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Zeros of the i.i.d. Gaussian power series: a conformally invariant determinantal process

by

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1. Introduction

Consider the random power series

$$f_{\mathbf{U}}(z)=\sum_{n=0}^{\infty} a_n z^n, \tag{1}$$

where $\{a_n\}_{n=0}^{\infty}$ are independent standard complex Gaussian random variables (with density $e^{-z\bar{z}}/\pi$). The radius of convergence of the series is a.s. 1, and the set of zeros forms a point process $Z_{\mathbf{U}}$ in the unit disk $\mathbf{U}$. Zeros of Gaussian power series have been studied starting with Offord [20], since these series are limits of random Gaussian polynomials. In the last decade, physicists have introduced a new perspective, by interpreting the zeros of a Gaussian polynomial as a gas of interacting particles, see Hannay [12], Lebœuf [15] and the references therein. Much of the recent interest in Gaussian analytic functions was spurred by the papers Edelman–Kostlan [9] and Bleher–Shiffman–Zelditch [4]. A fundamental property of $Z_{\mathbf{U}}$ is the invariance of its distribution under Möbius transformations that preserve the unit disk; see §2 for an explanation, and Sodin–Tsirelson [27] for references.

Our main new discovery is that the zeros $Z_{\mathbf{U}}$ form a *determinantal process*, and this yields an explicit formula for the distribution of the number of zeros in a disk. Furthermore, we show that the process $Z_{\mathbf{U}}$ admits a conformally invariant evolution which elucidates the repulsion between zeros.

Given a random function f and points $z_1, ..., z_n$, let $p_\varepsilon(z_1, ..., z_n)$ denote the probability that for all $1\leqslant i\leqslant n$, there is a zero of f in the disk of radius ε centered at z_i. The

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joint intensity of the point process of zeros of f, also known as the *n-point correlation function*, is defined by the limit

$$p(z_1, \ldots, z_n) = \lim_{\varepsilon \to 0} \frac{p_\varepsilon(z_1, \ldots, z_n)}{\pi^n \varepsilon^{2n}} \tag{2}$$

when it exists; see (10) for a related integral formula.

THEOREM 1. *The joint intensity of zeros for the i.i.d.* (*independent identically distributed*) *Gaussian power series* (1) *in the unit disk exists, and satisfies*

$$p(z_1, \ldots, z_n) = \pi^{-n} \det \left(\frac{1}{(1 - z_i \bar{z}_j)^2} \right)_{i,j}. \tag{3}$$

Thus the zero set of the i.i.d. series $f_\mathbf{U}(z)$ is a determinantal process in $\mathbf{U}$, governed by the *Bergman kernel* $K_\mathbf{U}(z, w) = \pi^{-1}(1 - z\bar{w})^{-2}$; see Soshnikov [28] for a survey of determinantal processes. In particular, (3) extends the known fact that $p(z_1, z_2) < p(z_1)p(z_2)$ for all $z_1, z_2 \in \mathbf{U}$, i.e., the zeros are negatively correlated. In fact, $Z_\mathbf{U}$ is the only process of zeros of a Gaussian analytic function which is negatively correlated and has a Möbius-invariant law; see §2.

The determinant formula for the joint intensity allows us to determine the distribution of the number of zeros of $f_\mathbf{U}$ in a disk, and identify the law of the moduli of the zeros.

THEOREM 2. (i) *The number* $N_r = |Z_\mathbf{U} \cap B_r(0)|$ *of zeros of* $f_\mathbf{U}$ *in the disk of Euclidean radius* r *about* 0 *satisfies*

$$\mathbf{E}(1+s)^{N_r} = \prod_{k=1}^{\infty} (1 + r^{2k} s) \tag{4}$$

for all real s. *Thus* N_r *has the same distribution as* $\sum_{k=1}^{\infty} X_k$, *where* $\{X_k\}_{k=1}^{\infty}$ *is a sequence of independent* $\{0,1\}$*-valued random variables with* $\mathbf{P}(X_k = 1) = r^{2k}$.

(ii) *Moreover, the set of moduli* $\{|z| : f_\mathbf{U}(z) = 0\}$ *has the same law as* $\{U_k^{1/2k}\}_{k=1}^{\infty}$, *where* $\{U_k\}_{k=1}^{\infty}$ *are i.i.d. random variables uniform in* $[0,1]$.

From Theorem 2 we readily obtain the asymptotics of the hole probability $\mathbf{P}(N_r = 0)$. Furthermore, the infinite product in (4) occurs in one of Euler's partition identities, see (36), and this connection yields part (ii) of the next corollary.

COROLLARY 3. (i) *Let* $h = 4\pi r^2/(1 - r^2)$, *the hyperbolic area of* $B_r(0)$. *As* $r \uparrow 1$, *we have*

$$\mathbf{P}(N_r = 0) = \exp\left(\frac{-\pi h + o(h)}{24} \right) = \exp\left(\frac{-\pi^2 + o(1)}{12(1-r)} \right).$$

(ii) *The binomial moments of N_r equal*

$$\mathbf{E}\binom{N_r}{k} = \frac{r^{k(k+1)}}{(1-r^2)(1-r^4)\dots(1-r^{2k})}.$$

(iii) *The ratio $(N_r-\mu_r)/\sigma_r$ converges in law to standard normal as $r\uparrow 1$, where*

$$\mu_r = \mathbf{E}N_r = \frac{r^2}{1-r^2} \quad \textit{and} \quad \sigma_r^2 = \operatorname{Var} N_r = \frac{r^2}{1-r^4}.$$

1.1. General domains

The covariance structure $\mathbf{E}(f_{\mathbf{U}}(z)\overline{f_{\mathbf{U}}(w)})=(1-z\bar{w})^{-1}$ equals 2π times the *Szegő kernel* $S_{\mathbf{U}}(z,w)=(2\pi)^{-1}(1-z\bar{w})^{-1}$ in the unit disk. The Szegő kernel $S_D(z,w)$ and the Bergman kernel $K_D(z,w)$ are defined, and positive definite, for any bounded planar domain D with a smooth boundary. (See the next section or Bell [2] for information on the Szegő and Bergman kernels.) For such domains we can consider the Gaussian analytic function $f_D(z)$ with covariance structure $2\pi S_D$ in D (an explicit formula for f_D is given in (12)). Recall that a *Gaussian analytic function* in D is a random analytic function f such that for any choice of $z_1,\dots,z_n$ in D, the random variables $f(z_1),\dots,f(z_n)$ have complex Gaussian joint distribution.

COROLLARY 4. *Let D be a simply-connected bounded planar domain, with a C^∞ smooth boundary. The joint intensity of zeros for the Gaussian analytic function f_D is given by the determinant of the Bergman kernel*:

$$p(z_1,\dots,z_n) = \det(K_D(z_i,z_j))_{i,j}.$$

Note that for simply-connected domains as in the corollary, the Bergman and Szegő kernels satisfy $K_D(z,w)=4\pi S_D(z,w)^2$, see Bell [2, Theorem 23.1].

1.2. The one-parameter family of Möbius-invariant zero sets

For $\varrho>0$, let $Z_{\mathbf{U},\varrho}$ denote the zero set of

$$f_{\mathbf{U},\varrho}(z) = \sum_{n=0}^{\infty} \binom{-\varrho}{n}^{1/2} a_n z^n, \tag{5}$$

where $\{a_n\}_{n=0}^{\infty}$ are i.i.d. standard complex Gaussians. In particular, $f_{\mathbf{U},1}$ has the same distribution as $f_{\mathbf{U}}$. As explained in Sodin–Tsirelson [27] (see also Bleher–Ridzal [3]), for any $\varrho>0$, the distribution of $Z_{\mathbf{U},\varrho}$ is invariant under Möbius transformations that preserve $\mathbf{U}$. Moreover, these are the only zero sets of Gaussian analytic functions with this invariance property. However, only $\varrho=1$ yields a determinantal zero process.

Taking $n=1$ in Theorem 1, one recovers the well-known formula $(1-|z|^2)^{-2}/\pi$ for the intensity of $Z_{\mathbf{U}}$. More generally, the intensity of $Z_{\mathbf{U},\varrho}$ in $\mathbf{U}$ is $\varrho/\pi(1-|z|^2)^2$, see Sodin [26]. It follows that the expected number of zeros in a Borel set $\Lambda\subset\mathbf{U}$ is $\varrho/4\pi$ times the hyperbolic area

$$A(\Lambda)=\int_\Lambda \frac{4\,dz}{(1-|z|^2)^2}.$$

(Integration is with respect to planar Lebesgue measure.) This can also be inferred from Proposition 8 below. In §5, we prove the following law of large numbers:

PROPOSITION 5. *Let $\varrho>0$, and suppose that $\{\Lambda_h\}_{h>0}$ is an increasing family of Borel sets in $\mathbf{U}$, parameterized by hyperbolic area $h=A(\Lambda_h)$. Then the number $N(h)=|Z_{\mathbf{U}}\cap\Lambda_h|$ of zeros of $f_{\mathbf{U},\varrho}$ in Λ_h satisfies*

$$\lim_{h\to\infty}\frac{N(h)}{h}=\frac{\varrho}{4\pi}\quad a.s.$$

1.3. Reconstruction of $|f_{\mathbf{U},\varrho}|$ from its zeros

THEOREM 6. (i) *Let $\varrho>0$. Consider the random function $f_{\mathbf{U},\varrho}$, and order its zero set $Z_{\mathbf{U},\varrho}$ in increasing absolute value as $\{z_k\}_{k=1}^\infty$. Then*

$$|f_{\mathbf{U},\varrho}(0)|=c_\varrho\prod_{k=1}^\infty e^{\varrho/2k}|z_k|\quad a.s., \tag{6}$$

where $c_\varrho=e^{(\varrho-\gamma-\gamma\varrho)/2}\varrho^{-\varrho/2}$ and $\gamma=\lim_{n\to\infty}\left(\sum_{k=1}^n 1/k-\log n\right)$ is Euler's constant.

(ii) *More generally, given $\zeta\in\mathbf{U}$, let $\{\zeta_k\}_{k=1}^\infty$ be $Z_{\mathbf{U},\varrho}$, ordered in increasing hyperbolic distance from ζ. Then*

$$|f_{\mathbf{U},\varrho}(\zeta)|=c_\varrho(1-|\zeta|^2)^{-\varrho/2}\prod_{k=1}^\infty e^{\varrho/2k}\left|\frac{\zeta_k-\zeta}{1-\bar\zeta\zeta_k}\right|. \tag{7}$$

Thus the analytic function $f_{\mathbf{U},\varrho}(z)$ is determined by its zero set, up to multiplication by a constant of modulus 1.

This theorem is proved in §6.

1.4. Dynamics

In order to understand the negative correlations for zeros of $f_{\mathbf{U}}$, we consider a dynamic version of the zero set $Z_{\mathbf{U}}$. Denote by $Z_{\mathbf{U}}(t)$ the zero set of the power series $\sum_{n=0}^{\infty} a_n(t) z^n$, where the coefficients $a_n(t)$ are independent stationary complex Ornstein–Uhlenbeck processes; in other words, $a_n(t)=e^{-t/2}W_n(e^t)$, where $\{W_n(\cdot)\}_{n=0}^{\infty}$ are independent complex Brownian motions.

A direct calculation gives that, for the process $Z_{\mathbf{U}}$, the intensity ratio

$$\frac{p(z_1, z_2)}{p(z_1)p(z_2)}$$

is strictly less than 1, and decreases to 0 as the hyperbolic distance between z_1 and z_2 tends to 0. This repulsion suggests that when two zeros get close, there is a drift in their motion that pushes them apart. However, this is not the case. Instead, we have the following result:

THEOREM 7. *Consider the process of zeros $\{Z_{\mathbf{U}}(t)\}$ in the unit disk, and condition on the event that at time $t=0$, there is a zero at the origin, i.e., $0 \in Z_{\mathbf{U}}(0)$. The movement of this zero is then described by a stochastic differential equation which at time $t=0$ has the form*

$$dz = \sigma\, dW,$$

where W is complex Brownian motion, there is no drift term, and

$$\frac{1}{\sigma} = |f'_{\mathbf{U}}(0)| = c_1 \prod_{k=2}^{\infty} e^{1/2k} |z_k| \quad a.s.$$

Heuristically, any zero of $f_{\mathbf{U}}$ oscillates faster when there are other zeros nearby; this causes repulsion.

Analogous processes $Z_D(t)$ can be defined in general domains, and we shall show in §7 that the family of processes $Z_D(t)$ is conformally invariant (no time change is needed). Theorem 7 can be extended to $\varrho \neq 1$ as well.

Conditioning to have a zero at a given location. It is important to note that the distribution of $f_{\mathbf{U}}$ given that its value is zero at 0 is *different* from the conditional distribution of $f_{\mathbf{U}}$ given that its zero set has a point at 0. In particular, in the second case the conditional distribution of the coefficient a_1 is not Gaussian. The reason for this is that the two ways of conditioning are defined by the limits as $\varepsilon \to 0$ of two different conditional distributions. In the first case, we condition on $|f_{\mathbf{U}}(0)| < \varepsilon$. In the second, we condition on $f_{\mathbf{U}}$ having a zero in the disk $B_\varepsilon(0)$ of radius ε about 0; the latter conditioning affects the distribution of a_1. See Lemma 18 in §6.

1.5. Hammersley's formula

The starting point of the proof of Theorem 1 is a general permanent formula for the joint intensity of zeros for Gaussian analytic functions. A version for polynomials is due to Hammersley [11] and Friedman [10]. The permanent form (9) for Gaussian polynomials appears in the physics literature (Hannay [12]). Closely related formulas for correlations between zeros of random sections of a positive holomorphic line bundle over a compact complex manifold were established by Bleher–Shiffman–Zelditch [4].

The version we need is for Gaussian analytic functions, that are not necessarily polynomials.

PROPOSITION 8. *Let f be a Gaussian analytic function in a planar domain D such that $\mathbf{E}f(z)=0$ for all $z\in D$. Given points $z_1,\dots,z_n\in D$, consider the matrices*

$$A=(\mathbf{E}f(z_i)\overline{f(z_j)}),\quad B=(\mathbf{E}f'(z_i)\overline{f(z_j)})\quad \text{and}\quad C=(\mathbf{E}f'(z_i)\overline{f'(z_j)}).$$

Assume that A is nonsingular.

(i) *The joint intensity for the zeros of f exists and satisfies*

$$p(z_1,\dots,z_n)=\frac{\mathbf{E}\big(|f'(z_1)\dots f'(z_n)|^2\,\big|\,f(z_1)=\dots=f(z_n)=0\big)}{\det(\pi A)}. \tag{8}$$

Consequently,

$$p(z_1,\dots,z_n)=\frac{\operatorname{perm}(C-BA^{-1}B^*)}{\det(\pi A)}. \tag{9}$$

(ii) *Assume that $A=A(z_1,\dots,z_n)$ is nonsingular when $z_1,\dots,z_n\in D$ are distinct. Let $Z_f^{\wedge n}$ denote the set of n-tuples of distinct zeros of f. Then for any Borel set $B\subset D^n$ we have*

$$\mathbf{E}\,\#(B\cap Z^{\wedge n})=\int_B p(z_1,\dots,z_n)\,dz_1\dots dz_n. \tag{10}$$

The proof of this proposition is given in §8.

In the derivation of Theorem 1 from Proposition 8, we use conformal invariance, the i.i.d. property of the coefficients, and the beautiful determinant-permanent identity (26) of Borchardt [6].

Remarks on the literature. A nice introduction to the theory of Gaussian analytic functions is given in Sodin [26]; for earlier results, see Hammersley [11], Friedman [10], Bogomolny–Bohigas–Lebœuf [5], Kostlan [14], Edelman–Kostlan [9] and Hannay [12]. Close to the topic of this paper are Shiffman–Zelditch [23] and Sodin–Tsirelson [27]. Determinantal processes are also being intensively studied, see Soshnikov [28]. Theorem 1 provides further evidence for the analogy, suggested in Lebœuf [15], between zeros of Gaussian polynomials and the Ginibre ensemble of eigenvalues of (non-Hermitian) random matrices with i.i.d. Gaussian entries, which is known to be determinantal.

2. Conformal invariance and preliminaries

Complex Gaussian random variables. Recall that a standard complex Gaussian random variable a has density $e^{-z\bar{z}}/\pi$, expected value 0 and variance $\mathbf{E}a\bar{a}=1$. A vector V of random variables has a complex Gaussian (joint) distribution if there is a deterministic vector V_0 such that $V-V_0$ is the image under a linear map of a vector of i.i.d. standard complex Gaussian random variables.

If X and Y are real Gaussian random variables of mean zero, then $X+iY$ is complex Gaussian if and only if X and Y are independent and have the same variance. A complex Gaussian random variable Z with $\mathbf{E}Z=0$ satisfies $\mathbf{E}Z^n=0$ for any integer $n\geqslant 1$.

The complex Gaussian power series. Recall the power series $f_{\mathbf{U}}$ in (1). A Borel–Cantelli argument shows that the radius of convergence of $f_{\mathbf{U}}$ equals 1 a.s. Clearly, the joint distributions of $f_{\mathbf{U}}(z_k)$ for any finite collection $\{z_k\}$ are complex Gaussian, so the values of $f_{\mathbf{U}}$ form a complex Gaussian ensemble. Since $f_{\mathbf{U}}$ is continuous, its distribution is determined by the covariance structure

$$\mathbf{E}f_{\mathbf{U}}(z)\overline{f_{\mathbf{U}}(w)}=\sum_{n=0}^{\infty}(z\bar{w})^n=(1-z\bar{w})^{-1}. \tag{11}$$

The right-hand side is 2π times the Szegő kernel in the unit disk; it suggests a natural way to generalize the power series $f_{\mathbf{U}}$.

The Szegő kernel. Let D be a bounded planar domain with a C^∞ smooth boundary (the regularity assumption can be weakened). Consider the set of complex analytic functions in D which extend continuously to the boundary ∂D. The classical Hardy space $H^2(D)$ is given by the L^2-closure of this set with respect to length measure on ∂D. Every element of $H^2(D)$ can be identified with a unique analytic function in D via the Cauchy integral (see Bell [2, §6]).

Consider an orthonormal basis $\{\psi_n\}_{n=0}^{\infty}$ for $H^2(D)$; e.g., in the unit disk, take $\psi_n(z)=z^n/\sqrt{2\pi}$ for $n\geqslant 0$. Use i.i.d. complex Gaussians $\{a_n\}_{n=0}^{\infty}$ to define the random analytic function

$$f_D(z)=\sqrt{2\pi}\sum_{n=0}^{\infty}a_n\psi_n(z) \tag{12}$$

(cf. (6) in Shiffman–Zelditch [23]). The factor of $\sqrt{2\pi}$ is included just to simplify formulas in the case where D is the unit disk. The covariance function of f_D is given by $2\pi S_D(z,w)$, where

$$S_D(z,w)=\sum_{n=0}^{\infty}\psi_n(z)\overline{\psi_n(w)} \tag{13}$$

is the Szegő kernel in D. The Szegő kernel S_D does not depend on the choice of orthonormal basis and is positive definite (i.e., for points $z_j \in D$ the matrix $(S_D(z_j, z_k))_{j,k}$ is positive definite).

Let $T:\Lambda \to D$ be a conformal homeomorphism between two bounded domains with C^∞ smooth boundary. The derivative T' of the conformal map has a well-defined square root, see Bell [2, p. 43]. If $\{\psi_n\}_{n=0}^\infty$ is an orthonormal basis for $H^2(D)$, then $\{\sqrt{T'}(\psi_n \circ T)\}_{n=0}^\infty$ forms an orthonormal basis for $H^2(\Lambda)$. In particular, the Szegő kernels satisfy

$$S_\Lambda(z,w) = T'(z)^{1/2}\,\overline{T'(w)^{1/2}}\,S_D(T(z),T(w)). \tag{14}$$

When D is a simply-connected domain, it follows from the transformation formula (14) that S_D does not vanish in the interior of D, so for arbitrary $\varrho > 0$ powers S_D^ϱ are defined.

Let $\{\eta_n\}_{n=0}^\infty$ be an orthonormal basis of the subspace of complex analytic functions in $L^2(D)$ with respect to Lebesgue area measure. The Bergman kernel

$$K_D(z,w) = \sum_{n=0}^\infty \eta_n(z)\overline{\eta_n(w)}$$

is independent of the basis chosen, see Nehari [18, formula (132)].

The Szegő random functions with parameter ϱ. Recall the one-parameter family of Gaussian analytic functions $f_{\mathbf{U},\varrho}$ defined in (5). The binomial expansion yields that the covariance structure $\mathbf{E} f_{\mathbf{U},\varrho}(z)\overline{f_{\mathbf{U},\varrho}(w)}$ equals

$$\sum_{n=0}^\infty \left|\binom{-\varrho}{n}\right| z^n \bar{w}^n = \sum_{n=0}^\infty \binom{-\varrho}{n}(-z\bar{w})^n = (1-z\bar{w})^{-\varrho} = [2\pi S_{\mathbf{U}}(z,w)]^\varrho. \tag{15}$$

The invariance of the distribution of $Z_{\mathbf{U},\varrho}$ under Möbius transformations of the unit disk is a special case of the following result:

PROPOSITION 9. *Let D be a bounded planar domain with a C^∞ boundary and let $\varrho > 0$. Suppose that either* (i) *D is simply-connected, or* (ii) *ϱ is an integer. Then there is a mean-zero Gaussian analytic function $f_{D,\varrho}$ in D with covariance structure*

$$\mathbf{E} f_{D,\varrho}(z)\overline{f_{D,\varrho}(w)} = [2\pi S_D(z,w)]^\varrho \quad \textit{for } z, w \in D.$$

The zero set $Z_{D,\varrho}$ of $f_{D,\varrho}$ has a conformally invariant distribution: if Λ is another bounded domain with a smooth boundary, and $T:\Lambda \to D$ is a conformal homeomorphism, then $T(Z_{\Lambda,\varrho})$ has the same distribution as $Z_{D,\varrho}$. Moreover, the following two random functions have the same distribution:

$$f_{\Lambda,\varrho}(z) \stackrel{d}{=} T'(z)^{\varrho/2}(f_{D,\varrho} \circ T)(z). \tag{16}$$

We call the Gaussian analytic function $f_{D,\varrho}$ described in the proposition the *Szegő random function with parameter* ϱ in D.

Proof. *Case* (i): D is simply-connected. Let $\Psi: D \to \mathbf{U}$ be a conformal map onto $\mathbf{U}$, and let $\{a_n\}_{n=0}^{\infty}$ be i.i.d. standard complex Gaussians. We claim that

$$f(z) = \Psi'(z)^{\varrho/2} \sum_{n=0}^{\infty} \binom{-\varrho}{n}^{1/2} a_n \Psi(z)^n \tag{17}$$

is a suitable candidate for $f_{D,\varrho}$. Indeed, repeating the calculation in (15), we find that

$$\begin{aligned}\mathbf{E}(f(z)\overline{f(w)}) &= [\Psi'(z)\overline{\Psi'(w)}]^{\varrho/2}[1-\Psi(z)\overline{\Psi(w)}]^{-\varrho} \\ &= [\Psi'(z)\overline{\Psi'(w)}]^{\varrho/2}[2\pi S_{\mathbf{U}}(\Psi(z),\Psi(w))]^{\varrho}.\end{aligned}$$

The last expression equals $[2\pi S_D(z,w)]^{\varrho}$ by the transformation formula (14). Thus we may define $f_{D,\varrho}$ by the right-hand side of (17). If $T: \Lambda \to D$ is a conformal homeomorphism, then $\Psi \circ T$ is a conformal map from Λ to $\mathbf{U}$, so (17) and the chain rule give the equality in law (16). Since T' does not have zeros in Λ, multiplying $f_{D,\varrho} \circ T$ by a power of T' does not change its zero set in Λ, and it follows that $T(Z_{\Lambda,\varrho})$ and $Z_{D,\varrho}$ have the same distribution.

Case (ii): ϱ is an integer. Let $\{\psi_n\}_{n=0}^{\infty}$ be an orthonormal basis for $H^2(D)$. Use i.i.d. complex Gaussians $\{a_{n_1,\ldots,n_\varrho}: n_1, \ldots, n_\varrho \geqslant 0\}$ to define the random analytic function

$$f_{D,\varrho}(z) = (2\pi)^{\varrho/2} \sum_{n_1,\ldots,n_\varrho \geqslant 0} a_{n_1,\ldots,n_\varrho} \psi_{n_1}(z) \ldots \psi_{n_\varrho}(z); \tag{18}$$

see Sodin [26] for convergence. A direct calculation shows that $f_{D,\varrho}$, thus defined, satisfies

$$\mathbf{E} f_{D,\varrho}(z)\overline{f_{D,\varrho}(w)} = (2\pi)^{\varrho} \sum_{n_1,\ldots,n_\varrho \geqslant 0} \psi_{n_1}(z)\overline{\psi_{n_1}(w)} \ldots \psi_{n_\varrho}(z)\overline{\psi_{n_\varrho}(w)} = [2\pi S_D(z,w)]^{\varrho}.$$

The transformation formula (14) implies that the two sides of (16) have the same covariance structure, $[2\pi S_\Lambda(z,w)]^{\varrho}$. This establishes (16) and completes the proof of the proposition. □

The general theory of Gaussian analytic functions implies that, up to multiplication by a deterministic analytic function, the random functions $f_{\mathbf{U},\varrho}$ are the only Gaussian analytic functions with marginal intensity of zeros proportional to the hyperbolic area element. See Sodin [26] for a proof.

Similarly, the zeros of the Gaussian analytic function

$$F_{\mathbf{C},\varrho}(z) = \sum_{n=0}^{\infty} \left(\frac{\varrho^n}{n!}\right)^{1/2} a_n z^n$$

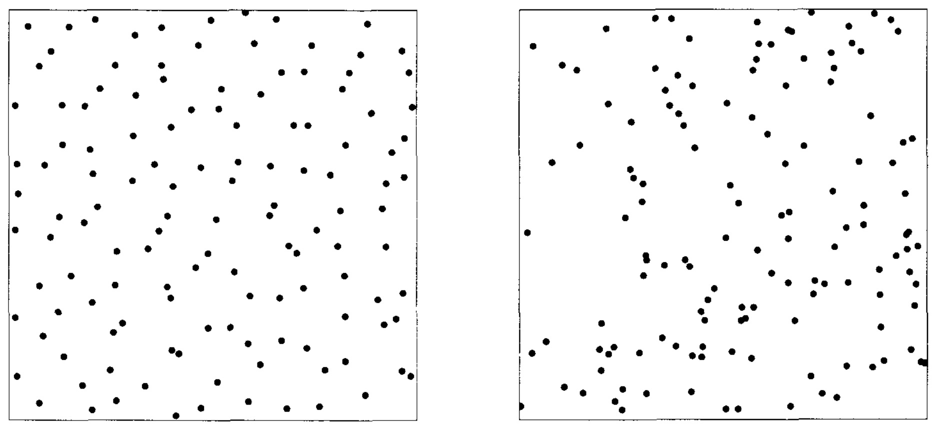

Fig. 1. The translation-invariant root process and a Poisson point process with the same intensity on the plane.

have distribution which is invariant under rotations and translations of the complex plane. Note that here ϱ is a simple scale parameter: $F_{\mathbf{C},\varrho}(z)=F_{\mathbf{C},1}(\sqrt{\varrho}\,z)$.

Letting $\varrho\to\infty$ in the definition of $f_{\mathbf{U},\varrho}$, one recovers that the limit of the rescaled point processes $\varrho^{1/2}Z_{\mathbf{U},\varrho}$ is the zero set of $F_{\mathbf{C},1}$; this phenomenon and its generalizations have been studied by Bleher–Shiffman–Zelditch [4].

Figure 1 shows a realization of the whole plane Gaussian zero process along with a Poisson point process of the same intensity. The orderliness of the zeros suggests that there is a local repulsion taking place. One gets similar pictures for the Szegő random functions in the unit disk. The two-point intensity for zeros at the points r and 0 is given by (9). The most revealing formula is the ratio $p(0,r)/p(0)p(r)$, which shows how far the point process is from a Poisson point process, where this ratio is identically 1. For general ϱ, with $s=1-r^2$, this ratio equals

$$\frac{1+(\varrho^2-2\varrho-2)(s^\varrho+s^{2+2\varrho})+(\varrho+1)^2(s^{2\varrho}+s^{2+\varrho})-2\varrho^2(s^{1+\varrho}+s^{1+2\varrho})+s^{2+3\varrho}}{(1-s^\varrho)^3}, \tag{19}$$

and in the case $\varrho=1$ it simplifies to $r^2(2-r^2)$. For every distance r, the correlation is minimal when $\varrho=1$ (see Figure 2). For all values of ϱ different from 1, for small distances zeros are negatively correlated, while for large distances the correlation is positive.

When $\varrho=1$, the zeros are purely negatively correlated: this special phenomenon is explained by the determinantal form of the joint intensity.

Remark. The Szegő random function for $\varrho=2$,

$$\sum_{n=0}^{\infty}\sqrt{n+1}\,a_n z^n,$$

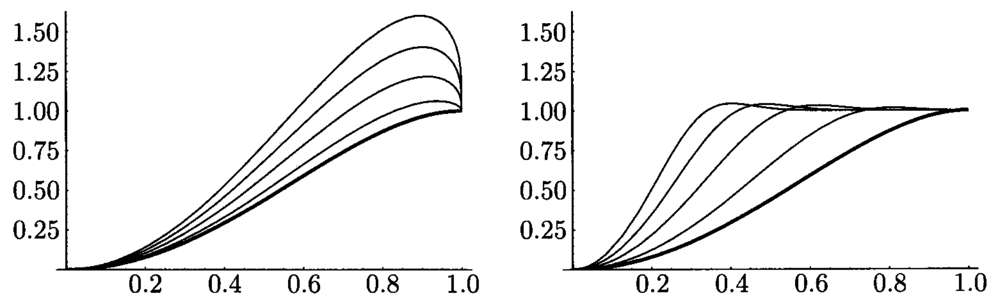

Fig. 2. Relative intensity at $z=0$ and $z=r$ as a function of r for $\varrho=1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$ and for $\varrho=1, 4, 9, 16, 25$.

coincides with the limit when $n \to \infty$ of the logarithmic derivative of the characteristic function of a random $n \times n$ unitary matrix (see Diaconis–Evans [8]).

The analytic extension of white noise. Next, we show that up to the constant term, the power series $f_{\mathbf{U}}$ has the same distribution as the analytic extension of white noise on the unit circle. Let $B(\cdot)$ be a standard real Brownian motion, and let

$$u(z) = \int_0^{2\pi} \mathrm{Poi}(z, e^{it})\, dB(t).$$

Here the integral with respect to B can be interpreted either as a stochastic integral, or as a Riemann–Stieltjes integral, using integration by parts and the smoothness of the Poisson kernel. Recall that the Poisson kernel

$$\mathrm{Poi}(z,w) = \frac{1}{2\pi}\,\mathrm{Re}\left(\frac{1+z\bar{w}}{1-z\bar{w}}\right) = \frac{1}{2\pi}\,\mathrm{Re}\left(\frac{2}{1-z\bar{w}}-1\right) = 2\,\mathrm{Re}\, S_{\mathbf{U}}(z,w) - \frac{1}{2\pi}$$

has the kernel property

$$\mathrm{Poi}(z,w) = \int_0^{2\pi} \mathrm{Poi}(z,e^{it})\,\mathrm{Poi}(e^{it},w)\,dt.$$

(This follows from the Poisson formula for harmonic functions, see Ahlfors [1, §6.3].) The white noise dB has the property that if f_1 and f_2 are smooth functions on an interval and $f_i^{\#} = \int f_i(t)\,dB(t)$, then $\mathbf{E} f_1^{\#} f_2^{\#} = \int f_1(t) f_2(t)\,dt$. By this and the kernel property we get $\mathbf{E} u(z)u(w) = \mathrm{Poi}(z,w)$. Therefore if b is a standard real Gaussian independent of $B(\cdot)$, then

$$\tilde{u}(z) = \sqrt{\frac{1}{2}\pi}\, u(z) + \frac{1}{2} b \tag{20}$$

has covariance structure $\mathbf{E}\tilde{u}(z)\tilde{u}(w) = \pi\,\mathrm{Re}\, S_{\mathbf{U}}(z,w)$. Now if ν and ν' are mean-zero complex Gaussians, then $\mathrm{Re}\,\mathbf{E}\nu\bar{\nu}' = 2\mathbf{E}\,\mathrm{Re}\,\nu\,\mathrm{Re}\,\nu'$; thus (11) implies that $\tilde{u}$ has the same distribution as $\mathrm{Re}\, f_{\mathbf{U}}$.

Remark. Similarly, since $f_{\mathbf{U},2}$ is the derivative of $\sum_{m=1}^{\infty} a_m z^m/\sqrt{m}$, the zero set $Z_{\mathbf{U},2}$ can be interpreted as the set of saddle points of the random harmonic function

$$u(z) = \sum_{m=1}^{\infty} \frac{\mathrm{Re}(a_m z^m)}{\sqrt{m}}$$

in $\mathbf{U}$. More generally, in any domain D, the zero set $Z_{D,2}$ can be interpreted as the set of saddle points of the Gaussian free field (with free boundary conditions) restricted to harmonic functions.

Joint moments of complex Gaussians. We will need the following known fact for the proofs of Hammersley's formula and Theorem 1.

FACT 10. *If $Z_1, ..., Z_n$ are mean-zero jointly complex Gaussian random variables with covariance matrix $\mathcal{M}_{jk} = \mathbf{E} Z_j \bar{Z}_k$, then $\mathbf{E}|Z_1 ... Z_n|^2 = \mathrm{perm}(\mathcal{M})$.*

Proof. We will check that in general for jointly complex normal, mean-zero random variables Z_j and W_j we have

$$\mathbf{E} Z_1 ... Z_n \overline{W}_1 ... \overline{W}_n = \sum_{\sigma} \prod_{j=1}^{k} \mathbf{E} Z_j \overline{W}_{\sigma(j)} = \mathrm{perm}(\mathbf{E} Z_j \overline{W}_k)_{j,k},$$

where the sum is over all permutations $\sigma \in S_n$. (See the book of Simon [25] for a similar statement in the real case.) Both sides are linear in each Z_j and $\overline{W}_j$, and we may assume that the Z_j and W_j are complex linear combinations of some finite i.i.d. standard complex Gaussian sequence $\{V_j\}_{j=1}^{n}$. The formula is proved by induction on the total number of nonzero coefficients that appear in the expression of the Z_j and W_j in terms of the V_j. If the number of nonzero coefficients is more than one for one of Z_j or W_j, then we may write that variable as a sum, and use induction and linearity. If it is 1 or 0 for all Z_j and W_j, then the formula is straightforward to verify; in fact, using independence it suffices to check that $V = V_j$ has $\mathbf{E} V^n \overline{V}^m = n! \mathbf{1}_{\{m=n\}}$. For $n \neq m$ this follows from the fact that V has a rotationally symmetric distribution. Otherwise, $|V|^{2n}$ has the distribution of the nth power of a rate-one exponential random variable, so its expectation equals $n!$. $\square$

3. A determinant formula in the i.i.d. case

The goal of this section is to prove Theorem 1 and Corollary 4. The proof relies on the i.i.d. nature of the coefficients of $f = f_{\mathbf{U}}$, Möbius invariance, Hammersley's formula and Borchardt's identity (26).

For $\beta\in\mathbf{U}$ let

$$T_\beta(z)=\frac{z-\beta}{1-\bar{\beta}z} \tag{21}$$

denote a Möbius transformation fixing the unit disk, and define

$$\tau_\beta(z)=\frac{(1-|\beta|^2)^{1/2}}{1-\bar{\beta}z},$$

so that $\tau_\beta^2(z)=T_\beta'(z)$.

Remark 11. Recall that for two jointly complex Gaussian random vectors X and Y, the distribution of Y given $X=0$ is the same as the distribution of Y with each entry projected to the orthocomplement (in L^2 of the underlying probability space) of the subspace spanned by the components X_i of X.

PROPOSITION 12. *Let $f=f_{\mathbf{U}}$ and $z_1,...,z_n\in\mathbf{U}$. The distribution of the random function*

$$T_{z_1}(z)\dots T_{z_n}(z)f(z) \tag{22}$$

is the same as the conditional distribution of $f(z)$ given $f(z_1)=...=f(z_n)=0$.

Proof. First consider $n=1$. The assertion is clear for $z_1=0$; here the i.i.d. property of the $\{a_k\}_{k=0}^\infty$ is crucial. More generally, for $z_1=\beta$, by (16) the random function $\tilde{f}=\tau_\beta(f\circ T_\beta)$ has the same distribution as f. Since $T_\beta(\beta)=0$, from the formula

$$\tilde{f}(z)=\tau_\beta(z)\sum_{k=0}^{\infty}a_k(T_\beta(z))^k$$

it is clear that the distribution of $T_\beta\tilde{f}$ is identical to the conditional distribution of $\tilde{f}$ given $\tilde{f}(\beta)=0$, whence the same must hold for f in place of $\tilde{f}$. The proposition for $n>1$ follows by induction: To go from n to $n+1$, we must condition $(f\mid f(z_1)=...=f(z_n)=0)$ on $f(z_{n+1})=0$. By the assumed identity for n points, this is equivalent to conditioning $(T_{z_1}\dots T_{z_n}f)(z)$ on $f(z_{n+1})=0$. By Remark 11, conditioning is a linear operator that commutes with multiplication by the deterministic functions T_{z_i}. Applying the equality of distributions $(f(z)\mid f(z_{n+1})=0)\overset{d}{=}T_{z_{n+1}}(z)f(z)$ completes the proof. □

Fix $z_1,...,z_n\in\mathbf{U}$ and let

$$\Upsilon(z)=\prod_{j=1}^{n}T_{z_j}(z). \tag{23}$$

Since $T_{z_k}(z_k)=0$ and $T_{z_k}'(z_k)=1/(1-z_k\bar{z}_k)$, we have

$$\Upsilon'(z_k)=T_{z_k}'(z_k)\prod_{j:j\neq k}T_{z_j}(z_k)=\prod_{j=1}^{n}\frac{1}{1-z_j\bar{z}_k}\prod_{j:j\neq k}(z_j-z_k) \tag{24}$$

for each $k\leqslant n$.

COROLLARY 13. *Let* $f=f_{\mathbf{U}}$ *and* $z_1, ..., z_n \in \mathbf{U}$. *The conditional joint distribution of the random variables* $(f'(z_k): k=1, ..., n)$ *given that* $f(z_1)=...=f(z_n)=0$ *is the same as the unconditional joint distribution of* $(\Upsilon'(z_k)f(z_k): k=1, ..., n)$.

Proof. The conditional distribution of f given that $f(z_j)=0$ for $1\leqslant j\leqslant n$ is the same as the unconditional distribution of Υf. Since $\Upsilon(z_k)=0$, the derivative of $\Upsilon(z)f(z)$ at $z=z_k$ equals $\Upsilon'(z_k)f(z_k)$. □

Consider the $n\times n$-matrices A and M, with entries

$$A_{j,k} = \mathbf{E}f(z_j)\overline{f(z_k)} = (1-z_j\bar{z}_k)^{-1},$$
$$M_{j,k} = (1-z_j\bar{z}_k)^{-2}.$$

By the classical Cauchy determinant formula, see Muir [17, p. 311], we have

$$\det A = \prod_{k,j} \frac{1}{1-z_j\bar{z}_k} \prod_{k<j} (z_j-z_k)(\bar{z}_j-\bar{z}_k).$$

Comparing this to (24), we see that

$$\det A = \prod_{k=1}^{n} |\Upsilon'(z_k)|. \tag{25}$$

We will need the classical identity of Borchardt [6] (see also Minc [16]):

$$\operatorname{perm}\left(\frac{1}{x_j+y_k}\right)_{j,k} \det\left(\frac{1}{x_j+y_k}\right)_{j,k} = \det\left(\frac{1}{(x_j+y_k)^2}\right)_{j,k}. \tag{26}$$

Setting $x_j=z_j^{-1}$, $y_k=-\bar{z}_k$ and dividing both sides by $\prod_{j=1}^{n} z_j^2$ gives that

$$\operatorname{perm}(A) \det A = \det M. \tag{27}$$

We are finally ready to prove Theorem 1. Corollary 4 is a direct consequence of the conformal invariance in Proposition 9 and the way the Szegő and Bergman kernels transform under conformal maps (see (14)).

Proof of Theorem 1. Recall from (8) that

$$p(z_1, ..., z_n) = \frac{\mathbf{E}\big(|f'(z_1) ... f'(z_n)|^2 \,|\, f(z_1) = ... = f(z_n) = 0\big)}{\pi^n \det A}. \tag{28}$$

By Corollary 13, the numerator of the right-hand side of (28) equals

$$\mathbf{E}(|f(z_1) ... f(z_n)|^2) \prod_{k=1}^{n} |\Upsilon'(z_k)|^2 = \operatorname{perm}(A) \det(A)^2, \tag{29}$$

where the last equality uses the Gaussian moment formula of Fact 10 and (25). Thus

$$p(z_1, ..., z_n) = \pi^{-n} \operatorname{perm}(A) \det A = \pi^{-n} \det M$$

by (27). □

4. The number of zeros of $f_{\mathbf{U}}$ in a disk

In this section we prove Theorem 2 and Corollary 3. In fact, the corollary only uses part (i) of the theorem, so we delay the proof of part (ii) to the end of the section.

LEMMA 14. *Let $r_1 \leqslant ... \leqslant r_m$, let $B_j = B_{r_j}(0)$ and let $\widetilde{N}_j = \#(Z_{\mathbf{U}} \cap B_j)$. Then*

$$\mathbf{E}\widetilde{N}_1(\widetilde{N}_2-1)\,...\,(\widetilde{N}_m-m+1) = \sum_{\sigma}\prod_{\nu\in\sigma}(-1)^{|\nu|+1}\frac{r_\nu^2}{1-r_\nu^2}, \tag{30}$$

where the sum is over all permutations σ of $\{1, ..., m\}$, the product is over all cycles ν of the permutation, $|\nu|$ is the length of ν, and $r_\nu = \prod_{j\in\nu} r_j$.

Proof. Applying Proposition 8 (ii) to the set $B_1 \times ... \times B_n$ we get

$$\begin{aligned}\mathbf{E}\widetilde{N}_1(\widetilde{N}_2-1)\,...\,(\widetilde{N}_m-m+1) &= \int_{B_1\times...\times B_m} p(z_1, ..., z_m)\, dz_1 \,...\, dz_m \\ &= \int_{B_1\times...\times B_m} \det(K(z_i, z_j))_{i,j}\, dz_1 \,...\, dz_m.\end{aligned}$$

Expanding the determinant and exchanging sums and integrals we get a sum over all permutations of m elements:

$$\sum_{\sigma} \operatorname{sgn}(\sigma) \int_{B_1\times...\times B_m} K(z_1, z_{\sigma_1}) \,...\, K(z_m, z_{\sigma_m})\, dz_1 \,...\, dz_m.$$

For each permutation σ, the corresponding integral is a product over cycles ν of σ of

$$I_\nu = \int K(z_1, z_2) K(z_2, z_3) \,...\, K(z_{|\nu|}, z_1)\, dz_1 \,...\, dz_{|\nu|}, \tag{31}$$

where ν is an ordered subset of $\{1, ..., m\}$ and each variable z_i ranges over the disk B_{ν_i} of radius r_{ν_i}. The formula for the Bergman kernel gives

$$K(z_1, z_2) = \frac{1}{\pi(1-z_1\bar{z}_2)^2} = \frac{1}{\pi}\sum_{n=0}^{\infty}(n+1)(z_1\bar{z}_2)^n.$$

Using this, we expand the product in (31) into a sum of monomials in the variables $\{z_j\}_{j=1}^n$ and $\{\bar{z}_j\}_{j=1}^n$; then we integrate term by term. Each monomial in which the exponents of z_j and $\bar{z}_j$ are different for some j integrates to 0. Thus in all remaining terms the exponents of z_j and $\bar{z}_j$ are the same. Since z_j always comes as part of a product $z_j\bar{z}_{j+1}$, the exponents of z_j and $\bar{z}_{j+1}$ have to be the same as well. This implies that in nonvanishing terms all exponents agree, and we are left with

$$I_\nu = \sum_{n=0}^{\infty} \pi^{-|\nu|} \int (n+1)^{|\nu|} (z_1\bar{z}_1 \,...\, z_{|\nu|}\bar{z}_{|\nu|})^n\, dz_1 \,...\, dz_{|\nu|}.$$

Since

$$(n+1)\int_{B_r(0)} |z|^{2n}\,dz = 2\pi(n+1)\int_0^r s^{2n+1}\,ds = \pi r^{2n+2},$$

setting $r_\nu = \prod_{j\in\nu} r_j$ we get

$$I_\nu = \sum_{n=0}^{\infty} r_\nu^{2(n+1)} = \frac{r_\nu^2}{1-r_\nu^2}.$$

Since $\operatorname{sgn}\sigma = \prod_{\nu\in\sigma}(-1)^{|\nu|+1}$, this completes the proof of the lemma. $\square$

Proof of Theorem 2 (i). Put

$$\beta_k = \mathbf{E}\binom{N_r}{k},$$

and let $\mathcal{P}$ be chosen uniformly at random from all permutations of $\{1, ..., k\}$. Let $q=r^2$. Then by (30), we have

$$\beta_k = \mathbf{E}\prod_{y\in\mathcal{P}} (-1)^{|y|+1}\frac{q^{|y|}}{1-q^{|y|}},$$

where the product is over cycles y of $\mathcal{P}$. Since the cycle containing 1 of $\mathcal{P}$ has length that is uniform on $\{1, ..., k\}$, and given that cycle, the other cycles form a uniform permutation on the rest of $\{1, ..., k\}$, we get the recursion

$$\beta_k = \frac{1}{k}\sum_{l=1}^{k} (-1)^{l+1}\frac{q^l}{1-q^l}\beta_{k-l}, \tag{32}$$

with $\beta_0=1$. Consider the generating function $\beta(s)=\sum_{k=0}^{\infty}\beta_k s^k$. Multiplying (32) by ks^k and summing over $k\geqslant 1$, we get

$$s\beta'(s) = \beta(s)\,s\psi(s), \tag{33}$$

where

$$\psi(s) = \sum_{l=1}^{\infty} (-s)^{l-1}\frac{q^l}{1-q^l} = \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} (-s)^{l-1} q^{kl}.$$

We write (33) in the form

$$(\log\beta(s))' = \psi(s),$$

which we integrate to get

$$\log\beta(s) = -\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\frac{(-q^k s)^l}{l} = \sum_{k=1}^{\infty}\log(1+q^k s),$$

where the constant term is zero as $\beta_0=\beta(0)=1$. Thus

$$\beta(s)=\prod_{k=1}^{\infty}(1+q^k s). \tag{34}$$

Taking expectations of the identity

$$(s+1)^{N_r}=\sum_{k=0}^{\infty}\binom{N_r}{k}s^k$$

gives

$$\mathbf{E}(s+1)^{N_r}=\sum_{k=0}^{\infty}\beta_k s^k=\beta(s),$$

and this concludes the proof of Theorem 2 (i). □

Proof of Corollary 3. (i) Theorem 2 implies that $\mathbf{P}(N_r=0)=\prod_{k=1}^{\infty}(1-r^{2k})$ and the asymptotics for the right-hand side are classical, see Newman [19, p. 19]. For the reader's convenience we indicate the argument. Let $L=\log\mathbf{P}(N_r=0)=\sum_{k=1}^{\infty}\log(1-r^{2k})$, which we compare to the integral

$$I=\int_1^{\infty}\log(1-r^{2k})\,dk=-\frac{1}{2\log r}\int_{-2\log r}^{\infty}\log(1-e^{-x})\,dx. \tag{35}$$

We have $I+\log(1-r^2)<L<I$, so $L=I+o(h)$. Since $-\log(1-e^{-x})=\sum_{n=1}^{\infty}e^{-nx}/n$, the integral in (35) converges to $-\frac{1}{6}\pi^2$. But

$$-\frac{1}{2\log r}=\frac{\frac{1}{2}+o(1)}{1-r}=\frac{h}{4\pi}+o(h),$$

and we get

$$L=-\frac{\frac{1}{12}\pi^2+o(1)}{1-r}=-\frac{\pi h}{24}+o(h),$$

as claimed.

(ii) One of Euler's partition identities (see Pak [21, §2.3.4]) gives

$$\prod_{k=1}^{\infty}(1+q^k s)=\sum_{k=0}^{\infty}\frac{q^{\binom{k+1}{2}}s^k}{(1-q)\dots(1-q^k)}, \tag{36}$$

so the claim follows from (34).

(iii) The formulas for the mean and variance of N_r follow from the binomial moment formula (30). Using the general central limit theorem due to Costin–Lebowitz [7] and Soshnikov [29, p. 497] for determinantal processes, we get that as $r\to 1$, the normalized distribution of N_r converges to standard normal, as required. Alternatively, the last claim can be easily verified by computing the asymptotics of the moment generating function directly. Yet another way is to apply Lindeberg's triangular array central limit theorem to the representation of N_r as the sum of independent random variables, as given in Theorem 2 (i). □

4.1. The joint distribution of the moduli of $Z_{\mathbf{U}}$

Proof of Theorem 2 (ii). The zero set of $f_{\mathbf{U}}$ is determinantal with the Bergman kernel $K(z,w)$. Let

$$K_n(z,w)=\frac{1}{\pi}\sum_{j=0}^{n-1}(j+1)(z\bar{w})^j.$$

Since $K_n(z,w)\to K(z,w)$, as $n\to\infty$, uniformly on compact subsets of $\mathbf{U}^2$, Proposition 3.10 of Shirai–Takahashi [24] yields that the determinantal point processes with kernels K_n converge weakly, as $n\to\infty$, to $Z_{\mathbf{U}}$. Thus it suffices to prove that the set of absolute values $\{|\zeta_j|\}_{j=1}^n$ of the n random points of the determinantal process with kernel K_n has the same law as $\{U_j^{1/2j}\}_{j=1}^n$, where U_j are i.i.d. uniform on [0,1].

For any $z_1,...,z_n$,

$$\begin{pmatrix} K_n(z_1,z_1) & \dots & K_n(z_1,z_n) \\ \vdots & \ddots & \vdots \\ K_n(z_n,z_1) & \dots & K_n(z_n,z_n) \end{pmatrix}$$

$$=\frac{1}{\pi^n}\begin{pmatrix} 1 & z_1 & \dots & z_1^{n-1} \\ 1 & z_2 & \dots & z_2^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & z_n & \dots & z_n^{n-1} \end{pmatrix}\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & n \end{pmatrix}\begin{pmatrix} 1 & \dots & 1 \\ \bar{z}_1 & \dots & \bar{z}_n \\ \vdots & \ddots & \vdots \\ \bar{z}_1^{n-1} & \dots & \bar{z}_n^{n-1} \end{pmatrix}.$$

Setting $z_j=r_je^{i\theta_j}$ we find that the joint intensity of $\{|\zeta_j|\}_{j=1}^n$, evaluated at $\{r_j\}_{j=1}^n$, equals

$$\begin{aligned}&\int_{[0,2\pi]^n}\det(K_n(z_j,z_k))_{j,k=1}^n\, r_1\,d\theta_1\dots r_n\,d\theta_n \\ &=\frac{n!}{\pi^n}\int_{[0,2\pi]^n}\Big(\sum_\sigma \operatorname{sgn}(\sigma)\prod_{j=1}^n z_j^{\sigma_j-1}\Big)\Big(\sum_\tau \operatorname{sgn}(\tau)\prod_{j=1}^n \bar{z}_j^{\tau_j-1}\Big)\, r_1\,d\theta_1\dots r_n\,d\theta_n.\end{aligned} \tag{37}$$

When we expand the sums, for $\sigma\neq\tau$ the integrand contains a factor of the form $z_j^p\bar{z}_j^q$ with $p\neq q$, and therefore the integral vanishes. When $\sigma=\tau$, we get $(2\pi)^n\prod_{j=1}^n r_j^{2\sigma_j-1}$. Thus (37) equals

$$2^n n!\sum_\sigma\prod_{j=1}^n r_j^{2\sigma_j-1}. \tag{38}$$

Now $U_j^{1/2j}$ has density $2jx^{2j-1}$ in $[0,1]$. Hence, the joint intensity of $\{U_1^{1/2},...,U_n^{1/2n}\}$ is precisely (38). This proves the theorem. □

Remark. The proof above is modeled after an argument of Kostlan [13] for the distribution of the eigenvalues of a random complex Gaussian matrix. It is simpler than our original proof that relied on random permutations.

5. Law of large numbers

The goal of this section is to prove Proposition 5, the law of large numbers for the number of zeros of $f_{\mathbf{U},\varrho}$. We will use the following lemma in the proof:

LEMMA 15. *Let μ be a Borel measure on a metric space S, and assume that all balls of the same radius have the same measure. Let $\psi: [0,\infty)\to[0,\infty)$ be a nonincreasing function. Let $A\subset S$ be a Borel set, and let $B=B_R(x)$ be a ball centered at $x\in S$ with $\mu(A)=\mu(B_R(x))$. Then for all $y\in S$,*

$$\int_A \psi(\operatorname{dist}(y,z))\,d\mu(z) \leqslant \int_B \psi(\operatorname{dist}(x,z))\,d\mu(z).$$

Proof. It suffices to check this claim for indicator functions $\psi(x)=\mathbf{1}_{\{x\leqslant r\}}$. In this case, the inequality reduces to

$$\mu(A\cap B_r(y)) \leqslant \mu(B_R(x)\cap B_r(x)),$$

which is clearly true both for $r\leqslant R$ and for $r>R$. □

Proof of Proposition 5. We have

$$\mathbf{E}N(h) = \int_\Lambda p(z)\,dz = \frac{4\varrho}{\pi}h.$$

Let $Q(z,w)=p(z,w)/p(z)p(w)$. Then by formula (19) we have

$$Q(0,w)-1 \leqslant C(1-|w|^2)^\varrho.$$

We denote the right-hand side by $\psi(0,w)$ and extend ψ to $\mathbf{U}^2$ so that it only depends on hyperbolic distance. We obtain

$$\begin{aligned}\mathbf{E}(N(h)(N(h)-1))-(\mathbf{E}N(h))^2 &= \int_\Lambda\int_\Lambda (p(z,w)-p(z)p(w))\,dw\,dz \\ &= \int_\Lambda\int_\Lambda (Q(z,w)-1)\,p(w)\,dw\,p(z)\,dz \\ &\leqslant \int_\Lambda\int_\Lambda \psi(z,w)\,p(w)\,dw\,p(z)\,dz.\end{aligned}$$

Let $B_R(0)$ be a ball with hyperbolic area $h=4\pi R^2/(1-R^2)$. Note that $p(w)\,dw$ is constant times the hyperbolic area element, so we may use Lemma 15 to bound the inner integral by

$$\int_{B_R(0)} \psi(0,w)p(w)\,dw = c\int_0^R (1-r^2)^\varrho(1-r^2)^{-2}r\,dr = \frac{c}{2}\int_S^1 s^{\varrho-2}\,ds$$

with $S=1-R^2$. Thus we get

$$\operatorname{Var} N(h) = \mathbf{E}N(h)+\mathbf{E}(N(h)(N(h)-1))-(\mathbf{E}N(h))^2 \leqslant \frac{h\varrho}{4\pi}+\frac{ch\varrho}{8\pi}\int_S^1 s^{\varrho-2}\,ds. \tag{39}$$

For $\varrho>1$ this is integrable, so $\operatorname{Var} N(h) \leqslant O(h)$. For $\varrho<1$ we can bound the right-hand side of (39) by $O(hS^{\varrho-1})=O(h^{2-\varrho})$. Thus in both cases, as well as when $\varrho=1$ (see Corollary 3 (iii)), we have

$$\operatorname{Var} N(h) \leqslant c(\mathbf{E}N(h))^{2-\beta}$$

with $\beta=\varrho\wedge 1>0$. For $\eta>1/\beta$, we find that

$$Y_k = \frac{N(k^\eta)-\mathbf{E}N(k^\eta)}{\mathbf{E}N(k^\eta)}$$

satisfies $\mathbf{E}Y_k^2=O(k^{-\eta\beta})$, whence $\mathbf{E}\sum_{k=1}^\infty Y_k^2<\infty$, so $Y_k\to 0$ a.s. Monotonicity and interpolation now give the desired result. □

6. Reconstructing the function from its zeros

The goal of this section is to prove Theorem 6. The main step in the proof is the following result:

PROPOSITION 16. *Let $c'_\varrho=e^{\varrho/2-\gamma/2}$. We have*

$$|f_{\mathbf{U},\varrho}(0)| = c'_\varrho \lim_{r\to 1}(1-r^2)^{-\varrho/2} \prod_{\substack{z\in Z_{\mathbf{U},\varrho}\\ |z|<r}} |z| \quad a.s.$$

We first need a simple lemma.

LEMMA 17. *If X and Y are jointly complex Gaussians with variance 1, then for some absolute constant c we have*

$$|\operatorname{Cov}(\log|X|, \log|Y|)| \leqslant c|\mathbf{E}X\bar{Y}|. \tag{40}$$

Proof. Write $Y=\alpha X+\beta Z$, where X and Z are i.i.d. standard complex Gaussian variables, $\alpha=\mathbf{E}X\bar{Y}$ and $|\alpha|^2+|\beta|^2=1$. It clearly suffices to consider $|\alpha|<\frac{1}{2}$. Since

$$\log|Y| = \log|\beta Z|+\log\left|1+\frac{\alpha X}{\beta Z}\right|,$$

the inequality (40) reduces to

$$\left|\operatorname{Cov}\left(\log|X|, \log\left|1+\frac{\alpha X}{\beta Z}\right|\right)\right| \leqslant c|\alpha|. \tag{41}$$

We will use the estimate

$$\mathbf{E}\left|\log\left|1+\frac{\lambda}{Z}\right|\right| \leqslant c_1|\lambda| \quad \text{for } \lambda\in\mathbf{C}, \tag{42}$$

which can be verified by considering the positive and negative parts of $\log|1+\lambda/Z|$ as follows. The positive part is handled using the numerical inequality $\log|1+w|\leqslant|w|$ and the integrability of $|Z|^{-1}$. For the negative part, when $|\lambda|\geqslant 1$, the density of $|1+\lambda/Z|$ is uniformly bounded in the disk of radius $\frac{1}{2}$, so it remains to consider the case $|\lambda|<1$. Then $\mathbf{E}\log_-|1+\lambda/Z|$ can be controlled by partitioning into the events

$$G_k=\{e^{-k}<|1+\lambda/Z|\leqslant e^{1-k}\}.$$

Since $\mathbf{P}(G_k)=O(|\lambda^2|e^{-2k})$, we get

$$\mathbf{E}\,\mathbf{1}_{G_k}\log_-\left|1+\frac{\lambda}{Z}\right| = O(k|\lambda^2|e^{-2k}).$$

Summing over k establishes (42).

By conditioning on X, (42) yields

$$\mathbf{E}\left(\log|X|\cdot\log\left|1+\frac{\alpha X}{\beta Z}\right|\right) \leqslant c_1\left|\frac{\alpha}{\beta}\right|\mathbf{E}\,|X\log|X||=c_2\left|\frac{\alpha}{\beta}\right|.$$

This bounds the first term (expectation of the product) in the covariance on the left-hand side of (41). The second term (product of expectations) can be bounded by the same argument. □

Proof of Proposition 16. Assume that $f=f_{\mathbf{U},\varrho}$ has no zeros at 0 or on the circle of radius r. Then Jensen's formula (Ahlfors [1, §5.3.1]) gives

$$\log|f(0)|=\frac{1}{2\pi}\int_0^{2\pi}\log|f(re^{i\alpha})|\,d\alpha+\sum_{\substack{z\in Z\\|z|<r}}\log\frac{|z|}{r},$$

where $Z=Z_{\mathbf{U},\varrho}$. Let $|f(re^{i\alpha})|^2=\sigma_r^2 Y$, where

$$\sigma_r^2=\operatorname{Var} f(re^{i\alpha})=[2\pi S_{\mathbf{U}}(r,r)]^{\varrho}=(1-r^2)^{-\varrho}$$

and Y is an exponential random variable with mean 1. We have

$$\mathbf{E}\log|f(re^{i\alpha})|=\tfrac{1}{2}(\log\sigma_r^2+\mathbf{E}\log Y)=\tfrac{1}{2}(-\varrho\log(1-r^2)-\gamma),$$

where the second equality follows from the integral formula for Euler's constant

$$\gamma = -\int_0^\infty e^{-x} \log x \, dx.$$

Introduce the notation

$$g_r(\alpha) = \log|f(re^{i\alpha})| + \tfrac{1}{2}(\varrho \log(1-r^2)+\gamma)$$

so that the distribution of $g_r(\alpha)$ does not depend on r and α, and $\mathbf{E}g_r(\alpha)=0$. Let

$$L_r = \frac{1}{2\pi}\int_0^{2\pi} g_r(\alpha)\, d\alpha.$$

We first prove that $L_r \to 0$ a.s. over a suitable deterministic sequence $r_n \uparrow 1$. We compute

$$\operatorname{Var} L_r = \mathbf{E}\,\frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} g_r(\alpha)\, g_r(\beta)\, d\beta\, d\alpha.$$

Since the above is absolutely integrable, we can exchange integral and expected value to get

$$\operatorname{Var} L_r = \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} \mathbf{E}(g_r(\alpha)\, g_r(\beta))\, d\beta\, d\alpha = \frac{1}{2\pi}\int_0^{2\pi} \mathbf{E}(g_r(\alpha)\, g_r(0))\, d\alpha,$$

where the second equality follows from rotational invariance. By Lemma 17, we have

$$\mathbf{E}\, g_r(\alpha)\, g_r(0) \leqslant c\,\frac{|\mathbf{E} f(re^{i\alpha})\overline{f(r)}|}{\operatorname{Var} f(r)} = c\left|\frac{1-r^2}{1-r^2 e^{i\alpha}}\right|^{\varrho}.$$

Let $\varepsilon = 1-r^2 < \frac{1}{2}$. Then for $\alpha \in [0,\pi]$ we get the bound

$$|1-r^2 e^{i\alpha}| \geqslant \begin{cases} \varepsilon, & |\alpha| \leqslant \varepsilon, \\ 2r^2 \sin\frac{1}{2}\alpha \geqslant \frac{1}{2}\alpha, & \varepsilon < \alpha < \frac{1}{2}\pi, \\ 1, & \frac{1}{2}\pi \leqslant \alpha \leqslant \pi, \end{cases}$$

which gives

$$\frac{1}{2c\varepsilon^\varrho}\operatorname{Var} L_r \leqslant \int_0^\pi \frac{d\alpha}{|1-r^2 e^{i\alpha}|^\varrho} \leqslant \varepsilon^{1-\varrho} + \frac{1}{2}\int_\varepsilon^{\pi/2} \alpha\, d\alpha + \frac{1}{2}\pi \leqslant \begin{cases} c', & \varrho < 1, \\ c'|\log\varepsilon|, & \varrho = 1, \\ c'\varepsilon^{1-\varrho}, & \varrho > 1. \end{cases}$$

By Chebyshev's inequality and the Borel–Cantelli lemma, this shows that, as $r \to 1$ over the sequence $r_n = 1-n^{-(1\vee(1/\varrho)+\delta)}$, we have a.s. $L_{r_n} \to 0$ and

$$\sum_{\substack{z\in Z \\ |z|<r}} \log\frac{|z|}{r} - \frac{\varrho\log(1-r^2)+\gamma}{2} \to \log|f(0)|,$$

or, exponentiating,

$$e^{-\gamma/2}(1-r^2)^{-\varrho/2}\prod_{\substack{z\in Z_{\mathbf{U},\varrho}\\|z|<r}}\frac{|z|}{r}\to|f(0)|. \tag{43}$$

Since the product is monotone decreasing and the ratio $(1-r_n^2)/(1-r_{n+1}^2)$ converges to 1, it follows that the limit is the same over every sequence $r_n\to 1$ a.s.

Finally, by the law of large numbers (Proposition 5), the number of zeros N_r in the ball of Euclidean radius r satisfies

$$N_r=\frac{r^2\varrho}{1-r^2}(1+o(1))=\frac{\varrho+o(1)}{1-r^2}\quad\text{a.s.}, \tag{44}$$

whence

$$r^{N_r}=\exp(N_r\log r)=e^{-\varrho/2+o(1)}\quad\text{a.s.}$$

Multiplying this with (43) yields the claim. □

Proof of Theorem 6. (i) By the law of large numbers for N_r (see also (44)),

$$\sum_{|z_k|\leqslant r}\frac{1}{k}=\gamma+\log N_r+o(1)=\gamma+\log\varrho-\log(1-r^2)+o(1). \tag{45}$$

Multiplying by $\frac{1}{2}\varrho$ and exponentiating, we get that

$$\prod_{|z_k|\leqslant r}e^{\varrho/2k}=e^{\gamma\varrho/2}\varrho^{\varrho/2}(1-r^2)^{-\varrho/2}(1+o(1)). \tag{46}$$

In conjunction with Proposition 16, this yields (6).

(ii) Let $f=f_{\mathbf{U},\varrho}$ and

$$T(z)=\frac{z-\zeta}{1-\bar{\zeta}z}.$$

By (16), f has the same law as

$$\tilde{f}=(T')^{\varrho/2}(f\circ T). \tag{47}$$

Now $T'(\zeta)=(1-|\zeta|^2)^{-1}$. Therefore

$$\tilde{f}(\zeta)=(1-|\zeta|^2)^{-\varrho/2}f(0)=c_\varrho\prod_{k=1}^{\infty}e^{\varrho/2k}|z_k|\quad\text{a.s.},$$

where $\{z_k\}_{k=1}^{\infty}$ are the zeros of f in increasing modulus. If $T(\zeta_k)=z_k$ then $\{\zeta_k\}_{k=1}^{\infty}$ are the zeros of $\tilde{f}$ in increasing hyperbolic distance from ζ. We conclude that

$$\tilde{f}(\zeta)=c_\varrho(1-|\zeta|^2)^{-\varrho/2}\prod_{k=1}^{\infty}e^{\varrho/2k}|T(\zeta_k)|\quad\text{a.s.}$$ □

For our study of the dynamics of zeros in the next section, we will need a reconstruction formula for $|f'_{\mathbf{U},\varrho}(0)|$ when we condition on the event $0\in Z_{\mathbf{U},\varrho}$.

LEMMA 18. *Denote by* Ω_ε *the event that the power series* $f_{\mathbf{U},\varrho}$ *defined in* (5) *has a zero in* $B_\varepsilon(0)$. *As* $\varepsilon\to 0$, *the conditional distribution of the coefficients* $a_1, a_2, a_3, \ldots$, *given* Ω_ε, *converges to a product law, where* a_1 *is rotationally symmetric,* $|a_1|$ *has density* $2r^3e^{-r^2}$, *and* $a_2, a_3, \ldots$ *are standard complex Gaussians.*

Proof. Let a_0 and a_1 be i.i.d. standard complex normal random variables, and $\varrho>0$. Consider first the limiting distribution, as $\varepsilon\to 0$, of a_1 given that the equation $a_0+a_1\sqrt{\varrho}\,z=0$ has a root Z in $B_\varepsilon(0)$. The limiting distribution must be rotationally symmetric, so it suffices to compute its radial part. If $S=|a_0|^2$ and $T=|a_1|^2$, set $U=\varrho|Z|^2=S/T$. The joint density of (S,T) is e^{-s-t}, so the joint density of (U,T) is $e^{-ut-t}t$. Thus as $\varepsilon\to 0$, the conditional density of T given $U<\varrho\varepsilon^2$ converges to the conditional density given $U=0$, i.e., te^{-t}. This means that the conditional distribution of a_1 is not normal; rather, its radial part has density $2r^3e^{-r^2}$.

We can now prove the lemma. The conditional density of the coefficients $a_1, a_2, \ldots$ given Ω_ε, with respect to their original product law, is given by the ratio

$$\frac{\mathbf{P}(\Omega_\varepsilon \mid a_1, a_2, \ldots)}{\mathbf{P}(\Omega_\varepsilon)}.$$

By Lemma 30, the limit of this ratio is not affected if we replace $f_{\mathbf{U},\varrho}$ by its linearization $a_0+a_1\sqrt{\varrho}\,z$. This yields the statement of the lemma. □

Kakutani's absolute continuity criterion (see Williams [31, Theorem 14.17]) applied to the coefficients gives the following lemma:

LEMMA 19. *The distributions of the random functions* $f_{\mathbf{U},\varrho}(z)$ *and* $(f_{\mathbf{U},\varrho}(z)-a_0)/z$ *are mutually absolutely continuous.*

Remark 20. By Lemma 18, conditioning on $0\in Z_{\mathbf{U},\varrho}$ amounts to setting $a_0=0$ and changing the distribution of a_1 in an absolutely continuous manner. Thus, by Lemma 19, given $0\in Z_{\mathbf{U},\varrho}$ the distribution of the random function $g(z)=f_{\mathbf{U},\varrho}(z)/z$ is absolutely continuous with respect to the distribution of the unconditioned $f_{\mathbf{U},\varrho}(z)$. Hence we may apply Theorem 6 to $g(z)$ and get that given $0\in Z_{\mathbf{U},\varrho}$, if we order the *other* zeros of $f_{\mathbf{U},\varrho}$ in increasing absolute value as $\{z_k\}_{k=1}^{\infty}$, then

$$|f'_{\mathbf{U},\varrho}(0)| = |g(0)| = c_\varrho \prod_{k=1}^{\infty} e^{\varrho/2k}|z_k| \quad \text{a.s.} \tag{48}$$

7. Dynamics of zeros

In order to understand the point process of zeros of $f_{\mathbf{U}}$ it is useful to think of it as a stationary distribution of a time-homogeneous Markov process.

Define the complex Ornstein–Uhlenbeck process

$$a(t) := e^{-t/2} W(e^t), \quad W(t) := \frac{B_1(t)+iB_2(t)}{\sqrt{2}},$$

where B_1 and B_2 are independent standard Brownian motions, and $W(t)$ is complex Brownian motion scaled so that $\mathbf{E} W(1)\overline{W(1)}=1$. The process $\{a(t)\}$ is then stationary Markov with the standard complex Gaussian as its stationary distribution. First we consider the process

$$\varphi_t(z)\,\varphi_t(z;D) = \sum_{n=0}^{\infty} W_n(t)\,\psi_n(z), \quad t>0,$$

where W_n are independent complex Brownian motions and $\{\psi_n(z)\}_{n=0}^{\infty}$ is an orthonormal basis for $H^2(D)$. With $t=e^{\tau}$ we get the time-homogeneous process

$$f_\tau(z) = e^{-\tau/2}\varphi_{e^\tau}(z) = \sum_{n=0}^{\infty} a_n(\tau)\,\psi_n(z).$$

Then the entire process $\varphi_t(z)$ (and so $f_\tau(z)$) is conformally covariant in the sense that if $T\colon \Lambda \to D$ is a conformal homeomorphism, then the process

$$\big\{\sqrt{T'(z)}\,\varphi_t(T(z))\big\}_{t>0}$$

has the same distribution as $\varphi_t(z;\Lambda)$, $t>0$. For this, by continuity, it suffices to check that the covariances agree. Indeed, for $s \leqslant t$,

$$\mathbf{E}\varphi_s(z)\,\overline{\varphi_t(w)} = \mathbf{E}\varphi_s(z)\,\overline{\varphi_s(w)},$$

so the problem is reduced to checking the equality of covariances for a fixed time, which has already been done in Proposition 9.

It follows automatically that the process $\{Z_D(t)\}$ of zeros of φ_t is conformally invariant. To check that it is a Markov process, recall from §2 that $Z_D(t)$ determines φ_t up to a multiplicative constant of modulus 1. It is easy to check that φ_t modulo such a constant is a Markov process; it follows that $Z_D(t)$ is a Markov process as well.

Remark 21. This argument works in the case $\varrho=1$. By replacing the i.i.d. coefficients a_n in (5) with Ornstein–Uhlenbeck processes, it is possible to define a dynamic version of the case $\varrho \neq 1$ in the unit disk. The same argument as above shows that these are Markov processes with distribution invariant under Möbius transformations of $\mathbf{U}$.

Finally, we give a stochastic differential equation description of the motion of zeros. Condition on starting at time 1 with a zero at the origin. This implies that $W_0(1)=0$, and by the Markov property all the W_i are complex Brownian motions started from some initial distribution at time 1. For t in a small time interval $(1,1+\varepsilon)$ and for z in the neighborhood of 0, we have

$$\varphi_t(z) = W_0(t)+W_1(t)z+W_2(t)z^2+O(z^3).$$

If $W_1(1)W_2(1)\neq 0$, then the movement of the root z_t of φ_t where $z_1=0$ is described by the movement of the solution of the equation $W_0(t)+W_1(t)z_t+W_2(t)z_t^2=O(z_t^3)$. Solving the quadratic gives

$$z_t = -\frac{W_1}{2W_2}\left(1-\sqrt{1-\frac{4W_0W_2}{W_1^2}}\right)+O(W_0^3).$$

Expanding the square root we get

$$z_t = -\frac{W_0}{W_1}+\frac{W_0^2W_2}{W_1^3}+O(W_0^3).$$

Since $W_0(t)$ is complex, $W_0^2(t)$ is a martingale, so there is no drift term. The noise term then has coefficient $-1/W_1$, so the movement of the zero at 0 is described by the stochastic differential equation (at $t=1$) $dz_t=-W_1(t)^{-1}\,dW_0(t)$, or, rescaling time for the time-homogeneous version, for any τ with $a_0(\tau)=0$ we get

$$dz_\tau = -\frac{1}{a_1(\tau)}\,da_0(\tau). \tag{49}$$

The absence of drift in (49) can be understood as follows: in the neighborhood we are interested in, this solution z_t will be an analytic function of the $\{W_n\}_{n=0}^{\infty}$, and therefore has no drift.

For other values of ϱ the same argument gives

$$dz_\tau = -\frac{1}{\sqrt{\varrho}\,a_1(\tau)}\,da_0(\tau).$$

Of course, it is more informative to describe this movement in terms of the relationship to other zeros, as opposed to the coefficient a_1. For this, we consider the reconstruction formula in Remark 20, which gives

$$|a_1| = |f'_{\mathbf{U},\varrho}(0)| = c_\varrho \prod_{k=1}^{\infty} e^{\varrho/2k}|z_k| \quad \text{a.s.}$$

This means that when there are many other zeros close to a zero, the noise term in its movement grows and it oscillates wildly. This produces a repulsion effect for zeros that we have already observed in the point process description. The equation (49) does not give a full description of the process as the noise terms for different zeros are correlated.

8. Hammersley's formula for Gaussian analytic functions

A version of the following theorem was proved by Hammersley [11]. The present version is from Friedman [10, Appendix B]. We say that a point process has *integral joint intensity* p if formula (10) holds for its counting function N.

THEOREM 22. *Let $f_n = a_n z^n + \ldots + a_0$ be a random polynomial so that $(a_0, \ldots, a_n)$ has an absolutely continuous distribution with respect to Lebesgue measure on $\mathbf{C}^{n+1}$. Then the integral joint intensity of zeros exists and equals*

$$p(z_1, \ldots, z_k) = \lim_{\varepsilon \to 0} (\pi \varepsilon^2)^{-k} \int_{\substack{f(z_i) \in B_\varepsilon(0) \\ i=1,\ldots,k}} |f'(z_1) \ldots f'(z_k)|^2 \, da_0 \ldots da_n. \tag{50}$$

We also need the following consequence of Cauchy's integral formula:

FACT 23. *Let D be a bounded domain, and let $B \subset D$ be a closed disk. Then for every $m \geqslant 0$ there exist constants c_m so that for every f analytic on D and every $z \in B$ the m-th derivative satisfies $|f^{(m)}(z)| \leqslant c_m (\int_D |f(w)|^2 \, dw)^{1/2}$.*

Proof. Cauchy's integral formula gives a uniform bound on $f^{(m)}(z)$ for $z \in B$ in terms of the L^1-norm of the function on any circle in D about B. Integration yields a bound in terms of the L^1-norm on an annulus, which is bounded above by the L^2-norm on D. □

Next, we note some consequences of the Taylor expansion for Gaussian analytic functions.

LEMMA 24. *Let f be a Gaussian analytic function defined on a domain D, and let $B \subset D$ be a closed disk about z_0. Consider the partial sums of the Taylor series expansion about z_0:*

$$f_n(z) = \sum_{k=0}^{n} a_k (z - z_0)^k.$$

Then for all $m \geqslant 0$, the m-th derivative satisfies

$$\sup_B \mathbf{E} |f_n^{(m)} - f^{(m)}|^2 \to 0 \quad \text{as } n \to \infty. \tag{51}$$

Consequently, for all $m_1, m_2 \geqslant 0$ the covariance function of the derivatives of orders m_1 and m_2 of f_n converges uniformly on B^2 to the covariance function of the corresponding derivatives of f.

Proof. Note that finite a.s. limits of jointly Gaussian random variables are jointly Gaussian with finite variance. This implies that the derivative of a Gaussian analytic function f is a Gaussian analytic function. Moreover, the Taylor series of f has jointly Gaussian coefficients. Consider the L^2-space of functions on the set $X = \Omega \times D$ with the product of $\mathbf{P}$ and Lebesgue area measure. Assume without loss of generality that B is centered at 0, and let $f_n(z) = \sum_{j=0}^n a_j z^j$. Since f_n is a projection of f in the space X to the subspace spanned by $f_0, ..., f_n$, it follows that $f_n \to f$ in $L^2(X)$. By Fact 23, we have

$$\mathbf{E} \sup_B |f_n^{(m)} - f^{(m)}|^2 \to 0$$

and therefore

$$\sup_B \mathbf{E} |f_n^{(m)} - f^{(m)}|^2 \to 0,$$

which implies the claim for the covariance functions. □

COROLLARY 25. *Let g_n be polynomial of degree n with i.i.d. standard Gaussian coefficients independent of f_n. Then $f_n + g_n/n!$ approximates f in the sense of* (51), *and for each n has coefficients with continuous joint density.*

We first show a preliminary version of Proposition 8; (i) will then follow from the integral formula and a general lemma about point processes.

PROPOSITION 26. *Using the notation of Proposition* 8, *assume that $A = A(z_1, ..., z_k)$ is nonsingular when $z_1, ..., z_n \in D$ are distinct. Denote by $N(\Lambda)$ the number of zeros of f in Λ. Then for any n disjoint bounded Borel subsets $\Lambda_1, ..., \Lambda_n$ of D, we have*

$$\mathbf{E} \prod_{i=1}^n N(\Lambda_i) = \int_{\Lambda_1 \times ... \times \Lambda_n} p(z_1, ..., z_n)\, dz_1 ... dz_n, \tag{52}$$

where the integrations are with respect to Lebesgue area measure.

Proof. Case 1: f is a polynomial whose coefficients have joint density. This is a consequence of Hammersley's formula, Theorem 22.

Case 2: D is the unit disk or the whole plane.

A Fubini argument implies that there is a dense set $\mathcal{R}_D$ of rectangles in D such that for $R \in \mathcal{R}_D$, almost surely f does not vanish on ∂R.

It clearly suffices to show the claim when the Λ_i are disjoint elements of $\mathcal{R}_D$.

Let $\{f_M\}_{M=1}^\infty$ denote the approximations of f by polynomials in Corollary 25.

For $\Lambda \in \mathcal{R}_D$, the argument principle implies that the number $N_M(\Lambda)$ of zeros of f_M in Λ, converges a.s. to the number $N(\Lambda)$ of zeros of f in Λ.

As M varies, the random variables $\prod_{i=1}^{n} N_M(\Lambda_i)$ are uniformly integrable, as they are uniformly bounded in L^2 by Lemma 27.

The covariance functions of f_M and f'_M converge uniformly on each $\Lambda_i \times \Lambda_j$ to those of f and f', whence the permanent-determinant formula (on the right of (9)) for f_M converges uniformly on $\Lambda_1 \times ... \times \Lambda_n$ to the permanent-determinant formula for f.

Applying formula (10) to f_M and letting $M \to \infty$ we see that it converges to the desired formula for f.

Case 3: D is simply-connected. The claim follows from the Riemann mapping theorem and Case 2.

Case 4: A general domain D. It suffices to prove (10) when each Λ_j is a closed square in D. Then we can find a simply-connected domain $D_0 \subset D$ that contains all the Λ_j, and we apply Case 3. □

For simple point processes, the following lemma implies Proposition 8 (ii).

LEMMA 27. *Consider a simple point process (a random subset) Z in a domain D with counting function $N(\Lambda) = \#(Z \cap \Lambda)$. Suppose that for any disjoint Borel subsets $\Lambda_1, ..., \Lambda_k$ of D, we have*

$$\mathbf{E} \prod_{i=1}^{k} N(\Lambda_i) = \int_{\Lambda_1 \times ... \times \Lambda_k} p(z_1, ..., z_k)\, dz_1 ... dz_k. \tag{53}$$

Let $Z^{\wedge k} \subset Z^k$ denote the set of k-tuples of distinct points. Then for any Borel set $B \subset D^k$, we have

$$\mathbf{E}\, \#(B \cap Z^{\wedge k}) = \int_B p(z_1, ..., z_k)\, dz_1 ... dz_k. \tag{54}$$

Proof. Note that both sides of (54) define a Borel measure on subsets $B \subset D^k$; thus it suffices to show the equivalence for the case when $B = B_1 \times ... \times B_k$ is a product set.

Consider a finite Borel partition $\mathcal{P}$ of D and let

$$M_k(\mathcal{P}) = \sum_{Q_1, ..., Q_k} \#(Q_1 \times ... \times Q_k \cap B \cap Z^k) = \sum_{Q_1, ..., Q_k} \prod_{i=1}^{k} N(Q_i \cap B_i),$$

where the sum is over ordered k-tuples $(Q_1, ..., Q_k)$ of distinct elements of $\mathcal{P}$. Then the hypothesis (53) implies that

$$\mathbf{E} M_k(\mathcal{P}) = \sum_{Q_1, ..., Q_k} \int_{Q_1 \times ... \times Q_k \cap B} p(z_1, ..., z_k)\, dz_1 ... dz_k, \tag{55}$$

where we sum over the same k-tuples as above. Consider a refining sequence of partitions $\mathcal{P}_j$ of D where the maximal diameter of the elements of $\mathcal{P}_j$ tends to 0 as $j\to\infty$. By definition, $M_k(\mathcal{P})$ counts the number of k-tuples $(z_1, ..., z_k)\in B$ where $z_1, ..., z_k$ are points of Z in distinct elements of $\mathcal{P}$. We deduce that

$$M_k(\mathcal{P}_j)\to|B\cap Z^{\wedge k}|$$

monotonically as $j\to\infty$. Taking expectations and letting $j\to\infty$ yields (54). □

We now proceed to analyze the behavior of Gaussian analytic functions near their zeros.

LEMMA 28. *Let f be a Gaussian analytic function in a domain D, and assume that for every $z\in D$ a.s. z is not a double zero of f. Then a.s. f has no double zeros.*

The Gaussian assumption is needed: consider $(z-\gamma)^2$ with γ a continuous random variable.

Proof. We may assume that there exists $z_0\in D$ such that $W=f(z_0)-\mathbf{E}f(z_0)$ is not identically zero (otherwise there is nothing to show). Let $g(z)=\mathbf{E}(f(z)\overline{W})/\mathbf{E}|W|^2$ and $h(z)=f(z)-Wg(z)$. Then g is a deterministic analytic function with $g(z_0)=1$, and $h(\cdot)$ is independent of W. By assumption all the zeros of g are not double zeros of f. Any other double zero of f would also be a double zero of $\psi=W+h/g$. If h/g is a random constant, then ψ a.s. has no zeros. Otherwise, a.s. $\psi'=(h/g)'$ has at most countably many zeros $\{\xi_j\}$, and they are a.s. not zeros of ψ since $W\neq-(h/g)(\xi_j)$ a.s. by independence. □

LEMMA 29. *Let f be a Gaussian analytic function (not necessarily of mean zero) with radius of convergence r_0, and let M_r be its maximum modulus over the closed disk of radius $r<r_0$. There exists $c,\gamma>0$ so that for all $x>0$ we have*

$$\mathbf{P}(M_r>x)\leqslant ce^{-\gamma x^2}.$$

Proof. Borell's Gaussian isoperimetric inequality (see Pollard [22]; the inequality was also shown independently by Tsirelson–Ibragimov–Sudakov [30]) implies that for any collection of mean-zero Gaussian variables with maximal standard deviation σ, the maximum M of the collection satisfies

$$\mathbf{P}(M>\operatorname{median}(M)+b\sigma)\leqslant\mathbf{P}(N>b), \tag{56}$$

where N is standard normal. Now the median of M_r is finite because $M_r<\infty$ a.s. Since the distribution of $f(z)$ is continuous as a function of z, the maximal standard deviation σ in the disk $B_r(0)$ is bounded. The mean-zero version of the lemma follows by applying (56) to the real and imaginary parts separately, and the general version follows easily. □

LEMMA 30. *Let $f(z)=a_0+a_1z+\ldots$ be a Gaussian analytic function. Assume that a_0 is nonconstant. Let A_ε denote the event that the number of zeros of $f(z)$ in the disk B_ε about 0 differs from the number of zeros of $h(z)=a_0+a_1z$ in B_ε.*

(i) *For all $\delta>0$ there is $c>0$ (depending continuously on the mean and covariance functions of f) so that for all $\varepsilon>0$ we have*

$$\mathbf{P}(A_\varepsilon)\leqslant c\varepsilon^{3-2\delta}.$$

(ii) $\mathbf{P}(A_\varepsilon\,|\,a_1,a_2,\ldots)\leqslant C\varepsilon^3$, *where C may depend on $(a_1,a_2,\ldots)$, but is finite almost surely.*

Proof. (i) By Rouché's theorem, if $|h|>|f-h|$ on the circle ∂B_ε, then f and h have the same number of zeros in B_ε. By Lemma 29 applied to $(f-h)/z^2$, we have

$$\mathbf{P}\big(\max_{z\in\partial B_\varepsilon}|f(z)-h(z)|>\varepsilon^{2-\delta}\big)<c_0\exp(-\gamma\varepsilon^{-2\delta})<c_1\varepsilon^3. \tag{57}$$

Let Θ be the annulus $\partial B_{a_1\varepsilon}+B_{\varepsilon^{2-\delta}}$, and consider the following events:

$$\begin{aligned} D_0&=\{|a_0|<2\varepsilon^{1-\delta}\},\\ E&=\{|a_1|<\varepsilon^{-\delta}\},\\ F&=\big\{\min_{z\in\partial B_\varepsilon}|h(z)|<\varepsilon^{2-\delta}\big\}=\{-a_0\in\Theta\}. \end{aligned}$$

Note that $\mathbf{P}(E^c)\leqslant c_2\varepsilon^3$ and that $E\cap F\subset D_0$. Given D_0, the distribution of a_0 is approximately uniform on $B_{2\varepsilon^{1-\delta}}$ (i.e., its conditional density is $O(\varepsilon^{2\delta-2})$). Since $\mathbf{P}(E)$ tends to one as $\varepsilon\to0$, this implies that

$$\mathbf{P}(F\cap E\,|\,D_0)=\mathbf{P}(-a_0\in\Theta,E\,|\,D_0)\leqslant c_3\frac{\mathbf{E}\,\mathrm{area}(\Theta)}{\mathrm{area}(B_{2\varepsilon^{1-\delta}})}\leqslant c_4\frac{\varepsilon^{2-\delta}}{\varepsilon^{1-\delta}}=c_4\varepsilon,$$

and therefore

$$\mathbf{P}(F)\leqslant\mathbf{P}(F\cap E\,|\,D_0)\,\mathbf{P}(D_0)+\mathbf{P}(E^c)\leqslant c_4\varepsilon c_5\varepsilon^{2-2\delta}+c_2\varepsilon^3\leqslant c_6\varepsilon^{3-2\delta}.$$

Together with (57), this gives the desired result. Since all our bounds depend continuously on the covariance function of f, we may choose c in a continuous manner, too.

(ii) The argument used to bound $\mathbf{P}(F)$ in (i) also yields that

$$\mathbf{P}\big(\min_{z\in\partial B_\varepsilon}|h(z)|<2|a_2|\varepsilon^2\,\big|\,\{a_j\}_{j=1}^\infty\big)\leqslant c_7\varepsilon^3.$$

An application of Rouché's theorem concludes the proof. $\square$

The following lemma relates the integral joint intensity to the pointwise (strong) version.

LEMMA 31. *Consider a simple point process on a domain D. Let $z_j \in D$, $j=1,\dots,n$. Assume that there exists disjoint neighborhoods D_j of z_j and a $\delta>0$ so that the integral version of the joint intensity satisfies*

$$p(z_1,\dots,z_n,z_*) < c_2|z_j-z_*|^{-2+\delta} \quad \text{on } D_1\times\dots\times D_n\times D_j \text{ for all } j.$$

Let $N_{j,\varepsilon}$ denote the number of points in the ball of radius ε about z_j. As $\varepsilon\to 0$, we have

$$\mathbf{P}(N_{1,\varepsilon}=\dots=N_{n,\varepsilon}=1) \leqslant \mathbf{E}(N_{1,\varepsilon}\dots N_{n,\varepsilon}) = \mathbf{P}(N_{1,\varepsilon}=\dots=N_{n,\varepsilon}=1)+o(\varepsilon^{2n}). \tag{58}$$

Proof. For nonnegative integers N_j we have

$$0 \leqslant \prod_{j=1}^n N_j - \prod_{j=1}^n \mathbf{1}_{\{N_j=1\}} \leqslant N_1\dots N_n \sum_{k=1}^n (N_k-1). \tag{59}$$

The left inequality is clear. For the right one, if for some k we have $N_k>1$ then the kth term on the right alone gives an upper bound. We apply (59) to the $N_{j,\varepsilon}$ with small ε and take expectations. We apply Lemma 27 to the set $B_\varepsilon(z_1)\times\dots\times B_\varepsilon(z_n)\times B_\varepsilon(z_k)$ to get

$$\begin{aligned}\mathbf{E}\, N_{1,\varepsilon}\dots N_{n,\varepsilon}(N_{k,\varepsilon}-1) &= \int_{B_\varepsilon(z_1)\times\dots\times B_\varepsilon(z_n)\times B_\varepsilon(z_k)} p(w_1,\dots,w_{n+1})\,dw_1\dots dw_{n+1} \\ &= o(\varepsilon^{2n}). \end{aligned}$$
□

LEMMA 32. *Consider a Gaussian analytic function f in a domain D with mean zero everywhere. Let $z_1,\dots,z_n\in D$, and assume that for each j, the random variables $f'(z_j), f(z_1),\dots,f(z_n)$ are linearly independent. Then there exist neighborhoods D_i of the z_i so that for each $1\leqslant j\leqslant n$ and $(w_1,\dots,w_n,w_*)\in D_1\times\dots\times D_n\times D_j$, the integral version of the joint intensity is defined and satisfies*

$$p(w_1,\dots,w_n,w_*) \leqslant c|w_j-w_*|^2.$$

Proof. By continuity, there exists bounded neighborhoods D_i of the z_i so that

(i) for all $w_*\in\bigcup_{i=1}^n D_i$ and $w_i\in D_i$, the random variables $f'(w_*)$ and $f(w_i)$, $1\leqslant i\leqslant n$, are linearly independent, and the determinant of the covariance matrix is bounded away from 0;

(ii) for all *distinct* points $w_*\in\bigcup_{i=1}^n D_i$ and $w_i\in D_i$, the random variables $f(w_*)$ and $f(w_i)$, $1\leqslant i\leqslant n$, are linearly independent.

Part (ii) follows by considering the Gaussian analytic function

$$\frac{f(w_*)-f(w_j)}{w_*-w_j},$$

which has a removable singularity at w_j. Taylor expansion at w implies that for $w, z\in\bigcup_{i=1}^n D_i$, the conditional distribution of $f'(w)$ given $f(w)=f(z)=0$ is Gaussian with variance bounded above by $c_1|z-w|^2$. Therefore,

$$\mathbf{E}\big(|f'(w_1)\dots f'(w_n)f'(w_*)|\;\big|\;f(w_1)=\dots=f(w_*)=0\big)\leqslant c_2\varepsilon^4,$$

where ε is the distance between w_* and the set $\{w_1,\dots,w_n\}$. Furthermore,

$$\frac{\partial^2}{\partial w_*^2}\det\operatorname{Cov}(f(w_1),\dots,f(w_n),f(w_*))\Big|_{w_*=w_j}=\det\operatorname{Cov}(f(w_1),\dots,f(w_n),f'(w_j)),$$

and since the right-hand side is bounded away from 0, we get

$$|\det\operatorname{Cov}(f(w_1),\dots f(w_n),f(w_*))|\geqslant c_3\varepsilon^2.$$

Now the permanent-determinant formula implies the claim of the lemma. □

Proof of Proposition 8. *Step* 1. We first verify the equivalence of (8) and (9). First note that when f has Gaussian coefficients, then the values and coefficients of f are jointly Gaussian. In particular, the expression (50) equals

$$\begin{aligned}\mathbf{E}\big(|f'(z_1)\dots f'(z_n)|^2\;\big|\;f(z_1)=\dots=f(z_n)=0\big)\,g(0,\dots,0)\\ =\mathbf{E}(|\mathcal{P}f'(z_1)\dots\mathcal{P}f'(z_n)|^2)\,g(0,\dots,0),\end{aligned}\tag{60}$$

where $g(0,\dots,0)=\pi^{-n}\det(A)^{-1}$ is the density of the Gaussian vector $X=\{f(z_j)\}_{j=1}^n$ at 0, and $\mathcal{P}$ is the projection to the orthocomplement of the subspace spanned by the entries of X. Setting $Y=\{f'(z_j)\}_{j=1}^n$, note that the projection $(I-\mathcal{P})$ of Y onto the subspace spanned by the entries of X is given by $BA^{-1}X$. For a column vector Z of mean zero, recall that $\operatorname{Cov}(Z)=\mathbf{E}(ZZ^*)$. Now

$$\operatorname{Cov}(\mathcal{P}Y)=\operatorname{Cov}(Y-BA^{-1}X)=\operatorname{Cov}(Y)-\operatorname{Cov}(BA^{-1}X)=C-BA^{-1}B^*.\tag{61}$$

This proves the equivalence of (8) and (9).

Step 2. Part (ii). By Lemma 28, the point process of zeros is simple. Part (ii) follows from Proposition 26 and Lemma 27.

Step 3. Part (i). Let F_ε denote the event that f has a zero in each of the disks $B_\varepsilon(z_1), ..., B_\varepsilon(z_n)$. Since the function $p(z_1, ..., z_n)$ is continuous, we have

$$\begin{aligned}\mathbf{P}(F_\varepsilon) &\leqslant \mathbf{E}N(B_\varepsilon(z_1)) \dots N(B_\varepsilon(z_n)) \\ &= \int_{B_\varepsilon(z_1)\times\dots\times B_\varepsilon(z_n)} p(z_1, ..., z_n)\, dz_1 \dots dz_n \\ &= p(z_1, ..., z_n)\, \pi^n \varepsilon^{2n} + o(\varepsilon^{2n}).\end{aligned} \tag{62}$$

If, for some j, the derivative $f'(z_j)$ is not linearly independent of $\{f(z_i): 1 \leqslant i \leqslant n\}$, then $p(z_1, ..., z_n) = 0$, and the claim follows from (62). Otherwise, the claim follows from (62) and Lemmas 31 and 32. □

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