Partition relations for partially ordered sets

by

STEVO TODORCEVIC

University of Colorado *Boulder, Colorado, U.S.A.* and *University of California* *Berkeley, California, U.S.A.*

Let P be a partially ordered set. If $r<\omega$, then $[P]^r$ denotes the set of all sequences $\langle a_1, ..., a_r \rangle$ such that $a_1, ..., a_r \in P$ and $a_1 <_P ... <_P a_r$. If γ is an ordinal and if α_i, $i<\gamma$ are order types (isomorphism types of linearly ordered sets), then the symbol $P \to (\alpha_i)^r_{i<\gamma}$ means that for any partition $[P]^r = \bigcup_{i<\gamma} K_i$ there exists an $i<\gamma$ and a chain $A \subseteq P$ such that $\operatorname{tp} A = \alpha_i$ and $[A]^r \subseteq K_i$. The negation of the partition symbol is written with $\nrightarrow$ instead of $\to$. Note that if P is a linearly ordered set, then $[P]^r$ and $P \to (\alpha_i)^r_{i<\gamma}$ have the usual meanings. If $\alpha_i = \alpha$ for all $i<\gamma$, then we write $P \to (\alpha)^r_\gamma$ instead of $P \to (\alpha_i)^r_{i<\gamma}$.

This paper is a study of the partition symbol $P \to (\alpha_i)^r_{i<\gamma}$ for partially ordered sets P such that $P \to (\varkappa)^1_\varkappa$ for some infinite cardinal $\varkappa$. Our main result for the case $\varkappa = \omega$ is the following theorem which proves a conjecture of Galvin [10; p. 718].

THEOREM 1. *Let P be a partially ordered set such that $P \to (\omega)^1_\omega$. Then*

$$P \to (\alpha)^2_k \quad \text{for all } \alpha < \omega_1 \text{ and } k < \omega.$$

This theorem completes a rather long list of weaker results: Erdös–Rado [7], [8], Hajnal [11], Galvin [9], Prikry [21], Baumgartner–Hajnal [1] and Galvin [10]. The history of the problem is discussed in [1; pp. 193–194], [4; pp. 271–272] and [10; pp. 711–712]. The most general results previously obtained in the direction of Theorem 1 are a result of Baumgartner and Hajnal [1] who proved Theorem 1 for the case when P is a linearly ordered set, and a result of Galvin [10; p. 714] who proved Theorem 1 under the stronger hypothesis $P \to (\eta)^1_\omega$. The hypothesis $P \to (\omega)^1_\omega$ in Theorem 1 is known to be necessary since $P \to (\omega, \omega+1)^2$ implies $P \to (\omega)^1_\omega$ (see [10; p. 718]).

Let us now consider a generalization of Theorem 1 to higher cardinals $\varkappa$. Unfortunately, there is a restriction to any such generalization since $\omega_2 \nrightarrow (\omega_1+\omega)_2^2$ is known to be consistent with GCH ([4, p. 272]). On the other hand, Erdös and Hajnal proved ([4; p. 273]) that $\omega_2 \to (\omega_1+n)_2^2$ for all finite n assuming CH. This was later generalized by Shelah [22] to all higher regular cardinals. In [4; p. 282] it is asked whether this result holds for all order types; i.e., whether $\varphi \to (\omega_1)^1_{2^{\aleph_0}}$ implies $\varphi \to (\omega_1+n)_k^2$ for all order types φ and all $k, n<\omega$. In a subsequent note we shall answer this question positively, but in this paper we present the following general answer in the case $k=2$.

THEOREM 2. *Suppose $\varkappa$ is a regular cardinal and $\lambda^\lambda<\varkappa$. Let P be a partially ordered set such that $P \to (\varkappa)^1_{2^\varkappa}$. Then*

$$P \to (\varkappa+\xi)_2^2 \quad \textit{for all } \xi<\lambda.$$

The next result is an analogue for partially ordered sets of a well-known partition relation for cardinals (see [5; § 17]).

THEOREM 3. *Assume $\lambda \geq \aleph_0$ and $\theta \geq 2$. Let $\varkappa=\theta^\lambda$ and let P be a partially ordered set such that $P \to (\varkappa)^1_\varkappa$. Then*

$$P \to (\alpha, (\mathrm{cf}\,\lambda+1)_\gamma)^2 \quad \textit{for all } \alpha<\varkappa^+ \textit{ and } \gamma<\mathrm{cf}\,\lambda.$$

Let P be a partially ordered set, let $[P]^{<\omega}=\bigcup_{r<\omega}[P]^r$, and let $f\colon [P]^{<\omega} \to \gamma$ be a given colouring. We say that a chain $A \subseteq P$ is end-homogeneous with respect to f if for every $s \in [A]^{<\omega}$ and for every $a, b \in A$ with $\max(s) <_P a, b$, we have $f(s^\frown a)=f(s^\frown b)$. If γ is an ordinal and if α is an order type, then the symbol $P \to \langle \alpha \rangle_\gamma^{<\omega}$ means that for every partition $f\colon [P]^{<\omega} \to \gamma$ there is an end-homogeneous chain of type α. The end-homogeneous chains are very useful in proving partition relations of the form $P \to (\alpha_\xi)^r_{\xi<\gamma}$, where $r>2$. But the partition relations of the of the form $P \to \langle \alpha \rangle_\gamma^{<\omega}$ have their own interest independent of this, and they are also very useful in many other combinatorial problem concerning partially ordered sets. The case $P=\varkappa^+$ of the following result is the well-known partition relation for cardinals (see [5; § 15]).

THEOREM 4. *Assume $\lambda \geq \aleph_0$ and $\theta \geq 2$. Let $\varkappa=\theta^\lambda$ and let P be a partially ordered set such that $P \to (\varkappa)^1_\varkappa$. Then*

$$P \to \langle \lambda+1 \rangle_\theta^{<\omega}.$$

One of the main points of our proof of the above theorems is the fact that we may restrict ourselves to trees. This fact is proved in § 1 by using the operation σP defined there and a result from [25].

In § 2 we prove Theorem 1 for partially ordered sets of bounded cardinality. In § 3 we eliminate this hypothesis using a forcing argument.

In § 4 we develop a technique for proving partition relations for trees which are now of great interest owing to the results of § 1. But these partition relations have their own interest which is independent of § 1. An example is the following relation which is a corollary of a result proved in § 4.

$$\text{nonspecial tree} \to (\text{nonspecial tree}, \omega+1)^2.$$

This relation means: If T is a tree which is not the union of countably many antichains and if $[T]^2=K_0\cup K_1$ is a given partition, then either there exists a set $X\subseteq T$ which is not the union of countably many antichains with $[X]^2\subseteq K_0$, or else there is a chain $A\subseteq T$ of type $\omega+1$ with $[A]^2\subseteq K_1$. Note that since ω_1 is a nonspecial tree, this result has as an immediate consequence the well-known relation $\omega_1\to(\omega_1, \omega+1)^2$ proved by Erdös and Rado [8; p. 459]. In § 4 we also give the proof of Theorem 3.

The proof of Theorem 2 is given in § 5.

The proof of Theorem 4 is given in § 6. In § 6 we also prove the Stepping-up Lemma for partially ordered sets and deduce several corollaries.

The technique developed in §§ 1 and 4 can also be used in several other combinatorial problems about partially ordered sets, e.g., set-mapping problems on partially ordered sets, etc. This approach will be discussed elsewhere.

We conclude the introduction with a few words about the notation. All undefined terms can be found in any standard text on set theory (e.g., [5]). The letters $\alpha, \beta, \gamma, \delta, \xi, \ldots$ are reserved for ordinals, and $\varkappa, \lambda, \theta, \ldots$ for infinite cardinals. $\mathscr{P}(A)$ is the set of all subsets of A considered as a partially ordered set ordered by $\subseteq$.

If P is a partially ordered set and if A and B are subsets of P, then $A<_P B$ means that $a<_P b$ for every $a\in A$ and $b\in B$. If $K\subseteq[P]^2$ and if $a\in P$, then $K(a)$ denotes the set $\{b\in P\colon \langle a, b\rangle\in K \text{ or } \langle b, a\rangle\in K\}$.

A tree is a partially ordered set T such that $\hat{t}=\{s\in T\colon s<_T t\}$ is well-ordered by $<_T$ for every $t\in T$. The order type of $\hat{t}$ is called the height of t in T, $\mathrm{ht}_T(t)$. The αth level of T is the set $T_\alpha=\{t\in T\colon \mathrm{ht}_T(t)=\alpha\}$, and $\mathrm{ht}(T)=\min\{\alpha\colon T_\alpha=\emptyset\}$ is the height of T. If A is a set of ordinals, then $T\restriction A=\bigcup_{\alpha\in A}T_\alpha$. If $t\in T$, then $T^t=\{s\in T\colon t\leqslant_T s\}$. For technical reasons, we shall assume that every tree has a minimal element denoted by $\emptyset$. If U is a subset of T then we say that $f\colon U\to T$ is regressive if for every $t\in U\setminus\{\emptyset\}$, $f(t)<_T t$.

Note that if T is a tree, then $T\not\to(\varkappa)^1_\varkappa$ is equivalent to saying that T is the union of

$\leq \varkappa$ antichains. In this case we call T a $\varkappa$-special tree. A special tree is an $\aleph_0$-special tree.

If α is an ordinal, then by α^+ we denote the minimal infinite cardinal above α, i.e., $\alpha^+ = \aleph_0 + |\alpha|^+$.

The result of Theorem 1 was announced in [26]. We would like to thank Professor Fred Galvin for many valuable communications concerning the problems considered in this paper.

§ 1. On the operation σP

Suppose that $\langle A, R\rangle$ and $\langle B, S\rangle$ are two given structures where R and S are binary relations. Then we say that $\langle A, R\rangle$ is $\langle B, S\rangle$-embeddable if there is a mapping $f: A \to B$ such that $f(a)\, S_{\neq}\, f(b)$ for all $a, b \in A$ with $a R_{\neq} b$. Note that f need not be one-to-one. The following simple fact about this notion will be used quite often in this paper.

LEMMA 1. *Suppose P and Q are partially ordered sets such that P is Q-embeddable. Let r be a positive integer, and let γ and α_i, $i<\gamma$ be ordinals. Then*

$$P \to (\alpha_i)^r_{i<\gamma} \quad \textit{implies} \quad Q \to (\alpha_i)^r_{i<\gamma}.$$

If $\mathcal{A} = \langle A, R\rangle$ is a structure with one binary relation R, then by $\sigma\mathcal{A}$ we denote the structure $\langle \sigma A, \subseteq\rangle$ where σA is the set of all one-to-one mappings s with domain $\alpha \in \mathrm{Ord}$ such that $\xi < \eta < \alpha$ implies $s(\xi)\, R\, s(\eta)$. Let $\sigma'\mathcal{A}$ denote the structure $\langle \sigma' A, \subseteq\rangle$ where $\sigma' A$ is the set of all s in σA with domain a successor ordinal. Note that $\sigma'\mathcal{A}$ is $\mathcal{A}$-embeddable. The following basic fact about $\sigma\mathcal{A}$ is proved in ZF alone.

THEOREM 5. *For any structure $\mathcal{A}$ with one binary relation, $\sigma\mathcal{A}$ is not $\mathcal{A}$-embeddable.*

Proof. Otherwise, let $f: \sigma A \to A$ be an embedding. Define $s: \mathrm{Ord} \to A$ by $s(\alpha) = f(s \restriction \alpha)$. Then s is well-defined and one-to-one, a contradiction.

Let $\mathcal{S}(A)$ denote the set of all one-to-one mappings from ordinals into A, and let $\mathcal{W}(A)$ denote the set of all well-orderable subsets of A. We consider $\mathcal{S}(A)$ and $\mathcal{W}(A)$ as partially ordered sets under the ordering $\subseteq$. It is clear that $\mathcal{S}(A)$ is $\mathcal{W}(A)$-embeddable via the mapping $s \mapsto \mathrm{range}(s)$, and that $\langle \mathcal{S}(A), \subseteq\rangle = \sigma\langle A, A^2\rangle$. So the following ZF-results are all consequences of Theorem 5.

COROLLARY 6. *There is no $f: \mathcal{W}(A) \to A$ such that $f(C) \neq f(D)$ for all $C \subset D$ in $\mathcal{W}(A)$.*

COROLLARY 7 (Galvin, Laver). *$\mathscr{S}(\varkappa)\to(\alpha)^1_\varkappa$ for all $\varkappa\geq\omega$ and $\alpha<\varkappa^+$.*

COROLLARY 8 (Galvin). *$\mathscr{P}(\varkappa)\to(\alpha)^1_\varkappa$ for all $\varkappa\geq\omega$ and $\alpha<\varkappa^+$.*

A structure related to $\mathscr{S}(A)$ and $\mathscr{W}(A)$ seems to have been first considered by F. Hartogs [12] who, via a similar argument, proved the weaker result that there is no one-to-one mapping from the height of $\mathscr{S}(A)$ into A. Corollary 6 improves a result of A. Tarski [23] to the effect that there is no one-to-one $f\colon \mathscr{W}(A)\to A$. The case $\varkappa=\omega$ of Corollary 7 was first proved by R. Laver (unpublished) answering a question of Galvin. The general result was since then proved by Galvin (unpublished). These results are included here with their kind permission. Let us note that Galvin's original proofs of Corollaries 7 and 8 used the Axiom of choice which from now on will be assumed in this paper.

If P is a partially ordered set then an element s of σP is uniquely determined by its range. So in this case we may and will consider σP to be the set of all well-ordered chains of P. The ordering on σP will be denoted by $\leq\cdot$ where $s\leq\cdot\, t$ means that s is an initial part of t. The operation σP for P a partially ordered set was first considered by D. Kurepa ([13], [15]) who proved Theorem 5 for $\mathscr{A}$ a partially ordered set ([14]; see also the paper of S. Ginsburg referred to in [15]). He also proved that $\sigma\mathbf{Q}$ is a nonspecial tree ([15]). Note that this is an immediate consequence of the fact that $\sigma\mathbf{Q}$ is not $\mathbf{Q}$-embeddable since $\mathbf{Q}$-embeddability for partially ordered sets is the same as being the union of $\leq\aleph_0$ antichains which is again an old result of Kurepa ([15]; see also [25]). The next result from [25; Theorem 5] is one of the key tools in our proofs of Theorems 1–4. Since [25] contains only the proof of the countable case of this result, for the convenience of the reader we sketch the proof of the general theorem.

THEOREM 9. *The following are equivalent for every partially ordered set P.*

(1) $P\not\to(\varkappa)^1_\varkappa$.

(2) *σP is $\mathscr{P}(\varkappa)$-embeddable.*

(3) *$\sigma' P$ is the union of $\leq\varkappa$ antichains.*

Proof. Only the implication (3)$\Rightarrow$(1) requires a proof (see [25]). Suppose $\sigma' P=\bigcup_{\alpha<\varkappa}A_\alpha$ where each A_α is an antichain of $\sigma' P$. For $\alpha<\varkappa$, let

$$B_\alpha=\{s\in\sigma' P\colon t\notin A_\alpha \text{ for all } t\in\sigma' P \text{ with } s<\cdot\, t\}.$$

Note that $A_\alpha\subseteq B_\alpha$ for all α. Let $\ll$ be a fixed well-ordering of $\sigma' P$. Now for each $a\in P$ we define $t_\alpha(a)\in\sigma' P$ by induction on $\alpha<\varkappa$ as follows. Suppose $t_\beta(a)$ has been defined for all $\beta<\alpha$ such that $t_\gamma(a)\leqslant\cdot\, t_\beta(a)$ and $\max t_\beta(a)\leqslant_P a$ for all $\gamma\leqslant\beta<\alpha$. Define

$$T_\alpha(a)=\{t\in\sigma'P\colon \max t\leqslant_P a \text{ and } t_\beta(a)\leqslant\cdot\, t \text{ for all } \beta<\alpha\}.$$

Then $T_\alpha(a)\neq\emptyset$ since $(\bigcup_{\beta<\alpha} t_\beta(a))\cup\{a\}\in T_\alpha(a)$. Let $t_\alpha(a)$ be the $\ll$-least member of $T_\alpha(a)\cap B_\alpha$ if this set is nonempty; otherwise let $t_\alpha(a)$ be the $\ll$-least element of $T_\alpha(a)$. This finishes the induction step.

For $a\in P$, let

$$t(a)=\left(\bigcup_{\alpha<\varkappa} t_\alpha(a)\right)\cup\{a\}.$$

Then $t(a)\in\sigma'P$, $\max t(a)=a$ and $t_\alpha(a)\leqslant\cdot\, t(a)$ for all $\alpha<\varkappa$.

Claim 1. If $t(a)\in A_\alpha$, then $t(a)=t_\alpha(a)$.

Proof. Note that $t(a)\in A_\alpha\subseteq B_\alpha$ and $t(a)\in T_\alpha(a)$. Hence $T_\alpha(a)\cap B_\alpha\neq\emptyset$, and so $t_\alpha(a)\in B_\alpha$. Since $t_\alpha(a)\in B_\alpha$ and $t_\alpha(a)\leqslant\cdot\, t(a)\in A_\alpha$, we must have $t(a)=t_\alpha(a)$ by the definition of B_α.

In proving $P\to(\varkappa)^1_\varkappa$ it suffices to show that for all $\alpha<\varkappa$, the set $\{a\in P\colon t(a)\in A_\alpha\}$ contains no chain of type $\varkappa$. So by Claim 1 it suffices to show the following:

Claim 2. For $\alpha<\varkappa$, let $P_\alpha=\{a\in P\colon t(a)=t_\alpha(a)\}$. Then P_α contains no chain of order type α^+.

Proof. Otherwise, let $\lambda=\alpha^+$ and let $\{a_\xi\colon \xi<\lambda\}$ be the increasing enumeration of a chain from P_α.

Note that $T_0(a_\xi)\subseteq T_0(a_\eta)$ for $\xi<\eta<\lambda$. Thus $t_0(a_\eta)\ll t_0(a_\xi)$. So there is a $\xi_0<\lambda$ and $t_0\in\sigma'P$ such that $t_0(a_\xi)=t_0$ for all $\xi_0\leqslant\xi<\lambda$. Now $\xi_0\leqslant\xi<\eta<\lambda$ implies $T_1(a_\xi)\subseteq T_1(a_\eta)$ and $t_1(a_\eta)\ll t_1(a_\xi)$. So we can find $\xi_0\leqslant\xi_1<\lambda$ and $t_1\in\sigma'P$ such that $t_1(a_\xi)=t_1$ for all $\xi_1\leqslant\xi<\lambda$. Continuing in this way, we obtain sequences $\langle\xi_\beta\colon\beta\leqslant\alpha\rangle$ and $\langle t_\beta\colon\beta\leqslant\alpha\rangle$ such that $t_\beta(a_\xi)=t_\beta$ for all $\xi_\beta\leqslant\xi<\lambda$. Let $\bar\xi=\sup\{\xi_\beta\colon\beta\leqslant\alpha\}$. Then $\bar\xi<\lambda$ and $t(a_\xi)=t_\alpha(a_\xi)=t_\alpha$ for all $\bar\xi\leqslant\xi<\lambda$. But this contradicts the fact that $\max t(a)=a$ for all $a\in P$. This finishes the proof of Claim 2 and also the proof of Theorem 9.

Theorem 9 is saying that $P\to(\varkappa)^1_\varkappa$ is equivalent to $\sigma'P\to(\varkappa)^1_\varkappa$ for any partially ordered set P. Since $\sigma'P$ is P-embeddable this shows that in proving Theorems 1–4 we may restrict ourselves to trees. This will be an essential point in our proofs of these results.

The next result shows that in considering partition relations for partially ordered sets P with the property $P \to (\varkappa)^1_\lambda$ we may restrict ourselves to the case $\lambda \geqslant \varkappa$. The proof of this result is contained in the above proof of Theorem 9.

THEOREM 10. $P \to (\varkappa)^1_\varkappa$ *iff* $P \to (\alpha^+)^1_{\alpha<\varkappa}$.

THEOREM 11. $P \to (\varkappa)^1_\varkappa$ *iff* $P \to (\varkappa)^1_{\mathrm{cf}\,\varkappa}$.

Let us also mention the following unpublished result of Galvin which is included here with his kind permission and which is an immediate consequence of Theorem 9.

THEOREM 12 (Galvin). *If* $\lambda \geqslant \varkappa \geqslant \omega$ *and if* $P \to (\varkappa)^1_\lambda$ *then* $P \to (\alpha)^1_\lambda$ *for all* $\alpha < \varkappa^+$.

Probably the most natural examples of partially ordered sets P such that $P \to (\omega)^1_\omega$ are ω_1, $\mathbf{R}$, $\sigma\mathbf{Q}$, $\sigma\mathbf{R}$, etc. In general, the most natural examples of partially ordered sets P such that $P \to (\varkappa)^1_\varkappa$ are $\varkappa^+$, $\mathscr{P}(\varkappa)$, and $\mathscr{S}(\varkappa)$. We shall see later that many partition relations for partially ordered sets P such that $P \to (\varkappa)^1_\lambda$ follow from the corresponding partition relations for partially ordered sets P such that $P \to (\varkappa)^1_\varkappa$.

§ 2. Constructing large homogeneous chains

Let $\mathfrak{p}$ be the least cardinal $\varkappa$ for which the following proposition does not hold: if $\langle A_\alpha : \alpha < \varkappa \rangle$ is a sequence of subsets of ω such that $\bigcap_{\alpha \in F} A_\alpha$ is infinite for every finite $F \subseteq \varkappa$, then there exists an infinite set $B \subseteq \omega$ such that $B \setminus A_\alpha$ is finite for every $\alpha < \varkappa$ (see [17; p. 154]). In this section we shall prove the relation

$$\text{nonspecial tree} \to (\alpha)^2_k$$

for nonspecial trees of cardinality $< \mathfrak{p}$.

The proof is given in a sequence of lemmas. Some of the lemmas are taken from [1], [10] and [16], but for the convenience of the reader we include proofs.

For each nonzero ordinal $\alpha < \omega_1$ we fix a sequence $\langle \alpha(n) : n < \omega \rangle$ of ordinals such that $\omega^\alpha = \Sigma_{n<\omega} \omega^{\alpha(n)}$ and $\alpha(n) \leqslant \alpha(n+1)$ for $n < \omega$. If $\alpha > 1$, then we choose $\alpha(0) \geqslant 1$. Thus for every well-ordered set A of type ω^α we have fixed decomposition $A = \bigcup_{n<\omega} A_n$ such that $A_m < A_n$ for $m < n$, and $\operatorname{tp} A_n = \omega^{\alpha(n)}$ for $n < \omega$. Let $A^n = \bigcup_{n \leqslant m < \omega} A_m$ for $n < \omega$.

Let $\mathscr{V}$ be a fixed nonprincipal ultrafilter on ω. By induction on $\alpha < \omega_1$, we define a uniform ultrafilter $\mathscr{U}_\alpha(A)$ on every well-ordered set A of type ω^α as follows. If $\alpha = 0$,

$\mathcal{U}_\alpha(A)$ is the unique ultrafilter on A. Let $1 \leq \alpha < \omega_1$ and let A be a well-ordered set of type ω^α. For $B \subseteq A$ we let

$$B \in \mathcal{U}_\alpha(A) \quad \text{iff} \quad \{n : B \cap A_n \in \mathcal{U}_{\alpha(n)}(A_n)\} \in \mathcal{V}.$$

An easy induction on $\alpha < \omega_1$ shows that $\operatorname{tp} B = \omega^\alpha$ for every $B \in \mathcal{U}_\alpha(A)$. The following lemma is taken from [16; p. 1031]. It first appeared implicitly in [1; p. 197].

LEMMA 2. *Assume $\varkappa < \mathfrak{p}$, $\alpha < \omega_1$, and A is a well-ordered set of type ω^α. Let $\langle B_\xi : \xi < \varkappa \rangle$ be a sequence of elements of $\mathcal{U}_\alpha(A)$. Then there is a $B \subseteq A$, with $\operatorname{tp} B = \omega^\alpha$ such that $B \setminus B_\xi$ is a bounded subset of B for every $\xi < \varkappa$.*

Proof. The proof is by induction on α. The case $\alpha = 0$ is trivial. Assume $\alpha > 0$ and that lemma holds for all $\beta < \alpha$. Thus for each $n < \omega$ we can find $C_n \subseteq A_n$ with $\operatorname{tp} C_n = \omega^{\alpha(n)}$ such that $C_n \setminus B_\xi$ is bounded in C_n for every $\xi < \varkappa$ such that $B_\xi \cap A_n \in \mathcal{U}_{\alpha(n)}(A_n)$. Since $\varkappa < \mathfrak{p}$, we can find an infinite $N \subseteq \omega$ which is almost included in each $N_\xi = \{n : B_\xi \cap A_n \in \mathcal{U}_{\alpha(n)}(A_n)\}$ for $\xi < \varkappa$. Now for $\xi < \varkappa$ we define $f_\xi \in {}^\omega\omega$ by

$$f_\xi(n) = \begin{cases} \min\{m : C_n \cap (A_n)^m \subseteq B_\xi\} & \text{if } n \in N_\xi \\ 0 & \text{otherwise.} \end{cases}$$

Since $\varkappa < \mathfrak{p}$, we can find $g \in {}^\omega\omega$ which eventually dominates each f_ξ, for $\xi < \varkappa$. Now for $n \in N$ we define $B_n = C_n \cap (A_n)^{g(n)}$. Let $B = \bigcup_{n \in N} B_n$. The $\operatorname{tp} B = \omega^\alpha$ and $B \setminus B_\xi$ is bounded in B for every $\xi < \varkappa$.

Let T be a fixed nonspecial tree of cardinality $< \mathfrak{p}$, and let $[T]^2 = K_1 \cup \ldots \cup K_k$ be a fixed disjoint partition. We have to show that for every $\alpha < \omega$ there exist an $i \in \{1, \ldots, k\}$ and a chain $A \subseteq T$ such that $\operatorname{tp} A = \omega^\alpha$ and $[A]^2 \subseteq K_i$. Instead of directly constructing arbitrarily large homogeneous chains in T we shall proceed as in [1] and [10] and construct arbitrarily large chains with the following property which is somewhat weaker than being homogeneous. A chain $A \subseteq T$ is called almost homogeneous if whenever $A' \subseteq A$ has type ω^α, then for all $\beta < \alpha$ there are $C, B \subseteq A'$ and $i \in \{1, \ldots, k\}$ such that $\operatorname{tp} C = \omega^\beta$, $\operatorname{tp} B = \omega^\alpha$, $C <_T B$ and $C \times B \subseteq K_i$. Let $\mathcal{H}$ denote the set of all almost homogeneous chains in T. The next lemma shows why it is more convenient to work with almost homogeneous chains.

LEMMA 3 ([10; p. 720]). *Suppose $\langle A_n : n < \omega \rangle$ is a sequence from $\mathcal{H}$ such that $A_m <_T A_n$ for $m < n$. Let $A = \bigcup_{n < \omega} A_n$. If for each $m < \omega$ there exists an $i_m \in \{1, \ldots, k\}$ such that $A_m \times A_n \subseteq K_{i_m}$ for all $n > m$, then $A \in \mathcal{H}$.*

Proof. Let $\beta<\alpha<\omega_1$, $A'\subseteq A$ and $\operatorname{tp} A'=\omega^\alpha$. If for some $n<\omega$, $A'\subseteq A_0\cup\ldots\cup A_n$ then $\operatorname{tp} A'\cap A_i=\omega^\alpha$ for some $i\leq n$, so we can use the fact that $A_i\in\mathscr{H}$. So assume A' is cofinal in A. Choose $C\subseteq A'$ so that $\operatorname{tp} C=\omega^\beta$ and $C\subseteq A_m$ for some $m<\omega$. Let $B=A'\setminus(A_0\cup\ldots\cup A_m)$. Then $\operatorname{tp} B=\omega^\alpha$ and $C\times B\subseteq K_{i_m}$.

The next lemma shows that in order to construct arbitrarily long homogeneous chains in T it suffices to construct arbitrarily long almost homogeneous chains in T.

LEMMA 4 ([10; p. 271]). *For every $\alpha<\omega_1$ there exists $\beta<\omega_1$ such that for every $B\in\mathscr{H}$ with $\operatorname{tp} B=\omega^\beta$ there is an $A\subseteq B$ with $\operatorname{tp} A=\omega^\alpha$ and $[A]^2\subseteq K_i$ for some $i\in\{1,\ldots,k\}$.*

Proof. For $\beta,\alpha_1,\ldots,\alpha_k<\omega_1$, the symbol $\beta\to(\alpha_1,\ldots,\alpha_k)^{\mathscr{H}}$ denotes the statement: for any $B\in\mathscr{H}$ with $\operatorname{tp} B=\omega^\beta$ there exists $i\in\{1,\ldots,k\}$ and $A\subseteq B$ such that $\operatorname{tp} A=\omega^{\alpha_i}$ and $[A]^2\subseteq K_i$. For the proof of Lemma 4 it suffices to prove the following: if $0<\alpha_1,\ldots,\alpha_k<\omega_1$ and $\beta<\omega_1$, and if for any $i\in\{1,\ldots,k\}$ and any $\gamma<\alpha_i$ there exists a $\beta_i(\gamma)<\beta$ such that $\beta_i(\gamma)\to(\alpha_1,\ldots,\alpha_{i-1},\gamma,\alpha_{i+1},\ldots,\alpha_k)^{\mathscr{H}}$, then $\beta\to(\alpha_1,\ldots,\alpha_k)^{\mathscr{H}}$. The details of the proof of the latter statement are left to the reader.

Let Σ denote the set of all nonspecial subtrees of T. For $X\in\Sigma$ let $\Sigma|X$ denote the set $\Sigma\cap\mathscr{P}(X)$. Then for every $X\in\Sigma$ and every $\beta<\omega_1$ we need to construct an almost homogeneous chain $B\subseteq X$ of type ω^β which is *bounded* in X. The boundedness is needed for the purpose of induction, and it is the new and essential difficulty which does not occur in the proofs from [1] and [10]. More particularly, fix $X\in\Sigma$ and $\beta<\omega_1$ and suppose we have proved that for every $Y\in\Sigma|X$ and every $\gamma<\beta$ we can find a bounded almost homogeneous chain in Y of type ω^γ. Using the induction hypothesis and some additional arguments we construct a sequence $\langle B_n\colon n<\omega\rangle$ of members of $\mathscr{H}$ such that $B_n\subseteq X$, $\operatorname{tp} B_n=\omega^{\beta(n)}$, $B_m<_T B_n$ and $B_m\times B_n\subseteq K_{i_m}$ for $n>m$. The new difficulty is that we must construct the sequence $\langle B_n\colon n<\omega\rangle$ in such a way that $B=\bigcup_{n<\omega}B_n$ is a bounded subset of X. At first sight this constraint seems to require some distributivity assumption on the tree T which rules out many examples of nonspecial trees. However, we shall show that such a sequence can be constructed without any additional assumption on T.

LEMMA 5. *Let $X\in\Sigma$ and let $f\colon X\to\mathrm{Ord}$ be such that $s\leq_T t$ implies $f(s)\geq f(t)$. Then f is constant on some $Y\in\Sigma|X$.*

Proof. Let $X'=\{t\in X\colon T^t\cap X$ is a special tree$\}$. Then X' is a special tree and $X_0=X\setminus X'$ has the property that for every $t\in X_0$, $T^t\cap X_0$ is a nonspecial tree. There must be a $t \in X_0$ such that f is constant on $Y = T^t \cap X_0$, since otherwise we would get a decreasing ω-sequence of ordinals.

The next lemma is the main result of this section. It completes the proof of $T \to (\alpha)^2_k$ for all $\alpha < \omega_1$ and $k < \omega$.

LEMMA 6. *For every $\beta < \omega_1$ and $X \in \Sigma$ there exists a bounded subset B of X such that $B \in \mathscr{H}$ and* $\operatorname{tp} B = \omega^\beta$.

Proof. The proof is by induction on β. The case $\beta = 0$ is trivial; so we assume $\beta > 0$. Let $\prec$ be a fixed well-ordering of $\mathscr{H}$.

Fix $t \in X$. By induction on $n < \omega$ we construct, if possible, sequences $\langle B_n(t) : n < \omega \rangle \in {}^\omega\mathscr{H}$ and $\langle i_n(t) : n < \omega \rangle \in {}^\omega\{1, \dots, k\}$ as follows. Let $n < \omega$ and assume that $B_m(t)$ and $i_m(t)$ are defined for every $m < n$. Let

$$\mathscr{H}_n(t) = \{A \in \mathscr{H} : A \subseteq X \cap \hat{t} \text{ and } \operatorname{tp} A = \omega^{\beta(n)} \text{ and } (\forall m < n)(B_m(t) \times A \subseteq K_{i_m(t)})\}.$$

If $\mathscr{H}_n(t) = \emptyset$, we stop the induction. So assume $\mathscr{H}_n(t) \neq \emptyset$. Let $A_n(t)$ be the $\prec$-least element of $\mathscr{H}_n(t)$ and let

$$X_n(t) = \{s \in X : s >_T A_n(t)\}.$$

For $s \in X_n(t)$ we let i_s be the unique $i \in \{1, \dots, k\}$ such that $K_i(s) \cap A_n(t) \in \mathscr{U}_{\beta(n)}(A_n(t))$. By Lemma 2, there exists a $C \subseteq A_n(t)$ such that $\operatorname{tp} C = \omega^{\beta(n)}$ and such that $C \setminus K_{i_s}(s)$ is bounded in C for every $s \in X_n(t)$. Let $C_n(t)$ be the $\prec$-least such C. Let $l_n(t)$ be the least $l < \omega$ such that $(C_n(t))^l \subseteq K_{i_t}(t)$. (Note that $t \in X_n(t)$.) Let $B_n(t) = (C_n(t))^{l_n(t)}$ and let $i_n(t) = i_t$. This completes the inductive definition.

Claim. For some $t \in X$, $B_n(t)$ and $i_n(t)$ are defined for every $n < \omega$.

Proof. Assume the contrary, i.e., that for every $t \in X$ there exists an $n(t) < \omega$ such that $B_m(t)$ and $i_m(t)$ are defined for every $m < n(t)$, but $\mathscr{H}_{n(t)}(t) = \emptyset$. Since $X \in \Sigma$ we can find $Y \in \Sigma | X$ and $n < \omega$ such that $n(t) = n$ for every $t \in Y$.

By induction on $m \leq n$ we define a decreasing sequence $\langle Y_m : m \leq n \rangle$ of members of $\Sigma | Y$ and sequences

$$\langle B_m : m < n \rangle \in {}^n\mathscr{H} \quad \text{and} \quad \langle i_m : m < n \rangle \in {}^n\{1, \dots, k\}$$

such that

$$B_m(t) = B_m \quad \text{and} \quad i_m(t) = i_m \quad \text{for all } m < n \text{ and } t \in Y_{m+1}.$$

Let $Y_0=Y$. Assume $m<n$ and Y_m is defined. Then for every $t\in Y_m$ we have that

$$\mathscr{H}_m(t)=\{A\in\mathscr{H}: A\subseteq X\cap\hat{t} \text{ and } \operatorname{tp}A=\omega^{\beta(m)} \text{ and } (\forall j<m)(B_j\times A\subseteq K_{i_j})\},$$

since $B_j(t)=B_j$ for every $j<m$. Thus we have that $\mathscr{H}_m(t)\subseteq\mathscr{H}_m(u)$ for $t\leqslant_T u$ in Y_m. Hence $A_m(t)\gtrsim A_m(u)$ for $t\leqslant_T u$ in Y_m. By Lemma 5, there exist $Y_{m+1}\in\Sigma|Y_m$ and $A_m\in\mathscr{H}$ such that $A_m(t)=A_m$ for every $t\in Y_{m+1}$. Furthermore we assume that for some $i_m\in\{1,\dots,k\}$ and $l_m<\omega$ we have $i_m(t)=i_m$ and $l_m(t)=l_m$ for every $t\in Y_{m+1}$. Note that $X_m(t)=\{s\in X: s>_T A_m\}$ for every $t\in Y_{m+1}$. Hence for some $C_m\in\mathscr{H}$, $C_m(t)=C_m$ for every $t\in Y_{m+1}$. Hence $B_m(t)=B_m=(C_m)^{l_m}$ for every $t\in Y_{m+1}$. This completes the inductive definition of $\langle Y_m: m\leqslant n\rangle$, $\langle B_m: m<n\rangle$ and $\langle i_m: m<n\rangle$. Thus in particular Y_n is defined and

$$B_m\times Y_n\subseteq K_{i_m} \quad \text{for every } m<n.$$

By the induction hypothesis, we can find a bounded subset A of Y_n such that $A\in\mathscr{H}$ and $\operatorname{tp}A=\omega^{\beta(n)}$. Let $t_0\in Y_n$ be a bound of A. Then $A\subseteq X\cap\hat{t}_0$ and $B_m(t_0)\times A\subseteq K_{i_m(t_0)}$ for every $m<n$, since $B_m(t_0)=B_m$ and $i_m(t_0)=i_m$ for every $m<n$. Hence $A\in\mathscr{H}_n(t_0)$. But this contradicts the fact that $\mathscr{H}_n(t)=\emptyset$ for every $t\in Y$.

Fix a $t\in X$ for which $B_n(t)$ and $i_n(t)$ are defined for every $n<\omega$. Let $B=\bigcup_{n<\omega}B_n(t)$. Then $\operatorname{tp}B=\omega^\beta$ and by the construction $B_m(t)\times B_n(t)\subseteq K_{i_m(t)}$ for every $n>m$. Hence by Lemma 3 we know that $B\in\mathscr{H}$. Since B is bounded by t in X this completes the proof of Lemma 6.

§ 3. Proof of Theorem 1

In this section we finish the proof of Theorem 1 by eliminating the assumption $|T|<\mathfrak{p}$ from § 2.

We say that a partially ordered set $\mathscr{C}$ satisfies the σ-finite chain condition if there is a partition $\mathscr{C}=\bigcup_{n<\omega}\mathscr{C}_n$ such that for every n, $\mathscr{C}_n$ contains no infinite set of pairwise incompatible (in $\mathscr{C}$) elements. The following lemma is well-known.

LEMMA 7. *For every cardinal $\varkappa$ there is a σ-finite chain condition poset $\mathscr{C}$ which forces $\varkappa<\mathfrak{p}$.*

Proof. Let $\bar{A}=\langle A_\alpha: \alpha<\varkappa\rangle$ be a sequence of subsets of ω such that $A_F=\bigcap_{\alpha\in F}A_\alpha$ is infinite for every finite $F\subseteq\varkappa$. Let $\mathscr{C}_{\bar{A}}$ denote the set of all pairs $p=\langle A_p, F_p\rangle$ where A_p and F_p are finite subsets of ω and $\varkappa$, respectively. For $p,q\in\mathscr{C}_{\bar{A}}$ define

$$p \leq q \quad \text{iff} \quad A_p \supseteq A_q,\ F_p \supseteq F_2 \text{ and } A_p \setminus A_q \subseteq A_{F_q}.$$

Then $\mathscr{C}_{\dot{A}}$ is the standard σ-centered poset which forces an infinite subset of ω almost included in any A_α ([17; p. 154]).

Choose a cardinal θ such that $\theta^\varkappa = \theta$. Let $\langle \mathscr{C}_\alpha : \alpha \leq \theta \rangle$ be a finite support interation of posets $\mathscr{C}_{\dot{A}}$ such that $\mathscr{C}_\theta$ forces $\varkappa < \mathfrak{p}$ ([17]). A simple and standard argument shows that $\mathscr{C}_\theta$ satisfies the σ-finite chain condition.

LEMMA 8. *Let $\mathscr{C}$ be a partially ordered set satisfying the σ-finite chain condition, and let P be a partially ordered set such that $P \to (\omega)^1_\omega$. Then P has the property $P \to (\omega)^1_\omega$ in any forcing extension by $\mathscr{C}$.*

Proof. By Theorem 9, we may assume that P is a tree. So let T be a nonspecial tree and let $\langle \dot{T}_n : n < \omega \rangle$ be a $\mathscr{C}$-name for a decomposition of T. Let $\mathscr{C} = \bigcup_{m<\omega} \mathscr{C}_m$ be a decomposition witnessing the σ-finite chain condition of $\mathscr{C}$. For each $t \in T$, we can fix $m_t, n_t \in \omega$ and $p_t \in \mathscr{C}_{m_t}$ such that p_t forces $t \in \dot{T}_{n_t}$. Since T is nonspecial we can find $m, n < \omega$ and an infinite chain $b \subseteq T$ such that $m_t = m$ and $n_t = n$ for all $t \in b$. By the property of $\mathscr{C}_m$ there must be $s < t$ in b such that p_s and p_t are compatible in $\mathscr{C}$. So any extension of p_s and p_t forces $\langle \dot{T}_n : n < \omega \rangle$ not to be an antichain-decomposition of T. This completes the proof.

Now we are ready to finish the proof of

$$\text{nonspecial tree} \to (\alpha)^2_k \text{ for all } \alpha < \omega_1 \text{ and } k < \omega.$$

So let T be a nonspecial tree, let $[T]^2 = K_1 \cup \ldots \cup K_k$ be a given disjoint partition, and let α be a fixed countable ordinal. By Lemmas 7 and 8 let $\mathscr{C}$ be a σ-finite chain condition poset which forces $|T| < \mathfrak{p}$ and $T \to (\omega)^1_\omega$. By § 2 we can find $i \in \{1, \ldots, k\}$, $p \in \mathscr{C}$, and a $\mathscr{C}$-name $\dot{A}$ for a chain of T such that p forces $\operatorname{tp} \dot{A} = \alpha$ and $[\dot{A}]^2 \subseteq K_i$. Let $\dot{h}$ be a $\mathscr{C}$-name for the unique isomorphism of α and $\dot{A}$, and let $\langle \alpha_n : n < \omega \rangle$ be an enumeration of α. By induction on $n < \omega$, choose a decreasing sequence $\langle p_n : n < \omega \rangle$ of elements of $\mathscr{C}$ and a sequence $\langle t_n : n < \omega \rangle$ of elements of T such that $p_0 = p$ and p_{n+1} forces $\dot{h}(\alpha_n) = t_n$. Then for each $m, n < \omega$, $t_m <_T t_n$ iff $\alpha_m < \alpha_n$. Hence $\{t_n : n < \omega\}$ is a chain of T of order type α such that $[\{t_n : n < \omega\}]^2 \subseteq K_i$.

§ 4. Partition relations for trees

The results of § 1 suggest a study of partition relations for trees in order to get corresponding partition relations for partially ordered sets in general. This section is devoted for such a study. It turns out that partition relations for trees are very natural generalizations of partition relations for cardinals and that several well-known partition relations for cardinals are straightforward consequences of the corresponding relations for trees.

Let T be a tree such that $\mathrm{ht}(T)=\varkappa$ is a regular uncountable cardinal. We say that a set $A\subseteq\varkappa$ is nonstationary with respect to T (see [24; p. 251]) iff there exists a regressive mapping $f\colon T\restriction A\to T$ such that $f^{-1}(s)$ is the union of $<\varkappa$ antichains for every $s\in T$. Let

$$NS_T=\{A\subseteq\varkappa\colon A \text{ is nonstationary with respect to } T\}.$$

Note that by the well-known theorem of Neumer [20], if T is a chain of length $\varkappa$ then $NS_T=NS_\varkappa$. It is clear that NS_T is a $\varkappa$-complete ideal on $\varkappa$. Note also that if T' is an initial part of T then $NS_T\subseteq NS_{T'}$. Hence if T has a chain of cardinality $\varkappa$, then $NS_T=NS_\varkappa$ since clearly $NS_\varkappa\subseteq NS_T$.

THEOREM 13 ([24; p. 251]). *NS_T is a normal ideal on $\varkappa$.*

The following result is the key in proving several partition relations for trees.

THEOREM 14. *Assume $\varkappa=\lambda^+$. Then the following are equivalent for every tree T of height $\varkappa$.*

(1) $\varkappa\in NS_T$.

(2) $\{\delta<\varkappa\colon \mathrm{cf}\,\delta=\mathrm{cf}\,\lambda\}\in NS_T$.

(3) *T is the union of $\leq\lambda$ antichains.*

Proof. The implications (3)⇒(1), (3)⇒(2) and (1)⇒(2) are trivial. So we have only to prove (2)⇒(3).

Let $E=\{\delta<\varkappa\colon\mathrm{cf}\,\delta=\mathrm{cf}\,\lambda\}$ and let $f\colon T\restriction E\to T$ be a regressive mapping such that $f^{-1}(s)$ is the union of $\leq\lambda$ antichains for every $s\in T$. For every $s\in T$ we fix a $g_s\colon f^{-1}(s)\to\lambda$ such that $g_s^{-1}(\gamma)$ is an antichain of T for every $\gamma<\lambda$. Also for every $T\setminus\{\emptyset\}$ we fix an enumeration $\langle s_\beta(t)\colon\beta<\lambda\rangle$ of $\{s\in T\colon s\leq_T t\}$. Let $j\colon\lambda\to\lambda\times\lambda\times\lambda$ be a bijection such that $j(\xi)=\langle\alpha,\beta,\gamma\rangle$ implies $\alpha\leq\xi$.

Fix $t\in T\setminus\{\emptyset\}$. By induction on $\xi<\lambda$ we define a sequence $\langle u_\xi(t)\colon\xi<\lambda\rangle$ of members of $\hat{t}\cup\{t\}$ as follows. Let $u_0(t)=\emptyset$. If ξ is a limit ordinal let $u_\xi(t)=\sup\{u_\alpha(t)\colon\alpha<\xi\}$. Suppose $u_\xi(t)$ has been defined. If $u_\xi(t)=t$, then let $u_{\xi+1}(t)=u_\xi(t)$. So assume $u_\xi(t)<_T t$. Let $j(\xi)=\langle\alpha,\beta,\gamma\rangle$. Then $\alpha\leq\xi$ and $s=s_\beta(u_\alpha(t))$ is defined. If the set

$$g_s^{-1}(\gamma)\cap\{v\in T\colon u_\xi(t)<_T v\leq_T t\}$$

is nonempty, let $u_{\xi+1}(t)$ be the unique element of this set. Otherwise let $u_{\xi+1}(t)$ be the minimal element of $\{v\in T: u_\xi(t)<_T v\leq_T t\}$.

Claim 1. For every $t\in T\setminus\{\emptyset\}$ there exists $\xi<\lambda$ such that $u_\xi(t)=t$.

Proof. Otherwise, let $t\in T\setminus\{\emptyset\}$ be such that $u_\xi(t)\neq t$ for every $\xi<\lambda$. Then $\langle u_\xi(t):\xi<\lambda\rangle$ is a strictly increasing sequence from $\hat{t}$. Let $u=\sup\{u_\xi(t):\xi<\lambda\}$, and let $\delta=\mathrm{ht}_T(u)$. Then $\delta\in E$, hence $f(u)=s$ is defined and $s<_T u$. Choose $\alpha<\lambda$ such that $s\leq_T u_\alpha(t)$. Choose $\beta<\lambda$ such that $s=s_\beta(u_\alpha(t))$. Let $\gamma=g_s(u)$ and $\xi=j^{-1}(\alpha,\beta,\gamma)$. Then

$$g_s^{-1}(\gamma)\cap\{v\in T: u_\xi(t)<_T v\leq_T t\}=\{u\}.$$

Hence $u_{\xi+1}(t)=u$ contradicting the fact that $u_{\xi+1}(t)<_T u$.

For $t\in T\setminus\{\emptyset\}$, let $\xi(t)=\min\{\xi<\lambda: u_\xi(t)=t\}$. By Claim 1, $\xi(t)$ exists for every $t\in T\setminus\{\emptyset\}$. Let $X_\xi=\{t\in T\setminus\{\emptyset\}:\xi(t)=\xi\}$. Then $T\setminus\{\emptyset\}=\bigcup_{\xi<\lambda}X_\xi$, so the following claim finishes the proof of Theorem 14.

Claim 2. For each $\xi<\lambda$, X_ξ contains no chain of cardinality ξ^+.

Proof. Assume the contrary and let $\xi_0<\lambda$ be such that X_{ξ_0} contains a chain A of type $\theta=\aleph_0+|\xi_0|^+$. Let $\langle t_\delta:\delta<\theta\rangle$ be the increasing enumeration of A.

By induction on $\xi\leq\xi_0$ we show that the sequence $\langle u_\xi(t_\delta):\delta<\theta\rangle$ is eventually constant. If $\xi=0$ or if ξ is a limit ordinal, the proof of this fact is straightforward. So we assume $\langle u_\xi(t_\delta):\delta<\theta\rangle$ is eventually constant and prove that $\langle u_{\xi+1}(t_\delta):\delta<\theta\rangle$ is eventually constant. Let δ_0 and u be such that $u_\xi(t_\delta)=u$ for every $\delta_0\leq\delta<\theta$. Let $j(\xi)=\langle\alpha,\beta,\gamma\rangle$.

Case I. There is a $\delta_1<\theta$ such that $\delta_1\geq\delta_0$ and

$$g_s^{-1}(\gamma)\cap\{v\in T: u<v\leq t_{\delta_1}\}\neq\emptyset,$$

where $s=s_\beta(u_\alpha(t_{\delta_1}))$. Then the unique element of this interesection is equal to $u_{\xi+1}(t_\delta)$ for every δ such that $\delta_1\leq\delta<\theta$.

Case II. Otherwise. Then for every δ with $\delta_0\leq\delta<\theta$ we have either $u_{\xi+1}(t_\delta)=u_\xi(t_\delta)$, or else $u_{\xi+1}(t_\delta)$ is the minimal element of $\{v\in T: u<v\leq t_{\delta_0}\}$. Hence $\langle u_{\xi+1}(t_\delta):\delta_0\leq\delta<\theta\}$ is constant.

In particular, $\langle u_{\xi_0}(t_\delta):\delta<\theta\rangle$ is eventually constant. But this is a contradiction since $u_{\xi_0}(t_\delta)=t_\delta$ for every $\delta<\theta$. This contradiction completes the proof of Claim 2.

THEOREM 15. *Assume* $\lambda \geq \aleph_0$, $\theta \geq 2$ *and* $\varkappa = \theta^\lambda$. *Then for all* $\gamma < \mathrm{cf}\,\lambda$,

$$\textit{non-}\varkappa\textit{-special tree} \to (\textit{non-}\varkappa\textit{-special tree},\ (\mathrm{cf}\,\lambda+1)_\gamma)^2.$$

Proof. Let T be a non-$\varkappa$-special tree. We may assume that $\mathrm{ht}(T) = \varkappa^+$. Let $\gamma < \lambda$ and let

$$[T]^2 = K_0 \cup \bigcup_{i<\gamma} J_i$$

be a given partition. Let $E = \{\delta < \varkappa^+ : \mathrm{cf}\,\delta = \mathrm{cf}\,\varkappa\}$. Then by Theorem 14, $E \notin NS_T$.

For every $t \in T \restriction E$ and for every $i < \gamma$, let $S_i(t)$ be a $\subseteq$-maximal subset of $\hat{t}$ such that $[S_i(t) \cup \{t\}]^2 \subseteq J_i$. If for some $t \in T \restriction E$ and $i < \gamma$ we have that $|S_i(t)| \geq \mathrm{cf}\,\lambda$, we are done, since $\mathrm{tp}(S_i(t) \cup \{t\}) \geq \mathrm{cf}\,\lambda + 1$ and $[S_i(t) \cup \{t\}]^2 \subseteq J_i$. So we may assume that $|S_i(t)| < \mathrm{cf}\,\lambda$ for every $t \in T \restriction E$ and every $i < \gamma$. Since $\mathrm{cf}\,\varkappa \geq \mathrm{cf}\,\lambda$, for every $t \in T \restriction E$ there exists $f(t) <_T t$ such that

$$\bigcup_{i<\gamma} S_i(t) <_T f(t).$$

By Theorem 14 we can find an $s \in T$ such that $f^{-1}(s)$ is not the union of $\leq \varkappa$ antichains. Since $\varkappa^{(\mathrm{cf}\,\lambda)} = \varkappa$, we can find $\langle S_i : i < \gamma \rangle$ and a non-$\varkappa$-special subtree X of $f^{-1}(s)$ such that

$$S_i(t) = S_i \quad \text{for all } t \in X \text{ and } i < \gamma.$$

The following claim finishes the proof of Theorem 15.

Claim. $[X]^2 \subseteq K_0$.

Proof. Otherwise there exists $u <_T t$ in X and $i < \gamma$ such that $\{u, t\} \in J_i$. Since $S_i(u) = S_i(t) = S_i$, we have that $u \notin S_i(t)$ and $[S_i(t) \cup \{u\} \cup \{t\}]^2 \subseteq J_i$, contradicting the $\subseteq$-maximality of $S_i(t)$.

COROLLARY 16 (see [5; § 17]). *Assume* $\lambda \geq \aleph_0$, $\theta \geq 2$, *and* $\gamma < \mathrm{cf}\,\lambda$. *Then*

$$(\theta^\lambda)^+ \to ((\theta^\lambda)^+, (\mathrm{cf}\,\lambda+1)_\gamma)^2.$$

The following consequence of Theorem 14, which we mention without proof, generalizes the well-known Δ-system lemma for cardinals ([5]).

THEOREM 17. *Assume* $\varkappa^\lambda = \varkappa$. *Let* T *be a non-*$\varkappa$*-special tree and let* $\langle F_t : t \in T \rangle$ *be a sequence of sets such that* $|F_t| < \lambda$ *for all* $t \in T$. *Then there exist a non-*$\varkappa$*-special subtree* T' *of* T *and a set* F *such that* $F_s \cap F_t = F$ *for all* $s, t \in T'$ *with* $s <_T t$.

It is clear that Theorem 4 follows directly from Theorem 15 using the results of § 1, so we shall now restrict our attention to some further applications of these two results.

THEOREM 18. *Let $\varkappa$ be a regular cardinal and let P be a partially ordered set such that $P\to(\varkappa)^1_{2^\varkappa}$. Then*

$$P\to(\alpha,(\varkappa+1)_\gamma)^2 \quad \textit{for all } \alpha<\varkappa^+ \textit{ and } \gamma<\varkappa.$$

Proof. Let $\lambda=2^\varkappa$ and let $\mathscr{C}_{\varkappa\lambda}$ be the standard $\varkappa$-closed poset which collapses λ to $\varkappa$. Then $|\mathscr{C}_{\varkappa\lambda}|=\lambda$, and so $\mathscr{C}_{\varkappa\lambda}$ forces $\varkappa=2^\varkappa=\lambda$ and $P\to(\varkappa)^1_\varkappa$. So by Theorem 4, $\mathscr{C}_{\varkappa\lambda}$ forces $P\to(\alpha,(\varkappa+1)_\gamma)^2$ for all $\alpha<\varkappa^+$ and $\gamma<\varkappa$. Since $\mathscr{C}_{\varkappa\lambda}$ is $\varkappa$-closed, an argument similar to that of § 3 shows that $P\to(\alpha,(\varkappa+1)_\gamma)^2$ is really true for all $\alpha<\varkappa^+$ and $\gamma<\varkappa$.

COROLLARY 19. $P\to(\varkappa^+)^1_{2^\varkappa}$ implies $P\to(\alpha,(\varkappa^++1)_\varkappa)^2$ *for all* $\alpha<\varkappa^{++}$.

Theorem 15 is not the strongest result that can be proved using the same methods. Namely, a similar proof shows that we can also have an analogue of Theorem 17.1 from [5].

Let us now consider the following corollary of Theorem 15.

COROLLARY 20. *Nonspecial tree$\to$(nonspecial tree, $\omega+1)^2$.*

Since ω_1 is a nonspecial tree, an immediate consequence of Corollary 20 is the well-known relation $\omega_1\to(\omega_1,\omega+1)^2$ proved by Erdös and Rado in [8; p. 459]. Concerning this Erdös–Rado result Hajnal [11; p. 283] showed that CH implies $\omega_1\not\to(\omega_1,\omega+2)^2$. On the other hand, the author [27] found a model of set theory in which $\omega_1\to(\omega_1,\alpha)^2$ holds for all $\alpha<\omega_1$. Hence it is natural to ask the following question.

Problem. Is nonspecial tree$\to$(nonspecial tree, $\omega+2)^2$ consistent?

The negative result of Hajnal has the following generalization.

THEOREM 21. *Assume* MA. *Suppose T is a nonspecial tree of cardinality $2^{\aleph_0}$ with no uncountable chains. Then there is a partition $[T]^2=K_0\cup K_1$ so that*

(1) *there is no nonspecial tree $X\subseteq T$ with $[X]^2\subseteq K_0$,*

(2) *there are no sets $A,B\subseteq T$ so that $A<_T B$, $\operatorname{tp}A=\omega$, $|B|=2$, and $A\times B\subseteq K_1$.*

COROLLARY 22. *Assume* MA. *Then*

$$\textit{nonspecial tree}\not\to(\textit{nonspecial tree},\ \omega+2)^2.$$

Proof of Theorem 21. Let T be a nonspecial tree of cardinality $\varkappa=2^{\aleph_0}$ with no uncountable chains. We may assume that the underlying set of T is $\varkappa$ and that $\alpha<_T\beta$ implies $\alpha<\beta$. Let $\langle A_\xi: \xi<\varkappa\rangle$ be a fixed enumeration of $[\varkappa]^{\aleph_0}$.

Using MA and induction on $\alpha<\varkappa$ we construct sets $S_\alpha\subseteq\hat{\alpha}=\{\beta:\beta<_T\alpha\}$ such that:

(i) $S_\alpha\cap S_\beta$ is finite for $\alpha\neq\beta$.

(ii) If $\xi<\alpha$ and if $A_\xi\cap\hat{\alpha}$ is not covered by finitely many S_β with $\beta<\alpha$, then $S_\alpha\cap A_\xi\neq\emptyset$.

Since $\hat{\alpha}$ is countable, MA applied to a standard σ-centered poset (see [17; p. 154]) will give us $S_\alpha\subseteq\hat{\alpha}$ satisfying the $<\varkappa$ requirements of (i) and (ii).

Define $[T]^2=K_0\cup K_1$ by

$$\{\beta,\alpha\}\in K_1 \quad\text{iff}\quad \beta\in S_\alpha.$$

For the proof of Theorem 21 it suffices to prove the following Claim.

Claim. There is no nonspecial subtree $X\subseteq T$ with $[X]^2\subseteq K_0$.

Proof. Suppose to the contrary that $X\subseteq T$ is a nonspecial tree such that $[X]^2\subseteq K_0$. We may assume that $T^\alpha\cap X$ is a nonspecial tree for every $\alpha\in X$. (Here $T^\alpha=\{\beta\in T: \alpha\leq_T\beta\}$.)

For $\beta\in X$, we define

$$C_\beta=\{\alpha\in T: S_\alpha\cap\hat{\beta}\cap X \text{ is infinite}\}.$$

Fact 1. C_β is finite for every $\beta\in X$.

Proof. Otherwise, let $\beta\in X$ be such that C_β is infinite. Choose $\xi<\varkappa$ such that $A_\xi=\hat{\beta}\cap X$. Since MA holds and since $T^\beta\cap X$ is a nonspecial tree, it has cardinality $\varkappa$ (see [2]). So we can find $\alpha\in T^\beta\cap X$ such that $\xi<\alpha$. By (i) and by the fact that C_β is infinite, it follows that $A_\xi\cap\hat{\alpha}$ is not covered by finitely many S_γ with $\gamma<\alpha$. Hence by (ii), $S_\alpha\cap A_\xi\neq\emptyset$. This contradicts the fact that $[X]^2\subseteq K_0$.

For $n<\omega$, let $X_n=\{\beta\in X: |C_\beta|=n\}$. Then by the Fact 1, $X=\bigcup_{n<\omega}X_n$. The next fact finishes the proof of the claim, since we are assuming that X is a nonspecial tree.

Fact 2. For each n, X_n is a special tree.

Proof. Suppose to the contrary that for some n, X_n is not a special tree. Pick $\alpha\in X_n$ such that $T^\alpha\cap X_n$ is a nonspecial tree. Let $\langle\beta_1,\ldots,\beta_n\rangle$ be the increasing enumeration of C_α. Choose $\alpha'\in T^\alpha\cap X_n$ such that $T^{\alpha'}\cap X_n$ is a nonspecial tree and such that $A=\{\gamma\in X_n:\beta_n<\gamma<_T\alpha'\}$ is infinite. Choose a $\xi<\varkappa$ such that $A_\xi=A$. Since $T^{\alpha'}\cap X_n$ is nonspecial, it has cardinality $\varkappa$ (see [2]). Hence we can find $\alpha''\in T^{\alpha'}\cap X_n$ such that $\xi<\alpha''$. Note that $C_{\alpha''}=\{\beta_1,\dots,\beta_n\}$, hence $S_\beta\cap A_\xi$ is finite for every $\beta<\alpha''$. By (ii), this means that $S_{\alpha''}\cap A_\xi\neq\emptyset$ contradicting the fact that $[X_n]^2\subseteq K_0$. This completes the proof.

Let us remark that the assumption that T has no uncountable chains cannot be dropped from Theorem 21, since in the model of $\omega_1\to(\omega_1,\alpha)^2$ constructed in [27] Martin's axiom holds.

§ 5. Proof of Theorem 2

Using the trick from the proof of Theorem 18 and results of § 1 it suffices to show under the assumption $\varkappa^{\aleph}=\varkappa$ and $\lambda^{\lambda}<\varkappa$ that

$$\text{non-}\varkappa\text{-special tree}\to(\varkappa+\xi)^2_2 \text{ for all } \xi<\lambda.$$

So let T be a fixed non-$\varkappa$-special tree, let $[T]^2=K_0\cup K_1$ be a given partition, and ξ be a given ordinal such that $\xi<\lambda<\varkappa$. We may assume that T contains no homogeneous non-$\varkappa$-special subtree, since otherwise we are done.

LEMMA 9. *Either T contains a homogeneous chain of type $\varkappa+\xi$ or else there is a sequence $\langle A_i: i<\xi\rangle$ of chains of T such that*

(1) $\operatorname{tp}A_i=\lambda$ *and* $[A_i]^2\subseteq K_0$

(2) $A_i<_TA_j$ *and* $A_i\times A_j\subseteq K_1$ *for* $i<j$.

Proof. Let $E=\{\delta<\varkappa^+:\operatorname{cf}\delta=\varkappa\}$. Let $<$ be a well-ordering of the set of all 0-homogeneous chains of T.

Fix $t\in T\restriction E$. By induction on $i<\xi$ we define, if possible 0-homogeneous sets $A_i^t\subseteq\hat{t}$ such that:

(a) $A_i^t\times\{t\}\subseteq K_1$, and $\operatorname{tp}A_i^t=\lambda$.

(b) $A_i^t<_TA_j^t$ and $A_i^t\times A_j^t\subseteq K_1$ for $i<j$.

Suppose $j<\xi$ is such that A_i^t is defined for every $i<j$. Let A_j^t be the $<$-least 0-homogeneous chain A of type λ such that $A\subseteq\hat{t}\cap K_1(t)$ and such that $A_i^t<_TA$ and $A_i^t\times A\subseteq K_1$ for all $i<j$, if such a set exists; otherwise we stop the induction at stage j.

If for some $t\in T\restriction E$ the induction has never stopped, we are done. Otherwise, for every $t\in T\restriction E$ there is a $j(t)<\xi$ such that A_i^t was defined for every $i<j(t)$ but $A_{j(t)}^t$ was

not. By Thoerem 14 and by the fact that $\varkappa^{\aleph}=\varkappa$, we can find a non-$\varkappa$-special subtree $X\subseteq T\upharpoonright E$, $j_0<\xi$ and $\langle A_i: i<j_0\rangle$ so that $j(t)=j_0$ and $A_i^t=A_i$ for all $t\in X$ and $i<j_0$. Choose $t_0\in X$ such that $Y=T^{t_0}\cap X$ is not $\varkappa$-special and there exists a 0-homogeneous set $B\subseteq\hat{t}_0\cap X$ such that $\operatorname{tp} B=\varkappa$. Such t_0 and B exist by Theorem 15 and our assumption that T contains no homogeneous non-$\varkappa$-special subtree.

Claim. $|K_1(t)\cap B|<\lambda$ for all $t\in Y$.

Proof. This claim follows from the fact that for every $t\in Y$, $A_{j_0}^t$ was not defined.

By $\varkappa^\lambda=\varkappa$, we can find a non-$\varkappa$-special tree $Z\subseteq Y$ and $C\subseteq B$ such that $K_1(t)\cap B=C$ for all $t\in Z$. Let $D=B\setminus C$. Then $\operatorname{tp} D=\varkappa$ and $D\times Z\subseteq K_0$. By Theorem 15 there is a 0-homogeneous chain $E\subseteq Z$ of type ξ. Then $D\cup E$ is a 0-homogeneous chain of type $\varkappa+\xi$. This finishes the proof of Lemma 9.

For each $t\in T\upharpoonright E$ we fix, if possible, a sequence $\langle B_i^t: i<\xi\rangle$ of chains from T^t such that

(c) $\operatorname{tp} B_i^t=\lambda$ and $[B_i^t]^2\subseteq K_0$,
(d) $B_i^t<_T B_j^t$ and $B_i^t\times B_j^t\subseteq K_1$ for $i<j$.

By Lemma 9, we may assume that the set of all $t\in T\upharpoonright E$ for which $\langle B_i^t: i<\xi\rangle$ does not exist is $\varkappa$-special. Now for every $t\in T\upharpoonright E$ for which $\langle B_i^t: i<\xi\rangle$ exists, and for every $i<\xi$, we fix a $\subseteq$-maximal set $S_i^t\subseteq\hat{t}$ such that:

(e) $[S_i^t\cup\{t\}]^2\subseteq K_0$,
(f) $\operatorname{tp}(K_0(s)\cap B_i^t)\geq\xi$ for every $s\in S_i^t$.

Assume first that there exist $t\in T\upharpoonright E$ and $i<\xi$ such that $|S_i^t|\geq\varkappa$. Since $\lambda^\lambda<\varkappa$, we can find $C\subseteq S_i^t$ and $D\subseteq B_i^t$ such that $\operatorname{tp} C=\varkappa$, $\operatorname{tp} D=\xi$ and for every $s\in C$, the inclusion $D\subseteq K_0(s)\cap B_i^t$ holds. Hence $C\cup D$ is a 0-homogeneous chain of type $\varkappa+\xi$, so we are done. Thus we may assume $|S_i^t|<\varkappa$ for every $t\in T\upharpoonright E$ and $i<\xi$ for which S_i^t is defined. By Theorem 14 and by the fact that $\varkappa^{\aleph}=\varkappa$, we can find a non-$\varkappa$-special subtree $X\subseteq T\upharpoonright E$ and a sequence $\langle S_i: i<\xi\rangle$ such that for every $t\in X$, $\langle B_i^t: i<\xi\rangle$ exists and

$$S_i^t=S_i \quad \text{for all } t\in X \text{ and } i<\xi.$$

By Theorem 15, we can find a chain $A\subseteq X$ of type $\varkappa$ such that $[A]^2\subseteq K_1$ and such that $Y=\{t\in X: t>_T A\}$ is not a $\varkappa$-special tree. Now we consider the following two cases.

Case I. For some $t\in Y$, $|K_0(t)\cap A|=\varkappa$. Let $s\in K_0(t)\cap A$. Then for every $i<\xi$ the set $S_i\cup\{s\}$ satisfies the first requirement (e) from the definitions of S_i^t, i.e., $[S_i\cup\{s,t\}]^2\subseteq K_0$. By the maximality of S_i^t we have that s does not satisfy the condition (f), i.e., $\operatorname{tp}(K_0(s)\cap B_i^t)<\xi$. Since $\lambda^\lambda<\varkappa$, we can find $C\subseteq K_0(t)\cap A$ with $|C|=\varkappa$ and $\langle D_i\colon i<\xi\rangle$ such that $K_0(s)\cap B_i^t=D_i$ for every $s\in C$ and $i<\xi$. For each $i<\xi$, we choose $t_i\in B_i^t\setminus D_i$ arbitrarily. Then $C\cup\{t_i\colon i<\xi\}$ is a 1-homogeneous chain of type $\varkappa+\xi$. Hence we are done.

Case II. For all $t\in Y$, $|K_0(t)\cap A|<\varkappa$. Since $\varkappa^\xi=\varkappa$ and since Y is a non-$\varkappa$-special tree, we can find a non-$\varkappa$-special subtree $Z\subseteq Y$ and a set $B\subseteq A$ such that $K_0(t)\cap A=B$ for all $t\in Z$. Let $C=A\setminus B$. Then $\operatorname{tp}C=\varkappa$, $[C]^2\subseteq K_1$ and $C\times Z\subseteq K_1$. By Theorem 15, there is a chain $D\subseteq Z$ such that $\operatorname{tp}D=\xi$ and $[D]^2\subseteq K_1$. Then $C\cup D$ is a 1-homogeneous chain of type $\varkappa+\xi$, so we are done also in this case.

This completes the proof of Theorem 2.

§ 6. Constructing end-homogeneous chains

In this section we give a proof of Theorem 4 and mention some applications of this theorem. So let $\lambda\geq\aleph_0$, $\theta\geq\aleph_0$, $\varkappa=\theta^\lambda$, and let P be a partially ordered set such that $P\to(\varkappa)^1_\varkappa$. We shall prove that $P\to\langle\lambda+1\rangle^{<\omega}_\theta$.

Let $f\colon[P]^{<\omega}\to\theta$ be a given partition. We consider the following two cases

Case I. For some $\lambda_0<\lambda$, $\theta^{\lambda_0}=\varkappa$. Then $\varkappa^\lambda=\varkappa$. By results of § 1 we may assume that $P=T$, where T is a non-$\varkappa$-special tree of height $\varkappa^+$. Let $E=\{\delta<\varkappa^+\colon \operatorname{cf}\delta=\operatorname{cf}\varkappa\}$. Then by Theorem 14, $E\notin NS_T$. For every $t\in T\restriction E$ we fix a $\subseteq$-maximal subset $S_t\subseteq\hat{t}$ such that $S_t\cup\{t\}$ is an end-homogeneous chain with respect to f, i.e., $f(x\frown s)=f(x\frown s')$ for all $x\in[S_t\cup\{t\}]^{<\omega}$ and $s,s'\in S_t\cup\{t\}$ with $\max(x)<_T s,s'$. If $|S_t|\geq\lambda$ for some $t\in T\restriction E$, we are done since $\operatorname{tp}(S_t\cup\{t\})\geq\lambda+1$. So we may assume $|S_t|<\lambda$ for all $t\in T\restriction E$. Since $\operatorname{cf}\varkappa\geq\lambda$, for every $t\in T\restriction E$ we can find $h(t)<_T t$ such that S_t is bounded in $\hat{t}$ by $h(t)$. By Theorem 14 we can find an $s\in T$ such that $h^{-1}(s)$ is not the union of $\leq\varkappa$ antichains. Since $\varkappa^\lambda=\varkappa$, we can find $S\subseteq\hat{s}$ and a non-$\varkappa$-special subtree $X\subseteq h^{-1}(s)$ such that

$$S_t=S\quad\text{for all } t\in X.$$

Furhermore, we may assume that

$$f(x\frown t)=f(x\frown t')\quad\text{for all } x\in[S]^{<\omega}\text{ and } t,t'\in X.$$

Pick $u <_T t$ in X. Then by the above properties of S and X we have that $S_t \cup \{u\} \cup \{t\}$ is an end-homogeneous chain contradicting the maximality of S_t.

Case II. $\theta^{\lambda'} < \varkappa$ for all $\lambda' < \lambda$. Fix $t \in \sigma P$. We shall define the standard tree ordering $<_t$ on t induced by the partition $f \upharpoonright [t]^{<\omega}$ (see [5; § 18]). In what follows $a, b, c, \ldots$ are elements of t and $<$ is the restriction of $<_P$ to t. If $a \in t$, then $\hat{a}$ denotes the set $\{b \in t: b < a\}$. For every $a \in t$ we define $h_a: \hat{a} \to 2$ by induction on $b < a$. Assume $b < a$ and $h_a(c)$ has been defined for every $c < b$. Let $A_{a,b} = \{c < b: h_a(c) = 1\}$. Then we put

$$h_a(b) = 1 \quad \text{iff} \quad f(x \frown b) = f(x \frown c) \text{ for all } x \in [A_{a,b}]^{<\omega}.$$

Let $A_a = \{b < a: h_a(b) = 1\}$. Now for $a, b \in t$ we define

$$b <_t a \quad \text{iff} \quad b \in A_a.$$

Then $<_t$ is a tree ordering on t since clearly $b \in A_a$ implies $A_b = A_a \cap \hat{a} = A_{a,b}$. Moreover, $A_a = \{b \in t: b <_t a\}$. The proofs of the following facts are straightforward.

Fact 1. Let $t \in \sigma P$ and let a and b have the same limit height in $\langle t, <_t \rangle$. Then $\{c \in t: c <_t a\} = \{c \in t: c <_t b\}$ implies $a = b$.

Fact 2. If $t \in \sigma P$ and if $a \in t$ has height α in $\langle t, <_t \rangle$, then a has at most $\theta^{|\omega+\alpha|}$ immediate $<_t$-successors.

Fact 3. If $s, t \in \sigma P$ and if $s \leqslant\cdot t$, then $<_s = <_t \upharpoonright s$, and moreover $\langle s, <_s \rangle$ is an initial part of $\langle t, <_t \rangle$.

Claim. For some $t \in \sigma P$, the height of $\langle t, <_t \rangle$ is $\geqslant \lambda + 1$.

Proof. Suppose to the contrary that $\operatorname{ht} \langle t, <_t \rangle \leqslant \lambda$ for every $t \in \sigma P$. For $\alpha < \lambda$, let $\varkappa_\alpha = (\theta^{|\omega+\alpha|})^+$. Then by the assumption, $\varkappa_\beta \leqslant \varkappa_\alpha < \varkappa$ for $\beta < \alpha < \lambda$. Let

$$W = \{u \in {}^{\lambda}\varkappa: u(\alpha) \in \varkappa_\alpha \text{ for all } \alpha \in \operatorname{dom}(u)\}.$$

Then $|W| = \varkappa$. We shall consider W as a tree ordered by $\subseteq$.

Fix $t \in \sigma P$. By induction on the levels we construct an isomorphical embedding $g_t: \langle t, <_t \rangle \to \langle W, \subset \rangle$ as follows. Let $a \in t$ and assume $g_t(b)$ is defined for every $b <_t a$. Let $\alpha < \lambda$ be the height of a in $\langle t, <_t \rangle$. If α is a limit ordinal we put $g_t(a) = \bigcup_{b <_t a} g_t(b)$. So assume now that α is a successor ordinal. Let $\beta = \alpha - 1$ and let b be the immediate predecessor of a. Then the set of all immediate successors of b in $\langle t, <_t \rangle$ is well-ordered by $<_P$, hence by Fact 2, there is a $\xi<\varkappa_\beta$ such that a is the ξth immediate successor of b in $\langle t, <_t\rangle$. Let $g_t(a)=g_t(b)^\frown\xi$. This completes the inductive definition.

Fact 4. If $s, t\in\sigma P$ and $s\leqslant\cdot t$, then $g_s=g_t\restriction t$.

Proof. Follows directly from Fact 3 and the definition of g_t for $t\in\sigma P$.

For $t\in P$, let $H(t)=\{g_t(a): a\in t\}$. Then by Fact 4, we have that $s<\cdot t$ implies $H(s)\subset H(t)$ for $s, t\in\sigma P$. Thus $H:\sigma P\to\mathcal{P}(W)$ is a strictly increasing mapping which contradicts Theorem 9, since $|W|=\varkappa$ and $P\to(\varkappa)^1_\varkappa$. This proves the Claim.

Fix $t\in\sigma P$ such that $\langle t, <_t\rangle$ has height $\geqslant\lambda+1$. Let $a\in t$ has height λ in $\langle t, <_t\rangle$. Then $\{b\in t: b<_t a\}=A_a$ has order type λ. By the definition of $<_t$, it follows that $A_a\cup\{a\}$ is an end-homogeneous chain with respect to f. Since $\operatorname{tp}(A_a\cup\{a\})=\lambda+1$ this completes the proof of Theorem 4.

COROLLARY 22. *Let P be a partially ordered set such that $P\to(2^\varkappa)^1_{2^\varkappa}$ then*

$$P\to\langle\varkappa^+ +1\rangle^{<\omega}_{2^\varkappa}.$$

COROLLARY 23. *If P is a partially ordered set such that $P\to(2^{\underset{\sim}{\varkappa}})^1_{2^{\underset{\sim}{\varkappa}}}$, then*

$$P\to\langle\varkappa+1\rangle^{<\omega}_\gamma \quad \textit{for all } \gamma<\varkappa.$$

The following lemma is a generalization of the well-known Stepping-up Lemma for cardinals (see [5; § 16]). It is the main tool for proving positive partition relations for partially ordered sets for exponent $r>2$.

THEOREM 24 (Stepping-up Lemma). *Let $\varkappa\geqslant\aleph_0$, $2\leqslant r<\omega$, and let γ and α_ξ, $\xi<\gamma$ be ordinals. Let P be a partially ordered set such that $P\to(2^{\underset{\sim}{\varkappa}})^1_{2^{\underset{\sim}{\varkappa}}}$. Then*

$$\varkappa\to(\alpha_\xi)^{r-1}_{\xi<\gamma} \quad \text{implies} \quad P\to(\alpha_\xi+1)^r_{\xi<\gamma}.$$

Proof. Let P be a partially ordered set such that $P\to(2^{\underset{\sim}{\varkappa}})^1_{2^{\underset{\sim}{\varkappa}}}$ and let $f:[P]^r\to\gamma$ be a given partition. We may assume $\alpha_\xi\geqslant r$ holds for all $\xi<\gamma$ in which case we must have $\gamma<\varkappa$ by the assumption $\varkappa\to(\alpha_\xi)^{r-1}_{\xi<\gamma}$. By Corollary 23 there is a chain $A\subseteq P$, end-homogeneous with respect to f, such that $\operatorname{tp}A=\varkappa+1$. Let a be the maximal point of A. If $\{b_1, \ldots, b_{r-1}\}\in A\setminus\{a\}$ we put

$$g(\{b_1, \ldots, b_{r-1}\})=f(\langle b_1, \ldots, b_{r-1}, a\rangle).$$

Since $\varkappa\to(\alpha_\xi)^{r-1}_{\xi<\gamma}$ there exist $\xi<\gamma$ and $B\subseteq A$ such that $\operatorname{tp}B=\alpha_\xi$ and $g''[B]^{r-1}=\{\xi\}$. Since

A is an end-homogeneous chain, this implies $f''[B\cup\{a\}]^r=\{\xi\}$. This completes the proof since $\operatorname{tp}(B\cup\{a\})=\alpha_\xi+1$.

COROLLARY 25. *If P is a partially ordered set such that $P\to(2^\varkappa)^1_{2^\varkappa}$, then*

$$P\to(\varkappa+1)^2_\gamma \quad \text{for } \gamma<\operatorname{cf}\varkappa.$$

Let $\varkappa$ be a cardinal and let $n<\omega$. Then by induction on n we define $\exp_0(\varkappa)=\varkappa$ and $\exp_{n+1}(\varkappa)=\exp_n(2^\varkappa)$.

COROLLARY 26. *Assume $\lambda\geq\aleph_0$, $\theta\geq2$* and $2<r<\omega$. *Let P be a partially ordered set such that $P\to(\varkappa)^1_\varkappa$, where $\varkappa=\exp_{r-2}(\theta^\lambda)$. Then*

$$P\to((\theta^\lambda)^+,\ (\operatorname{cf}\lambda)_\gamma)^r \quad \text{for } \gamma<\operatorname{cf}\lambda.$$

Proof. By Corollary 16, we have $(\theta^\lambda)^+\to((\theta^\lambda)^+,\ (\operatorname{cf}\lambda+1)_\gamma)^2$. Now by induction on $2<r<\omega$, using the Stepping-up Lemma, we actually get the stronger result

$$P\to((\theta^\lambda)^++r-2,\ (\operatorname{cf}\lambda+r-1)_\gamma)^r.$$

COROLLARY 27. *Assume $\lambda\geq\aleph_0$, and $2<r<\omega$. Let P be a partially ordered set such that $P\to(\varkappa)^1_\varkappa$, where $\varkappa=\exp_{r-1}(\lambda)$.* Then

$$P\to((2^\lambda)^+,\ (\lambda^+)_\lambda)^r.$$

For a set A and $n=1,2,\dots$ we define $\mathscr{P}^1(A)=\mathscr{P}(A)$ and $\mathscr{P}^{n+1}(A)=\mathscr{P}(\mathscr{P}^n(A))$. We consider $\mathscr{P}^n(A)$ as a partially ordered set under the ordering $\subseteq$. The following result is an immediate consequence of Theorem 1 and the Stepping-up Lemma.

COROLLARY 28. $\mathscr{P}^n(\omega)\to(\alpha)^{n+1}_k$ *for all $\alpha<\omega_1$ and nonzero $k,n<\omega$.*

Using Ramsey's theorem and the Stepping-up Lemma we have also the following result of Galvin announced in [10; p. 718].

THEOREM 29 (Galvin). *Let P be a partially ordered set such that $P\to(\omega)^1_\omega$. Then*

$$P\to(\omega+1)^r_k \quad \textit{for all } r,k<\omega.$$

Galvin remarks that Theorem 29 is in a sense best possible since $\omega_1\nrightarrow(\omega,\omega+2)^3$. He also remarks that by using Nash-Williams' generalization of Ramsey's theorem [19; p. 33] in the above proof of Theorem 29 one obtains the following stronger result.

THEOREM 30 (Galvin). *Let P be a partially ordered set such that $P \to (\omega)^1_\omega$ and let $\mathscr{C}$ be a collection of finite chains in P such that no element of $\mathscr{C}$ is an intitial segment of another. Then for every $k<\omega$ and every function $f\colon \mathscr{C} \to k$ there is a chain $A \subseteq P$ such that* $\operatorname{tp} A=\omega+1$ *and f is constant on $\mathscr{C} \cap [A]^{<\omega}$.*

References

[1] BAUMGARTNER, J. & HAJNAL, A., A proof (involving Martin's axiom) of a partition relation. *Fund. Math.*, 73 (1973), 193–203.

[2] BAUMGARTNER, J., MALITZ, J. & REINHARDT, W., Embedding trees in the rationals. *Proc. Nat. Acad. Sci. U.S.A.*, 67 (1970), 1748–1753.

[3] ERDÖS, P. & HAJNAL, A., Unsolved problems in set theory. *Amer. Math. Soc. Proc. Symp. Pure Math.*, 13, Part 1 (1971), 17–48.

[4] — Unsolved and solved problems in set theory. *Amer. Math. Soc. Proc. Symp. Pure Math.*, 25 (1974), 267–287.

[5] ERDÖS, P., HAJNAL, A., MÁTÉ, A. & RADO, R., *Combinatorial set theory: Partition relations for cardinals*. North-Holland, 1984.

[6] ERDÖS, P., HAJNAL, A. & RADO, R., Partition relations for cardinal numbers. *Acta Math. Acad. Sci. Hungar.*, 16 (1965), 93–196.

[7] ERDÖS, P. & RADO, R., Combinatorial theorems on classifications of subsets of a given set. *Proc. London Math. Soc.*, 2 (1952), 417–439.

[8] — A partition calculus in set theory. *Bull. Amer. Math. Soc.*, 62 (1956), 427–489.

[9] GALVIN, F., Partition theorems for the real line. *Notices Amer. Math. Soc.*, 15 (1968), 660; Erratum 16 (1969), 1095.

[10] — On a partition theorem of Baumgartner and Hajnal. *Coll. Math. Soc., János Bolyai, 10. Infinite and finite sets*. Keszthely (Hungary) 1973, North-Holland (1975), 711–729.

[11] HAJNAL, A., Some results and problems on set theory. *Acta Math. Acad. Sci. Hungary.*, 11 (1960), 277–298.

[12] HARTOGS, F., Über das Problem der Wohlordnung. *Math. Ann.*, 76 (1915), 438–443.

[13] KUREPA, D., Ensembles ordonnés et ramifiés. *Publ. Math. Univ. Belgrade*, 4 (1935), 1–138.

[14] — Ensembles ordonnés et leurs sous-ensembles bien ordonnés. *C. R. Acad. Sci. Paris Ser. A*, 242 (1956), 2202–2203.

[15] — Monotone mappings between some kinds of ordered sets. *Glasnik Mat.-Fiz. Astr.*, 19 (1964), 175–186.

[16] LAVER, R., Partition relations for uncountable cardinals $\leqslant 2^{\aleph_0}$. *Coll. Math. Soc. János Bolyai, 10. Infinite and finite sets*. Keszthely (Hungary), 1973, North-Holland (1975), 1029–1042.

[17] MARTIN, D. A. & SOLOVAY, R. M., Internal Cohen extensions. *Ann. Math. Logic*, 2 (1970), 143–178.

[18] MÁTÉ, A., A partition relation for Souslin trees, *Trans. Amer. Math. Soc.*, 261 (1981), 143–149.

[19] NASH-WILLIAMS, C. ST. J. A., On well-quasi-ordering transfinite sequences. *Proc. Cambridge Phil. Soc.*, 61 (1965), 33–39.

[20] NEUMER, J., Verallgemeinerung eines Satzes von Alexandroff und Urysohn. *Math. Z.*, 54 (1951), 254–261.

[21] PRIKRY, K., On a set-theoretic partition problem. *Duke Math. J.*, 39 (1972), 77–83.

[22] SHELAH, S., Notes on combinatorial set theory. *Israel J. Math.*, 14 (1973), 262–277.

[23] TARSKI, A., On well-ordered subsets of any set. *Fund. Math.*, 32 (1939), 176–183.

[24] TODORCEVIC, S., Stationary sets, trees and continuums. *Publ. Inst. Math. Beograd*, 27 (41) (1981), 249–262.
[25] — Real functions on the family of all well-ordered subsets of a partially ordered set. *J. Symbolic Logic*, 48 (1983), 91–96.
[26] — A partition relation for partially ordered sets. *Abstracts Amer. Math. Soc.*, 2 (1981), A-362.
[27] — Forcing positive partition relations. *Trans. Amer. Math. Soc.*, 280 (1983), 703–720.

Received December 20, 1982