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Wave breaking for nonlinear nonlocal shallow water equations

by

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1. Introduction

A basic question in the theory of nonlinear partial differential equations is: when can a singularity form and what is its nature? The typical well-posedness result (see e.g. [23]) asserts that either a solution of a PDE exists for all time or else there is a time $T<\infty$ such that some norm of the solution becomes unbounded as $t\uparrow T$; the latter phenomena is called (finite-time) blow-up.

The behavior of the solution as the blow-up time is approached is of particular interest. A simple kind of singularity occurs when the solution itself becomes unbounded in a finite time. For models describing water waves we say that wave breaking holds if the solution (representing the wave) remains bounded but its slope becomes infinite in finite time: the profile will gradually steepen as it propagates until it finally develops a point where the slope is vertical and the wave is said to have broken, cf. [35].

Blow-up techniques are quite particular to each type of equation; there is no general method [34]. We present now a quite representative sample of methods to accomplish blow-up for nonlinear wave equations; see also the recent surveys [2], [3], [34].

The functional method (see [17], [22]) consists in introducing an appropriate functional F of a solution, depending on time, and using the PDE to get a (first- or second-order) differential inequality for F which implies finite-time blow-up for well-chosen initial data; surveys can be found in [21], [34].

A more sophisticated method than the functional method is the averaging method: introducing appropriate coordinates it is sometimes possible to prove the breakdown of certain averaged quantities (see [2], [33]).

For semilinear equations it is possible to define the maximal domain of existence of a solution and to try to understand the behavior of the solution near the boundary of this domain; for the equation $u_{tt}-u_{xx}-|u|^p=0$ with $1<p<\infty$, any solution which blows up in fact becomes infinite on a noncharacteristic C^1-curve in the (t,x)-plane [5]: near this blow-up curve the solutions are governed by the ODE $u_{tt}=|u|^p$ (see also [2]).

By analyzing the behavior of solutions of the scalar conservation law $u_t+f(u)_x=0$ along characteristics and using blow-up results for ODE's, a proof of the blow-up of the first derivative (even if the initial data are C^∞) was given by Lax [26] and developed in [20] and [27]; a somehow similar idea was used in [25].

Another well-established method refers to the search of blow-up for solutions of a certain given form containing undetermined coefficients or functions and a small parameter (see [2], [18], [19]).

We need a different approach to prove blow-up (in the form of wave breaking) for some physically relevant equations modelling shallow water waves and for which wave breaking was conjectured. Let us first present the equations under consideration.

The problem of long water waves dates back to the experimental work of Russell (1844). The Korteweg–de Vries equation (KdV) was introduced in 1895 to model the behavior of long waves on shallow water in close agreement with the observations of Russell:

$$\begin{cases} u_t+6uu_x+u_{xxx}=0, \\ u(0,x)=u_0(x), \end{cases} \tag{1.1}$$

with $u(t,x)$ representing the wave height above a flat bottom. The KdV model admits solitary waves which present soliton interaction: two solitary waves keep their shape and size after interaction although the ultimate position of each wave has been affected by the nonlinear interaction (see [35]). KdV has a bi-Hamiltonian structure which permits to obtain very precise information about the structure of the equation by the inverse scattering method, the equation being integrable (see [28]).

However, as soon as $u_0\in H^1(\mathbf{R})$, the solutions of (1.1) are global, cf. [24], and it is known that some shallow water waves break! Whitham [35] emphasized that the breaking phenomena is one of the most intriguing long-standing problems of water wave theory, and since the KdV equation can not describe breaking, he suggested the equation

$$\begin{cases} u_t+uu_x+\displaystyle\int_{\mathbf{R}} K_0(x-\xi)\,u_x(t,\xi)\,d\xi=0, \\ u(0,x)=u_0(x), \end{cases} \tag{1.2}$$

with the singular kernel

$$K_0(x)=\frac{1}{2\pi}\int_{\mathbf{R}}\left(\frac{\tanh\xi}{\xi}\right)^{1/2} e^{i\xi x}\,d\xi,$$

as a relatively simple model equation combining full linear dispersion with long wave nonlinearity, and conjectured that it describes the effect of breaking of waves.

It seems quite natural to generalize (1.2) by replacing Whitham's integral operator by an arbitrary pseudodifferential operator, considering instead of (1.2) the equation

$$\begin{cases} u_t + uu_x + \mathbf{K}[u] = 0, \\ u(0,x) = u_0(x), \end{cases} \tag{1.3}$$

where the operator $\mathbf{K}$ is given by

$$\mathbf{K}[u](x) = \frac{1}{2\pi} \int_{\mathbf{R}} \varkappa(\xi)\, \hat{u}(t,\xi)\, e^{i\xi x}\, d\xi,$$

$\hat{u}(t,\xi)$ being the Fourier transform of $u(t,x)$. The function $\varkappa$, defining the operator $\mathbf{K}$, is called the symbol of the operator. In general, if $\varkappa$ is a polynomial then $\mathbf{K}$ will be a constant-coefficient differential operator, while if $\varkappa$ is not a polynomial then $\mathbf{K}$ will be nonlocal, in the sense that a change in the values of a function g inside an open set U will affect the values of $\mathbf{K}[g]$ at points outside U. For $\varkappa(\xi) = i(\xi(\tanh\xi))^{1/2}$ we obtain Whitham's original equation (1.2). With this approach the Whitham-type equations (1.3) represent, as particular cases, many equations that are of great interest in problems of modern mathematical physics. For an outline of a number of physical problems leading to nonlinear equations of type (1.3) we refer to [29].

If, in contrast to KdV, the equation (1.2) describes the phenomenon of wave breaking, the numerical calculations carried out for the Whitham equation do not support any strong claim that soliton interaction can be expected, cf. [13].

Recently (see [7]), R. Camassa and D. Holm derived an equation modelling the same phenomena as KdV and Whitham's equation:

$$\begin{cases} u_t - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \\ u(0,x) = u_0(x). \end{cases} \tag{1.4}$$

Unlike KdV (which is an approximation to the equations of motion), equation (1.4) is obtained by approximating directly in the Hamiltonian for Euler's equations in the shallow water regime, cf. [8]. It is a good approximation for the full inviscid water wave equation—just as consistent in the small-amplitude shallow water regime as KdV or Whitham's equation, cf. [4], [8].

Actually, equation (1.4) was obtained formally more than 15 years ago by Fuchssteiner and Fokas (see [15], [16]) as a bi-Hamiltonian generalization of KdV; they also proved that (1.4) is formally integrable. As noted in [31], the novelty of Camassa and Holm's work was that they gave a physical derivation of (1.4) and found that the equation has solitons. This led to numerous papers devoted to the study of (1.4), see [1], [4], [6], [11], [31], and the citations therein.

In [7] Camassa and Holm conjectured that an initial profile having a sufficiently negative slope steepens and verticality develops in finite time—wave breaking occurs. §2 will provide the key result to prove this conjecture in §4.

As noted by Whitham [35], it is intriguing to know what mathematical models for shallow water waves could include both the phenomena of soliton interaction and wave breaking. The Camassa–Holm equation reconciles these properties which have been known for different models (KdV and Whitham's equation, respectively) and has the potential to become the new master equation for shallow water wave theory, cf. [16].

The formal approach to prove wave breaking for Whitham's equation (1.2) and the Camassa–Holm model (1.4) originates in an idea of Seliger [32]. For the nonlinear nonlocal equation of type (1.3),

$$\begin{cases} u_t+uu_x+\displaystyle\int_{\mathbf{R}} K(x-\xi)\,u_\xi(t,\xi)\,d\xi=0, \\ u(0,x)=u_0(x), \end{cases} \tag{1.5}$$

where K is a regular (continuous and integrable over $\mathbf{R}$), symmetric kernel, monotonically decreasing on $\mathbf{R}_+$, Seliger [32] was able to formally show wave breaking by a rather ingenious argument: consider $m_1(t):=\min_{x\in\mathbf{R}}[u_x(t,x)]$, $m_2(t):=\max_{x\in\mathbf{R}}[u_x(t,x)]$ attained at $x=\xi_1(t)$ and $x=\xi_2(t)$, respectively. By differentiating (1.5) and setting $x=\xi_i(t)$, $i=1,2$, he obtained formally differential inequalities for m_1 and m_2 which yield the desired wave breaking. A similar idea was used recently in [29] to settle the long-standing conjecture of Whitham [35] regarding wave breaking for the equation (1.2). This formal analysis is however not mathematically rigorous: it is impossible to guarantee the smoothness of the curve $\xi_1(t)$ on which $\min_{x\in\mathbf{R}}[u_x(t,x)]$ is attained. Therefore, it seems that one has to assume in addition that the curves $\xi_1(t)$ and $\xi_2(t)$ are smooth (see the results for wave breaking in [29, Chapter 1], for example).

The aim of the present paper is to show how one can avoid this additional strong assumption and therefore to prove rigorously the breaking of waves property conjectured for various models of type (1.3). We also apply the method to the Camassa–Holm equation.

2. The abstract key result

The result in this section regards the time evolution of the slope (in x) at an inflection point for a function $v(t,\cdot)\in H^2(\mathbf{R})$ with a C^1-dependence on the time parameter t; later on, $v(t,x)$ will be a solution of a Whitham-type equation (1.3) or of the Camassa–Holm equation (1.4).

THEOREM 2.1. *Let $T>0$ and $v\in C^1([0,T);H^2(\mathbf{R}))$. Then for every $t\in[0,T)$ there exists at least one point $\xi(t)\in\mathbf{R}$ with*

$$m(t):=\inf_{x\in\mathbf{R}}[v_x(t,x)]=v_x(t,\xi(t)),$$

and the function m is almost everywhere differentiable on $(0,T)$ with

$$\frac{dm}{dt}(t)=v_{tx}(t,\xi(t)) \quad a.e. \ on \ (0,T).$$

Proof. Let $c>0$ stand for a generic constant.

Fix $t\in[0,T)$ and define $m(t):=\inf_{x\in\mathbf{R}}[v_x(t,x)]$. If $m(t)\geqslant 0$ we have that $v(t,\cdot)$ is nondecreasing on $\mathbf{R}$ and therefore $v(t,\cdot)\equiv 0$ (recall $v(t,\cdot)\in L^2(\mathbf{R})$), so that we may assume $m(t)<0$. Since $v_x(t,\cdot)\in H^1(\mathbf{R})$ we see that $\lim_{|x|\to\infty} v_x(t,x)=0$ so that there exists at least a $\xi(t)\in\mathbf{R}$ with $m(t)=v_x(t,\xi(t))$.

Let now $s,t\in[0,T)$ be fixed. If $m(t)\leqslant m(s)$ we have

$$0\leqslant m(s)-m(t)=\inf_{x\in\mathbf{R}}[v_x(s,x)]-v_x(t,\xi(t))\leqslant v_x(s,\xi(t))-v_x(t,\xi(t)),$$

and by the Sobolev embedding $H^1(\mathbf{R})\subset L^\infty(\mathbf{R})$ we conclude that

$$|m(s)-m(t)|\leqslant|v_x(t)-v_x(s)|_{L^\infty(\mathbf{R})}\leqslant c\,|v_x(t)-v_x(s)|_{H^1(\mathbf{R})}.$$

Hence the mean-value theorem for functions with values in Banach spaces—$H^1(\mathbf{R})$ in the present case—yields (see [12])

$$|m(t)-m(s)|\leqslant c\,|t-s|\max_{0\leqslant\tau\leqslant\max\{s,t\}}[|v_{tx}(\tau)|_{H^1(\mathbf{R})}], \quad t,s\in[0,T).$$

Since $v_{tx}\in C([0,T),H^1(\mathbf{R}))$, we see that m is locally Lipschitz on $[0,T)$ and therefore Rademacher's theorem (cf. [14]) implies that m is almost everywhere differentiable on $(0,T)$.

Fix $t\in(0,T)$. We have that

$$\left|\frac{v_x(t+h)-v_x(t)}{h}-v_{tx}(t)\right|_{H^1(\mathbf{R})}\to 0 \quad \text{as } h\to 0,$$

and therefore

$$\sup_{y\in\mathbf{R}}\left|\frac{v_x(t+h,y)-v_x(t,y)}{h}-v_{tx}(t,y)\right|\to 0 \quad \text{as } h\to 0, \tag{2.1}$$

in view of the continuous embedding $H^1(\mathbf{R})\subset L^\infty(\mathbf{R})$.

By the definition of m,

$$m(t+h)=v_x(t+h,\xi(t+h))\leqslant v_x(t+h,\xi(t)).$$

Consequently, given $h>0$, we obtain

$$\frac{m(t+h)-m(t)}{h}\leqslant\frac{v_x(t+h,\xi(t))-v_x(t,\xi(t))}{h}.$$

Letting $h\to 0^+$ and using (2.1), we find

$$\limsup_{h\to 0^+}\frac{m(t+h)-m(t)}{h}\leqslant v_{tx}(t,\xi(t)),\quad t\in(0,T). \tag{2.2}$$

On the other hand,

$$m(t-h)=v_x(t-h,\xi(t-h))\leqslant v_x(t-h,\xi(t)),$$

and thus

$$\frac{m(t)-m(t-h)}{h}\geqslant\frac{v_x(t,\xi(t))-v_x(t-h,\xi(t))}{h},\quad h>0.$$

Letting $h\to 0^+$ and using (2.1), we find

$$\liminf_{h\to 0^+}\frac{m(t)-m(t-h)}{h}\geqslant v_{tx}(t,\xi(t)),\quad t\in(0,T). \tag{2.3}$$

Since m is almost everywhere differentiable, relations (2.2) and (2.3) enable us to conclude

$$\frac{dm}{dt}(t)=v_{tx}(t,\xi(t))\quad\text{a.e. on }(0,T),$$

and the proof is complete. □

Remark 2.2. In addition to the hypotheses of Theorem 2.1, let v be of class C^2 in the space variable x and assume that there is a C^1-curve $\xi(t)$ on which $m(t):=\inf_{x\in\mathbf{R}}[v_x(t,x)]$ is attained. We would have

$$\frac{dm}{dt}(t)=v_{tx}(t,\xi(t))+v_{xx}(t,\xi(t))\,\frac{d\xi}{dt}(t)=v_{tx}(t,\xi(t))$$

since $v_{xx}(t,\xi(t))=0$, $m(t)$ being a minimum for $v_x(t,\cdot)\in C^2$. However, cf. [29], it is impossible to guarantee the smoothness of such a curve $\xi(t)$. Moreover, it is not hard to see (by means of some pictures) that there is no reason why the minimum should be attained along a curve. In this context, Theorem 2.1 shows that the general situation is not far from the optimal one described before—what we are really interested in is the time evolution of the slope at the inflection point: $dm(t)/dt=v_{tx}(t,\xi(t))$ a.e. □

3. Breaking of waves for Whitham-type equations

Various results of wave breaking for equations of Whitham type (1.3) were recently obtained in [29] under the assumption that the minimum of the slope of the solution is attained along a smooth curve. We will show now how Theorem 2.1 enables us to eliminate this assumption.

We want to emphasize the method and not the technicalities. For this reason we focus on the case of a regular kernel (Seliger's formal approach); it is not hard to see then how the method applies to the more complicated cases handled in [29] (including Whitham's equation (1.2)).

Let us now give a rigorous mathematical proof of the occurence of wave breaking (for formal proofs see [32], [35]) for the equation (1.5).

The method of successive approximations yields the following result regarding the Cauchy problem for the initial value problem (1.5):

THEOREM 3.1 [29]. *Assume that $K\in C(\mathbf{R})\cap L^1(\mathbf{R})$ is symmetric and monotonically decreasing on $\mathbf{R}_+$, $K\not\equiv 0$, and let $u_0\in H^\infty(\mathbf{R})$. Then for some $t_0>0$ there is a unique solution $u(t,x)$ of (1.5) in the class $C^\infty([0,t_0];H^\infty(\mathbf{R}))$. The solution can destruct only as a result of breaking of waves.*

The destruction of the solution as a result of breaking of waves should be understood as follows: if the maximal existence time $T>0$ of the solution is finite, we have $\sup_{(t,x)\in[0,T)\times\mathbf{R}}|u(t,x)|<\infty$ while $\sup_{x\in\mathbf{R}}|u_x(t,x)|\to\infty$ as $t\uparrow T$.

We will now prove the following blow-up result for (1.5):

THEOREM 3.2. *A sufficiently asymmetric initial profile yields wave breaking. More precisely, if $u_0\in H^\infty(\mathbf{R})$ satisfies*

$$\inf_{x\in\mathbf{R}}[u_0'(x)]+\sup_{x\in\mathbf{R}}[u_0'(x)]\leqslant -2K(0)<0,$$

then for the solution of (1.5) with initial data u_0 we observe wave breaking.

Proof. Let $T>0$ be the maximal existence time of the solution $u(t,x)$ of (1.5) with initial data $u_0\in H^\infty(\mathbf{R})$, as given by Theorem 3.1. Define for $t\in[0,T)$,

$$m_1(t):=\inf_{x\in\mathbf{R}}[u_x(t,x)]=u_x(t,\xi_1(t)),$$

$$m_2(t):=\sup_{x\in\mathbf{R}}[u_x(t,x)]=u_x(t,\xi_2(t)),$$

where $\xi_i(t)$, $i=1,2$, are some points in $\mathbf{R}$; see Theorem 2.1 for the existence of $\xi_1(t)$—an analogous result clearly yields the existence of $\xi_2(t)$.

Differentiating (1.5) with respect to x and evaluating the resulting equation at $x=\xi_i(t)$, $i=1,2$, we obtain

$$\frac{dm_i}{dt}+m_i^2+\int_{\mathbf{R}} K(\eta)u_{xx}(t,\xi_i(t)-\eta)\,d\eta=0 \quad \text{a.e. on } (0,T), \tag{3.1}$$

taking into account Theorem 2.1 and the fact that $u_{xx}(t,\xi_i(t))=0$ for $i=1,2$.

By Lebesgue's dominated convergence theorem we have that

$$\lim_{n\to\infty}\int_{-n}^{n} K(\eta)u_{xx}(t,\xi_i(t)-\eta)\,d\eta=\int_{\mathbf{R}} K(\eta)u_{xx}(t,\xi_i(t)-\eta)\,d\eta. \tag{3.2}$$

If $[a,b]\subset\mathbf{R}$ is an interval where K is monotone and $f\colon[a,b]\to\mathbf{R}$ is continuous, by the second mean-value theorem (cf. [30]) there is some $c\in[a,b]$ such that

$$\int_a^b K(x)f(x)\,dx=K(a)\int_a^c f(x)\,dx+K(b)\int_c^b f(x)\,dx.$$

We therefore find, for $n\geqslant 1$ and $i=1,2$, points $\alpha_i^n\in[-n,0]$, $\beta_i^n\in[0,n]$ such that

$$\begin{aligned}\int_{-n}^{0} K(\eta)u_{xx}(t,\xi_i(t)-\eta)\,d\eta=&-K(-n)[u_x(t,\xi_i(t)-\alpha_i^n)-u_x(t,\xi_i(t)+n)]\\ &-K(0)[u_x(t,\xi_i(t))-u_x(t,\xi_i(t)-\alpha_i^n)]\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}\int_{0}^{n} K(\eta)u_{xx}(t,\xi_i(t)-\eta)\,d\eta=&-K(0)[u_x(t,\xi_i(t)-\beta_i^n)-u_x(t,\xi_i(t))]\\ &-K(n)[u_x(t,\xi_i(t)-n)-u_x(t,\xi_i(t)-\beta_i^n)],\end{aligned} \tag{3.4}$$

respectively. Recalling the definition of $m_1(t)$ and $m_2(t)$ we deduce by adding (3.3) and (3.4) that

$$\begin{aligned}\int_{-n}^{n} K(\eta)u_{xx}(t,\xi_i(t)-\eta)\,d\eta=&\,K(0)[u_x(t,\xi_i(t)-\alpha_i^n)-u_x(t,\xi_i(t)-\beta_i^n)]\\ &+K(-n)[u_x(t,\xi_i(t)+n)-u_x(t,\xi_i(t)-\alpha_i^n)]\\ &+K(n)[u_x(t,\xi_i(t)-\beta_i^n)-u_x(t,\xi_i(t)-n)]\\ \leqslant&\,K(0)[m_2(t)-m_1(t)]+2K(n)[m_2(t)-m_1(t)], \quad n\geqslant 1,\end{aligned}$$

since K is nonnegative on $\mathbf{R}$ as a consequence of the monotonicity of $K\in L^1(\mathbf{R})$ on $\mathbf{R}_+$, K being also symmetric. Furthermore, the conditions on K force $\lim_{n\to\infty}K(n)=0$. Letting $n\to\infty$ in the previous inequality we therefore obtain in view of (3.2) the estimate

$$\int_{\mathbf{R}} K(\eta)u_{xx}(t,\xi_i(t)-\eta)\,d\eta\leqslant K(0)[m_2(t)-m_1(t)], \quad t\in(0,T),\ i=1,2,$$

and from (3.1) we infer the inequalities

$$\frac{dm_1}{dt} \leqslant -m_1^2 + K(0)(m_2 - m_1) \quad \text{a.e. on } (0,T),$$

$$\frac{dm_2}{dt} \leqslant -m_2^2 + K(0)(m_2 - m_1) \quad \text{a.e. on } (0,T).$$

Summing up, we get

$$\frac{d}{dt}(m_1+m_2) \leqslant (m_2-m_1)[2K(0)+(m_1+m_2)] - 2m_2^2 \quad \text{a.e. on } (0,T).$$

Since $m_1(0)+m_2(0) \leqslant -2K(0)$ at time $t=0$, we see from the previous relation that it remains so for all time (the proof of Theorem 2.1 shows that m_1 is locally Lipschitz and therefore also absolutely continuous, cf. [14], and the same is true for m_2). We obtain that

$$\frac{dm_1}{dt} \leqslant -(m_1+K(0))^2 - K^2(0) \leqslant -(m_1+K(0))^2 \quad \text{a.e. on } (0,T).$$

Defining $m(t) := m_1(t) + K(0)$, $t \in [0,T)$, we see that $m(0) < 0$ and

$$\frac{dm}{dt} \leqslant -m^2(t) \quad \text{a.e. on } (0,T). \tag{3.5}$$

As noted before, m_1 is locally Lipschitz so that m is too, and therefore $m \in W^{1,\infty}_{\text{loc}}(0,T)$, cf. [14]. Since $m(t) < 0$ on $[0,T)$, it is not hard to check that $1/m$ is locally Lipschitz, and thus $1/m \in W^{1,\infty}_{\text{loc}}(0,T)$. Differentiating the relation $m \cdot (1/m) = 1$, we see that

$$\frac{d}{dt}\frac{1}{m} = -\frac{1}{m^2}\frac{dm}{dt} \quad \text{a.e. on } (0,T).$$

From relation (3.5) we find now

$$\frac{d}{dt}\frac{1}{m} \geqslant 1 \quad \text{a.e. on } (0,T),$$

and integration yields

$$\frac{1}{m(t)} \geqslant \frac{1}{m(0)} + t, \quad 0 \leqslant t < T,$$

so that $m(t) \to -\infty$ before t reaches $1/|m(0)|$. This proves that the wave $u(t,x)$ breaks in finite time. □

4. Wave breaking for the Camassa–Holm equation

In this section we will use Theorem 2.1 to prove the wave-breaking result for the model (1.4) conjectured in [7], [8].

We first recall the following result regarding the Cauchy problem for the initial value problem (1.4):

THEOREM 4.1 [9]. *Given $u_0 \in H^3(\mathbf{R})$, there exists a maximal $T = T(u_0) > 0$ and a unique solution*

$$u = u(\cdot, u_0) \in C([0,T); H^3(\mathbf{R})) \cap C^1([0,T); H^2(\mathbf{R}))$$

to problem (1.4). Moreover, the solution depends continuously on the initial data, i.e., the mapping $u_0 \mapsto u(\cdot, u_0): H^3(\mathbf{R}) \to C([0,T); H^3(\mathbf{R})) \cap C^1([0,T); H^2(\mathbf{R}))$ is continuous. The solution can destruct only as a result of breaking of waves.

For the precise meaning of wave breaking throughout this section we refer to the definition of this term given in the context of Whitham-type equations (see §3).

At this point, let us note for further considerations an important conservation law for (1.4): if $u_0 \in H^3(\mathbf{R})$, then $|u(t)|^2_{H^1(\mathbf{R})} = \int_{\mathbf{R}} (u^2 + u_x^2)\, dx$ is conserved in time as long as the solution exists (see [7], [10]).

To prove wave breaking, it is convenient to write equation (1.4) in the form

$$u_t + uu_x + \mathbf{K}[f(u, u_x)] = 0, \tag{4.1}$$

where $\mathbf{K}$ is the pseudodifferential operator with symbol $k(\xi) = (-i\xi)/(1+\xi^2)$ and $f(u,v) := u^2 + \frac{1}{2}v^2$. Note that the nonlocal term in (4.1) is nonlinear in u and u_x whereas in the Whitham-type equations (1.3) the nonlocal term is linear in u.

With $p(x) := \exp(-|x|)$, the resolvent $(1-\partial_x^2)^{-1}$ can be represented as the convolution operator

$$Q^{-1} f := (1-\partial_x^2)^{-1} f = \tfrac{1}{2} p * f, \quad f \in L^2(\mathbf{R}),$$

where Q denotes the operator $1 - \partial_x^2$ acting in $L^2(\mathbf{R})$ with $\mathrm{dom}(Q) = H^2(\mathbf{R})$.

Assume now that $u_0 \in H^3$ and let $u \in C([0,T); H^3(\mathbf{R})) \cap C^1([0,T); H^2(\mathbf{R}))$ be the corresponding strong solution of (1.4). We write (1.4) as

$$(1-\partial_x^2)(u_t + uu_x) = -2uu_x - u_x u_{xx} = -\partial_x\big(u^2 + \tfrac{1}{2}u_x^2\big).$$

Applying $(1-\partial_x^2)^{-1}$ to both sides, we get

$$u_t + uu_x = -\partial_x\big(\tfrac{1}{2} p * \big(u^2 + \tfrac{1}{2}u_x^2\big)\big) \quad \text{in } C([0,T); H^1(\mathbf{R})).$$

Differentiating this relation with respect to x, we find

$$\begin{aligned} u_{tx} + uu_{xx} + u_x^2 &= -\partial_x^2\big(\tfrac{1}{2} p * \big(u^2 + \tfrac{1}{2}u_x^2\big)\big) \\ &= (Q - I)\big(\tfrac{1}{2} p * \big(u^2 + \tfrac{1}{2}u_x^2\big)\big) \\ &= u^2 + \tfrac{1}{2}u_x^2 - \tfrac{1}{2} p * \big(u^2 + \tfrac{1}{2}u_x^2\big), \end{aligned}$$

that is,

$$u_{tx} + uu_{xx} + \tfrac{1}{2}u_x^2 = u^2 - \tfrac{1}{2} p * \big(u^2 + \tfrac{1}{2}u_x^2\big) \quad \text{in } C([0,T), L^2(\mathbf{R})). \tag{4.2}$$

We prove now the following blow-up result for (1.4):

THEOREM 4.2. *Assume that the initial profile* $u_0\in H^3(\mathbf{R})$ *has at some point a slope which is less than* $-(1/\sqrt{2})|u_0|_{H^1(\mathbf{R})}$. *Then wave breaking for the corresponding solution of* (1.4) *occurs.*

Proof. Let $x_0\in\mathbf{R}$ with $\sqrt{2}\,u_0'(x_0)<-|u_0|_{H^1(\mathbf{R})}$ and consider the corresponding solution $u\in C([0,T),H^3(\mathbf{R}))\cap C^1([0,T),H^2(\mathbf{R}))$ of the initial value problem (1.4). In addition, choose $\varepsilon\in(0,1)$ such that

$$(2-2\varepsilon)[u_0'(x_0)]^2\geqslant|u_0|^2_{H^1(\mathbf{R})}. \tag{4.3}$$

First, note that

$$\begin{aligned}2u^2(t,x)&=2\left(\int_{-\infty}^x uu_x-\int_x^\infty uu_x\right)\\&<\int_{-\infty}^x(u^2+u_x^2)+\int_x^\infty(u^2+u_x^2)=|u|^2_{H^1(\mathbf{R})}=|u_0|^2_{H^1(\mathbf{R})},\end{aligned} \tag{4.4}$$

for all $(t,x)\in[0,T)\times\mathbf{R}$, since $\int_{\mathbf{R}}(u^2+u_x^2)\,dx$ is a conservation law for (1.4). The fact that for $u(t,\cdot)\in H^2(\mathbf{R})$, $u(t,\cdot)\not\equiv0$, we can not have $u=\pm u_x$ on $\mathbf{R}$ (as one can easily check) justifies the strict inequality in (4.4).

Define now $m(t):=\inf_{x\in\mathbf{R}}[u_x(t,x)]$ and let $\xi(t)\in\mathbf{R}$ be a point where this infimum is attained (see Theorem 2.1). Clearly $u_{xx}(t,\xi(t))=0$, by the definition of $\xi(t)$ since $u(t)\in H^3(\mathbf{R})\subset C^2(\mathbf{R})$. Hence, setting $x=\xi(t)$ in (4.2), we obtain from Theorem 2.1 the relation

$$\frac{dm}{dt}+\frac{1}{2}m^2=u^2(t,\xi(t))-\frac{1}{2}\int_{\mathbf{R}}p(\xi(t)-\eta)\left[u^2(t,\eta)+\tfrac{1}{2}u_x^2(t,\eta)\right]d\eta\quad\text{a.e. on }(0,T). \tag{4.5}$$

Observe that the inequality

$$\begin{aligned}e^{-x}\int_{-\infty}^x e^\eta[u^2(t,\eta)+u_x^2(t,\eta)]\,d\eta&\geqslant2e^{-x}\int_{-\infty}^x e^\eta u(t,\eta)\,u_x(t,\eta)\,d\eta\\&=e^{-x}\int_{-\infty}^x e^\eta\frac{d}{d\eta}[u^2(t,\eta)]\,d\eta\\&=u^2(t,x)-e^{-x}\int_{-\infty}^x e^\eta u^2(t,\eta)\,d\eta\end{aligned}$$

yields

$$e^{-x}\int_{-\infty}^x e^\eta[2u^2(t,\eta)+u_x^2(t,\eta)]\,d\eta\geqslant u^2(t,x), \tag{4.6}$$

whereas

$$e^x\int_x^\infty e^{-\eta}[u^2(t,\eta)+u_x^2(t,\eta)]\,d\eta \geqslant -2e^x\int_x^\infty e^{-\eta}u(t,\eta)u_x(t,\eta)\,d\eta$$
$$= -e^x\int_x^\infty e^{-\eta}\frac{d}{d\eta}[u^2(t,\eta)]\,d\eta$$
$$= u^2(t,x)-e^x\int_x^\infty e^{-\eta}u^2(t,\eta)\,d\eta$$

leads to

$$e^x\int_x^\infty e^{-\eta}[2u^2(t,\eta)+u_x^2(t,\eta)]\,d\eta \geqslant u^2(t,x). \tag{4.7}$$

Since $p(x)=e^{-|x|}$, $x\in\mathbf{R}$, we obtain from (4.5)–(4.7) that

$$\frac{dm}{dt} \leqslant -\tfrac{1}{2}m^2+\tfrac{1}{2}u^2(t,\xi(t)) \quad \text{a.e. on } (0,T). \tag{4.8}$$

On the other hand, (4.3) and (4.4) imply

$$u^2(t,\xi(t)) < \tfrac{1}{2}|u_0|^2_{H^1(\mathbf{R})} \leqslant (1-\varepsilon)[u_0'(x_0)]^2 \leqslant (1-\varepsilon)m^2(0), \quad t\in(0,T), \tag{4.9}$$

recalling the definition of m.

Observe that m is continuous on $[0,T)$ and absolutely continuous on $(0,T)$ since $m\in W^{1,\infty}_{\text{loc}}(0,T)$, cf. the proof of Theorem 2.1 and [14]. We now claim that

$$m^2(t) > \left(1-\tfrac{1}{2}\varepsilon\right)m^2(0), \quad t\in[0,T). \tag{4.10}$$

If this would not be true, there is some $t_0\in(0,T)$ with $m^2(t)>\left(1-\frac{1}{2}\varepsilon\right)m^2(0)$ on $[0,t_0)$ and $m^2(t_0)=\left(1-\frac{1}{2}\varepsilon\right)m^2(0)$ by the continuity of m: note that $m(0)<0$. But in this case, a combination of (4.8) and (4.9) would give

$$\frac{dm}{dt} \leqslant -\tfrac{1}{4}\varepsilon m^2(0) < 0 \quad \text{a.e. on } (0,t_0).$$

An integration—recall that m is absolutely continuous on $(0,T)$—yields $m(0)\geqslant m(t_0)$. On the other hand, $m(0)<0$ and $m^2(t)>\left(1-\frac{1}{2}\varepsilon\right)m^2(0)$ on $(0,t_0)$ force m to be negative on $(0,t_0)$, and therefore we would obtain $m^2(t_0)\geqslant m^2(0)$, in contradiction with our assumption $m^2(t_0)=\left(1-\frac{1}{2}\varepsilon\right)m^2(0)$. This proves that (4.10) holds.

Combining (4.10) with (4.8) and (4.9), we obtain

$$\frac{dm}{dt}(t) \leqslant -\tfrac{1}{4}\varepsilon m^2(t) \quad \text{a.e. on } (0,T).$$

The same approach used in the last part of the proof of Theorem 3.2 enables us to conclude from here that $m(t)\to-\infty$ in finite time. □

Example 4.3. Let $f_n(x):=\exp(-nx^2)$ for $n\geqslant 1$, $x\in\mathbf{R}$, and note that

$$\inf_{x\in\mathbf{R}}[f_{n,x}(x)]=-\sqrt{2n}\,e^{-1/2}\quad\text{and}\quad |f_n|^2_{H^1(\mathbf{R})}=\sqrt{\pi}\left(\frac{1}{\sqrt{2}}\sqrt{n}+\frac{1}{\sqrt{2n}}\right).$$

Choosing n sufficiently large and setting $u_0:=f_n$, an application of Theorem 4.2 shows that the corresponding solution of (1.4) blows up in finite time. □

Remark 4.4. In [9] we obtained a wave-breaking result for (1.4) for certain antisymmetric initial data. The method of the proof was different from the one we use here: it relied on the fact that antisymmetry is preserved by the Camassa–Holm flow, cf. [7]. The above example shows that symmetric initial profiles can also yield wave breaking. □

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