{0\})\times\mathcal{B}^+(E_2,\{0\}).$$

Choose inner products in $E_1$ and $E_2$; we will use "polar coordinates" $r_i, p_i$ for $E_i$, where $r_i \in [0,\infty)$ and $p_i \in S(E_i)$ is a vector of norm 1 (here and later $S(E)$ denotes the unit sphere of the inner-product space $E$). Choose orthonormal bases $v_1, \ldots, v_n$ for $E_1$ and $w_1, \ldots, w_m$ for $E_2$. Then declaring $v_i \otimes w_j$ to be an orthonormal basis defines an inner product on $E_1 \otimes E_2$, independent of the choice of bases, such that $\|p \otimes q\| = \|p\| \, \|q\|$.

The classical Veronese embedding $\mathbf{P}(E_1) \times \mathbf{P}(E_2) \to \mathbf{P}(E_1 \otimes E_2)$ is the map which takes lines $l_1 \subset E_1$, $l_2 \subset E_2$ to the line $l_1 \otimes l_2 \subset E_1 \otimes E_2$.

This mapping has an oriented analog, which is no longer an embedding, but a double cover of its image. Again we leave the proof of the following lemma to the reader.

LEMMA 5.5. *The mapping*

$$\phi: S(E_1) \times S(E_2) \to S(E_1 \otimes E_2)$$

*given by* $\phi(p_1, p_2) = p_1 \otimes p_2$ *is a double cover of its image*: $\phi(-p_1, -p_2) = \phi(p_1, p_2)$.

Now to prove Theorem 5.3. The map $F: E_1 \times E_2 \to E_1 \otimes E_2$ given by $(x, y) \mapsto x \otimes y$ defines the locus $E_1 \times \{0\} \cup \{0\} \times E_2$ to be blown up, so to construct

$$\mathcal{B}^+(E_1 \times E_2, (E_1 \times \{0\}) \cup (\{0\} \times E_2))$$

we need to compute the closure $\bar{\Gamma} \subset E_1 \times E_2 \times S(E_1 \otimes E_2)$ of the graph $\Gamma$ of

$$\frac{F}{|F|}: (E_1 - \{0\}) \times (E_2 - \{0\}) \to S(E_1 \otimes E_2).$$

Note that

$$F(r_1 p_1, r_2 p_2) = r_1 r_2 \phi(p_1, p_2).$$

Consider the semi-algebraic mapping

$$\Phi: [0,\infty) \times S(E_1) \times [0,\infty) \times S(E_2) \to E_1 \times E_2 \times S(E_1 \otimes E_2)$$

given by

$$\Phi: ((r_1, p_1), (r_2, p_2)) \mapsto (r_1 p_1, r_2 p_2, \phi(p_1, p_2)).$$

Observe that this mapping is proper, so its image is closed. Moreover, the image of $\Phi$ contains $\Gamma$ as a dense open set, so the image of $\Phi$ is the closure of $\Gamma$. Moreover, $\Phi$ is injective on the locus where $(r_1, r_2) \neq (0,0)$. This is clear from the first two coordinates of $\Phi$ if both $r_1$ and $r_2$ do not vanish, and is still true if exactly one vanishes. Indeed, if $r_1 \neq 0$, then $\Phi((r_1, p_1), (r_2, p_2))$ certainly determines $p_1$, and using the third coordinate, $p_2$ also.

Therefore $\Phi$ induces a semi-analytic isomorphism between the quotient

$$\mathcal{B}^+(E_1,\{0\})\times\mathcal{B}^+(E_2,\{0\})/\sim, \quad \text{where } ((0,p_1),(0,p_2))\sim((0,-p_1),(0,-p_2)),$$

and $\bar{\Gamma}$. Thus every point $(0,0,\phi(p_1,p_2))\in\bar{\Gamma}$ corresponds to two ends, and $\Phi$ lifts to a semi-analytic isomorphism

$$\widetilde{\Phi}: \mathcal{B}^+(E_1,\{0\})\times\mathcal{B}^+(E_2,\{0\})\to\mathcal{B}^+(E_1\times E_2,(E_1\times\{0\})\cup(\{0\}\times E_2)). \qquad \square$$

Theorem 5.3 allows us to understand the fibers of the projection

$$p: \mathcal{B}^+(X_1\times X_2,(X_1\times Y_2)\cup(Y_1\times X_2)):$$

if $(x_1,x_2)\in(X_1-Y_1)\times(X_2-Y_2)$, then $p^{-1}(x_1,x_2)$ is a point, of course;
if $(x_1,x_2)\in Y_1\times(X_2-Y_2)$, the fiber $p^{-1}(x_1,x_2)$ is canonically $S(T_{x_1}X_1/T_{x_1}Y_1)$;
if $(x_1,x_2)\in(X_1-Y_1)\times Y_2$, the fiber $p^{-1}(x_1,x_2)$ is canonically $S(T_{x_2}X_2/T_{x_2}Y_2)$;
if $(x_1,x_2)\in Y_1\times Y_2$, the fiber $p^{-1}(x_1,x_2)$ is canonically

$$S(T_{x_1}X_1/T_{x_1}Y_1)\times S(T_{x_2}X_2/T_{x_2}Y_2).$$

*The real-oriented blow-up of a complex manifold along a complex subspace.* In our applications of the real-oriented blow-up, the spaces $Y\subset X$ will be the underlying real-analytic spaces $X_{\mathbf{R}}$, $Y_{\mathbf{R}}$ of complex-analytic spaces $X$, $Y$. Rather than write $\mathcal{B}^+(X_{\mathbf{R}},Y_{\mathbf{R}})$, we will denote the real-oriented blow-up by $\mathcal{B}^+_{\mathbf{R}}(X,Y)$. The space $\mathcal{B}^+_{\mathbf{R}}(X,Y)$ has extra structure in that case: the natural action $\theta * p=e^{i\theta}p$ of the circle $\mathbf{R}/2\pi\mathbf{Z}$ on the unit sphere of a complex inner-product vector space induces an action of $\mathbf{R}/2\pi\mathbf{Z}$ on $\mathcal{B}^+_{\mathbf{R}}(Y)$, and when $Y$ is of the form $E_1\times\{0\}\cup\{0\}\times E_2\subset E_1\times E_2$ as above, where both $E_1$ and $E_2$ are complex vector spaces, the same natural action of $\mathbf{R}/2\pi\mathbf{Z}$ on $S(E_1)$ and $S(E_2)$ gives an action of the 2-dimensional torus $(\mathbf{R}/2\pi\mathbf{Z})^2$ on $p^{-1}(x_1,x_2)$ when $(x_1,x_2)\in Y_1\times Y_2$.

We will mainly be interested in the case where $X$ is a complex surface, and $Z\subset X$ is a divisor with normal crossings. Let us denote by $Z'$ the smooth part of $Z$, i.e., the complement of the set of double points. The following statement then summarizes our discussion.

THEOREM 5.6. (a) *The real-oriented blow-up $\mathcal{B}^+_{\mathbf{R}}(X,Z)$ is then a 4-dimensional manifold with boundary, and this boundary is a 3-dimensional manifold with corners corresponding to the double points of $Z$.*

(b) *The group $\mathbf{R}/2\pi\mathbf{Z}$ acts naturally on $p^{-1}(Z')$, making it into a principal circle bundle.*

(c) *The fibers above the double points are naturally principal under the group $(\mathbf{R}/2\pi\mathbf{Z})^2$.*

*The real-oriented blow-up of a complex blow-up.* The crucial question for us will be the following. Let $X$ be a complex surface, $Z\subset X$ a divisor with normal crossings, and $z\in Z$ a point. What relation is there between the real-oriented blow-ups $\mathcal{B}^+_{\mathbf{R}}(X,Z)$ and $\mathcal{B}^+_{\mathbf{R}}(\widetilde{X}_z,\pi^{-1}(Z))$? The answer is contained in the following two theorems. Let

$$p:\mathcal{B}^+_{\mathbf{R}}(X,Z)\to X \quad \text{and} \quad \tilde{p}:\mathcal{B}^+_{\mathbf{R}}(\widetilde{X}_z,\pi^{-1}(Z))\to \widetilde{X}_z$$

be the canonical projections.

A first thing to notice is that the topological pairs $\mathcal{B}^+_{\mathbf{R}}(Z)\subset\mathcal{B}^+_{\mathbf{R}}(X,Z)$ and $\mathcal{B}^+_{\mathbf{R}}(\widetilde{Z})\subset\mathcal{B}^+_{\mathbf{R}}(\widetilde{X}_z,\widetilde{Z})$ are homeomorphic. After all, the boundary of a tubular neighborhood of $Z$ in $X$ is still the boundary of a tubular neighborhood of $\widetilde{Z}$ in $\widetilde{X}_z$. What has changed is the pattern of tori in the 3-dimensional manifold $\mathcal{B}^+_{\mathbf{R}}(Z)$, and the circle and torus actions on the tori and regions between the tori respectively.

*The real-oriented blow-up of a complex blow-up at a simple point of a divisor.* Let $z\in Z$ be a simple point, and let $\tilde{z}\in\widetilde{X}_z$ be the new double point of $\widetilde{Z}$, i.e., the intersection of the proper transform of $Z$ with the exceptional divisor. Then $\mathcal{B}^+_{\mathbf{R}}(\widetilde{X}_z,\widetilde{Z})$ can be understood as follows. The fiber $p^{-1}(z)$ is a circle; thicken this circle to make a solid torus, invariant under the existing circle action. Keep the old circle action on the outside of the solid torus, and modify it inside, so that the oriented circle orbits on the boundary of the solid torus are the "sums" of the old ones and of boundaries of discs $\Delta$ in the solid torus, which are oriented so that $p:\Delta\to Z$ is orientation-preserving. Theorem 5.7 makes this precise.

Call $Y$ the space $\mathcal{B}^+_{\mathbf{R}}(X,Z)$, but with the following modified circle action on part of the boundary $\mathcal{B}^+_{\mathbf{R}}(Z)$. Choose a disc $D\subset Z$ centered at $z$, parametrized by $w$ with $|w|<R$ and a section $\sigma: D\to\mathcal{B}^+_{\mathbf{R}}(D)$, giving us an isomorphism $\mathcal{B}^+_{\mathbf{R}}(D)\to D\times(\mathbf{R}/2\pi\mathbf{Z})$, allowing us to parametrize $\mathcal{B}^+_{\mathbf{R}}(D)\subset\mathcal{B}^+_{\mathbf{R}}(X,Z)$ by $w$ and $\theta\in\mathbf{R}/2\pi\mathbf{Z}$. Choose $r<R$, and define a new circle action on $\mathcal{B}^+_{\mathbf{R}}(D_r)$ by the formula

$$\Theta*(w,0,\theta)=(we^{i\Theta},0,\theta+\Theta). \tag{5.1}$$

THEOREM 5.7. *There exists a homeomorphism $h:\mathcal{B}^+_{\mathbf{R}}(\widetilde{X}_z,\widetilde{Z})\to Y$ which coincides with the identity outside of $p^{-1}(U)$ for some neighborhood $U$ of $D$ in $X$, which takes $\tilde{p}^{-1}(\tilde{z})$ to the torus $|w|=r$, and which respects the circle and torus actions.*

For the proof, see [HPV, pp. 50–57].

*The real-oriented blow-up of a complex blow-up at a double point of a divisor.* Let $z$ be a double point of the divisor $Z\subset X$, and call $\tilde{z}_1,\tilde{z}_2$ the two double points above $z$ in $\widetilde{X}_z$, i.e., the two intersections of the proper transform of $Z$ with the new exceptional divisor.

Again we will construct a model $Y$ for $\mathcal{B}^+_{\mathbf{R}}(\widetilde{X}_z, \widetilde{Z})$ whose underlying space is $\mathcal{B}^+_{\mathbf{R}}(X,Z)$, but where we modify the tori and group action on $\mathcal{B}^+_{\mathbf{R}}(Z)$. Roughly, $Y$ is constructed from $\mathcal{B}^+_{\mathbf{R}}(X,Z)$ as follows. Thicken the torus $p^{-1}(z)$ so that the thickened torus is invariant under the circle actions. Keep the old circle actions on the outside of the thickened torus, and modify it inside, so that the oriented circle orbits on the boundary torus are the "sums" of an orbit of $\mathbf{R}/2\pi\mathbf{Z}\times\{0\}$ and an orbit of $\{0\}\times\mathbf{R}/2\pi\mathbf{Z}$. Theorem 5.8 makes this precise.

Choose local coordinates $w_1$ and $w_2$ on a neighborhood $U$ of $z$ in $X$, such that $|w_1|, |w_2|<R$, and $Z$ is given in $U$ by the equation $w_1w_2=0$. This gives an isomorphism of $\mathcal{B}^+_{\mathbf{R}}(X,Z)$ near $p^{-1}(z)$ with the standard model $\mathcal{B}^+_{\mathbf{R}}(D,0)\times\mathcal{B}^+_{\mathbf{R}}(D,0)$, which can be described as the set of

$$(r_1, r_2, \theta_1, \theta_2)\in[0,R)^2\times(\mathbf{R}/2\pi\mathbf{Z})^2.$$

The projection $\pi$ is given by $w_1=r_1e^{i\theta_1}$, $w_2=r_2e^{i\theta_2}$, and the set $\mathcal{B}^+_{\mathbf{R}}(Z)$ corresponds to the subset where $r_1r_2=0$.

Now choose $r<R$, and modify the circle action on the subset

$$P=\{(r_1,r_2,\theta_1,\theta_2)\in[0,R)^2\times(\mathbf{R}/2\pi\mathbf{Z})^2 \mid |r_1|,|r_2|<r \text{ and } r_1r_2=0\}$$

by setting

$$\begin{aligned}\Theta*(r_1,0,\theta_1,\theta_2)&=(r_1,0,\theta_1+\Theta,\theta_2+\Theta),\\ \Theta*(0,r_2,\theta_1,\theta_2)&=(0,r_2,\theta_1+\Theta,\theta_2+\Theta).\end{aligned}$$

Keep the previous circle action outside $P$; this gives as it should two circle actions on the two tori $|w_1|=r$ and $|w_2|=r$.

THEOREM 5.8. *There exists a homeomorphism $h: \mathcal{B}^+_{\mathbf{R}}(\widetilde{X}_z, \widetilde{Z})\to Y$ which is the identity outside of $p^{-1}(U)$, which maps the torus $\tilde{p}^{-1}(\tilde{z}_1)$ to the torus $r_1=r$, $r_2=0$, which maps the torus $\tilde{p}^{-1}(\tilde{z}_2)$ to the torus $r_1=0$, $r_2=r$, and which respects all the circle and torus actions.*

For the proof, see [HPV, pp. 58–64].

*Naturality.* Let $X$ be a smooth surface, $Z\subset X$ a divisor with normal crossings, and $z\in D$ a point.

THEOREM 5.9. (a) *The mapping $\pi: \widetilde{X}_z\to X$ lifts to a unique mapping*

$$\tilde{\pi}: \mathcal{B}^+_{\mathbf{R}}(\widetilde{X}_z, \widetilde{Z})\to\mathcal{B}^+_{\mathbf{R}}(X,Z)$$

*such that the diagram*

$$\begin{tikzcd} \mathcal{B}^+_{\mathbf{R}}(\widetilde{X}_z,\widetilde{Z}) \arrow[r, "\tilde{\pi}"] \arrow[d, "\tilde{p}"'] & \mathcal{B}^+_{\mathbf{R}}(X,Z) \arrow[d, "p"] \\ \widetilde{X}_z \arrow[r, "\pi"] & X \end{tikzcd}$$

*commutes.*

(b) *If $z$ is a simple point of $Z$, then the mapping $\tilde{\pi}$ maps the torus $\tilde{p}^{-1}(\tilde{z})$ (parametrized by $(\theta_1,\phi_2)$) to the circle $p^{-1}(z)$ (parametrized by $\theta_2$), by the mapping*

$$(\theta_1,\phi_2)\mapsto\theta_1+\phi_2.$$

(c) *If $z$ is a double point of $Z$, then the mapping $\tilde{p}$, mapping the torus $\tilde{p}^{-1}(z_1)$, parametrized by $\theta_1,\phi_2$, to the torus $p^{-1}(z)$, parametrized by $\theta_1$ and $\theta_2$, is given by the formula*

$$\begin{bmatrix}\theta_1\\ \phi_2\end{bmatrix}\mapsto\begin{bmatrix}\theta_1\\ \theta_1+\phi_2\end{bmatrix}.$$

This is proved in [HPV, part (a) on p. 51, part (b) on p. 52, and part (c) on p. 59].

*Infinitely many blow-ups.* Suppose that we repeat infinitely many times the following procedure, as in §3.

Take a surface $X_0$ containing a divisor with normal crossings $Z_0\subset X_0$; although it is not essential, we will assume that $X_0$ is compact. Choose a point $z_0\in Z_0$, blow it up to create a surface $X_1=(\widetilde{X_0})_{z_0}$, with a projection $\pi_1\colon X_1\to X_0$; set $Z_1=\pi_1^{-1}(Z_0)$. Now choose a new point $z_1\in Z_1$, etc. Denote by

$$\tilde{\pi}_i\colon\mathcal{B}^+_{\mathbf{R}}(X_i,Z_i)\to\mathcal{B}^+_{\mathbf{R}}(X_{i-1},Z_{i-1})$$

the map induced from $\pi_i$.

Theorems 5.7 and 5.8 assert that at each stage the pair $\mathcal{B}^+_{\mathbf{R}}(Z_i)\subset\mathcal{B}^+_{\mathbf{R}}(X_i,Z_i)$ is homeomorphic to the pair $\mathcal{B}^+_{\mathbf{R}}(Z_0)\subset\mathcal{B}^+_{\mathbf{R}}(X_0,Z_0)$. It seems reasonable to think that the same will remain true in the limit, and it is.

THEOREM 5.10. *The projective limit of pairs*

$$\varprojlim(\mathcal{B}^+_{\mathbf{R}}(Z_i),\tilde{\pi}_i)\subset\varprojlim(\mathcal{B}^+_{\mathbf{R}}(X_i,Z_i),\tilde{\pi}_i)$$

*is homeomorphic to the pair $\mathcal{B}^+_{\mathbf{R}}(Z_0)\subset\mathcal{B}^+_{\mathbf{R}}(X_0,Z_0)$, and the canonical map*

$$\varprojlim(\mathcal{B}^+_{\mathbf{R}}(X_i,Z_i),\tilde{\pi}_i)\to\mathcal{B}^+_{\mathbf{R}}(X_0,Z_0)$$

*can be approximated by homeomorphisms.*

This theorem is proved in [HPV, pp. 64–65].

## 6. Real-oriented blow-ups for Hénon mappings

The Hopf fibration $p: S^3 \to S^2$ is a famous example from topology, where the circle acts on the 3-sphere, and the quotient is the 2-sphere. If we think of $S^3$ as the unit sphere in $\mathbf{C}^2$, and $S^2$ as $\mathbf{P}^1_{\mathbf{C}}$, then it can be written as

$$p: (z_1, z_2) \mapsto [z_1 : z_2].$$

It can also be thought of as the quotient of the 3-sphere by the circle action

$$\mathbf{R}/2\pi\mathbf{Z} \times S^3 \to S^3, \quad \Theta * (z_1, z_2) = (e^{i\Theta} z_1, e^{i\Theta} z_2).$$

The Hopf fibration seems a natural candidate to be the real-oriented blow-up of a projective line in a surface, and it is.

*The real-oriented blow-up of the line at infinity in* $\mathbf{P}^2_{\mathbf{C}}$. We will begin our construction of real-oriented blow-ups constructing $\mathcal{B}^+_{\mathbf{R}}(\mathbf{P}^2_{\mathbf{C}}, l_\infty)$, and then seeing the effect on this real-oriented blow-up of the complex blow-ups described in §2. Local coordinates on a neighborhood of $l_\infty \subset \mathbf{P}^2_{\mathbf{C}}$ are

$$u_1 = \frac{1}{x}, \quad v_1 = \frac{y}{x} \quad \text{and} \quad u_2 = \frac{1}{y}, \quad v_2 = \frac{x}{y}.$$

This leads to the charts $\mathbf{C} \times (\mathbf{R}/2\pi\mathbf{Z}) \to \mathcal{B}^+_{\mathbf{R}}(l_\infty)$ given by $(v_j, \theta_j)$, $j = 1, 2$, where $\theta_j = -\arg u_j$.

On the overlap $v_1, v_2 \neq 0$ these coordinates are identified by

$$v_2 = \frac{1}{v_1}, \quad \theta_2 = \theta_1 + \arg v_1.$$

This is a variant of the Hopf fibration.

PROPOSITION 6.1. (a) *The mapping* $S^3 \to \mathcal{B}^+_{\mathbf{R}}(l_\infty)$ *given by*

$$(z_1, z_2) \mapsto \begin{cases} v_1 = z_2/z_1,\ \theta_1 = -\arg z_1, & \text{if } z_1 \neq 0, \\ v_2 = z_1/z_2,\ \theta_2 = -\arg z_2, & \text{if } z_2 \neq 0, \end{cases}$$

*is a diffeomorphism which carries the orientation of* $S^3$ *as the boundary of the complement of the* 4*-ball to the standard orientation of* $\mathcal{B}^+_{\mathbf{R}}(l_\infty)$.

(b) *This diffeomorphism transforms the circle action* $\Theta * (z_1, z_2) = (e^{-i\Theta} z_1, e^{-i\Theta} z_2)$ *into the canonical circle action*

$$\Theta * (v_j, \theta_j) = (v_j, \theta_j + \Theta).$$

*Proof.* The main thing to check is that the mapping is compatible with the identification $\theta_2=\theta_1+\arg v_1$, which becomes $\arg z_2=\arg z_1+\arg(z_2/z_1)$.

The map is injective: if we know $v_1$ and $\theta_1$, then from $z_2=v_1z_1$ and the equation $|z_1|^2+|z_2|^2=1$ we see

$$|z_1|^2=\frac{1}{1+|v_1|^2},$$

and since we also know the argument of $z_1$, we know $z_1$, hence $z_2$.

The surjectivity is also clear from the argument above.

The compatibility with the circle action is

$$\frac{e^{i\Theta}z_1}{e^{i\Theta}z_2}=\frac{z_1}{z_2}, \quad \arg e^{i\Theta}z_1=\arg z_1+\Theta.$$

The vectors

$$\begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \quad \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \quad \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

form a direct basis of $T_{(1,0)}S^3$ if we orient $S^3$ as the boundary of the ball. The first of these vectors is tangent to the oriented orbit through $(1,0)$, whereas the last two project under $p$ to a direct basis of $T_{p(1,0)}\mathbf{P}^1_{\mathbf{C}}$. □

Each space $X_{[-N-1,N+1]}$ is obtained from $X_{[-N,N]}$ by a sequence of blow-ups, first at $\mathbf{q}_N$ and $\mathbf{p}_{N+1}$, and then at points of the most recent exceptional divisor; let

$$\pi_{N+1,N}\colon X_{[-N-1,N+1]}\to X_{[-N,N]}$$

denote the blow-down mapping. By Proposition 5.9, the mapping $\pi_{N+1,N}$ induces a projection

$$\tilde{\pi}_{N+1,N}\colon \mathcal{B}^+_{\mathbf{R}}(X_{[-(N+1),N+1]},D_{[-(N+1),N+1]})\to\mathcal{B}^+_{\mathbf{R}}(X_{[-N,N]},D_{[-N,N]}),$$

which allows us to consider the projective limit

$$\mathcal{B}^+_{\mathbf{R}}(X_\infty,D_\infty)=\varprojlim\,(\mathcal{B}^+_{\mathbf{R}}(X_{[-N,N]},D_{[-N,N]});\tilde{\pi}_{N+1,N}).$$

There is a canonical inclusion $j_N\colon\mathbf{C}^2\to X_{[-N,N]}$ (as in Proposition 3.6 (b)), which lifts to $\tilde{\jmath}_N\colon\mathbf{C}^2\to\mathcal{B}^+_{\mathbf{R}}(X_{[-N,N]},D_{[-N,N]})$ since the real-oriented blow-up is taken along $D_{[-N,N]}=X_{[-N,N]}-\mathbf{C}^2$. These inclusions are compatible with $\tilde{\pi}_{N+1,N}$, leading to an inclusion $\tilde{\jmath}_\infty\colon\mathbf{C}^2\to\mathcal{B}^+_{\mathbf{R}}(X_\infty,D_\infty)$.

THEOREM 6.2. (a) *The mapping $\tilde{\jmath}_\infty$ is injective, with dense image, allowing us to think of $\mathbf{C}^2$ as a subset of $\mathcal{B}^+_{\mathbf{R}}(X_\infty, D_\infty)$.*

(b) *The Hénon mapping $H: \mathbf{C}^2 \to \mathbf{C}^2$ extends continuously to an automorphism*

$$\mathcal{B}^+_{\mathbf{R}}(H_\infty): \mathcal{B}^+_{\mathbf{R}}(X_\infty, D_\infty) \to \mathcal{B}^+_{\mathbf{R}}(X_\infty, D_\infty).$$

*Proof.* (a) The injectivity of $\tilde{\jmath}_\infty$ is clear since all the $\tilde{\jmath}_N$ are injective. Moreover, all the $j_N$ have dense image by Proposition 3.6, and so do the $\tilde{\jmath}_N$ since the interior of a manifold with boundary is dense in the manifold. The density of the image of $\tilde{\jmath}_\infty$ follows immediately, since the topology on the projective limit is inherited from the topology of the product. (This also follows from the much more general Theorem 5.10.)

(b) Clearly the Hénon mapping, i.e., the shift, induces an isomorphism

$$X_{[-N,N+1]} \to X_{[-(N+1),N]},$$

hence a homeomorphism

$$\mathcal{B}^+_{\mathbf{R}}(X_{[-N,N+1]}, D_{[-N,N+1]}) \to \mathcal{B}^+_{\mathbf{R}}(X_{[-(N+1),N]}, D_{[-(N+1),N]})$$

by Proposition 5.2. The result will follow since

$$\mathcal{B}^+_{\mathbf{R}}(X_\infty, D_\infty) = \varprojlim_{M,N\to\infty} \mathcal{B}^+_{\mathbf{R}}(X_{[-N,M]}, D_{[-N,M]})$$

when $N$ and $M$ can go to infinity in any way one wants: the pairs $[-N,N]$ are cofinal in the projective system of pairs $[-N,M]$. $\square$

The remainder of this section is devoted to understanding the structure of $\mathcal{B}^+_{\mathbf{R}}(X_\infty, D_\infty)$ in detail, as this is equivalent to understanding the dynamics of Hénon mappings at infinity.

THEOREM 6.3. (a) *The pair $(\mathcal{B}^+_{\mathbf{R}}(X_\infty, D_\infty), \mathcal{B}^+_{\mathbf{R}}(D_\infty))$ is homeomorphic to the pair $(B^4, S^3)$, the closed 4-ball bounded by the 3-sphere.*

(b) *The mapping $p: \mathcal{B}^+_{\mathbf{R}}(D_\infty) \to D_\infty$ has as its fibers:*

- *a circle above ordinary points;*
- *a torus above double points;*
- *a d-adic solenoid $\Sigma^-$ above $\mathbf{p}^\infty$, and a d-adic solenoid $\Sigma^+$ above $\mathbf{q}^\infty$.*

*Proof.* Part (a) was proved in Theorem 5.10. More precisely, we saw that the real-oriented blow-up of $\mathbf{P}^2$ along the line at infinity is a 4-ball bounded by a 3-sphere, and so $\varprojlim \mathcal{B}^+_{\mathbf{R}}(X_{[-N,N]}, D_{[-N,N]})$ is also. Indeed, $\varprojlim \mathcal{B}^+_{\mathbf{R}}(X_{[-N,N]}, D_{[-N,N]})$ is obtained by infinitely many times making a blow-up of a surface $X$ at a point $z$ of a divisor $D$, then taking the real-oriented blow-up of the resulting surface $\widetilde{X}_z$ along the inverse image $\widetilde{D}$ of the divisor $D$. That is the situation of Theorem 5.10. Moreover, the first two cases of part (b) follow immediately from Theorem 5.6.

The third statement of part (b) is a bit more delicate. The point $\mathbf{p}^\infty$ is represented by the sequence

$$\mathbf{p}_1 \in X_{[0,0]}, \quad \mathbf{p}_2 \in X_{[-1,1]}, \quad \mathbf{p}_3 \in X_{[-2,2]}, \quad \dots,$$

and above this point we see the projective limit of the system of circles

$$p_0^{-1}(\mathbf{p}_1) \leftarrow p_1^{-1}(\mathbf{p}_2) \leftarrow p_2^{-1}(\mathbf{p}_3) \leftarrow \dots .$$

*Remark.* We are adding an index to avoid ambiguity, calling

$$p_N : \mathcal{B}^+_{\mathbf{R}}(X_{[-N,N]}, D_{[-N,N]}) \to X_{[-N,N]}$$

the canonical projection; the added notation is necessary as the $\mathbf{p}_{N+1}$ can be viewed as points in all $X_{[-M,M]}$ with $M>N$ (or in $X_\infty$), but only in $X_{[-N,N]}$ are $\mathbf{p}_{N+1}$ and $\mathbf{q}_N$ simple points of $D_{[-N,N]}$.

There is a canonical parametrization of $p_N^{-1}(\mathbf{p}_{N+1}) \subset \mathcal{B}^+_{\mathbf{R}}(X_{[-N,N]}, D_{[-N,N]})$, obtained from the composition

$$\begin{array}{ccccccc}
p_N^{-1}(\mathbf{p}_{N+1}) & \xrightarrow[N\text{th coordinate}]{\text{projection to}} & p^{-1}(\widetilde{\mathbf{p}}) & \xrightarrow{\text{applying } \widetilde{H}} & p^{-1}(\mathbf{p}) & \xrightarrow{\arg(1/y)} & \mathbf{R}/2\pi\mathbf{Z}. \\
\cap & & \cap & & \cap & & \\
\mathcal{B}^+_{\mathbf{R}}(X_{[-N,N]}, D_{[-N,N]}) & & \mathcal{B}^+_{\mathbf{R}}(\widetilde{X}, \widetilde{D}) & & \mathcal{B}^+_{\mathbf{R}}(\mathbf{P}^2, l_\infty) & &
\end{array}$$

We will denote this coordinate by $\theta_N$.

There is also a natural parametrization of $p_N^{-1}(\mathbf{q}_{-N}) \subset \mathcal{B}^+_{\mathbf{R}}(X_{[-N,N]}, D_{[-N,N]})$, which is a bit simpler, obtained from the similar composition

$$\begin{array}{ccccccc}
p_N^{-1}(\mathbf{q}_{-N}) & \xrightarrow[-N\text{th coordinate}]{\text{projection to}} & p^{-1}(\mathbf{q}') & \simeq & p^{-1}(\mathbf{q}) & \xrightarrow{\arg(1/x)} & \mathbf{R}/2\pi\mathbf{Z}. \\
\cap & & \cap & & \cap & & \\
\mathcal{B}^+_{\mathbf{R}}(X_{[-N,N]}, D_{[-N,N]}) & & \mathcal{B}^+_{\mathbf{R}}(\widetilde{X}, \widetilde{D}) & & \mathcal{B}^+_{\mathbf{R}}(\mathbf{P}^2, l_\infty) & &
\end{array}$$

We will denote this coordinate by $\phi_N$.

Now part (c) follows from Proposition 6.4.

PROPOSITION 6.4. *We have*

$$\tilde{\pi}_{N+1,N}(\theta_{N+1}) = d\theta_{N+1} + \arg a \quad \textit{and} \quad \tilde{\pi}_{N+1,N}(\phi_{-N-1}) = d\phi_{-N-1}.$$

*End of proof of Theorem* 6.3 *using Proposition* 6.4. One description (see [HO1, §3]) of the $d$-adic solenoid $\Sigma_d$ is as

$$\Sigma_d = \varprojlim (\mathbf{R}/2\pi\mathbf{Z}, \theta \mapsto d\theta).$$

Clearly the second part of Proposition 6.4 shows that precisely the space $\Sigma^+$ above $\mathbf{q}^\infty$ is canonically the $d$-adic solenoid.

For the point $\mathbf{p}^\infty$, observe that if we set $\psi = \theta + \arg a/(d-1)$, then the mapping $\theta \mapsto d\theta + \arg a$ becomes

$$\psi \mapsto d\left(\psi - \frac{\arg a}{d-1}\right) + \arg a + \frac{\arg a}{d-1} = d\psi.$$

Thus the subset $\Sigma^- \subset \mathcal{B}^+_{\mathbf{R}}(X_\infty, D_\infty)$ above $\mathbf{q}^\infty$ is also a $d$-adic solenoid.

*Proof of Proposition* 6.4. Let us first compute the map

$$\begin{array}{cc} p^{-1}(\widetilde{\mathbf{p}}) \to & p^{-1}(\mathbf{p}) \\ \cap & \cap \\ \mathcal{B}^+_{\mathbf{R}}(\widetilde{X}, \widetilde{D}) & \mathcal{B}^+_{\mathbf{R}}(\mathbf{P}^2, l_\infty) \end{array}$$

induced by the blow-down mapping $\widetilde{X}_H \to \mathbf{P}^2$, where both domain and range are identified with $\mathbf{R}/2\pi\mathbf{Z}$:

- $p^{-1}(\mathbf{p})$ using $\arg(1/y)$;
- $p^{-1}(\widetilde{\mathbf{p}})$ using $(\arg(1/y)) \circ \widetilde{H}$.

In §2, we began by using the coordinates $u = x/y$, $v = 1/y$ near $\mathbf{p}$, so we see that $\arg v$ is our parameter for $p^{-1}(\mathbf{p})$. Still in the notation of §2, $\arg X_1$ gives the parametrization of $p^{-1}(\widetilde{\mathbf{p}})$. Formula (2.6), for the case $k = d-1$, tells us that the point $\arg X_1 = \theta$ of $p^{-1}(\widetilde{\mathbf{p}})$ is mapped by $\mathcal{B}^+_{\mathbf{R}}(\widetilde{H})$ to the point where $v$ has argument

$$\arg \frac{X_1\left(X_1^{d-1} Y_{d-1} + a \sum_{j=0}^{d-2} Q_j X_1^j\right)}{X_1^{d-1} Y_{d-1} + a \sum_{j=0}^{d-2} Q_j X_1^j} = \arg X_1.$$

Thus we must see how the blow-down maps $p^{-1}(\widetilde{\mathbf{p}})$ to $p^{-1}(\mathbf{p})$, working in the coordinate $\theta = \arg X_1$ in the domain, and $\theta = \arg v$ in the range, since these correspond under the Hénon mapping.

More precisely, consider the mapping from the circle $p^{-1}(\widetilde{\mathbf{p}})\subset\mathcal{B}^+_{\mathbf{R}}(\widetilde{X}_H,\widetilde{D})$ to the circle $p^{-1}(\mathbf{p})\subset\mathcal{B}^+_{\mathbf{R}}(\mathbf{P}^2,l_\infty)$. Let $\mathbf{a}_0=\mathbf{p}$, $\mathbf{a}_1,\dots,\mathbf{a}_{2d-2}$ be the successive points at which we performed blow-ups to get from $\mathbf{P}^2$ to $\widetilde{X}_H$, and finally set $\mathbf{a}_{2d-1}=\widetilde{\mathbf{p}}$. The point $\mathbf{a}_0$ and the points $\mathbf{a}_d,\dots,\mathbf{a}_{2d-1}$ are simple points of the divisor constructed so far; the points $\mathbf{a}_1,\dots,\mathbf{a}_{d-1}$ are double points. The inverse images of these points are parametrized by

$$\begin{aligned}&\arg v \ \text{ at } \mathbf{a}_0,\\ &\begin{pmatrix}\arg X_1\\ \arg u\end{pmatrix} \text{ at } \mathbf{a}_1,\\ &\begin{pmatrix}\arg X_1\\ \arg X_k\end{pmatrix} \text{ at } \mathbf{a}_k, \quad k=2,\dots,d-1,\\ &\arg X_1 \ \text{ at } \mathbf{a}_k, \quad k=d,\dots,2d-1.\end{aligned} \tag{6.1}$$

In these coordinates, the blow-down mapping $\widetilde{X}_H\to\mathbf{P}^2$ induces the composition

$$\underset{\mathbf{a}_{2d-1}}{\theta}\mapsto\dots\mapsto\underset{\mathbf{a}_d}{\theta}\mapsto\underset{\mathbf{a}_{d-1}}{\begin{pmatrix}\theta\\ \theta+\arg a\end{pmatrix}}\mapsto\underset{\mathbf{a}_{d-2}}{\begin{pmatrix}\theta\\ 2\theta+\arg a\end{pmatrix}}\mapsto\dots\mapsto\underset{\mathbf{a}_1}{\begin{pmatrix}\theta\\ (d-1)\theta+\arg a\end{pmatrix}}\mapsto\underset{\mathbf{a}_0}{d\theta+\arg a}. \tag{6.2}$$

These are all straightforward applications of Theorem 5.2, parts (b) and (c). Let us spell out the mapping which takes $\mathbf{a}_d$ to $\mathbf{a}_{d-1}$. In that case we are taking the circle above the point $X_1=0, X_d=a$, parametrized by $\arg X_1$, to the torus above $X_1=0, X_{d-1}=0$. The blow-down mapping is

$$\begin{pmatrix}X_1\\ X_d\end{pmatrix}\mapsto\begin{pmatrix}X_1\\ X_{d-1}\end{pmatrix}=\begin{pmatrix}X_1\\ X_1X_d\end{pmatrix},$$

and in particular the circle $X_1=\varrho e^{i\theta}, X_d=a$ is mapped to the circle $X_1=\varrho e^{i\theta}, X_{d-1}=\varrho a e^{i\theta}$. If we let $\varrho\to 0$ and remember only the arguments, we get the desired formula.

This proves the first part of Proposition 6.4: by definition, the mapping $p_N^{-1}(\mathbf{p}_{N+1})\to p_{N-1}^{-1}(\mathbf{p}_N)$ is precisely the mapping above, the domain and range being identified with $\mathbf{R}/2\pi\mathbf{Z}$ by $H^{\circ N}$ and $H^{\circ(N-1)}$ respectively.

There are two ways to approach the second part of Proposition 6.4: to make the sequence of blow-ups at $\mathbf{q}$, repeating the material of §2 to make $H^{-1}$ well-defined, or to make a change of variables to make $H^{-1}$ conjugate to a Hénon mapping, for a new polynomial $p_1$ and a new Jacobian $a_1$, of course. Remember that we used the fact that the polynomial $p$ is monic in §2, so $p_1$ must also be monic. We invite the reader to show that in the variables $x_1, y_1$, where $\zeta x_1=y$ and $\zeta y_1=x$ with $\zeta^{d-1}=a$, we have

$$H^{-1}\colon\begin{pmatrix}x_1\\ y_1\end{pmatrix}\mapsto\begin{pmatrix}p_1(x_1)-y_1/a\\ x_1\end{pmatrix}$$

with $p_1$ monic. Thus, in these coordinates the blow-down takes

$$\arg(1/y_1) \mapsto d\arg(1/y_1) + \arg(1/a).$$

We invite the reader to check that this means exactly that in the original variables, the second formula of Proposition 6.4 is satisfied. $\square$

## 7. The topology of $\mathcal{B}^+_{\mathbf{R}}(X_\infty, D_\infty)$

Solenoidal mappings of the solid torus to itself are among the basic examples of dynamical systems. The typical example is the map

$$\tau_{d,k}: S^1 \times D \to S^1 \times D$$

given by

$$\tau_{d,k}(\zeta, z) = \left(\zeta^d, \tfrac{1}{2}\zeta + \varepsilon z \zeta^{1-d+k}\right),$$

where $S^1$ is the unit circle in $\mathbf{C}$, $D \subset \mathbf{C}$ is the unit disc, and $\varepsilon > 0$ is chosen so that the map is injective.

We will define a mapping from a solid torus to itself to be an *unbraided solenoidal mapping of degree $d$* precisely if it is topologically conjugate to $\tau_{d,k}$ for some $k$.

This definition is justified by Theorem 3.11 of [HO1], which asserts that any injective map $f: S^1 \times D \to S^1 \times D$ in the correct isotopy class, which expands in the circle direction and contracts in the disc direction, is conjugate to precisely one of the $\tau_{d,k}$. The labeling is justified by Theorems 4.1 and 4.6 of [HO1], which assert that if the solid torus is embedded in $S^3$ in the standard way, then only $\tau_{d,0}$ extends to $S^3$.

We will now describe the mapping $\mathcal{B}^+_{\mathbf{R}}(H_\infty)$ more precisely. A first statement says that appropriate restrictions of $\mathcal{B}^+_{\mathbf{R}}(H_\infty)$ and $\mathcal{B}^+_{\mathbf{R}}(H_\infty)^{-1}$ are solenoidal.

Let us denote by $T_{\mathbf{p}_i}$ and $T_{\mathbf{q}_i}$ the tori $\tilde{p}^{-1}(\mathbf{p}_i)$ and $\tilde{p}^{-1}(\mathbf{q}_i)$. Each separates $\mathcal{B}^+_{\mathbf{R}}(D_\infty)$ into two pieces. We will denote by $T^+_{\mathbf{p}_i}$ the one that contains the attracting solenoid $\Sigma^+$, and by $T^-_{\mathbf{p}_i}$ the one that contains the repelling solenoid $\Sigma^-$, and similarly for $\mathbf{q}_i$.

PROPOSITION 7.1. (a) *The mapping $\mathcal{B}^+_{\mathbf{R}}(H_\infty)$ maps $T^+_{\mathbf{p}_i}$ (resp. $T^+_{\mathbf{q}_i}$) into itself, and its restriction to $T^+_{\mathbf{p}_i}$ is conjugate to $\tau_{d,0}$.*

(b) *The map $\mathcal{B}^+_{\mathbf{R}}(H_\infty)^{-1}$ maps $T^-_{\mathbf{p}_i}$ (resp. $T^-_{\mathbf{q}_i}$) into itself, and its restriction to $T^-_{\mathbf{p}_i}$ is also conjugate to $\tau_{d,0}$.*

*Proof.* We need to examine carefully the sequence of blow-ups that makes $H$ well-defined, to understand how the tori corresponding to the double points of $D_\infty$ are embedded in the 3-sphere $\mathcal{B}^+_{\mathbf{R}}(D_\infty)$. Recall that we called $\mathbf{a}_0 = \mathbf{p}$, $\mathbf{a}_1, \ldots, \mathbf{a}_{2d-2}$ the successive points at which we performed blow-ups to get from $\mathbf{P}^2$ to $\widetilde{X}_H$, and finally set $\mathbf{a}_{2d-1} = \widetilde{\mathbf{p}}$.

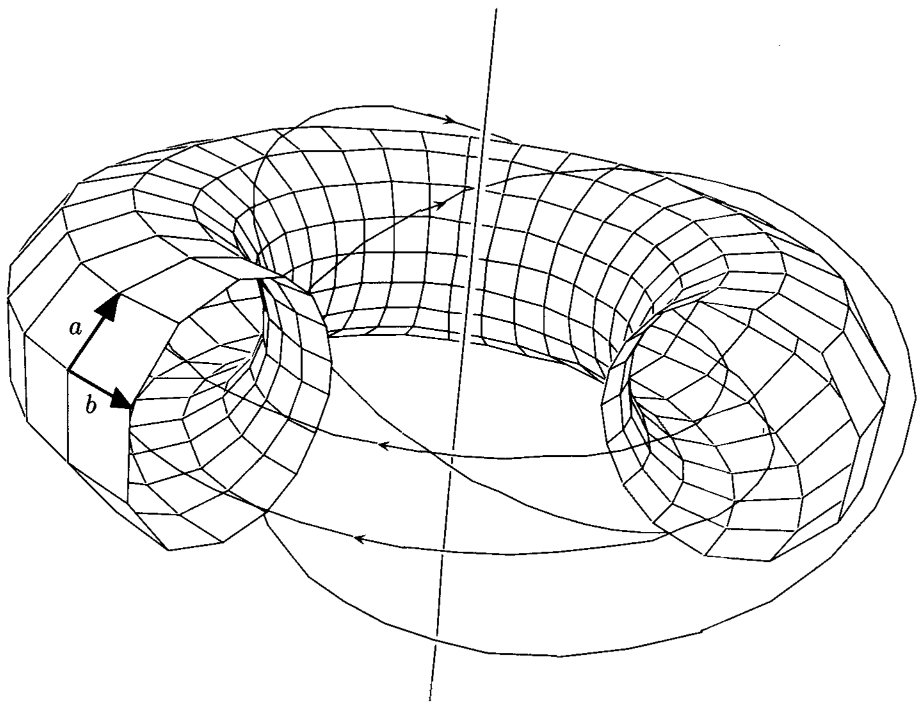



Fig. 15. Blow up $\mathbf{P}^2$ at a point $z \in l_\infty$, and then take the real-oriented blow-up of the divisor consisting of $l_\infty$ and the exceptional divisor. You obtain a 3-sphere containing a torus, and a circle action on both components of the complement of the torus. This picture represents the stereographic projection of this space, with the inside of the torus corresponding to the exceptional divisor, where the circle orbits do not link, and the outside corresponding to the line at infinity, where the circles link with linking number 1. Curves describing the homology classes $a$ and $b$ are drawn on the torus; note that the circle orbits on the outside are in the class $-a-b$ and those inside are in the class $-b$.

In §6, we showed how to start, creating a torus $T_{\mathbf{a}_0}$ in the 3-sphere, separating the solid torus corresponding to $l_\infty$ from the solid torus corresponding to the first exceptional divisor (which will eventually be the irreducible component $B$ of $\widetilde{D}$). Figure 15 shows how these solid tori are placed after stereographic projection; the words "inner" and "outer" will refer to this picture.

The next $d-1$ blow-ups are fairly easy to understand, now that we have started right. We thicken the torus $T_{\mathbf{a}_0}$, creating an inner torus $T_{\mathbf{a}_1}$ and an outer torus $T_{\mathbf{p}'} = T_{\mathbf{p}_0}$ (which we can call by its final name, since it will not be affected by further blow-ups). Then thicken the inner torus $T_{\mathbf{a}_1}$, creating an inner torus $T_{\mathbf{a}_2}$, and one which corresponds to $L_1 \cap L_2$ (between $T_{\mathbf{a}_2}$ and $T_{\mathbf{p}_0}$). Then thicken the inner torus $T_{\mathbf{a}_2}$ again, $d-1$ times in all. The inside of the torus $T_{\mathbf{p}_0}$ is $T^-_{\mathbf{p}_0}$. See Figure 16 for the case $d=3$.

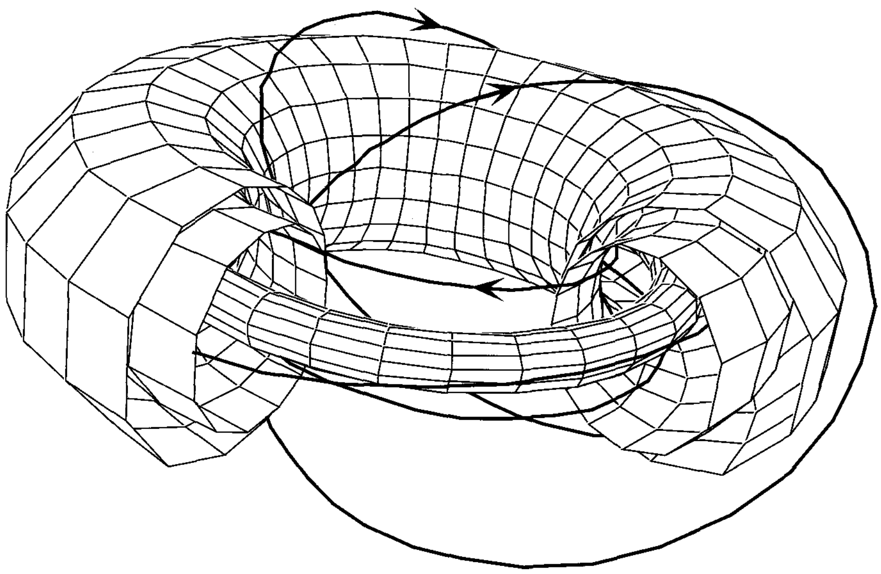


Fig. 16. The picture above corresponds to the situation after $d$ blow-ups when $d=3$, and after stereographic projection. The outside corresponds to the real-oriented blow-up of the line at infinity (now $A'$), with the Hopf circle action, as shown. The inner torus corresponds to $B$, with the circle action where the orbits are not linked. The region between the inner torus and the next corresponds to $L_{d-1}$; that is where all the further action will take place, thickening a circle orbit (represented on the drawing, and going around 3 times in one direction as it turns once in the other direction). The region between the outer torus and the next corresponds to $L_1$; the circle action there has orbits which turn twice in one direction as they turn once in the other; we have not drawn them to keep the drawing simpler.

The circle orbits fibering the regions between the successive tori are contained in $T^-_{\mathbf{p}_0}$ with the innermost torus (corresponding to the component $B$) removed, which is a space with homology $\mathbf{Z}^2$, generated by $a$ and $b$ (see Figure 15). At the first thickening, an oriented circle orbit between the two tori is in the homology class $-a-2b$, at the next the new thickened torus is fibered by curves with homology class $-a-3b$, etc., ending up with a thickened torus fibered by circles in the homology class $-a-db$, and an inner solid torus (corresponding to $B$) with fibers in the homology class $-b$.

In summary, after the first $d$ blow-ups, we have an inner solid torus with fibers in the homology class $-b$, then a succession of thickened tori with fibers in the homology classes

$$-a-db, \quad -a-(d-1)b, \quad ..., \quad -a-2b, \tag{7.1}$$

and finally the region $T^+_{\mathbf{p}_0}$ corresponding to $l_\infty$, fibered (by the old Hopf fibers) in the homology class $-a-b$. See Figures 16 and 17.

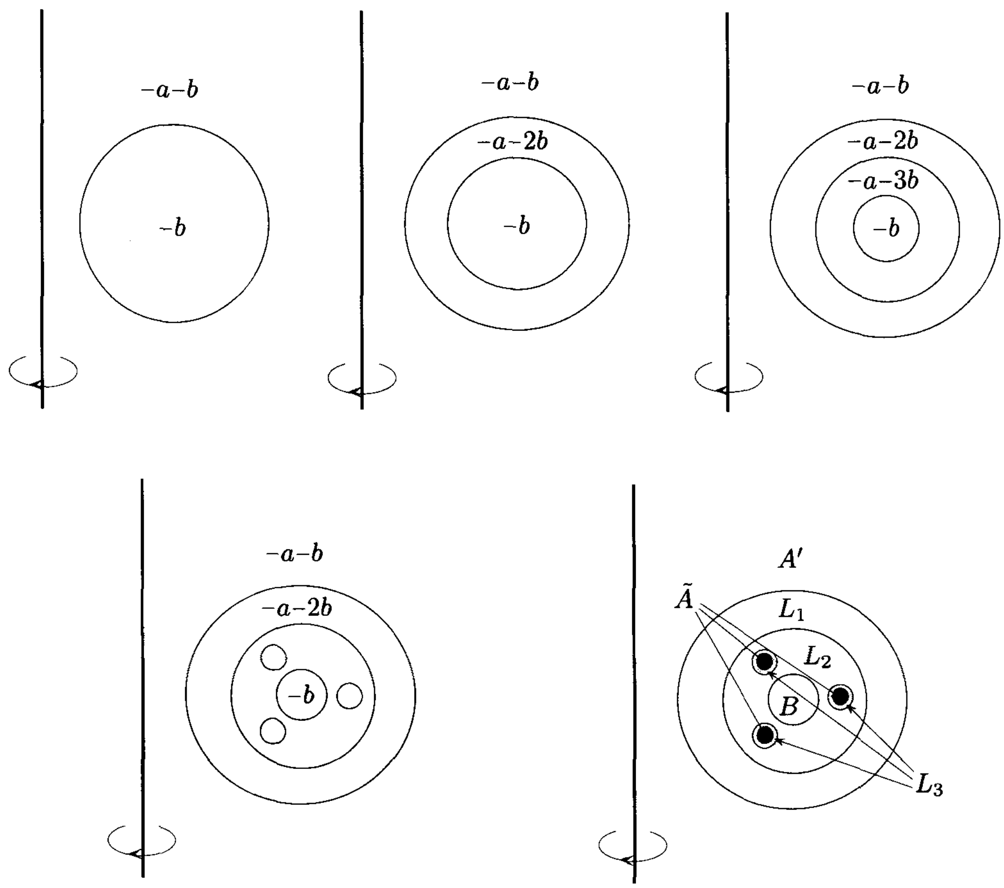



Fig. 17. You can almost imagine constructing the pattern of tori in $S^3$ by rotating the figure above around the $z$-axis (shown as a heavy line). The case above corresponds to $d=3$. The "almost" is because the small circles are not actually rotated: as they turn around the $z$-axis they also turn in their annulus, so as to connect up, forming a single torus, as shown in Figure 16.

We must now make $d-1$ more blow-ups of ordinary points. We make the first by thickening a circle orbit in the region corresponding to $L_{d-1}$, which is fibered by circles in the homology class $-a-db$. We then thicken a circle inside this torus, which we may take to be the "core circle", and repeat this $d-2$ times. The final torus created this way is $T_{\mathbf{q}_1}$. All the solid tori are thickenings of the original circle, and hence in the homology class $-a-db$.

We need to start making the second series of blow-ups, as we do not yet have the torus $T_{\mathbf{p}_1}$. So thicken a fiber inside $T^-_{\mathbf{q}_1}$, creating a solid torus still in the homology class

$-a-db$, and thicken it again $d-1$ times; the outermost torus of the series just created is $T_{\mathbf{p}_1}$. We will not need to describe the further blow-ups.

Moreover, $\mathcal{B}^+_{\mathbf{R}}(H_\infty)^{-1}$ is a homeomorphism which maps the solid torus $T^-_{\mathbf{p}_0}$ to the solid torus $T^-_{\mathbf{p}_1}$. We claim that as a map $T^-_{\mathbf{p}_0}\to T^-_{\mathbf{p}_0}$ it is conjugate to the mapping $\tau_{d,0}$, where $\tau_{d,k}$ is given by the formula

$$\tau_{d,k}(\zeta,z)=\left(\zeta^d,\tfrac{1}{2}\zeta+\varepsilon z\zeta^{k+1-d}\right).$$

Certainly $\mathcal{B}^+_{\mathbf{R}}(H_\infty)$ expands in the circle direction; in fact it is $\beta\mapsto d\beta$ (this is the coordinate $\beta$ of the stereographic projection). By choosing our thickenings sufficiently small, the mapping will be contracting on the discs.

Theorem 3.11 of [HO1] asserts that every injective map from a solid torus to itself whose image is in the homotopy class of the $(1,d)$-torus (un)knot and which is expanding in the circle direction and contracting in the disc direction is conjugate to precisely one of the $\tau_{d,k}$, and Propositions 4.1 and 4.6 of [HO1] assert that only $\tau_{d,0}$ extends to the 3-sphere. □

By Theorem 3.1 of [HO1], there are maps $\pi_i^+\colon T^+_{\mathbf{p}_i}\to\mathbf{R}/2\pi\mathbf{Z}$ and $\pi_i^-\colon T^-_{\mathbf{p}_i}\to\mathbf{R}/2\pi\mathbf{Z}$ such that the following diagrams commute:

$$\begin{tikzcd} T^+_{\mathbf{p}_i} \arrow[r, "\mathcal{B}^+_{\mathbf{R}}(H_\infty)"] \arrow[d, "\pi_i^+"'] & T^+_{\mathbf{p}_i} \arrow[d, "\pi_i^+"'] \\ \mathbf{R}/2\pi\mathbf{Z} \arrow[r, "\theta\mapsto d\theta"] & \mathbf{R}/2\pi\mathbf{Z} \end{tikzcd} \quad\text{and}\quad \begin{tikzcd} T^-_{\mathbf{p}_i} \arrow[r, "\mathcal{B}^+_{\mathbf{R}}(H_\infty)^{-1}"] \arrow[d, "\pi_i^-"'] & T^-_{\mathbf{p}_i} \arrow[d, "\pi_i^-"'] \\ \mathbf{R}/2\pi\mathbf{Z} \arrow[r, "\theta\mapsto d\theta"] & \mathbf{R}/2\pi\mathbf{Z} \end{tikzcd}$$

In our case, these functions $\pi_i^+$ and $\pi_i^-$ can be computed explicitly; they are given by Proposition 7.2. Before stating this proposition, notice that

- $T^+_{\mathbf{q}_i}$ is the set of $x\in\mathcal{B}^+_{\mathbf{R}}(X_\infty,D_\infty)$ such that $(p(x))_i=\mathbf{q}'$;
- $T^-_{\mathbf{q}_i}$ is the set of $x\in\mathcal{B}^+_{\mathbf{R}}(X_\infty,D_\infty)$ such that $(p(x))_{i-1}\in\tilde{A}$;
- $T^-_{\mathbf{p}_i}$ is the set of $x\in\mathcal{B}^+_{\mathbf{R}}(X_\infty,D_\infty)$ such that $(p(x))_{i-1}=\tilde{\mathbf{p}}$;
- $T^+_{\mathbf{p}_i}$ is the set of $x\in\mathcal{B}^+_{\mathbf{R}}(X_\infty,D_\infty)$ such that $(p(x))_i\in A'$.

There is a natural mapping $Q_i\colon X_\infty\to X_{[i,\infty)}$, which blows $D_\infty$ down onto $D_{[i,\infty)}$. Since $Q_i$ is a blow-down, it induces a mapping

$$\mathcal{B}^+_{\mathbf{R}}(Q_i)\colon\mathcal{B}^+_{\mathbf{R}}(X_\infty,D_\infty)\to\mathcal{B}^+_{\mathbf{R}}(X_{[i,\infty)},D_{[i,\infty)}).$$

More precisely, the blow-downs $X_{[-N,N]}\to X_{[i,N]}$, $i<N$, induce mappings on the real-oriented blow-ups $\mathcal{B}^+_{\mathbf{R}}(X_{[-N,N]},D_{[-N,N]})\to\mathcal{B}^+_{\mathbf{R}}(X_{[i,N]},D_{[i,N]})$; to construct $Q_i$, we must pass to the projective limit.

The mapping $Q_i$ maps $T^+_{\mathbf{q}_i}$ to the circle above $\mathbf{q}_i$ in $\mathcal{B}^+_{\mathbf{R}}(D_{[i,\infty)})$. This circle is canonically parametrized by $\phi_i$. Let us denote the composition

$$\Phi_i = \phi_i \circ \mathcal{B}^+_{\mathbf{R}}(Q_i)|_{T^+_{\mathbf{q}_i}} : T^+_{\mathbf{q}_i} \to \mathbf{R}/2\pi\mathbf{Z}.$$

Exactly analogously, there is a natural blow-down $P_i: X_\infty \to X_{(-\infty,i]}$, which blows $D_\infty$ down onto $D_{(-\infty,i]}$. Again, since $P_i$ is a projective limit of blow-downs, it induces a mapping

$$\mathcal{B}^+_{\mathbf{R}}(P_i): \mathcal{B}^+_{\mathbf{R}}(X_\infty, D_\infty) \to \mathcal{B}^+_{\mathbf{R}}(X_{(-\infty,i]}, D_{(-\infty,i]})$$

which maps $T^-_{\mathbf{p}_i}$ to the circle above $\mathbf{p}_i$. This circle is canonically parametrized by $\psi_i$ (remember that $\psi_i = \theta_i + \arg(a/(d-1))$). Let us denote the composition

$$\Psi_i = \psi_i \circ \mathcal{B}^+_{\mathbf{R}}(P_i)|_{T^-_{\mathbf{p}_i}} : T^-_{\mathbf{p}_i} \to \mathbf{R}/2\pi\mathbf{Z}.$$

PROPOSITION 7.2. (a) *We may choose* $\pi^+_i = \Phi_i$.

(b) *We may choose* $\pi^-_i = \Psi_i$.

*Proof.* (a) Clearly $\Phi_{i-1}(\mathcal{B}^+_{\mathbf{R}}(H_\infty)(x)) = \Phi_i(x)$ when $x \in T^+_{\mathbf{q}_i}$, as the left-hand side is just the right-hand side shifted one to the left. Moreover, Proposition 6.4 says that $\Phi_i(y) = d\Phi_{i-1}(y)$ for $y \in T^+_{\mathbf{q}_{i-1}}$. So

$$\Phi_i(\mathcal{B}^+_{\mathbf{R}}(H_\infty)(x)) = d\Phi_{i-1}(\mathcal{B}^+_{\mathbf{R}}(H_\infty)(x)) = d\Phi_i(x).$$

The argument for part (b) is similar. □

*Remark.* What happened to the $d-1$ choices of $\pi^+$ and $\pi^-$? For $\pi^+$, our particular choice was given by the coordinate system in $\mathbf{C}^2$, because ultimately, $\psi = \arg(1/x)$. If we conjugate a Hénon mapping by setting $x_1 = \zeta x$, $y_1 = \zeta y$ where $\zeta^{d-1} = 1$, it is easy to show that the Hénon mapping remains of the same form (the polynomial remains monic and the number $a$ is not changed). For $\pi^-$, we do not actually have a canonical choice. The coordinate $\theta$, which ultimately comes from $\arg(1/y)$, is canonical, but $\psi = \theta + \arg(a/(d-1))$ is exactly ambiguous by a $(d-1)$-root of 1, as one would expect.

The point of these computations is that since $\pi^+$ and $\pi^-$ are conjugacy invariants of the mapping $\mathcal{B}^+_{\mathbf{R}}(H_\infty)$ (up to the ambiguity above), we can use them to find a condition for when the restrictions of $\mathcal{B}^+_{\mathbf{R}}((H_1)_\infty)$ and $\mathcal{B}^+_{\mathbf{R}}((H_2)_\infty)$ to the spheres at infinity are conjugate, where $H_1$ and $H_2$ are Hénon mappings with corresponding polynomials $p_1$ and $p_2$, and Jacobians $a_1$ and $a_2$. In order to pin down our result, we need to know something about toroidal decompositions of 3-manifolds.

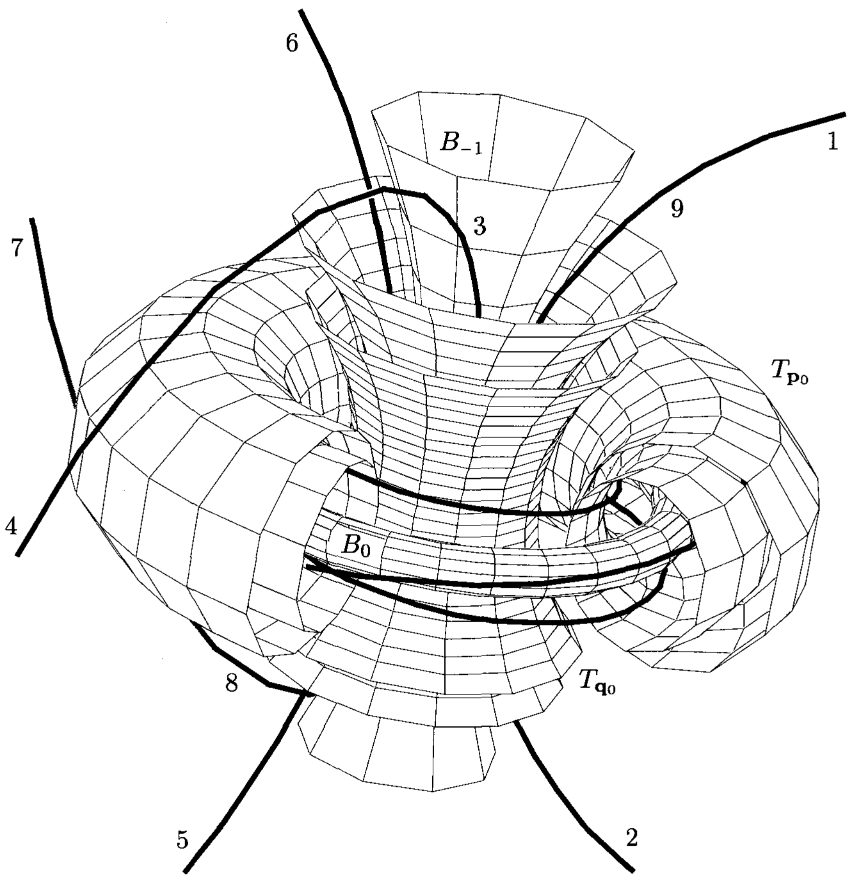



Fig. 18. The configuration of tori in the 3-sphere at infinity between the torus corresponding to $B_0$ and $B_{-1}$, in the case $d=3$. The torus corresponding to $L_{-1,1}\cap L_{-1,2}$, shown as a heavy curve, winds three times around the torus corresponding to $B_0$, in the figure a small thickening of the unit circle in the $(x,y)$-plane. The torus corresponding to $L_{0,2}\cap L_{0,3}$, also shown as a heavy curve, winds three times around the torus $B_{-1}$, represented in the figure as a thickening of the $z$-axis.

*Toroidal decompositions.* The 3-manifold $\mathcal{B}^+_{\mathbf{R}}(D_\infty)-(\Sigma^+\cup\Sigma^-)$ has an interesting toroidal decomposition, a special case of which is illustrated by Figure 18. We will give the definitions and basic properties, largely due to Jaco, Johannson, Shalen and Waldhausen. Our sources are [Hem] and especially [Hat2].

Let $M$ be an orientable irreducible 3-manifold with boundary. A properly embedded surface $S\subset M$ is *incompressible* if for any closed embedded disc in $D\subset M$ with $\partial D\subset S$, there is a disc $D'\subset S$ with $\partial D=\partial D'$. The manifold $M$ is atoroidal if each incompressible torus is isotopic to a boundary component.

Let $D\subset\mathbf{C}$ be the open unit disc. A *Seifert manifold* is a 3-dimensional manifold, foliated by circles, such that each leaf has a neighborhood homeomorphic to the quotient of $D\times[0,1]$ by the equivalence relation which identifies $(0,z)$ with $(1,e^{2\pi ip/q}z)$ for some rational number $p/q$, with the foliation induced by the lines $\{z\}\times[0,1]$. The set of leaves is then a surface with boundary $\Omega$, and the canonical mapping $M\to\Omega$ is referred to as a *Seifert fibration.* This is a locally trivial fibration over the subset $\Omega'\subset\Omega$ corresponding to the regular leaves; the singular leaves (like the one corresponding to $z=0$ in the model) correspond to the discrete set $\Omega-\Omega'$.

The key results for us are the following. Theorem 7.3 is exactly Theorem 3.3 of [Hat2].

THEOREM 7.3. *Let $M$ be a 3-dimensional compact orientable manifold with boundary. Then there exists a collection of disjoint incompressible tori $T_i\subset M$ such that each component of $M-\bigcup_i T_i$ is either atoroidal or a Seifert manifold, and a minimal such collection is unique up to isotopy.*

THEOREM 7.4. *A Seifert manifold with at least two boundary components has a unique Seifert fibration up to isomorphism.*

This follows immediately from Theorem 4.3 of [Hat2]. Indeed, Hatcher shows that the Seifert fibration is unique except for a list of exceptions, and all these exceptions have 1 or 0 boundary components.

THEOREM 7.5. *Let $f\colon M\to\Omega$ be a Seifert fibration, and let $\Omega'\subset\Omega$ be the complement of the points corresponding to singular fibers. Suppose that $M$ is connected and that $\partial M\neq\varnothing$. Then every incompressible surface in $M$ without boundary is isotopic to a surface of the form $f^{-1}(\gamma)$ for some curve $\gamma\subset\Omega'$, and the isotopy classes of such surfaces correspond exactly to the isotopy classes of such curves.*

This follows from Proposition 3.5 of [Hat2]. Hatcher proves that every incompressible and boundary-incompressible surface is isotopic to either a vertical or a horizontal surface. Horizontal surfaces have non-empty boundary, and surfaces without boundary are vacuously boundary-incompressible. So our surfaces are isotopic to vertical surfaces, i.e., surfaces of the form $f^{-1}(\gamma)$.

We will be interested in applying these notions to the manifold $\mathcal{B}^+_{\mathbf{R}}(D_\infty)-(\Sigma^+\cup\Sigma^-)$, which comes with a family of tori $T_{\mathbf{p}_i}$, which we will see are incompressible.

THEOREM 7.6. (a) *The tori $T_{\mathbf{p}_i}\subset\mathcal{B}^+_{\mathbf{R}}(D_\infty)-(\Sigma^+\cup\Sigma^-)$ are incompressible.*

(b) *Every incompressible torus in $\mathcal{B}^+_{\mathbf{R}}(D_\infty)-(\Sigma^+\cup\Sigma^-)$ is isotopic to exactly one of the $T_{\mathbf{p}_i}$.*

*Proof.* (a) The homology $H_1(\mathcal{B}^+_{\mathbf{R}}(D_\infty)-(\Sigma^+\cup\Sigma^-))$ is isomorphic to $\mathbf{Z}[1/d]\oplus\mathbf{Z}[1/d]$. This is proved e.g. using the Alexander duality theorem [Sp, 6.2, Theorem 16], which asserts that

$$H_1(\mathcal{B}^+_{\mathbf{R}}(D_\infty)-(\Sigma^+\cup\Sigma^-))=H^1(\Sigma^+)\oplus H^1(\Sigma^-).$$

The cohomology above is Čech cohomology (isomorphic to Alexander–Spanier cohomology), given by the inductive limit of the singular cohomology of a basis of neighborhoods of $\Sigma^\pm$. Using the system of neighborhoods $T^+_{\mathbf{p}_{-i}}$ of $\Sigma^+$, we see that

$$H^1(\Sigma^+)=\varinjlim(\mathbf{Z}, n\mapsto dn)=\mathbf{Z}[1/d].$$

(This is similar to but simpler than Example 4.3.)

An isomorphism is specified by sending the generators $a, b$ (see Figure 15) of $H_1(T_{\mathbf{p}_0})$ to $(0,1)$ and $(1,0)$ in $\mathbf{Z}[1/d]\oplus\mathbf{Z}[1/d]$. Under this isomorphism, the corresponding generators of $H_1(T_{\mathbf{p}_i})$ are sent to $(0,d^{-i})$ and $(d^i,0)$. In particular, the inclusion is injective on the homology of such a torus. If a disc in $\mathcal{B}^+_{\mathbf{R}}(D_\infty)-(\Sigma^+\cup\Sigma^-)$ bounds a disc in $T_{\mathbf{p}_i}$, then the homology class of this boundary is zero in $H_1(\mathcal{B}^+_{\mathbf{R}}(D_\infty)-(\Sigma^+\cup\Sigma^-))$, hence also in $H_1(T_{\mathbf{p}_i})$, so the curve bounds a disc in the torus, since any simple closed curve in a torus which is trivial in the homology bounds a disc. (This is not true on surfaces of higher genus, which is why incompressibility is not defined using injectivity of the inclusion on homology.) This is the definition of an incompressible torus.

(b) Let $T'$ be such a torus. It is contained in the compact manifold $T^-_{\mathbf{p}_i}\cap T^+_{\mathbf{p}_j}$ for $i$ sufficiently small and $j$ sufficiently large, which allows us to apply Theorem 7.3, where $M$ must be compact. So it is enough to prove that $T_{\mathbf{p}_{i+1}},\ldots,T_{\mathbf{p}_{j-1}}$ is a minimal family of incompressible tori in $T^-_{\mathbf{p}_i}\cap T^+_{\mathbf{p}_j}$ such that the components of the boundary are atoroidal or Seifert manifolds.

First observe that the components of

$$(T^-_{\mathbf{p}_i}\cap T^+_{\mathbf{p}_j})-\left(\bigcup_{n=i+1}^{j-1}T_{\mathbf{p}_n}\right)$$

are both atoroidal and Seifert manifolds. Indeed, the region $T^-_{\mathbf{p}_i}\cap T^+_{\mathbf{p}_{i+1}}$ is homeomorphic to the region $M_i$ bounded by the tori corresponding to $L_{i,d-2}\cap L_{i,d-1}$ and $L_{i,d}\cap L_{i,d-1}$. This region contains the solid torus corresponding to $B_i$, but that torus can be collapsed onto a circle without changing the homeomorphism type; call $M_i'$ the resulting manifold.

The manifold $M_i'$ is fibered by the natural circle action, and the circle corresponding to $B_i$ becomes a singular circle of type $(1,d)$. Thus $T^-_{\mathbf{p}_i}\cap T^+_{\mathbf{p}_{i+1}}$ is a Seifert manifold; let $f_i\colon T^-_{\mathbf{p}_i}\cap T^+_{\mathbf{p}_{i+1}}\to\Omega_i$ be the corresponding projection to the set of leaves (the base). It is also atoroidal, by Theorem 7.5, since $\Omega_i$ is an annulus with one distinguished point

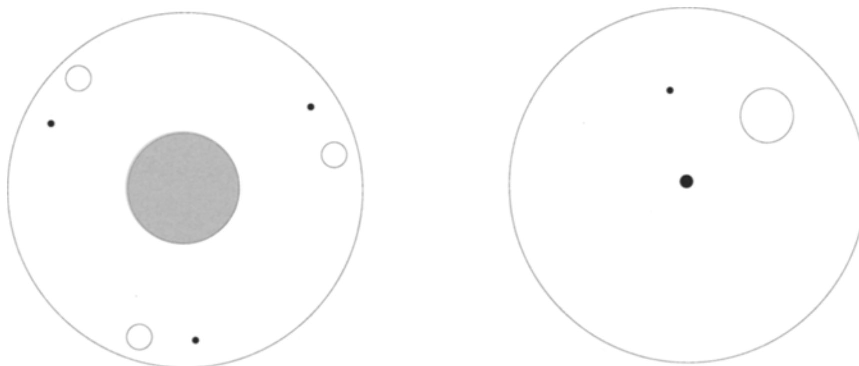


Fig. 19. On the left, we have repeated the relevant part of Figure 17, showing the annulus corresponding to $M_i$ when $d=3$; the three dots represent the intersection of the plane of the figure with a circle orbit in $M_i$. The right-hand side represents the first, after applying $z \mapsto z^d$ and collapsing the central disc to a point. Every point on the right corresponds to a unique circle orbit, i.e., the base is an annulus with a single singular fiber (corresponding to the central point).

corresponding to the unique singular fiber. This is seen as follows. In Figure 17, the manifold $M_i$ corresponds to the annulus between the center circle and the next, with the $d$ small circles removed.

If we parametrize the disc by $z$, and compose $z \mapsto z^d$ with a collapse of the central disc to a point (corresponding to collapsing the solid torus corresponding to $B$ to a circle), you manufacture a space which corresponds exactly to the set of leaves. See Figure 19.

Any simple closed curve on an annulus with a puncture is homotopic to a point or to a boundary component, so there are no incompressible tori in $M_i'$ by Theorem 7.5.

Now we need to show that our family $T_{\mathbf{p}_{i+1}}, ..., T_{\mathbf{p}_{j-1}}$ is minimal. It is clearly enough to show that $T^-_{\mathbf{p}_i} \cap T^+_{\mathbf{p}_{i+2}}$ is neither atoroidal nor Seifert. It clearly is not atoroidal, since it contains $T_{\mathbf{p}_{i+1}}$, so we must show that it is not Seifert. Suppose that $f: T^-_{\mathbf{p}_i} \cap T^+_{\mathbf{p}_{i+2}} \to \Omega$ is a Seifert fibration, where $\Omega$ is some surface. The surface $\Omega$ must have a boundary consisting of two components, but otherwise we do not know much about it. By Theorem 7.5, there is a curve $\gamma \subset \Omega'$, where $\Omega'$ is the complement of the projections of the singular fibers, such that $T_{\mathbf{p}_i}$ is isotopic to $T_i' = f^{-1}(\gamma)$; in particular, the restriction of $f$ to the components of $(T^-_{\mathbf{p}_i} \cap T^+_{\mathbf{p}_{i+2}}) - T_i'$ is a Seifert fibration.

But each of these is already a Seifert fibration, in fact in a unique way by Theorem 7.4. It is then enough to show that the fibers of $f_i$ and $f_{i+1}$ on the torus $T_{\mathbf{p}_i}$ which is the intersection of their domains are not homotopic curves; since they should both be homotopic to the fibers of $f$, this contradicts the existence of such an $f$.

In the basis $a, b$ for $H_1(T_{\mathbf{p}_i})$, we saw that the homology class of a fiber of $f_0$ is $-a-db$.

We claim that a fiber of $f_1$ has homology class $-da-b$; knowing this will end the proof.

We need to repeat the construction of Proposition 7.1 to understand the sequence of tori corresponding to the block $B_1$ of $D_\infty$. Take the 3-sphere, with the sequence of real-oriented blow-ups corresponding to $\mathcal{B}^+_{\mathbf{R}}(\widetilde{D})$, as shown in Figure 16.

The fiber above $\mathbf{q}$ is a circle orbit outside the torus corresponding to $\mathbf{q}'$, which we may take to be the $z$-axis. Thicken this torus; the fibers inside the thickened torus will now have the homology class $-a-b+b=-a$ by Theorem 5.7. Indeed, if you choose a small disc transverse to the $z$-axis, it projects to $\widetilde{D}$ (in fact, to a neighborhood of $\mathbf{q}'$ in the line at infinity $A'$), and the induced orientation gives its boundary the orientation $+b$. Now, when we make $d-1$ more blow-ups, always of double points, the regions between the tori created have circle orbits in the classes $-2a-b, ..., -da-b$. The region foliated by curves in the class $-da-b$ corresponds to the 3-manifold $M_1$. □

*Conjugacy invariants of* $\mathcal{B}^+_{\mathbf{R}}(H_\infty)$. Let $H_1$ and $H_2$ be Hénon mappings. We will give a necessary condition for when the restrictions of $\mathcal{B}^+_{\mathbf{R}}((H_1)_\infty)$ and $\mathcal{B}^+_{\mathbf{R}}((H_2)_\infty)$ to the spheres at infinity $\mathcal{B}^+_{\mathbf{R}}(D_{1,\infty})$ and $\mathcal{B}^+_{\mathbf{R}}(D_{2,\infty})$ are conjugate. To (sort of) lighten notation, we will call these restrictions $\mathcal{B}^+_{\mathbf{R}}((H_i)'_\infty)$.

THEOREM 7.7. *In order for $\mathcal{B}^+_{\mathbf{R}}((H_1)'_\infty)$ and $\mathcal{B}^+_{\mathbf{R}}((H_2)'_\infty)$ to be topologically conjugate, it is necessary that they have the same degree, and that*

$$\arg a_1 \equiv \arg a_2 \pmod{2\pi/(d-1)}.$$

*Proof.* That they must have the same degree is clear by counting fixed points in the solenoids. We will first investigate the "critical locus" of the map

$$(\pi_1^+, \pi_0^-)\colon T^-_{\mathbf{p}_0} \cap T^+_{\mathbf{p}_1} \to (\mathbf{R}/2\pi\mathbf{Z})^2.$$

*Remark.* The notion of "critical locus" is not quite right: $\mathcal{B}^+_{\mathbf{R}}(D_\infty)$ is not naturally a differentiable manifold, it is naturally a *manifold with corners*, almost an object of the piecewise-linear category. What we will find is more PL than $C^1$: the set of points which have neighborhoods on which $\pi_1^+$ and $\pi_0^-$ differ by a constant is non-empty. In our setting, the critical locus will be the closure of this open set by definition. This is a much stronger notion of "critical" than one would expect: generically for a differentiable mapping from a 3-dimensional manifold to a surface, the critical locus should be a curve.

LEMMA 7.8. *On $p_\infty^{-1}(B_0)$, we have the identity*

$$\pi_1^+ - \pi_0^- = -\arg a + \pi - \frac{\arg a}{d-1} = \pi - \frac{d}{d-1}\arg a.$$

*Proof.* The parametrized path

$$t \mapsto \begin{bmatrix} c \\ te^{-i\alpha} \end{bmatrix}, \quad t>0,$$

thought of as a path in $X_\infty$, approaches a specific point of $B_0$ with coordinate $c$. Thought of as a path in $\mathcal{B}^+_{\mathbf{R}}(X_\infty, D_\infty)$, it approaches the point $x$ above $c$ where $\Psi_0 = \alpha + (\arg a)/(d-1)$. Thus $\pi_0^-(x) = \alpha + (\arg a)/(d-1)$.

To compute $\pi_1^+(x)$, apply $H$ to the path

$$t \mapsto \begin{pmatrix} p(c) - ate^{-i\alpha} \\ c \end{pmatrix}, \quad t>0,$$

which approaches $\mathcal{B}^+_{\mathbf{R}}(H_\infty)(x) \in p_\infty^{-1}(B_{-1})$. Just as we needed the argument of $1/y$ to compute $\Psi_0$ (adjusted by $(\arg a)/(d-1)$), we need the argument of $1/x$ (unadjusted) to compute $\Phi_0(H(x)) = \Phi_1(x)$; clearly this argument is $\alpha - \arg a + \pi$. Thus $\pi_1^+(x) = \alpha - \arg a + \pi$. □

Thus the two functions $\pi_1^+$ and $\pi_0^-$ differ by the constant $(d/(d-1)) \arg a$ on this solid torus.

LEMMA 7.9. *The solid torus $p_\infty^{-1}(B_0)$ is the critical locus of $(\pi_1^+, \pi_0^-)$.*

*Proof.* We will only outline how to do this for points above $L_1$. Choose as above a curve in $X_\infty$ tending in $\mathcal{B}^+_{\mathbf{R}}(X_\infty, D_\infty)$ to a point above a point $c \in L_1$. For instance,

$$t \mapsto \begin{pmatrix} t^{-i\alpha} \\ cte^{-2i\alpha} \end{pmatrix}, \quad t>0,$$

is a curve approaching a point $x$ above $L_1$. At this point, we have $\pi_0^-(x) = 2\alpha - \arg c + (\arg a)/(d-1)$. If we apply $H$ to this curve and compute $\arg(1/x)$, we find

$$\pi_1^+(x) = \begin{cases} d\alpha & \text{if } d>2, \\ 2\alpha - \arg(1-ac) & \text{if } d=2. \end{cases}$$

Since $\arg c$ shows up explicitly in the formula for $\pi_1^+ - \pi_0^-$, such a point is not critical. □

*Remark.* The point $ac=1$ in the case $d=2$ above corresponds to $B \cap L_1$; a similar point will show up above $L_{d-1}$ for every $d$.

We now need to see that the number $(d/(d-1)) \arg a$ from Lemma 7.8 is (almost) a conjugacy invariant of $\mathcal{B}^+_{\mathbf{R}}(H_\infty)$.

Suppose that $F: \mathcal{B}^+_{\mathbf{R}}(D_{1,\infty}) \to \mathcal{B}^+_{\mathbf{R}}(D_{2,\infty})$ conjugates $\mathcal{B}^+_{\mathbf{R}}(H_{1,\infty}) \to \mathcal{B}^+_{\mathbf{R}}(H_{2,\infty})$, where we have used indices 1 and 2 to distinguish the objects created from $H_1$ and $H_2$. Then $F$

must send $\Sigma_1^\pm$ to $\Sigma_2^\pm$ since these are the closures of the set of periodic points of $\mathcal{B}^+_{\mathbf{R}}(H_{1,\infty})$ and $\mathcal{B}^+_{\mathbf{R}}(H_{2,\infty})$ respectively, so it must also send incompressible tori in the complement of the solenoids $\Sigma_1^\pm$ to incompressible tori in the complement of the solenoids $\Sigma_2^\pm$. Since $T_{\mathbf{p}_i}$ separates the $T_{\mathbf{p}_j}$ with $j>i$ from the $T_{\mathbf{p}_j}$ with $j<i$, we see that the order of the tori must be preserved, and there exists $k$ such that

$$F(T_{1,\mathbf{p}_i}) \text{ is isotopic to } T_{2,\mathbf{p}_{i+k}}.$$

By composing $F$ with $\mathcal{B}^+_{\mathbf{R}}((H_2)'_\infty)^{\circ k}$, we may assume that $k=0$, and that

$$F(T_{1,\mathbf{p}_i}) \text{ is isotopic to } T_{2,\mathbf{p}_i}.$$

LEMMA 7.10. *The functions* $\pi^+_{1,i}$ *and* $\pi^+_{2,i}\circ F$ *must differ by a multiple of* $2\pi/(d-1)$ *on their common domain of definition.*

*Proof.* By composing $\pi^+_{1,i}$ with an appropriate multiple of $2\pi/(d-1)$, we may assume that if $x_1\in\Sigma_1^+$ is the fixed point of $\mathcal{B}^+_{\mathbf{R}}((H_1)'_\infty)$ with $\pi^+_{1,i}(x_1)=0$, then $F(x_1)=x_2$ is the fixed point of $\mathcal{B}^+_{\mathbf{R}}((H_2)'_\infty)$ with $\pi^+_{2,i}(x_2)=0$. After this change, we must show that $\pi^+_{1,i}$ and $\pi^+_{2,i}\circ F$ coincide on their common domain.

Choose $j$ sufficiently small so that

$$T^+_{2,\mathbf{p}_j}\subset T^+_{2,\mathbf{p}_i}\cap F(T^+_{1,\mathbf{p}_i}).$$

Now both diagrams

$$\begin{tikzcd} T^+_{2,\mathbf{p}_j} \arrow[r, "\mathcal{B}^+_{\mathbf{R}}(H_{2,\infty})"] \arrow[d, "\pi^+_{2,i}"'] & T^+_{2,\mathbf{p}_j} \arrow[d, "\pi^+_{2,i}"'] \\ \mathbf{R}/2\pi\mathbf{Z} \arrow[r, "\theta\mapsto d\theta"] & \mathbf{R}/2\pi\mathbf{Z} \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} T^+_{2,\mathbf{p}_j} \arrow[r, "\mathcal{B}^+_{\mathbf{R}}(H_{2,\infty})"] \arrow[d, "\pi^+_{1,i}\circ F^{-1}"'] & T^+_{2,\mathbf{p}_j} \arrow[d, "\pi^+_{1,i}\circ F^{-1}"'] \\ \mathbf{R}/2\pi\mathbf{Z} \arrow[r, "\theta\mapsto d\theta"] & \mathbf{R}/2\pi\mathbf{Z} \end{tikzcd}$$

commute. The uniqueness statement [HO1, Theorem 3.1] is not quite enough to guarantee that the vertical arrows coincide, since they are not of degree 1. In that case, the proof guarantees that such maps differ by a multiple of $1/(d^{i-j}-1)$. This is enough to guarantee that if $\pi^+_{2,i}$ and $\pi^+_{1,i}\circ F^{-1}$ coincide at a single point, then they coincide everywhere; indeed, they do coincide at $x_2$. Now

$$\pi^+_{2,i}=\frac{1}{d^m}\circ\pi^+_{2,i}\circ\mathcal{B}^+_{\mathbf{R}}(H_{2,\infty}) \quad\text{and}\quad \pi^+_{1,i}\circ F^{-1}=\frac{1}{d^m}\circ\pi^+_{2,i}\circ\mathcal{B}^+_{\mathbf{R}}(H_{2,\infty});$$

any difference comes from different branches of $1/d^m$. Since $F(T_{1,\mathbf{p}_i})$ is isotopic to $T_{2,\mathbf{p}_i}$, and we can choose a branch continuously during the isotopy, we see that $\pi^+_{2,i}$ and $\pi^+_{1,i}\circ F^{-1}$ (the latter adjusted at the beginning of the proof) must agree on their common domain of definition. □

Now we need to know something about this common domain of definition.

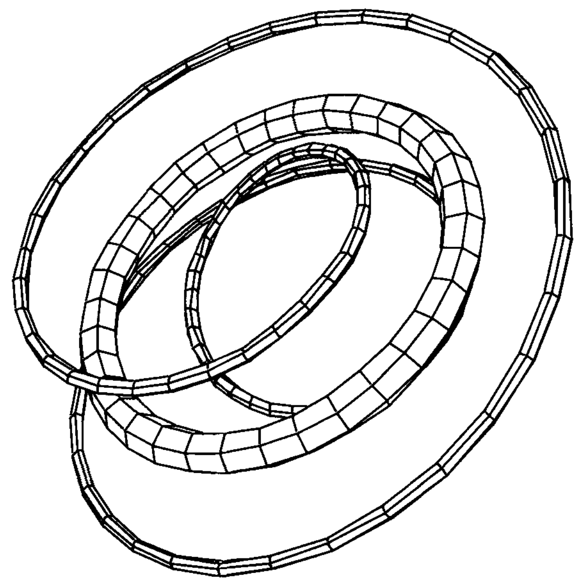


Fig. 20. A $(3,1)$-curve on the boundary of a solid torus, corresponding to 3 times the generator of the homology of the solid torus, and the core curve of the solid torus. Clearly they link with linking number 1, whereas any curve outside the torus links with the $(3,1)$-curve with linking number some multiple of 3. Thus this figure represents the case $i-j=1$ and $d=3$.

LEMMA 7.11. *The interiors of the torus $F(T^+_{1,\mathbf{p}_i})$ and the interior of the torus $\mathcal{B}_{i-1}$ corresponding to $B_{2,i-1}$ must have non-empty intersection.*

*Proof.* Another way to say this is to say that $\mathcal{B}_{i-1}$ cannot be isotoped, in $\mathcal{B}^+_{\mathbf{R}}(D_{2,\infty})-(\Sigma^+_2\cup\Sigma^-_2)$, to a torus outside of $F(T^+_{1,\mathbf{p}_i})$. This can be seen from linking numbers. The presumed isotopy will take place in the complement of $T^+_{2,\mathbf{p}_j}$ for $j$ sufficiently small. All curves (or unknotted solid tori) outside $T^+_{2,\mathbf{p}_i}$ have linking number some integer multiple of $d^{i-j}$ with $T^+_{2,\mathbf{p}_j}$. But $\mathcal{B}_{i-1}$ has linking number $d^{i-j-1}$ with $T^+_{2,\mathbf{p}_j}$ (see Figure 20).

Since the linking number must be constant during the isotopy, this is a contradiction. □

Thus $F$ must map some open subset of the torus corresponding to $B_{1,i}$ to some open subset of the torus corresponding to $B_{2,i}$, in such a way that

$$\pi^{\pm}_{2,i}\circ F=\pi^{\pm}_{1,i}$$

up to a multiple of $1/(d-1)$. This proves Theorem 7.7. □

*Remark.* The condition $\arg a_1=\arg a_2$ is also sufficient for conjugacy (see [HO4]). It turns out that the conjugacy properties of mappings like $\mathcal{B}^+_{\mathbf{R}}((H)_\infty)$ (or the mapping

$h_d$ of [HO1]) are quite subtle; and that there is an infinite-dimensional moduli space, even when the maps are hyperbolic on a neighborhood of the solenoids.

## 8. The compactification of compositions of Hénon mappings

A theorem of Friedland and Milnor [FM] asserts that any polynomial automorphism of $\mathbf{C}^2$ is either elementary, in the sense that we can find one variable that depends only on itself, or conjugate to a composition of Hénon maps. Therefore, understanding the appropriate compactification of $\mathbf{C}^2$ to which such a composition extends is evidently important.

*A conjecture by Milnor.* In a personal communication, Milnor suggested what the 3-sphere at infinity should look like; we will now state and prove this conjecture.

Let

$$H_i\colon \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} p_i(x)-a_i y \\ x \end{pmatrix}, \quad i=1,\dots,k,$$

be $k$ Hénon mappings, with $a_i\neq 0$ and $p_i$ of degree $d_i\geqslant 2$. We will consider $G=H_k\circ\dots\circ H_1$, which is a polynomial mapping of algebraic degree $d=d_1\cdot\ldots\cdot d_k$.

Recall that $\Sigma_d=\varprojlim(\mathbf{R}/\mathbf{Z}, t\mapsto dt)$ is the $d$-adic solenoid, and that $\sigma_d\colon\Sigma_d\to\Sigma_d$ is the map induced by $t\mapsto dt$.

We will call the simplest link of two circles with linking number $d$ the one formed by the circles

$$\begin{pmatrix} \cos t \\ \sin t \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \left(1+\frac{1}{2}\cos dt\right)\cos t \\ \left(1+\frac{1}{2}\cos dt\right)\sin t \\ \frac{1}{2}\sin dt \end{pmatrix}, \quad 0\leqslant t\leqslant 2\pi,$$

as represented in Figure 21.

THEOREM 8.1. (a) *There exists a topology on $\mathbf{C}^2\sqcup S^3$ homeomorphic to the 4-ball, with $S^3$ corresponding to the boundary, such that $G$ extends continuously to a homeomorphism $g\colon S^3\to S^3$.*

(b) *The homeomorphism $g$ has two invariant solenoids $\Sigma^+,\Sigma^-$, one attracting and one repelling, and both homeomorphic to $\Sigma_d$, and the homeomorphisms can be chosen to be conjugacies between the restriction $g|_{\Sigma^+}$ and $\sigma_d$, and between $g|_{\Sigma^-}$ and $\sigma_d^{-1}$.*

(c) *The complement $M=S^3-(\Sigma^+\cup\Sigma^-)$ has a decomposition by incompressible tori $(T_i)_{i\in\mathbf{Z}}$, unique up to isotopy, into pieces $M_i$ bounded by $T_i$ and $T_{i+1}$, homeomorphic to the complement of the simplest link of two circles with linking number $d_{i \bmod k}$. Moreover, $M_i\cap M_j=\varnothing$ unless $|i-j|\leqslant 1$. The tori can be chosen so that $g(T_i)=T_{i+k}$.*

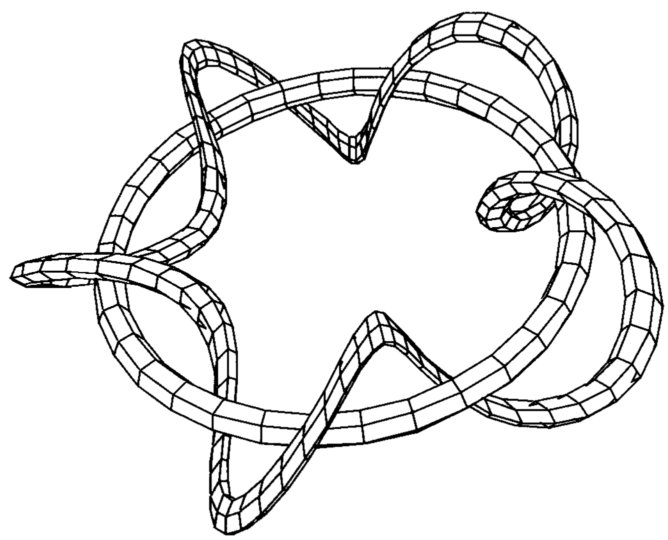


Fig. 21. The simplest link of two circles linking with linking number 5.

In particular, the topology of the sphere at infinity is different for a composition of Hénon maps with total degree $d$ and for a single such mapping: the solenoids are the same but they are embedded differently in the 3-sphere.

*Proof.* Let $\widetilde{X}_G$ be the minimal blow-up of $\mathbf{P}^2$ on which $\widetilde{G}: \widetilde{X}_G \to \mathbf{P}^2$ is well-defined.

It can be constructed as follows: set $G_m = H_m \circ \ldots \circ H_1$, so that $G_1 = H_1$ and $G_k = G$, and define $\widetilde{X}_{G_m}$ to be the minimal blow-up of $\mathbf{P}^2$ on which $\widetilde{G}_m: \widetilde{X}_{G_m} \to \mathbf{P}^2$ is well-defined. Further denote by $\pi_{G_m}: \widetilde{X}_{G_m} \to \mathbf{P}^2$ the canonical projection, and by $\widetilde{D}_{G_m} = \pi_{G_M}^{-1}(l_\infty)$ the divisor at infinity of $\widetilde{X}_{G_m}$.

We will construct $\widetilde{X}_{G_m}$ by induction. Clearly $\widetilde{X}_{G_1} = \widetilde{X}_{H_1}$ is the space constructed in §2. Suppose that we have constructed $\widetilde{X}_{G_{m-1}}$, together with $\widetilde{G}_{m-1}$ and $\pi_{G_{m-1}}$.

Set $\widetilde{X}_{G_m}$ to be such that the upper left-hand square of the diagram

$$\begin{tikzcd} \widetilde{X}_{G_m} \arrow[r] \arrow[d] & \widetilde{X}_{H_m} \arrow[r, "\widetilde{H}_m"] \arrow[d, "\pi_{H_m}"'] & \mathbf{P}^2 \\ \widetilde{X}_{G_{m-1}} \arrow[r, "\widetilde{G}_{m-1}"] \arrow[d, "\pi_{G_{m-1}}"'] & \mathbf{P}^2 & \\ \mathbf{P}^2 & & \end{tikzcd}$$

is a fiber product in the category of analytic spaces. Then the top line is a mapping $\widetilde{G}_m: \widetilde{X}_{G_m} \to \mathbf{P}^2$, whereas the left-hand column represents $\widetilde{X}_{G_m}$ as a modification

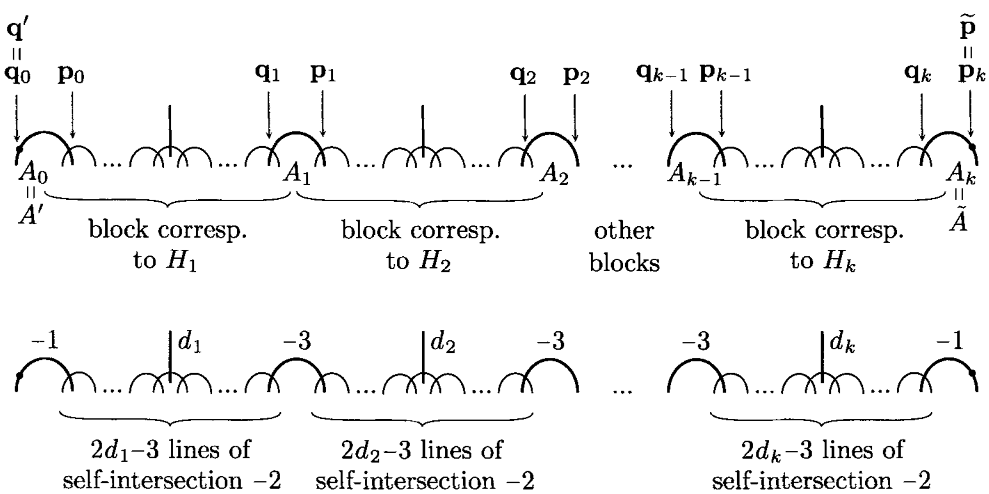



Fig. 22. The divisor $\widetilde{D}_G$. The top figure gives the labels of all the components and points to which we will need to refer. The bottom figure gives the self-intersections of all these components.

of $\mathbf{P}^2$ at $\mathbf{p}$. Thus $\mathbf{C}^2$ is dense in $\widetilde{X}_{G_m}$, and it is clear by induction that $\widetilde{G}_m$ extends $G_m\colon \mathbf{C}^2 \to \mathbf{C}^2$. That it is the minimal modification of $\mathbf{P}^2$ to which $G_m$ extends follows from the fact that the construction of §2 is the minimal modification to which the individual Hénon maps extend. Thus $\widetilde{X}_{G_m}$ is the required minimal blow-up.

The divisor above infinity

$$\widetilde{D}_G = \widetilde{X}_G - \mathbf{C}^2 = \pi_G^{-1}(l_\infty)$$

looks as in Figure 22.

As before, we will avoid an infinite sequence of blow-ups by considering a sequence space. Consider the rational mapping $G^\sharp\colon \widetilde{X}_G \rightsquigarrow \widetilde{X}_G$, which is $\widetilde{G}$ wherever it is defined, and define

$$\bar{\Gamma}_G \subset \widetilde{X}_G \times \widetilde{X}_G$$

to be the closure of the graph $\Gamma_{G^\sharp} \subset \widetilde{X}_G \times \widetilde{X}_G$ of $G$.

LEMMA 8.2. *A pair $(x,y)$ belongs to $\bar{\Gamma}_G$ if and only if either*

- *it is in $\Gamma_{G^\sharp}$, or*
- *$x=\widetilde{\mathbf{p}}$, $y \in (\widetilde{D} - A') \cup \{\mathbf{p}'\}$.*

The proof is analogous to that of Theorem 2.4.

Now define the natural compactification of the composition of Hénon mappings $G$ as

$$X_\infty(G) = \{(\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots) \in (\widetilde{X}_G)^{\mathbf{Z}} \mid (x_n, x_{n+1}) \in \bar{\Gamma}_G \text{ for all } n \in \mathbf{Z}\}.$$

Using Lemma 8.2, this space is not so difficult to understand.

PROPOSITION 8.3. *The space $X_\infty(G)$ is compact. The complement of two points*

$$\mathbf{q}^\infty = (\ldots, \mathbf{q}', \mathbf{q}', \mathbf{q}', \ldots) \quad \textit{and} \quad \mathbf{p}^\infty = (\ldots, \widetilde{\mathbf{p}}, \widetilde{\mathbf{p}}, \widetilde{\mathbf{p}}, \ldots)$$

*is an algebraic manifold.*

*Proof.* The proof is the same as that of Proposition 3.8.

*Proof of Theorem* 8.1. Again, to understand the structure of the bad points, we will pass to the real-oriented blow-ups.

We can define spaces $X_{[-N,M]}(G)$ analogously to the construction in §4, with the divisors $D_{[-N,M]}(G)$; next we construct the real-oriented blow-ups

$$\mathcal{B}^+_{\mathbf{R}}(X_{[-N,M]}(G), D_{[-N,M]}(G))$$

and take their projective limit

$$\mathcal{B}^+_{\mathbf{R}}(X_\infty(G), D_\infty(G)) = \varprojlim_{N\to\infty} (\mathcal{B}^+_{\mathbf{R}}(X_{[-N,N]}(G), D_{[-N,N]}(G)).$$

Note that again we are using Proposition 5.9 to construct the mappings implicit in the projective limit.

The pair $\mathcal{B}^+_{\mathbf{R}}(D_\infty(G)) \subset \mathcal{B}^+_{\mathbf{R}}(X_\infty(G), D_\infty(G))$ is the compactification of $\mathbf{C}^2$ promised in Theorem 8.1 (a). By Theorem 5.10, the pair is homeomorphic to $(B^4, S^3)$, exactly as in the proof of Theorem 6.3. Moreover, the inclusion of $\mathbf{C}^2 \subset \mathcal{B}^+_{\mathbf{R}}(X_\infty(G), D_\infty(G))$ and the extension of $G$ to the real-oriented blow-up are constructed exactly as in Theorem 6.2.

The proof of Theorem 8.1 (b) is closely analogous to Proposition 6.4, but requires a bit of terminology. First, label components and points of $\widetilde{D}_\infty$ as follows: let $A_0 = l_\infty \subset \mathbf{P}^2$, and define recursively $A_i \subset \widetilde{X}_{G_i}$ to be the component of $H_i^{-1}(A_{i-1})$ which is sent by $H_i$ isomorphically onto $A_{i-1}$. Finally, let $\mathbf{p} = \mathbf{p}_0$ and $\mathbf{q} = \mathbf{q}_0$; by induction each $A_i$ contains the two points $\mathbf{p}_i, \mathbf{q}_i$ which under the isomorphism $G_i|_{A_i}$ map to $\mathbf{p}_{i-1}$ and $\mathbf{q}_{i-1}$, as in Figure 22.

Next, we will label $\mathbf{p}_{m,i}$ and $\mathbf{q}_{m,i}$ the points of $D_\infty(G)$ whose $m$th entry is $\mathbf{p}_i$. This requires a bit of care when $i=0$ and $i=k$, which we will leave to the reader.

Proposition 6.4 tells us that there are natural angles $\theta_j$ parametrizing $p^{-1}(\mathbf{p}_j) \subset \widetilde{X}_{G_j}$, and that these angles correspond under the Hénon mappings (see equation (6.1), where this angle appears as the argument of $v$ and the argument of $X_1$). Note that we are considering these fibers at the moment when they are created by the blow-up, so that each lies above a simple point of the divisor defined so far. Moreover, the same proposition (specifically, see equation (6.2)) says that the composition of the Hénon mappings takes angles $\theta_j$ to angles $\theta_{j-1}$ as indicated in the diagram

$$\theta_k \xmapsto[\tilde{X}_{G_k}\to\tilde{X}_{G_{k-1}}]{\text{blow-down}} d_k\theta_k+\arg a_k \xmapsto[\tilde{X}_{G_{k-1}}\to\tilde{X}_{G_{k-2}}]{\text{blow-down}} d_k d_{k-1}\theta_k+d_{k-1}\arg a_k+\arg a_{k-1}$$

$$\xmapsto[\tilde{X}_{G_{k-2}}\to\tilde{X}_{G_{k-3}}]{\text{blow-down}} \ \dots\ \xmapsto[\tilde{X}_{G_1}\to\tilde{X}_{G_0}]{\text{blow-down}} d\theta_k+\beta,$$

where $d=d_k\cdot\ldots\cdot d_1$ and

$$\beta=d_{k-1}d_{k-2}\ldots d_1\arg a_k+d_{k-2}d_{k-3}\ldots d_1\arg a_{k-1}+\ldots+d_1\arg a_2+\arg a_1.$$

An analogous argument, using appropriate conjugates of the inverses of the $H_j$, shows that the similar parameter $\phi_j$ of $p^{-1}(\mathbf{q}_j)$ is simply multiplied by $d$. Now the proof ends in the same way as the proof of Proposition 6.4, showing that the fibers above $\mathbf{p}^\infty$ and $\mathbf{q}^\infty$ are both $d$-adic solenoids, in one case using the angles $\phi_n$, and in the other $\psi_n=\theta_n+\beta/(d-1)$.

Part (c) has substantially already been proved: The tori corresponding to the $\mathbf{p}_{n,j}$ do form a sequence of incompressible tori in $\mathcal{B}^+_{\mathbf{R}}(D_\infty(G))-(\Sigma^+\cup\Sigma^-)$, and the components of the complements are homeomorphic to the simplest link of two circles which link with linking number $d_j$. Moreover, the proof we have given in Theorem 7.6 that this is the unique toroidal decomposition of $\mathcal{B}^+_{\mathbf{R}}(D_\infty(G))-(\Sigma^+\cup\Sigma^-)$ goes through without change. □

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