A GENERAL FIRST MAIN THEOREM OF VALUE DISTRIBUTION. I

BY

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Dedicated to Marilyn Stoll

The theory of value distribution of functions of one complex variable is well developed and has yielded beautiful results and applications. An important aspect is that a non-constant meromorphic function can be considered as an open map into the Riemann sphere $\mathbf{P}$, so providing a ramified covering of an open subset of $\mathbf{P}$.

Compared to the theory for one variable, the theory of value distribution for functions of several complex variables is still in its infant stage. The dual aspect seems to be lost. One way is to study the value distribution of a holomorphic function. Here H. Kneser [7] (1936) and [8] (1938) initiated a theory of value distribution for a meromorphic function of finite order in $\mathbf{C}^n$. This theory was later completed in [19] (1953) and gave an analogon to the theory of functions of finite order of one variable. The Kneser integral(2) substituted for the Weierstrass-product. In [20] (1953), these results were applied to construct Jacobian and abelian functions similar to the construction of the σ, ζ and $\wp$ function in one variable. Also the Kneser integral was extended to functions of infinite order in [19] (1953) and was applied in [14] (1964) to the theory of normal families of divisors in $\mathbf{C}^n$.

Ahlfors, H. Weyl and J. Weyl developed a theory of value distribution of a holomorphic map $f: M \to \mathbf{P}^n$ of a Riemann surface M into the n-dimensional, complex projective space $\mathbf{P}^n$. They were able to prove both Main Theorems and the Defect Relation. These results can be found in [29] (1943). In [21] (1953) and [22] (1954), this theory was united with the theory of Kneser to a value distribution of meromorphic maps $f: M \to \mathbf{P}^n$ where M is a pure m-dimensional complex manifold. Both Main Theorems and the Defect

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(2) A different integral representation was used by Lelong [9], [10] and [11], which enables him to give a new proof of the Cousin II Theorem for positive divisors of finite order.

Relation could be proved. In [25] (1964) and [26] (1964), the First Main Theorem was extended to the case, where M is a pure m-dimensional analytic set in a complex manifold. An application showed, that an analytic set M of pure dimension m in $\mathbf{C}^n$ is algebraic if and only if the $2m$-dimensional Hausdorff measure of $M \cap \{z \mid |z| < r\}$ is $O(r^{2m})$ for $r \to \infty$.

The other aspect of value distribution in one variable leads to the study of open, holomorphic maps $f: M \to \mathbf{C}^m$ where M is a pure m-dimensional complex manifold. The fibers are discrete. Fatou [4] (1922) and Bieberbach [1] (1933) found a biholomorphic map of $\mathbf{C}^m$ onto an open, non-dense subset of $\mathbf{C}^m$, whose Jacobian was constant. This discouraged the "map" approach. However, Schwartz [15] (1954) and [16] (1954) used topological methods to study this case and generalized the Ahlfors theory of covering spaces. Levine [12] (1960) found an unintegrated First Main Theorem.

Chern [3] (1960) integrated this theorem and obtained a result (Chern-Picard Theorem) which assures that an open holomorphic map $f: \mathbf{C}^m \to \mathbf{P}^m$ assumes almost every point in $\mathbf{P}^m$ if the characteristic of f grows stronger than the new "deficit" term which appeared in his First Main Theorem. Hence, the difficulties which the Bieberbach map posed were overcome.

Recently, Bott and Chern [2] (1965) have extended these methods to the study of sections in fiber bundles and obtained deep results.

Comparing the formulas in Levine [12] and Chern [3] with the formulas in [21], [22] and Kneser [8], a great similarity becomes apparent. Therefore, the question of an united theory arises. Such an unification will be provided here.

Let M and P be connected complex manifolds with $\dim M = m$ and $\dim P = n$. Let γ be a family of pure p-dimensional analytic subsets of P. Let $f: M \to P$ be an holomorphic map. Suppose that f is general of order $r = n - p$ in respect to γ, that means, that $f^{-1}(S)$ is empty or analytic of pure dimension $q = m - r$ for every $S \in \gamma$. The theory of value distribution is concerned with the size of $f^{-1}(S)$ for $S \in \gamma$. A typical statement would be: If f "grows" strongly enough, $f(M)$ intersects "most" elements of γ. Usually, the "*most*" which results from a "*First Main Theorem*" would be in the Lebesgue measure sense, where upon the "*most*" which results from a "*Second Main Theorem*" would mean "all up to finitely many". The theory of Kneser and Stoll belong to the codimension $r = 1$, the theory of Chern and Levine belong to the codimension $r = n$. Here, both theories will be special case of an unified theory for codimension r where $1 \leqslant r \leqslant \operatorname{Min}(m, n)$, at least so far the "*First Main Theorem*" is concerned. P will be the n-dimensional complex projective space, and γ will be the Grassmann manifold of all p-dimensional complex planes in P.

Let V be an $(n+1)$-dimensional complex vector space with a given Hermitian product $(\,|\,)$. On every exterior product $V[p] = V \wedge \ldots \wedge V$ (p-times) an associated Hermitian product is induced if $0 < p \leqslant n+1$. It defines a norm on $V[p]$. For $\mathfrak{a} \in V - \{0\}$, define $\varrho(\mathfrak{a}) = \{z\mathfrak{a} \mid z \in \mathbf{C}\}$. Then

$$\mathbf{P}(V) = \{\varrho(\mathfrak{a}) \mid \mathfrak{a} \in V - \{0\}\}$$

is the complex projective space of V and $\varrho: V - \{0\} \to \mathbf{P}(V)$ is the natural projection. Denote the natural projection of $V[p] - \{0\}$ onto $\mathbf{P}(V[p])$ also by ϱ. Define

$$\tilde{\mathfrak{G}}^{(p+1)}(V) = \{\mathfrak{c}_0 \wedge \ldots \wedge \mathfrak{c}_p \mid \mathfrak{c}_\mu \in V\} \subseteq V[p+1].$$

Then $$\mathfrak{G}^p(V) = \varrho(\tilde{\mathfrak{G}}^{p+1}(V) - \{0\})$$

is the Grassmann manifold of the p-dimensional complex planes in $\mathbf{P}(V)$. Of course $\mathfrak{G}^0(V) = \mathbf{P}(V)$. If $\alpha \in \mathfrak{G}^p(V)$, then $E(\alpha)$ and $\ddot{E}(\alpha)$ are well defined by

$$E(\alpha) = \{\mathfrak{z} \mid \mathfrak{z} \wedge \mathfrak{a} = 0\}, \quad \text{where} \quad \mathfrak{a} \in \varrho^{-1}(\alpha)$$
$$\ddot{E}(\alpha) = \varrho(E(\alpha) - \{0\}).$$

If $\alpha \in \mathfrak{G}^p(V)$ and $\beta \in \mathfrak{G}^q(V)$, then $\|\alpha : \beta\|$ are well defined by

$$\|\alpha : \beta\| = \frac{|\mathfrak{a} \wedge \mathfrak{b}|}{|\mathfrak{a}|\,|\mathfrak{b}|}, \quad \text{where } \mathfrak{a} \in \varrho^{-1}(\alpha),\ \mathfrak{b} \in \varrho^{-1}(\beta).$$

Moreover, $0 \leqslant \|\alpha : \beta\| \leqslant 1$.

On any complex manifold, let $d = \partial + \bar{\partial}$ be the exterior derivative with ∂ as the complex component and $\bar{\partial}$ as the conjugate complex component. Define $d^\perp = i(\partial - \bar{\partial})$. On $V - \{0\}$, define the non-negative "*projective*" form ω by

$$\omega(\mathfrak{z}) = \tfrac{1}{2} d^\perp d \log |\mathfrak{z}|.$$

Then one and only one positive "*projective*" form $\ddot{\omega}$ of bidegree (1, 1) exists on $\mathbf{P}(V)$ such that $\varrho^*(\ddot{\omega}) = \omega$. Here $\ddot{\omega}$ is the exterior form of a Kaehler metric on $\mathbf{P}(V)$. Define the projective forms of bidegree (p, p) by

$$\omega_p = \frac{1}{p!}\, \omega \wedge \ldots \wedge \omega; \quad \ddot{\omega}_p = \frac{1}{p!}\, \ddot{\omega} \wedge \ldots \wedge \ddot{\omega} \quad (p\text{-times}).$$

Define the "*euclidean*" forms on V by

$$v = \frac{i}{2}\, \partial\bar{\partial}(\mathfrak{z} \mid \mathfrak{z}); \quad v_p = \frac{1}{p!}\, v \wedge \ldots \wedge v \quad (p\text{-times})$$

Let $0 \leqslant p < n$ and $\alpha \in \mathfrak{G}^p(V)$. On $V - E(\alpha)$, a form $\Phi(\alpha)$ of bidegree (1, 1) is well-defined by

$$\tilde{\Phi}(\alpha)(\mathfrak{z}) = \tfrac{1}{2} d^\perp d \log |\mathfrak{a} \wedge \mathfrak{z}|,$$

where $\mathfrak{a} \in \varrho^{-1}(\alpha)$. One and only one non-negative form $\Phi(\alpha)$ of bidegree (1, 1) exists on $\mathbf{P}(V) - \ddot{E}(\alpha)$ such that

$$\varrho^*(\Phi(\alpha)) = \tilde{\Phi}(\alpha).$$

Define $r = n - p$. On $\mathbf{P}(V) - \ddot{E}(\alpha)$, define the Levine form of order r by

$$\Lambda(\alpha) = \frac{1}{(r-1)!} \sum_{\mu=0}^{r-1} \Phi(\alpha)^{\mu} \wedge \ddot{\omega}^{r-1-\mu}.$$

The form $\Lambda(\alpha)$ is non-negative and has bidegree $(r-1, r-1)$. For $r=1$ is $\Lambda(\alpha)=1$. Define

$$W(s) = \frac{\pi^s}{s!} \quad \text{if } s \in \mathbf{N}.$$

Let M be a complex manifold of pure dimension m. Let χ be a differential form of bidegree (q, q) and class C^1 on M with $d\chi = 0$. Suppose that

$$0 \leqslant m - q = n - p = r \leqslant \operatorname{Min}(m, n).$$

Let $f: M \to N$ be a holomorphic map, which is general of order r (in respect to $\mathfrak{G}^p(V)$). For every $z \in f^{-1}(\ddot{E}(\alpha))$, an intersection multiplicity $\nu_f(z; \alpha)$ is defined, which is a positive integer. The function $\nu_f(z; \alpha)$ of z is constant on every connectivity component of the set of simple points of $f^{-1}(\ddot{E}(\alpha))$. As a consequence of a residue formula (Theorem 4.4) the following result is obtained.

THE UNINTEGRATED FIRST MAIN THEOREM (1). *Let H be an open, relative compact subset of M whose boundary S is either empty or a smooth, $(2m-1)$-dimensional submanifold of M which is oriented to the exterior of H. Take $\alpha \in \mathfrak{G}^p(V)$. Suppose that*

$$\xi(\alpha) = d^{\perp} \log \|f : \alpha\| \wedge f^*(\Lambda(\alpha)) \wedge \chi$$

is integrable over S. Then

$$\frac{1}{2\pi} \frac{1}{W(r-1)} \int_S \xi(\alpha) + \int_{H \cap f^{-1}(\ddot{E}(\alpha))} \nu_f(z; \alpha)\, \chi = \frac{1}{W(r)} \int_H f^*(\ddot{\omega}_r) \wedge \chi.$$

For $r = m$ (i.e., $q = 0$) and $\chi = 1$, this is the First Main Theorem of Levine [12]. If M is compact and $H = M$, then $S = \emptyset$ and

$$\int_{f^{-1}(\ddot{E}(\alpha))} \nu_f(z; \alpha)\, \chi = \frac{1}{W(r)} \int_H f^*(\ddot{\omega}_r) \wedge \chi$$

is constant on $\mathfrak{G}^p(V)$. If $r = n$ (i.e., $p = 0$), then $\mathfrak{G}^p(V) = \mathbf{P}(V)$ and $f: M \to \mathbf{P}(V)$ is an open

(1) See Theorem 4.5.

holomorphic map of pure fiber dimension q(1). In *II*,(2) the continuity of the fiber integral

$$n_f(G;\alpha)=\int_{f^{-1}(\ddot{E}(\alpha))}\nu_f(z;\alpha)\,\chi$$

was proved. Here, it turns out that $n_f(G;\alpha)$ is constant, if $d\chi=0$ and if the image manifold is $\mathbf{P}(V)$.

Now, suppose that χ is non-negative on M. Let G and g be non-empty, open, relative compact subsets of M with smooth C^∞-boundaries Γ of G and γ of g, both of which are oriented to the exterior. Assume that $\bar{g}\subset G$. Let ψ be any continuous, non-negative function on $\bar{G}$, such that $\psi|\Gamma=0$ and $\psi|\bar{g}=R>0$ are constant, such that ψ is of class C^∞ on $\bar{G}-g$ and such that $0\leqslant\psi(z)\leqslant R$ for $z\in\bar{G}$. Define the *characteristic* of f by

$$T_f(G)=\frac{1}{W(r)}\int_G \psi f^*(\omega_r)\wedge\chi.$$

For $\alpha\in\mathfrak{G}^p(V)$, define the *compensation functions* by

$$m_f(\Gamma;\alpha)=\frac{1}{2\pi}\frac{1}{W(r-1)}\int_\Gamma \log\frac{1}{\|f:\alpha\|}f^*(\Lambda(\alpha))\wedge d^\perp\psi\wedge\chi,$$

$$m_f(\gamma;\alpha)=\frac{1}{2\pi}\frac{1}{W(r-1)}\int_\gamma \log\frac{1}{\|f:\alpha\|}f^*(\Lambda(\alpha))\wedge d^\perp\psi\wedge\chi.$$

Define the *valence function* by(3)

$$N_f(G;\alpha)=\int_{f^{-1}(\ddot{E}(\alpha))}\nu_f(z:\alpha)\,\psi\chi.$$

The integrands of these integrals are non-negative. This is not true for the *deficit*

$$\Delta_f(G;\alpha)=\frac{1}{2\pi}\frac{1}{W(r-1)}\int_G \log\frac{1}{\|f:\alpha\|}f^*(\Lambda(\alpha))\wedge dd^\perp\psi\wedge\chi.$$

These assumptions imply the *First Main Theorem*(4)

$$N_f(G;\alpha)+m_f(\Gamma;\alpha)-m_f(\gamma;\alpha)=T_f(G)+\Delta_f(G;\alpha).$$

The theory of [21] and [22] is obtained by choosing $r=1(q=m-1)$ and $dd^\perp\psi\wedge\chi=0$

(1) Such a map is also called q-fibering.

(2) Stoll [27] is denoted by I and Stoll [28] is denoted by II, because these papers are necessary preparations for this paper and were written simultaneously.

(3) If $q=0$, the integral is a finite sum.

(4) See Theorem 4.8.

on $G-\bar{g}$. The theory of H. Weyl and J. Weyl [29] is obtained by choosing $m=r=1$ and $\chi=1$ with $dd^{\perp}\psi=0$ on $G-\bar{g}$. The theory of H. Kneser [8] is obtained by choosing $r=1=n(q=m-1, p=0)$ and $M=\mathbf{C}^m$ with $\chi=v_{m-1}$, where $G=\{\mathfrak{z} \mid |\mathfrak{z}|<r\}$ and $g=\{\mathfrak{z} \mid |\mathfrak{z}|<r_0\}$ with $0<r_0<r$ and

$$\psi(\mathfrak{z})=\frac{1}{2m-2}\left(\frac{1}{|\mathfrak{z}|^{2m-1}}-\frac{1}{r^{2m-2}}\right) \text{ on } G-\bar{g}.$$

The theory of Chern [3] is obtained by choosing $r=m=n(q=0=p)$ and $M=\mathbf{C}^m$ with $\chi=1$. Moreover, G, g and ψ are chosen as in the theory of H. Kneser. However Theorem 5.12 shows that the Chern–Picard Theorem can be obtained for other choices of ψ.

If $f:M\to\mathbf{P}(V)$ is an open map, then f is general of order s for every s in $0<s\leqslant n$. In addition suppose that M is a connected, non-compact, pseudoconvex manifold, where pseudoconvex means, that a non-negative function h of class C^∞ exists on M such that $d^{\perp}dh\geqslant 0$ and such that

$$G(r)=\{z \mid h(z)<r\}$$

is a relative compact subset of M if $r>0$.(¹) Choose any such a plurisubharmonic function h. Define

$$\chi_s=\frac{1}{s!}\,d^{\perp}dh\wedge\ldots\wedge d^{\perp}dh \quad (s\text{-times}).$$

Take $r_0>0$ such that $g=G(r_0)\neq\emptyset$ and such that $dh\neq 0$ on $\overline{G(r_0)}-G(r_0)=\gamma$. For $r>r_0$, define

$$\psi(z)=\begin{cases}0 & \text{if } z\in M-G(r),\\ r-h(z) & \text{if } z\in G(r)-g,\\ r-r_0 & \text{if } z\in g.\end{cases}$$

The characteristic of f for the order s is

$$T_{s,f}(r)=\frac{1}{W(s)}\int_{G(r)}\psi_r f^*(\ddot{\omega}_s)\wedge\chi_{m-s}$$

if $r\geqslant r_0$. The function $T_{s,f}$ is differentiable for $r\geqslant r_0$ and its derivative is

$$A_{s,f}(r)=\frac{1}{W(s)}\int_{G(r)}f^*(\ddot{\omega}_s)\wedge\chi_{m-s}.$$

Define

$$\delta_f=\limsup_{r\to\infty}\frac{A_{n-1,f}(r)}{T_{n,f}(r)},$$

$$b(M)=\frac{1}{W(n)}\int_{f(M)}\ddot{\omega}_n.$$

(¹) M is a Stein manifold if and only if such a function h with the stronger condition $d^{\perp}dh>0$ exists. See Grauert [5], [6] and Narasimhan [13].

Then $b(M)$ is the normalized measure of the image of f in $\mathbf{P}(V)$. Obviously $0 \leqslant b(M) \leqslant 1$. Then the following generalization of the Chern–Picard Theorem [3] holds:(1)

THEOREM. *Suppose that* $T_{n,f}(r) \to \infty$ *for* $r \to \infty$. *Then*

$$0 \leqslant (1 - b(M)) \leqslant \frac{m-n+1}{4\pi} \sum_{\nu=0}^{n-1} \frac{1}{\nu+1} \delta_f.$$

Especially, if $\delta_f = 0$, *then* f *assumes almost every "value" of* $\mathbf{P}(V)$.

As a preparation for the proof of the First Main Theorem, the limit of certain integrals is obtained in section 1. The intersection number $\nu_f(z; \alpha)$ is introduced and studied in section 2. Its definition is reduced to the multiplicity of a certain open holomorphic map π_f. In section 3, the Levine form is studied. In section 4, two integral theorems are proved which yield the First Main Theorem in its unintegrated and its integrated form. In section 5, the "*spherical*" representation (Theorem 5.8) of the characteristic and of the deficit is proved for open maps. The Chern–Picard Theorem [3] is extended to open maps.

The results of Bott and Chern [2] have not been considered in this paper, because almost all of this research had been done as [2] became available to me.

§ 1. The existence and the limit of certain integrals

The proof of the First Main Theorem requires some highly technical results about the existence and the limit of certain integrals. The proof of these results shall be given here to avoid an undue interruption of the later representation.

The concepts and notations of *I* and *II* shall be used. For convenience sake, some are recalled here.

1. Let V be a complex vector space of dimension m. Define $V^m = V \times \ldots \times V$ (m-times). Then the general linear groups

$$GL(V) = \{(\mathfrak{c}_1, \ldots, \mathfrak{c}_m) \mid \mathfrak{c}_1 \wedge \ldots \wedge \mathfrak{c}_m \neq 0\}$$

which is the set of all bases of V, is open and dense in V^m.

2. A function $(\,|\,): V \times V \to \mathbf{C}$ is a *Hermitian product* on V, if and only if it is linear in the first variable, if $(\mathfrak{x} \mid \mathfrak{y}) = \overline{(\mathfrak{y} \mid \mathfrak{x})}$ for all $\mathfrak{x} \in V$ and $\mathfrak{y} \in V$ and if $(\mathfrak{x} \mid \mathfrak{x}) > 0$ if $0 \neq \mathfrak{x} \in V$. Denote $|\mathfrak{x}| = \sqrt{(\mathfrak{x} \mid \mathfrak{x})}$. Then V becomes a complex Hilbert space. An element $(\mathfrak{c}_1, \ldots, \mathfrak{c}_m) \in V^m$ is said to be an orthonormal base if and only if $(\mathfrak{c}_\mu \mid \mathfrak{c}_\nu) = 0$ for all $\mu \neq \nu$ as $|\mathfrak{c}_\mu| = 1$ for all μ. The set $\mathfrak{U}(V)$ of all orthonormal bases of V is a non-empty, connected, real analytic submanifold of $GL(V)$.

(1) See Theorem 5.13.

3. If not otherwise directed, the space $\mathbf{C}^m = \mathbf{C} \times \ldots \times \mathbf{C}$ is thought to be endowed with the Hermitian product defined by

$$(\mathfrak{x} \mid \mathfrak{y}) = \sum_{\mu=1}^{m} x_\mu \bar{y}_\mu \quad \text{if } \mathfrak{x} = (x_1, \ldots, x_m),\ \mathfrak{y} = (y_1, \ldots, y_m).$$

Then the base $(\mathfrak{e}_1, \ldots, \mathfrak{e}_m)$ is orthonormal, where $\mathfrak{e}_\mu = (\delta_{1\mu}, \ldots, \delta_{m\mu})$ and $\delta_{\mu\nu} = 0$ if $\mu \neq \nu$ and $\delta_{\mu\mu} = 1$.

4. The group $\mathfrak{S}(m)$ of all permutations π of $\{1, \ldots, m\}$ operates effectively on V^m, $GL(V)$ and $\mathfrak{U}(V)$ by setting

$$\pi(\mathfrak{c}_1, \ldots, \mathfrak{c}_m) = (\mathfrak{c}_{\pi(1)}, \ldots, \mathfrak{c}_{\pi(m)}).$$

Let $\mathfrak{T}(q, m)$ be the set of all injective and increasing maps

$$\varphi: \{1, \ldots, q\} \to \{1, \ldots, m\}$$

Then $\varphi \in \mathfrak{T}(q, m)$ defines one and only one permutation $\underline{\varphi} \in \mathfrak{S}(m)$ such that $\underline{\varphi}(\nu) = \varphi(\nu)$ for $\nu = 1, \ldots, q$ and $\underline{\varphi}(\nu) < \underline{\varphi}(\nu+1)$ for $\nu = q+1, \ldots, m-1$.

5. Let M be a pure m-dimensional complex manifold. The set $\mathfrak{S}_M$ of all biholomorphic map $\alpha: U_\alpha \to U'_\alpha$ of an open subset U_α of M onto an open subset U'_α of $\mathbf{C}^m$ is the complex structure of M. Then $\alpha = (z_1^\alpha, \ldots, z_m^\alpha)$, where $z_\mu = z_\mu^\alpha$ are holomorphic functions on U_α. Let d be the exterior derivative on M. For $\varphi \in \mathfrak{T}(q, m)$ and $\psi \in \mathfrak{T}(q, m)$ define

$$\zeta_\varphi^\alpha = \zeta_\varphi = dz_{\varphi(1)} \wedge \ldots \wedge dz_{\varphi(q)},$$

$$\eta_{\varphi\psi}^\alpha = \eta_{\varphi\psi} = \left(\frac{i}{2}\right)^q dz_{\varphi(1)} \wedge d\bar{z}_{\psi(1)} \wedge \ldots \wedge dz_{\varphi(q)} \wedge d\bar{z}_{\psi(q)}$$

on U_α. Then

$$\bar{\zeta}_\varphi^\alpha = \bar{\zeta}_\varphi = \bar{z}_{\varphi(1)} \wedge \ldots \wedge d\bar{z}_{\varphi(q)},$$

$$\eta_{\varphi\psi} = \bar{\eta}_{\psi\varphi} = (-1)^{q(q-1)/2} \left(\frac{i}{2}\right)^q \zeta_\varphi \wedge \bar{\zeta}_\psi.$$

6. If χ is a differential form of bidegree (q, q) on a subset A of M, if $\alpha \in \mathfrak{S}_M$ with $A \cap U_\alpha \neq \emptyset$, then

$$\chi = \sum_{\varphi \in \mathfrak{T}(q,m)} \sum_{\psi \in \mathfrak{T}(q,m)} \chi_{\varphi\psi}^\alpha \eta_{\varphi\psi}^\alpha$$

on $A \cap U_\alpha$, where $\chi_{\varphi\psi}^\alpha$ are uniquely defined functions on $A \cap U_\alpha$. χ is said to be real, if $\chi = \bar{\chi}$; this is the case, if and only if $\bar{\chi}_{\varphi\psi}^\alpha = \chi_{\psi\varphi}^\alpha$ for all ψ, φ, α. If χ has bidegree (m, m) on A, then $\{\iota\} = \mathfrak{T}(m, m)$, where ι is the identity and $\chi = \chi_{\iota\iota}^\alpha \eta_{\iota\iota}$. The set of all real forms of bidegree (m, m) on A is ordered at $a \in A$ such that $\chi \leqslant \xi$ at a if and only if $\chi_{\iota\iota}^\alpha(a) \leqslant \xi_{\iota\iota}^\alpha(a)$ when ever $\alpha \in \mathfrak{S}_M$ with $a \in U_\alpha$.

7. Let M and N be complex manifolds. Let $f: N \to M$ be holomorphic. Then, every form χ on M induces a form $f^*(\chi)$ on N. The map σ^* is a homomorphism, which preserves degrees, bidegrees and conjugation and commutes with d, ∂, $\bar{\partial}$ and $d^\perp$, where ∂ is the complex component and $\bar{\partial}$ is the conjugate complex component of d, and where

$$d^\perp = i(\partial - \bar{\partial}).$$

8. Let M be a pure m-dimensional complex manifold. Let $0 < q < m$. Let χ and ξ be real differential forms of bidegree (q, q) on M. Then $\chi \leqslant \xi$ at $a \in M$, if and only if for every smooth, pure q-dimensional complex submanifold N of M with inclusion $j_N: N \to M$, the inequality $j_N^*(\chi) \leqslant j_N^*(\xi)$ holds at a. (Observe, that 6. applies, because $j_N^*(\chi)$ and $j_N^*(\xi)$ are again real and of bidegree (q, q).) A partial order on the set of all real differential forms of bidegree (q, q) on M is defined, especially, the concept of positive, negative, non-negative and non-positive forms at a are defined. If $A \subseteq M$ then $\chi \leqslant \xi$ (resp. $\chi < \xi$) on A if $\chi \leqslant \xi$ (resp. $\chi < \xi$) at every $a \in A$. Hence χ is positive (negative, non-negative or non-positive on A) if this is true at every point of A. The forms $\eta_{\varphi\psi}^\alpha$ are non-negative on U_α.

9. Let M be a pure m-dimensional complex manifold. Let ψ and χ be forms of bidegree (q, q) on M with $0 < q \leqslant m$. Let χ be non-negative on M. Then $|\psi| \leqslant \chi$ means, that for every smooth, pure q-dimensional, complex submanifold N of M with inclusion $j_N: N \to M$, the inequality $|j_N^*(\psi)| \leqslant j_N^*(\chi)$ holds.

10. Let V be a complex vector space of dimension m with a Hermitian product $(\,|\,)$. For $\mathfrak{z} \in V$, define

$$(d\mathfrak{z}\,|\,\mathfrak{z}) = \partial(\mathfrak{z}\,|\,\mathfrak{z}) \quad (\mathfrak{z}\,|\,d\mathfrak{z}) = \bar{\partial}(\mathfrak{z}\,|\,\mathfrak{z})$$

$$(d\mathfrak{z}\,|\,d\mathfrak{z}) = \partial(\mathfrak{z}\,|\,d\mathfrak{z}) = -\bar{\partial}(d\mathfrak{z}\,|\,\mathfrak{z})$$

$$v = \frac{i}{2}(d\mathfrak{z}\,|\,d\mathfrak{z})$$

$$v_p = \frac{1}{p!}\, v \wedge \ldots \wedge v \quad (p\text{-times}).$$

Then v is the associated form of the Kähler metric defined by $(\,|\,)$. Moreover, v_p is positive if $0 < p \leqslant m$. The forms v and v_p are called the *euclidean forms* to $(\,|\,)$. If $\mathfrak{c} = (\mathfrak{c}_1, \ldots, \mathfrak{c}_m)$ is a base of V, define $\alpha: V \to \mathbf{C}^m$ by $\alpha(\mathfrak{c}_\mu) = \mathfrak{e}_\mu$ for $\mu = 1, \ldots, m$.(1) Then $\pi_\mu: \mathbf{C}^m \to \mathbf{C}$ is defined by $\pi_\mu(z_1, \ldots, z_m) = z_\mu$. Then $z_\mu = \pi_\mu \circ \alpha: V \to \mathbf{C}$ are the coordinates associated to $\mathfrak{c}$. It is

$$\mathfrak{z} = \sum_{\mu=1}^{m} z_\mu(\mathfrak{z})\, \mathfrak{c}_\mu \quad \text{for } \mathfrak{z} \in V.$$

For $\varphi \in \mathfrak{T}(p, m)$ and $\psi \in \mathfrak{T}(p, m)$, define $\zeta_\varphi^{\mathfrak{c}} = \zeta_\varphi^\alpha$ and $\eta_{\varphi\psi}^{\mathfrak{c}} = \eta_{\varphi\psi}^\alpha$; if clear, write $\zeta_\varphi = \zeta_\varphi^{\mathfrak{c}}$ and $\eta_{\varphi\psi} = \eta_{\varphi\psi}^{\mathfrak{c}}$ and call these the forms associated to $\mathfrak{c}$. Define $g_{\mu\nu} = (\mathfrak{c}_\mu\,|\,\mathfrak{c}_\nu)$ then

(1) Here $\mathfrak{e}_\mu = (\delta_{\mu 1}, \ldots, \delta_{\mu m})$ where $\delta_{\mu\nu} = 0$ if $\mu \neq \nu$ and where $\delta_{\mu\mu} = 1$.

$$v=\frac{i}{2}\sum_{\mu,\nu=1}^{m} g_{\mu\nu}\,dz_\mu\wedge d\bar{z}_\nu.$$

If and only if $\mathfrak{c}\in\mathfrak{U}(V)$ is orthonormal, then $g_{\mu\nu}=\delta_{\mu\nu}$ and

$$v=\frac{i}{2}\sum_{\mu=1}^{m} dz_\mu\wedge d\bar{z}_\mu$$

$$v_p=\left(\frac{i}{2}\right)^p\sum_{\varphi\in\mathfrak{T}(p,m)}\eta^{\mathfrak{c}}_{\varphi\varphi}.$$

11. Let V be a complex vector space of dimension $n+1$ with a Hermitian product $(\,|\,)$. Let $\mathbf{P}(V)$ be the associated projective space and let $\varrho: V-\{0\}\to\mathbf{P}(V)$ be the residual map. Then $\varrho^{-1}(\varrho(\mathfrak{a}))=\{z\mathfrak{a}\,|\,0\neq z\in\mathbf{C}\}$, where $\mathfrak{a}\neq 0$. On $V-\{0\}$, define

$$\omega=\frac{1}{2}\,dd^{\perp}\log\frac{1}{|\mathfrak{z}|}=\frac{i}{2}\,\partial\bar{\partial}\log(\mathfrak{z}\,|\,\mathfrak{z}).$$

Then $d\omega=0$ and
$$\omega=\frac{i}{2}\,\frac{(\mathfrak{z}|\mathfrak{z})\,(d\mathfrak{z}|d\mathfrak{z})-(d\mathfrak{z}|\mathfrak{z})\wedge(\mathfrak{z}|d\mathfrak{z})}{|\mathfrak{z}|^4}$$

This form is non-negative and called the *projective form* to $(\,|\,)$ on V. Define

$$\omega_p=\frac{1}{p!}\,\omega\wedge\ldots\wedge\omega\quad(p\text{-times})$$

Then ω_p is non-negative and called the projective form of bidegree (p,p) on V. On $\mathbf{P}(V)$, one and only one exterior form $\ddot{\omega}$ of bidegree $(1,1)$ exists such that $\varrho^*(\ddot{\omega})=\omega$ on $V-\{0\}$. The form $\ddot{\omega}$ is real analytic, positive definit and $d\ddot{\omega}=0$. It is the exterior form associated to a Kaehler metric on $\mathbf{P}(V)$. Define

$$\ddot{\omega}_p=\frac{1}{p!}\,\ddot{\omega}\wedge\ldots\wedge\ddot{\omega}\quad(p\text{-times})$$

Then $\varrho^*(\ddot{\omega}_p)=\omega_p$ on $V-\{0\}$.

12. Let M be an oriented, real manifold of class C^∞. Let N be an oriented, real submanifold of pure dimension n and class C^k with $k\geqslant 1$. Let $j_N: N\to M$ be the inclusion map. Let ψ be a form of degree n on a subset A of M such that $j_N^*(\psi)$ is integrable over $A\cap N$. Define

$$\int_{A\cap N}\psi=\int_{A\cap N} j_N^*(\psi),\qquad \int_{A\cap N}|\psi|=\int_{A\cap N}|j_N^*(\psi)|.$$

13. Let M be a complex manifold. Let N be a pure n-dimensional analytic subset of

M. Let $\dot{N}$ be the set of simple points of M. Let ψ be a form of bidegree (n, n) on a subset A of M, which is integrable over $A \cap \dot{N}$. Define

$$\int_{A\cap N} \psi = \int_{A\cap \dot{N}} \psi, \qquad \int_{A\cap N} |\psi| = \int_{A\cap \dot{N}} |\psi|.$$

If A is compact and if ψ is continuous on A, these integrals exist.

In the following lemmata, the spaces $\mathbf{C}^m$ and $\mathbf{C}^p$ with $0 < p \leqslant m$ appear often. The forms v_q, ω_q, $\eta_{\varphi\psi}$ and ζ_φ are formed for the natural coordinates. Those on $\mathbf{C}^m$ shall be distinguished from those on $\mathbf{C}^p$ by a lower bar: $\underline{v}_q$, $\underline{\omega}_q$, $\underline{\eta}_{\varphi\psi}$ and $\underline{\zeta}_\varphi$.

LEMMA 1.1. *Let M be an open, non-empty subset of $\mathbf{C}^m$. Let $f: M \to \mathbf{C}^p$ be a holomorphic, q-fibering map.*[1] *Let $s \in \mathbf{N}$. Suppose that $1 \leqslant s < p = m - q \leqslant m$. Let $\varkappa$ be a non-negative integer. Let χ be a differential form of bidegree $(m-s, m-s)$ on a compact subset K of M. Suppose that the coefficients of χ are measurable and bounded on K. Take $\varphi \in \mathfrak{T}(s, p)$. For $\varrho \in \mathbf{R}$ with $0 < \varrho < 1$, define*

$$L(\varrho) = \{z \mid \tfrac{1}{2}\varrho \leqslant |f(z)| \leqslant \varrho \text{ with } z \in K\},$$

$$I(\varrho) = \int_{L(\varrho)} \left(\log \frac{1}{|f|}\right)^{\varkappa} \frac{1}{|f|^{2s}} |f^*(\eta_{\varphi\varphi}) \wedge \chi|.$$

Then $I(\varrho) \to 0$ for $\varrho \to 0$.

Proof. Without loss of generality, $\varphi(\nu) = \nu$ for $\nu = 1, 2, \ldots, s$ can be assumed. Now, $f = (f_1, \ldots, f_p)$, where f_μ is holomorphic on M. Then

$$f^*(\eta_{\varphi\varphi}) = (\tfrac{1}{2}i)^s \, df_1 \wedge d\bar{f}_1 \wedge \ldots \wedge df_s \wedge d\bar{f}_s.$$

An open neighborhood H of K exists, such that $\bar{H}$ is compact and contained in M and such that H is a finite union of balls. On $M - K$, define χ by setting $\chi(z) = 0$. Then

$$\chi = \sum_{\alpha \in \mathfrak{T}(m-s, m)} \sum_{\beta \in \mathfrak{T}(m-s, m)} \chi_{\alpha\beta} \underline{\eta}_{\alpha\beta}.$$

A constant $B > 0$ exists such that $|\chi_{\alpha\beta}(\mathfrak{z})| \leqslant B$ if $\mathfrak{z} \in M$ for all α and β in $\mathfrak{T}(m-s, m)$. Define

$$F_\alpha = \frac{\partial(f_1, \ldots\ldots\ldots\ldots, f_s)}{\partial(z_{\underline{\alpha}(m-s+1)}, \ldots, z_{\underline{\alpha}(m)})} \quad \text{for } \alpha \in \mathfrak{T}(m-s, m).$$

Then

$$|f^*(\eta_{\varphi\varphi}) \wedge \chi| = \Big| \sum_{\alpha \in \mathfrak{T}(m-s, m)} \sum_{\beta \in \mathfrak{T}(m-s, m)} \operatorname{sign} \alpha \operatorname{sign} \beta F_\alpha \bar{F}_\beta \chi_{\alpha\beta} \Big| \underline{v}_m$$

$$\leqslant \frac{B}{2} \sum_{\alpha \in \mathfrak{T}(m-s, m)} \sum_{\beta \in \mathfrak{T}(m-s, m)} (|F_\alpha|^2 + |F_\beta|^2) \underline{v}_m$$

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[1] A holomorphic map $f: M \to N$ is said to be q-fibering if and only if $f^{-1}(f(a))$ is a pure q-dimensional analytic set for every $a \in M$.

$$=\binom{m}{s}B\sum_{\alpha\in\mathfrak{T}(m-s,m)}|F_\alpha|^2\underline{v}_m$$

$$=\binom{m}{s}B\left(\frac{i}{2}\right)^s df_1\wedge d\bar{f}_1\wedge\ldots\wedge df_s\wedge d\bar{f}_s\wedge\underline{v}_{m-s}$$

$$=\binom{m}{s}Bf^*(\eta_{\varphi\varphi})\wedge\underline{v}_{m-s}.$$

For $\varrho\in\mathbf{R}$ with $0<\varrho<1$ define

$$T(\varrho)=\left\{\mathfrak{z}\,|\,\mathfrak{z}\in\bar{H},\ \frac{\varrho}{2}\leqslant|f(\mathfrak{z})|\leqslant\varrho\right\},$$

$$J(\varrho)=\int_{T(\varrho)}\left(\log\frac{1}{|f|}\right)^{\varkappa}\frac{1}{|f|^{2s}}f^*(\eta_{\varphi\varphi})\wedge\underline{v}_{m-s}.$$

Then $L(\varrho)\subseteq T(\varrho)$ and

$$I(\varrho)\leqslant\binom{m}{s}B\cdot J(\varrho),\quad\text{if } 0<\varrho<1.$$

Therefore, it suffices to prove

$$J(\varrho)\to 0\quad\text{for } \varrho\to 0.$$

Define $\pi:\mathbf{C}^p\to\mathbf{C}^s$ by $\pi(z_1,\ldots,z_p)=(z_1,\ldots,z_s)$. On $\mathbf{C}^s$ distinguish the standard forms by a dash: v_r', ω_r', $\eta_{\varphi\psi}'$ and ζ_φ'. Then $g=\pi\circ f:M\to\mathbf{C}^s$ is a map of pure fiber dimension $q+p-s=m-s$, hence $(m-s)$-fibering. Moreover, $\eta_{\varphi\varphi}=\pi^*(v_s')$ and $f^*(\eta_{\varphi\varphi})=g^*(v_s')$. According to II Proposition 2.9 is

$$J(\varrho)=\int_{T(\varrho)}\left(\log\frac{1}{|f|}\right)^{\varkappa}\frac{1}{|f|^{2s}}\underline{v}_{m-s}\wedge g^*(v_s')$$

$$=\int_{\mathfrak{w}\in\mathbf{C}^s}\left(\int_{g^{-1}(\mathfrak{w})\cap T(\varrho)}\left(\log\frac{1}{|f|}\right)^{\varkappa}\frac{1}{|f|^{2s}}\underline{v}_{m-s}\right)v_s'.$$

If $\mathfrak{z}\in g^{-1}(\mathfrak{w})\cap T(\varrho)$, then $|\mathfrak{w}|=|g(\mathfrak{z})|\leqslant|f(\mathfrak{z})|\leqslant\varrho$. Hence

$$0\leqslant J(\varrho)=\int_{|\mathfrak{w}|\leqslant\varrho}\left(\int_{g^{-1}(\mathfrak{w})\cap T(\varrho)}\nu_g\left(\log\frac{1}{|f|}\right)^{\varkappa}\frac{1}{|f|^{2s}}\underline{v}_{m-s}\right)v_s'$$

$$\leqslant\frac{4^s}{\varrho^{2s}}\int_{|\mathfrak{w}|\leqslant\varrho}\left(\int_{g^{-1}(\mathfrak{w})\cap T(\varrho)}\nu_g\left(\log\frac{1}{|f|}\right)^{\varkappa}\underline{v}_{m-s}\right)v_s' \tag{*}$$

$$=4^s\int_{|\mathfrak{w}|\leqslant 1}\left(\int_{g^{-1}(\varrho\mathfrak{w})\cap T(\varrho)}\nu_g\left(\log\frac{1}{|f|}\right)^{\varkappa}\underline{v}_{m-s}\right)v_s'$$

for $0<\varrho<1$.

Because $g^{-1}(0)\supseteq f^{-1}(0)$ with $\dim g^{-1}(0)=m-s$ and $\dim f^{-1}(0)=m-p$ and $m-p<m-s$,

f does not vanish identically on any branch of $g^{-1}(\mathfrak{w})$ if $\mathfrak{w}\in\mathbf{C}^s$. A constant $F>1$ exists such that $|f(\mathfrak{z})|\leqslant F$ if $\mathfrak{z}\in\bar{H}$. According to Theorem 4.9, the integral

$$G(w)=\int_{g^{-1}(w)\cap\bar{H}}\left(\log\frac{F}{|f|}\right)^{\times}\nu_g\underline{v}_{m-s}$$

is a continuous function on $\mathbf{C}^s$. Therefore, a constant $C>0$ exists such that $0\leqslant G(\mathfrak{w})\leqslant C$ if $|\mathfrak{w}|\leqslant 1$. Let $0\leqslant\varrho\leqslant 1$ and $\mathfrak{w}\in\mathbf{C}^s$ with $|\mathfrak{w}|\leqslant 1$. Then

$$\int_{g^{-1}(\varrho\mathfrak{w})\cap T(\varrho)}\nu_f\left(\log\frac{1}{|f|}\right)^{\times}\underline{v}_{m-s}\leqslant\int_{g^{-1}(\varrho\mathfrak{w})\cap T(\varrho)}\nu_f\left(\log\frac{F}{|f|}\right)^{\times}\underline{v}_{m-s}\leqslant G(\varrho\mathfrak{w})\leqslant C.$$

Now,

$$\int_{g^{-1}(\varrho\mathfrak{w})\cap T(\varrho)}\nu_g\left(\log\frac{1}{|f|}\right)^{\times}\underline{v}_{m-s}\to 0$$

for $\varrho\to 0$ shall be proved.

Take $\varepsilon>0$. Define

$$A(\varepsilon)=\bigcup_{\mathfrak{y}\in\bar{H}\cap f^{-1}(0)}\{\mathfrak{z}\,|\,|\mathfrak{z}-\mathfrak{y}|<\varepsilon\}.$$

Because $\bar{H}$ is compact and contained in M, an open neighborhood H_0 of $\bar{H}$ exists such that $\bar{H}_0$ is compact and contained in M. Then $\varepsilon_0>0$ exists such that

$$\overline{A(\varepsilon)}\subseteq\overline{A(\varepsilon_0)}\subseteq H_0\quad\text{if } 0<\varepsilon\leqslant\varepsilon_0.$$

Therefore $\overline{A(\varepsilon)}$ is compact if $0<\varepsilon\leqslant\varepsilon_0$. Take $\mathfrak{z}\in\bar{A}(\frac{1}{2}\varepsilon)$. A sequence $\{\mathfrak{z}_\nu\}_{\nu\in\mathbf{N}}$ of points in $A(\frac{1}{2}\varepsilon)$ converges to $\mathfrak{z}$. Hence $\mathfrak{y}_\nu\in\bar{H}\wedge f^{-1}(0)$ exists such that $|\mathfrak{z}_\nu-\mathfrak{y}_\nu|<\frac{1}{2}\varepsilon$. Because $\bar{H}$ is compact, a convergent subsequence $\{\mathfrak{y}_{\nu_\lambda}\}_{\lambda\in\mathbf{N}}$ exists, where $\nu_\lambda\to\infty$ for $\lambda\to\infty$ and where $\mathfrak{y}=\lim_{\lambda\to\infty}\mathfrak{y}_{\nu_\lambda}\in\bar{H}\cap f^{-1}(0)$. Then $|\mathfrak{z}-\mathfrak{y}|\leqslant\frac{1}{2}\varepsilon<\varepsilon$. Hence $\mathfrak{z}\in A(\varepsilon)$. Therefore $\bar{A}(\frac{1}{2}\varepsilon)\subseteq A(\varepsilon)$. Take a continuous function λ_ε on $\mathbf{C}^m$ with $0\leqslant\lambda_\varepsilon\leqslant 1$ such that λ_ε has compact support in $A(\varepsilon)$ and such that $\lambda_\varepsilon(\mathfrak{z})=1$ if $\mathfrak{z}\in A(\frac{1}{2}\varepsilon)$.

Suppose, that a sequence $\{\varrho_\nu\}_{\nu\in\mathbf{N}}$ of real numbers with $0<\varrho_\nu<1$ converges to zero such that $T(\varrho_\nu)-A(\frac{1}{2}\varepsilon)\neq\varnothing$ for each $\nu\in\mathbf{N}$. Pick $\mathfrak{z}_\nu\in T(\varrho_\nu)-A(\frac{1}{2}\varepsilon)$ for each $\nu\in\mathbf{N}$. Then $\mathfrak{z}_\nu\in\bar{H}$. A subsequence $\{\mathfrak{z}_{\nu_\lambda}\}_{\lambda\in\mathbf{N}}$, with $\nu_\lambda\to\infty$ for $\lambda\to\infty$, converges to a point $\mathfrak{y}\in\bar{H}-A(\frac{1}{2}\varepsilon)$. Moreover, $|f(\mathfrak{z}_{\nu_\lambda})|\leqslant\varrho_{\nu_\lambda}$ for $\lambda\in\mathbf{N}$. Hence $f(\mathfrak{y})=0$. Therefore, $\mathfrak{y}\in\bar{H}\cap f^{-1}(0)-A(\frac{1}{2}\varepsilon)=\varnothing$. Contradiction! Hence, for each ε in $0<\varepsilon\leqslant\varepsilon_0$; a number $\varrho_0(\varepsilon)$ with $0<\varrho_0(\varepsilon)<1$ exists such that $T(\varrho)\subseteq A(\frac{1}{2}\varepsilon)$ if $0<\varrho\leqslant\varrho_0(\varepsilon)$. Consequently,

$$\int_{g^{-1}(\varrho\mathfrak{w})\cap T(\varrho)}\nu_g\left(\log\frac{1}{|f|}\right)^{\times}\underline{v}_{m-s}\leqslant\int_{g^{-1}(\varrho\mathfrak{w})\cap\bar{H}}\nu_g\left(\log\frac{F}{|f|}\right)^{\times}\lambda_\varepsilon\underline{v}_{m-s}$$

if $|\mathfrak{w}|\leqslant 1$ and $0<\varrho<\varrho_0(\varepsilon)$.

Because λ_ε is continuous, because H is the union of balls and because f is not identically zero on any branch of $g^{-1}(0)$, II Theorem 4.9 implies

$$\overline{\lim_{\varrho\to 0}} \int_{g^{-1}(\varrho\mathfrak{w})\cap T(\varrho)} \nu_g \left(\log\frac{1}{|f|}\right)^{\varkappa} \underline{v}_{m-s} \leqslant \int_{g^{-1}(0)\cap\bar{H}} \nu_g \lambda_\varepsilon \left(\log\frac{F}{|f|}\right)^{\varkappa} \underline{v}_{m-s}$$

$$\leqslant \int_{g^{-1}(0)\cap\bar{H}\cap A(\varepsilon)} \nu_g \left(\log\frac{F}{|f|}\right)^{\varkappa} \underline{v}_{m-s}.$$

If $0<\varepsilon'<\varepsilon$, then $A(\varepsilon')\subseteq A(\varepsilon)$. Moreover, $\bigcap_{0<\varepsilon\leqslant\varepsilon_0} A(\varepsilon)=\bar{H}\cap f^{-1}(0)$, which is a subset of measure zero on $g^{-1}(0)$. Hence

$$\int_{g^{-1}(0)\cap\bar{H}\cap A(\varepsilon)} \nu_f \left(\log\frac{F}{|f|}\right)^{\varkappa} \underline{v}_{m-s}\to 0$$

for $\varepsilon\to 0$. Hence

$$\overline{\lim_{\varrho\to 0}} \int_{g^{-1}(\varrho\mathfrak{w})\cap T(\varrho)} \nu_g \left(\log\frac{1}{|f|}\right)^{\varkappa} \underline{v}_{m-s}=0.$$

Because the integral is non-negative, $\overline{\lim}_{\varrho\to 0}$ can be replaced by $\lim_{\varrho\to 0}$. Because the integral is uniformly bounded, the theorem of bounded convergence implies in (*) that $J(\varrho)\to 0$ for $\varrho\to 0$, q.e.d.

LEMMA 1.2. *Let $M\neq\emptyset$ be an open subset of $\mathbb{C}^m$. Let $f: M\to\mathbb{C}^p$ be a holomorphic, q-fibering map with $q=m-p$ and $0<p\leqslant m$. Let K be a compact subset of M. Let χ be a differential form of bidegree (q, q) on K with bounded and measurable coefficients on K. For $0<\varrho<1$, define*

$$L(\varrho)=\left\{\mathfrak{z}\,|\,\mathfrak{z}\in K,\ \frac{\varrho}{2}\leqslant|f(\mathfrak{z})|\leqslant\varrho\right\},$$

$$I(\varrho)=\int_{L(\varrho)} \frac{1}{|f|^{2p}}\,|\chi\wedge f^*(v_p)|.$$

Then a constant $D>0$ exists such that $|I(\varrho)|\leqslant D$ if $0<\varrho<1$.

Proof. At first, consider the case $q>0$. Let $\varphi\in\mathfrak{T}(p, p)$ be the identity. Continue χ onto M by setting $\chi(z)=0$ if $z\in M-K$. Denote $f=(f_1, ..., f_p)$. Construct H, B, $T(\varrho)$, $J(\varrho)$, $\eta_{\varphi\varphi}=v_p$, $F>1$, $\pi=\mathrm{Id}:\mathbb{C}^p\to\mathbb{C}^p$ and $g=f$ as in the proof of Lemma 1.1, where $s=p$ and $\varkappa=0$. Then

$$I(\varrho)\leqslant\binom{m}{p}BJ(\varrho)\quad \text{if } 0<\varrho<1$$

and
$$J(\varrho)=\int_{T(\varrho)}\frac{1}{|f|^{2p}}f^*(v)\wedge\underline{v}_q$$
$$=\int_{\varrho/2\leqslant|\mathfrak{w}|\leqslant\varrho}\frac{1}{|\mathfrak{w}|^{2p}}\left(\int_{f^{-1}(\mathfrak{w})\cap\bar{H}}\nu_f\underline{v}_q\right)v_p$$
$$\leqslant 4^p\int_{\frac{1}{2}\leqslant|\mathfrak{w}|\leqslant 1}\left(\int_{f^{-1}(\varrho\mathfrak{w})\cap\bar{H}}\nu_f\underline{v}_q\right)v_p.$$

Because H is the union of finitely many balls, $\int_{f^{-1}(\mathfrak{w})\cap\bar{H}}\nu_f\underline{v}_q$ is continuous on $\mathbf{C}^p$, hence bounded by a constant $D_0>0$ on the unit ball. Hence

$$0\leqslant\int_{f^{-1}(\varrho\mathfrak{w})\cap\bar{H}}\nu_f\underline{v}_q\leqslant D_0$$

if $|\mathfrak{w}|\leqslant 1$ and $0<\varrho<1$. Therefore

$$0\leqslant J(\varrho)\leqslant 4^p\cdot\frac{\pi^p}{p!}D_0=D_1$$

and
$$I(\varrho)\leqslant\binom{m}{p}BD_1=D\quad\text{if } 0<\varrho<1.$$

As the second case, consider $q=0$. Then $p=m$. The map f is light. According to I Lemma 2.5 a constant $D_0>0$ exists such that

$$n_f(K;\mathfrak{w})=\sum_{\mathfrak{z}\in K}\nu_f(\mathfrak{z};\mathfrak{w})\leqslant D_0\quad\text{if }\mathfrak{w}\in\mathbf{C}^p.$$

Because $q=0$, the form χ is a function, which is bounded and measurable on K. Hence $B>0$ exists such that $|\chi(\mathfrak{z})|\leqslant B$ if $\mathfrak{z}\in K$. Therefore

$$\sum_{\mathfrak{z}\in K}\nu_f(\mathfrak{z};\mathfrak{w})\,|\chi(\mathfrak{z})|\leqslant D_0B.$$

According to II Lemma 2.8 is

$$I(\varrho)=\int_{L(\varrho)}|\chi|\frac{1}{|f|^{2m}}|f^*(v_m)|=\int_{\varrho/2\leqslant|\mathfrak{w}|\leqslant\varrho}\Big(\sum_{\mathfrak{z}\in K}\nu_f(\mathfrak{z};\mathfrak{w})\,|\chi(\mathfrak{z})|\Big)\frac{1}{|\mathfrak{w}|^{2m}}v_m$$
$$\leqslant\frac{4^m}{\varrho^{2m}}D_0B\int_{|\mathfrak{w}|\leqslant\varrho}v_m=4^mD_0B\frac{\pi^m}{m!}=D$$

if $0<\varrho<1$, q.e.d.

Lemma 1.3. *Let $M\neq\emptyset$ be open in $\mathbf{C}^m$. Let $f:M\to\mathbf{C}^p$ be a holomorphic, q-fibering map with $q=m-p$. Let $\varkappa$, s and t non-negative integers such that*

$$1\leqslant s\leqslant p\leqslant m,\quad 1\leqslant t\leqslant p\leqslant m,\quad s+t<2p.$$

Let φ be in $\mathfrak{T}(s, p)$ and let ψ be in $\mathfrak{T}(t, p)$. Let K be a compact subset of M. Let χ be a differential form on K with bounded and measurable coefficients on K. Define $\sigma = m - s$ and $\tau = m - t$. Suppose that χ has the bidegree (σ, τ). For $0 < \varrho < 1$, define

$$L(\varrho) = \left\{\mathfrak{z} \mid \mathfrak{z} \in K, \frac{\varrho}{2} \leqslant |f(\mathfrak{z})| \leqslant \varrho\right\},$$

$$I(\varrho) = \int_{L(\varrho)} \left(\log \frac{1}{|f|}\right)^{\varkappa} \frac{1}{|f|^{s+t}} |f^*(\zeta_\varphi \wedge \bar{\zeta}_\psi) \wedge \chi|.$$

Then $I(\varrho) \to 0$ for $\varrho \to 0$.

Proof. Without loss of generality, it can be assumed, that $1 \leqslant s < p$ and $1 \leqslant t \leqslant p$. Then, holomorphic function f_μ exist such that $f = (f_1, \ldots, f_p)$. Then

$$f^*(\zeta_\varphi) = df_{\varphi(1)} \wedge \ldots \wedge df_{\varphi(s)} \text{ and } f^*(\bar{\zeta}_\psi) = d\bar{f}_{\psi(1)} \wedge \ldots \wedge d\bar{f}_{\psi(t)}.$$

Bounded and measurable function $\chi_{\alpha\beta}$ exist on K such that

$$\chi = \sum_{\alpha \in \mathfrak{T}(\sigma, m)} \sum_{\beta \in \mathfrak{T}(\tau, m)} \chi_{\alpha\beta} \zeta_\alpha \wedge \bar{\zeta}_\beta.$$

A constant $B \geqslant 0$ exists such that $|\chi_{\alpha\beta}(\mathfrak{z})| \leqslant B$ for $\mathfrak{z} \in K$ and $\alpha \in \mathfrak{T}(\sigma, m)$ and $\beta \in \mathfrak{T}(\tau, m)$. Define

$$F_\alpha = \frac{\partial(f_{\varphi(1)}, \ldots, f_{\varphi(s)})}{\partial(z_{\underline{\alpha}(\sigma+1)}, \ldots, z_{\underline{\alpha}(m)})} \quad \text{if } \alpha \in \mathfrak{T}(\sigma, m),$$

$$G_\beta = \frac{\partial(f_{\psi(1)}, \ldots, f_{\psi(t)})}{\partial(z_{\underline{\beta}(\tau+1)}, \ldots, z_{\underline{\beta}(m)})} \quad \text{if } \beta \in \mathfrak{T}(\tau, m).$$

Then
$$|f^*(\zeta_\varphi \wedge \bar{\zeta}_\psi) \wedge \chi| = \Big|\sum_{\alpha \in \mathfrak{T}(\sigma, m)} \sum_{\beta \in \mathfrak{T}(\tau, m)} \operatorname{sign} \alpha \operatorname{sign} \beta F_\alpha \bar{G}_\beta \chi_{\alpha\beta}\Big| \underline{v}_m$$
$$\leqslant B \sum_{\alpha \in \mathfrak{T}(\sigma, m)} \sum_{\beta \in \mathfrak{T}(\tau, m)} |F_\alpha| \, |G_\beta| \, \underline{v}_m.$$

Define
$$I_\alpha(\varrho) = \int_{L(\varrho)} \left(\log \frac{1}{|f|}\right)^{2\varkappa} \frac{1}{|f|^{2s}} |F_\alpha|^2 \underline{v}_m$$
$$= \int_{L(\varrho)} \left(\log \frac{1}{|f|}\right)^{2\varkappa} \frac{1}{|f|^{2s}} f^*(\eta_{\varphi\varphi}) \wedge \underline{\eta}_{\alpha\alpha},$$

$$J_\beta(\varrho) = \int_{L(\varrho)} \frac{1}{|f|^{2t}} |G_\beta|^2 \underline{v}_m = \int_{L(\varrho)} \frac{1}{|f|^{2t}} f^*(\eta_{\psi\psi}) \wedge \underline{\eta}_{\beta\beta}.$$

Then
$$0 \leqslant I(\varrho) \leqslant B \sum_{\alpha \in \mathfrak{T}(\sigma, m)} \sum_{\beta \in \mathfrak{T}(\tau, m)} \int_{L(\varrho)} \left(\log \frac{1}{|f|}\right)^{\varkappa} \frac{1}{|f|^{t+s}} |F_\alpha| \, |G_\beta| \, \underline{v}_m$$
$$\leqslant B \sum_{\alpha \in \mathfrak{T}(\sigma, m)} \sum_{\beta \in \mathfrak{T}(\tau, m)} I_\alpha(\varrho)^{\frac{1}{2}} \cdot J_\beta(\varrho)^{\frac{1}{2}}.$$

According to Lemma 1.1, $I_\alpha(\varrho)\to 0$ for $\varrho\to 0$. If $t<p$, then $J_\beta(\varrho)\to 0$ for $\varrho>0$ according to Lemma 1.1. If $t=p$, then $J_\beta(\varrho)$ is bounded on $0<\varrho<1$. Hence $I(\varrho)\to 0$ for $\varrho>0$ in both cases; q.e.d.

LEMMA 1.4. *Under the same assumptions as in Lemma* 1.3, *the integral*

$$\int_K \left(\log\frac{1}{|f|}\right)^{\varkappa}\frac{1}{|f|^{s+t}} f^*(\zeta_\varphi\wedge\bar{\zeta}_\psi)\wedge\chi$$

exists.

Proof. Without loss of generality, $0<s<p$ can be assumed. Define $L(\varrho)$, $I(\varrho)$, F_α, G_β, $I_\alpha(\varrho)$, $J_\beta(\varrho)$ as in Lemma 1.3. Then $J_\beta(\varrho)\to 0$ for $\varrho\to 0$ or $J_\beta(\varrho)$ is bounded in $0<\varrho<1$. In either case, a constant $D_1>0$ exists such that $|J_\beta(\varrho)|\leqslant D_1$ for $0<\varrho<1$ and $\beta\in\mathfrak{T}(\tau,m)$. Moreover,

$$I_\alpha(\varrho)=\int_{L(\varrho)}\left(\log\frac{1}{|f|}\right)^{2\varkappa}\frac{1}{|f|^{2s}} f^*(\eta_{\varphi\varphi})\wedge\underline{\eta}_{\alpha\alpha}$$

$$\leqslant\frac{1}{(\log 1/\varrho)^4}\int_{L(\varrho)}\left(\log\frac{1}{|f|}\right)^{2k+4}\frac{1}{|f|^{2s}} f^*(\eta_{\varphi\varphi})\wedge\underline{\eta}_{\alpha\alpha},$$

where

$$\tilde{I}_\alpha(\varrho)=\int_{L(\varrho)}\left(\log\frac{1}{|f|}\right)^{2\varkappa+4}\frac{1}{|f|^{2s}} f^*(\eta_{\varphi\varphi})\wedge\underline{\eta}_{\alpha\alpha}\to 0$$

for $\varrho\to 0$ according to Lemma 1.1. Hence a constant $D_2>0$ exists such that $\tilde{I}_\alpha(\varrho)\leqslant D_2$ if $0<\varrho<1$ and $\alpha\in\mathfrak{T}(\sigma,m)$. Hence $I_\alpha(\varrho)\leqslant(\log 1/\varrho)^{-4}D_2$ for $0<\varrho<1$. Then

$$0\leqslant I(\varrho)\leqslant B\sum_{\alpha\in\mathfrak{T}(\sigma,m)}\ \sum_{\beta\in\mathfrak{T}(\tau,m)} I_\alpha(\varrho)^{\frac12} J_\beta(\varrho)^{\frac12}$$

$$\leqslant\binom{m}{\sigma}\binom{m}{\tau}B\sqrt{D_1}\sqrt{D_2}\cdot\frac{1}{(\log 1/\varrho)^2}=\frac{B_1}{(\log 1/\varrho)^2}$$

for $0<\varrho<1$, where B_1 is a positive constant. If $1<n\in\mathbf{N}$, then

$$I\left(\frac{1}{2^n}\right)\leqslant B_1\frac{1}{(\log 2)^2}\frac{1}{n^2}=B_2\frac{1}{n^2},$$

where B_2 is constant. Define $K_2=\{\mathfrak{z}\,|\,\mathfrak{z}\in K \text{ and } |f(\mathfrak{z})|\leqslant\frac12\}$ and $K_1=K-K_2$. Then $K_2=\bigcup_{n=1}^{\infty} L(1/2^n)$ where $L(1/2^{n+1})\cap L(1/2^n)=\{\mathfrak{z}\,|\ |f(\mathfrak{z})|=1/2^{n+1}\}\cap K$ is a set of measure zero. Hence

$$\int_{K_2}\left(\log\frac{1}{|f|}\right)^{\varkappa}\frac{1}{|f|^{s+t}}|f^*(\zeta_\varphi\wedge\bar{\zeta}_\psi)\wedge\chi|=\sum_{n=1}^{\infty} I\left(\frac{1}{2^n}\right)<B_2\sum_{n=1}^{\infty}\frac{1}{n^2}<\infty.$$

Hence $(\log 1/|f|)^{\varkappa}\, 1/|f|^{s+t}\, f^*(\zeta_\varphi \wedge \bar{\zeta}_\psi) \wedge \chi$ is integrable over K_2. Clearly, it is integrable over K_1. Hence, it is integrable over K, q.e.d.

Now, the case $s=0$ has to be treated.

LEMMA 1.5. *Let $M \neq \emptyset$ be open in $\mathbf{C}^m$. Let $f: M \to \mathbf{C}^p$ be a holomorphic, q-fibering map. Let $0 < p = m-q \leqslant m$. Let $\varkappa$ be a non-negative integer. Let χ be bounded measurable form of bidegree (m, m) on a compact subset K of M. For $\varrho \in \mathbf{R}$ with $0 < \varrho < 1$, define*

$$L(\varrho) = \left\{ \mathfrak{z} \,\middle|\, \mathfrak{z} \in K, \frac{\varrho}{2} \leqslant |f(\mathfrak{z})| \leqslant \varrho \right\},$$

$$I(\varrho) = \int_{L(\varrho)} \left(\log \frac{1}{|f|} \right)^{\varkappa} |\chi|.$$

Then $I(\varrho) \to 0$ for $\varrho \to 0$. Moreover, the integral

$$\int_K \left(\log \frac{1}{|f|} \right)^{\varkappa} \chi$$

exists.

Proof. If $f(\mathfrak{z}) \neq 0$ for all $\mathfrak{z} \in K$, the lemma is trivial. Suppose that $A = K \cap f^{-1}(0) \neq \emptyset$. A constant $B > 0$ exists such that $|\chi| \leqslant B \underline{v}_m$ on K. Define $E = \{ w \mid |w| \leqslant 1 \}$. Define $\varphi: M \times \mathbf{C} \to \mathbf{C}$ and $\pi: M \times \mathbf{C} \to M$ as the natural projections. Then $\nu_\varphi(\mathfrak{z}, 0) = 1$ for $\mathfrak{z} \in M$. Moreover, $\varphi^{-1}(0) = M \times \{0\}$ and $\varphi^{-1}(0) \cap (K \times E) = K \times \{0\}$. Define $j: M \to M \times \mathbf{C}$ by $j(\mathfrak{z}) = (\mathfrak{z}, 0)$. Then $\pi \circ j = \mathrm{Id}: M \to M$ is the identity. Define $g = f \circ \pi$. Then $g \circ j = f$. Moreover,

$$K_1 = \varphi^{-1}(0) \cap g^{-1}(0) \cap (K \times E) = A \times \{0\} \neq \emptyset.$$

Apply II Lemma 4.8 using the table

There	M	m	N	p	q	f	$\varkappa$	g	b	χ	K	K_1
Here	$M \times \mathbf{C}$	$m+1$	$\mathbf{C}$	1	m	φ	$\varkappa$	g	0	$\pi^*(\underline{v}_m)$	$K \times E$	K_1

Hence
$$J = \int_{\varphi^{-1}(0) \cap (K \times E)} \nu_\varphi |\log |g||^{\varkappa}\, \pi^*(\underline{v}_m)$$

exists. Because $j: M \to M \times \{0\}$ is biholomorphic,

$$J = \int_K |\log |f||^{\varkappa}\, \underline{v}_m$$

exists. Because χ is measurable and $|\chi| \leqslant B \underline{v}_m$. The functions $|(\log 1/|f|)|^{\varkappa} |\chi|$ and $(\log 1/|f|)^{\varkappa} \chi$ are integrable over K.

Define $A(\varrho)=\{\mathfrak{z}\,|\,\mathfrak{z}\in K,\ |f(\mathfrak{z})|\leqslant\varrho\}$ for $0<\varrho<1$. Define

$$S(\varrho)=\int_{A(\varrho)}\left(\log\frac{1}{|f|}\right)^{\varkappa}|\chi|.$$

Because $\bigcap_{0<\varrho<1}A(\varrho)=A$ is a set of measure zero, and because $A(\varrho')\subseteq A(\varrho)$ if $0<\varrho'<\varrho<1$, the integral $S(\varrho)\to 0$ for $\varrho\to 0$. Because $0\leqslant I(\varrho)\leqslant S(\varrho)$, also $I(\varrho)\to 0$ for $\varrho\to 0$, q.e.d.

Of course, this lemma can also be proved directly.

LEMMA 1.6. *Let $M\neq\emptyset$ be open in $\mathbf{C}^m$. Let $f:M\to\mathbf{C}^p$ be a holomorphic, q-fibering map with $q=m-p$. Let $\varkappa$ be a non-negative integer. Let $t\in\mathbf{N}$ such that $t\leqslant p\leqslant m$. Take ψ in $\mathfrak{T}(t,m)$. Define $\tau=m-t$. Let χ be a differential form on the compact subset K of M with bounded and measurable coefficients on K. Suppose that χ has bidegree (m,τ). For $0<\varrho<1$, define*

$$L(\varrho)=\left\{\mathfrak{z}\,\middle|\,\mathfrak{z}\in K,\ \frac{\varrho}{2}\leqslant|f(\mathfrak{z})|\leqslant\varrho\right\},$$

$$I(\varrho)=\int_{L(\varrho)}\left(\log\frac{1}{|f|}\right)^{\varkappa}\frac{1}{|f|^{t}}f^{*}(\xi_{\psi})\wedge\chi.$$

Then $I(\varrho)\to 0$ for $\varrho\to 0$. Moreover,

$$\int_{K}\left(\log\frac{1}{|f|}\right)^{\varkappa}\frac{1}{|f|^{t}}f^{*}(\xi_{\psi})\wedge\chi\quad\text{and}\quad\int_{K}\left(\log\frac{1}{|f|}\right)^{\varkappa}\frac{1}{|f|^{t}}f^{*}(\zeta_{\psi})\wedge\bar{\chi}$$

exist.

Proof. Let $\iota\in\mathfrak{T}(m,m)$ be the identity $\iota(\nu)=\nu$ for $\nu=1,\ldots,m$. Bounded and measurable function χ_{β} exist on K such that

$$\chi=\sum_{\beta\in\mathfrak{T}(\tau,m)}\chi_{\beta}\zeta_{\iota}\wedge\bar{\zeta}_{\beta}$$

A constant $B>0$ exists such that $|\chi_{\beta}|\leqslant B$ on K. Holomorphic functions f_{μ} exist such that $f=(f_1,\ldots,f_p)$. Then

$$f^{*}(\xi_{\psi})=d\bar{f}_{\psi(1)}\wedge\ldots\wedge d\bar{f}_{\psi(t)}.$$

Define
$$G_{\beta}=\frac{\partial(f_{\psi(1)},\ldots,f_{\psi(t)})}{\partial(z_{\beta(\tau+1)},\ldots,z_{\beta(m)})}\quad\text{if }\beta\in\mathfrak{T}(\tau,m).$$

Then
$$|f^{*}(\xi_{\psi})\wedge\chi|=\Big|\sum_{\beta\in\mathfrak{T}(\tau,m)}\operatorname{sign}\beta\,\bar{G}_{\beta}\chi_{\beta}\Big|\,\underline{v}_m\leqslant B\sum_{\beta\in\mathfrak{T}(\tau,m)}|\bar{G}_{\beta}|\,\underline{v}_m.$$

Define
$$J_\beta(\varrho)=\int_{L(\varrho)}\frac{1}{|f|^{2t}}|G_\beta|^2\,\underline{v}_m=\int_{L(\varrho)}\frac{1}{|f|^{2t}}f^*(\eta_{\psi\psi})\wedge\underline{\eta}_{\beta\beta},$$
$$I_t(\varrho)=\int_{L(\varrho)}\left(\log\frac{1}{|f|}\right)^{2\varkappa}\underline{v}_m,$$
$$\tilde{I}_t(\varrho)=\int_{L(\varrho)}\left(\log\frac{1}{|f|}\right)^{2\varkappa+4}\underline{v}_m.$$

Then
$$I_t(\varrho)\to 0 \text{ and } \tilde{I}_t(\varrho)\to 0 \text{ for } \varrho\to 0.$$

If $t<p$, then $J_\beta(\varrho)\to 0$ for $\varrho\to 0$; if $t=p$, then $J_\beta(\varrho)$ is bounded in $0<\varrho<1$. In either case, a constant $D_1>0$ exists such that $J_\beta(\varrho)\leqslant D_1$ for $0<\varrho<1$. A constant $D_2>0$ exists such that $\tilde{I}_t(\varrho)\leqslant D_2$ if $0<\varrho<1$. Then
$$0\leqslant I(\varrho)\leqslant B\sum_{\beta\in\mathfrak{T}(\tau,m)}\int_{L(\varrho)}\left(\log\frac{1}{|f|}\right)^{\varkappa}\frac{1}{|f|^t}|G_\beta|\,\underline{v}_m$$
$$\leqslant B\sum_{\beta\in\mathfrak{T}(\tau,m)}I_t(\varrho)^{\frac{1}{2}}J_\beta(\varrho)^{\frac{1}{2}}.$$

Hence, $I(\varrho)\to 0$ for $\varrho\to\infty$. Moreover,
$$I_t(\varrho)\leqslant\frac{1}{(\log 1/\varrho)^4}\tilde{I}_t(\varrho)\leqslant\frac{D_2}{(\log 1/\varrho)^4}$$
for $0<\varrho<1$. Hence
$$I(\varrho)\leqslant\binom{m}{t}B\sqrt{D_1}\sqrt{D_2}\frac{1}{(\log 1/\varrho)^2}=\frac{B_1}{(\log 1/\varrho)^2}$$

for $0<\varrho<1$, where B_1 is constant. If $1<n\in\mathbf{N}$, then
$$I\left(\frac{1}{2^n}\right)\leqslant B_1\frac{1}{(\log 2)^2}\frac{1}{n^2}=\frac{B_2}{n^2}$$

where B_2 is constant. Define $K_2=\{\mathfrak{z}\mid\mathfrak{z}\in K \text{ and } |f(\mathfrak{z})|\leqslant\frac{1}{2}\}$ and $K_1=K-K_2$. Then $K_2=\bigcup_{n=1}^{\infty}L(1/2^n)$ where $L(1/2^n)\cap L(1/2^m)$ is a set of measure zero if $n\neq m$. Hence
$$\int_{K_2}\left(\log\frac{1}{|f|}\right)^{\varkappa}\frac{1}{|f|^t}|f^*(\xi_\psi)\wedge\chi|=\sum_{n=1}^{\infty}I\left(\frac{1}{2^n}\right)<B_2\sum_{n=1}^{\infty}\frac{1}{n^2}<\infty.$$

Therefore $(\log|1/f|)^{\varkappa}|1/f|^t f^*(\xi_\psi)\wedge\chi$ and its conjugate are integrable over K_2. Clearly, they are integrable over K_1. Hence, they are integrable over $K=K_1\cup K_2$, q.e.d.

Proposition 1.7. *Let M be a complex manifold of pure dimension m. Let $f:M\to\mathbf{C}^p$ be a holomorphic, q-fibering map with $q=m-p$, where $0<p\leqslant m$. Let $\varkappa$, s, t be non-negative integers such that*

$$0 \leqslant s \leqslant p, \quad 0 \leqslant t \leqslant p, \quad s+t<2p.$$

Define $\sigma=m-s$ and $\tau=m-t$. Let K be a compact subset of M. Let χ be a differential form on K whose coefficients are measurable and locally bounded. Suppose that χ has bidegree (σ, τ). Let φ be a differential form of bidegree (s, t) on $\mathbf{C}^p$ with measurable and locally bounded coefficients. Suppose that for every $\varrho>0$ a measurable function h_ϱ is given on M. Suppose that these functions are uniformly bounded by a constant B on K, that is $|h_\varrho(z)| \leqslant B$ if $\varrho>0$ and $z \in K$. For $\varrho>0$, define

$$L(\varrho)=\left\{z \,\middle|\, z \in K, \frac{\varrho}{2} \leqslant |f(z)| \leqslant \varrho\right\},$$

$$I(\varrho)=\int_{L(\varrho)} |h_\varrho| \left(\log \frac{1}{|f|}\right)^{\varkappa} \frac{1}{|f|^{s+t}} |f^*(\varphi) \wedge \chi|.$$

Then $I(\varrho) \rightarrow 0$ for $\varrho \rightarrow 0$ and

$$\tilde{\chi}=\left(\log \frac{1}{|f|}\right)^{\varkappa} \frac{1}{|f|^{s+t}} f^*(\varphi) \wedge \chi$$

is integrable over K.

Proof. The form χ is integrable over K, if and only if every point $z_0 \in K$ has a compact neighborhood U such that χ is integrable over $K \cap U$. Therefore, it can be assumed that $M \neq \emptyset$ is open in $\mathbf{C}^m$.

1. Case: $s=t=0$. Then φ is a function and $f^*(\varphi)=\varphi \circ f$ is a function on M, which is bounded and measurable on K. Hence $\tilde{\chi}$ is integrable over K according to Lemma 1.5.

2. Case: $0=s<t$. Then measurable and locally bounded function φ_β exist on $\mathbf{C}^p$ such that

$$\varphi=\sum_{\beta \in \mathfrak{T}(t, p)} \varphi_\beta \zeta_\beta.$$

Then $\varphi_\beta \circ f$ is a measurable and bounded function on K and

$$f^*(\varphi)=\sum_{\beta \in \mathfrak{T}(t, p)} (\varphi_\beta \circ f) f^*(\zeta_\beta).$$

According to Lemma 1.6, the differential

$$\tilde{\chi}_\beta=\left(\log \frac{1}{|f|}\right)^{\varkappa} \frac{1}{|f|^t} (\varphi_\beta \circ f) f^*(\zeta_\beta)$$

is integrable over K. Hence $\tilde{\chi}=\sum_{\beta \in \mathfrak{T}(t, p)} \tilde{\chi}_\beta$ is integrable over K.

3. Case: $0=t<s$. By conjugation, this follows from case 2.

4. Case: $0<s$ and $0<t$: Measurable and locally bounded function $\varphi_{\alpha\beta}$ exist on $\mathbf{C}^p$ such that

$$\varphi=\sum_{\alpha\in\mathfrak{T}(s,p)}\sum_{\beta\in\mathfrak{T}(t,p)}\varphi_{\alpha\beta}\zeta_\alpha\wedge\bar{\zeta}_\beta.$$

Then

$$f^*(\varphi)=\sum_{\alpha\in\mathfrak{T}(s,p)}\sum_{\beta\in\mathfrak{T}(t,p)}(\varphi_{\alpha\beta}\circ f)\,f^*(\zeta_\alpha\wedge\bar{\zeta}_\beta),$$

where each $\varphi_{\alpha\beta}\circ f$ is bounded and measurable on K. According to Lemma 1.4 is

$$\tilde{\chi}_{\alpha\beta}=\left(\log\frac{1}{|f|}\right)^{\varkappa}\frac{1}{|f|^{s+t}}(\varphi_{\alpha\beta}\circ f)\cdot f^*(\zeta_\alpha\circ\bar{\zeta}_\beta)\circ\chi$$

integrable over K. Hence $\tilde{\chi}=\sum_{\alpha\in\mathfrak{T}(s,p)}\sum_{\beta\in\mathfrak{T}(t,p)}\tilde{\chi}_{\alpha\beta}$ is integrable over K.

Return to the original assumptions that K is a compact subset of a complex manifold M. Define

$$A(\varrho)=\{z\,|\,z\in K \text{ with } |f(z)|\leqslant\varrho\}.$$

If $0<\varrho<1$, then

$$S(\varrho)=\int_{A(\varrho)}|\tilde{\chi}|$$

exists. Because $A(\varrho')\subseteq A(\varrho)$ if $0<\varrho'<\varrho<1$ and because $\bigcap_{0<\varrho<1}A(\varrho)=f^{-1}(0)\cap K$ is a set of measure zero. $S(\varrho)\to 0$ for $\varrho\to 0$. Because

$$|I(\varrho)|\leqslant\int_{L(\varrho)}|h_\varrho|\,|\tilde{\chi}|\leqslant B\int_{A(\varrho)}|\tilde{\chi}|=S(\varrho)$$

also $I(\varrho)\to 0$ for $\varrho\to 0$; q.e.d.

Now, a type of residue formula shall be proved, which will be very helpful later on. Let V be a complex vector space of dimension m with an Hermitian product $(\,|\,)$. Then the projective forms ω and ω_p are defined on $V-\{0\}$. On $\mathbf{P}(V)$ the forms $\ddot{\omega}$ and $\ddot{\omega}_p$ are defined. Let $\varrho: V-\{0\}\to\mathbf{P}(V)$ be the residual map. Then $\omega_p=\varrho^*(\ddot{\omega}_p)$. Because degree $\ddot{\omega}_p=2p$, for $p\geqslant m$ is $\ddot{\omega}_p=0$. Therefore,

$$\omega_p=0 \text{ on } V-\{0\} \text{ if } p\geqslant m.$$

On $V-\{0\}$ is

$$d^\perp\log|\mathfrak{z}|=i(\partial-\bar{\partial})\,\tfrac{1}{2}\log(\mathfrak{z}|\mathfrak{z})=\frac{i}{2}\frac{1}{|\mathfrak{z}|^2}((d\mathfrak{z}|\mathfrak{z})-(\mathfrak{z}|d\mathfrak{z})).$$

Moreover,

$$\omega_{m-1}=\left(\frac{i}{2}\right)^{m-1}\frac{1}{(m-1)!}\left(\frac{(d\mathfrak{z}|d\mathfrak{z})}{|\mathfrak{z}|^2}-\frac{(d\mathfrak{z}|\mathfrak{z})\wedge(\mathfrak{z}|d\mathfrak{z})}{|\mathfrak{z}|^4}\right)^{m-1}$$

$$=\left(\frac{i}{2}\right)^{m-1}\frac{1}{(m-1)!}\frac{(d\mathfrak{z}|d\mathfrak{z})^{m-1}}{|\mathfrak{z}|^{2m-2}}-\left(\frac{i}{2}\right)^{m-1}\frac{1}{(m-2)!}\frac{(d\mathfrak{z}|d\mathfrak{z})^{m-2}}{|\mathfrak{z}|^{2m}}\wedge(d\mathfrak{z}|\mathfrak{z})\wedge(\mathfrak{z}|d\mathfrak{z})$$

$$=|\mathfrak{z}|^{2-2m}v_{m-1}-|\mathfrak{z}|^{-2m}(d\mathfrak{z}|\mathfrak{z})\wedge(\mathfrak{z}|d\mathfrak{z})\wedge v_{m-2}.$$

Hence $$d^{\perp}\log|\mathfrak{z}|\wedge\omega_{m-1}=|\mathfrak{z}|^{2-2m}d^{\perp}\log|\mathfrak{z}|\wedge v_{m-1}.$$

LEMMA 1.8. *Let M be a complex manifold of pure dimension m. Let $f:M\to\mathbf{C}^p$ be a holomorphic, q fibering map with $q=m-p$ and $0<p\leqslant m$. Let H be an open subset of M with compact closure $\bar{H}$. Let χ be a continuous differential form of bidegree (q,q) on M. Let T be the support of χ in $\bar{H}-H$. Suppose that $f^{-1}(0)\cap T$ is a set of measure zero on $f^{-1}(0)$ if $q>0$ and that $f^{-1}(0)\cap T=\varnothing$ if $q=0$. Suppose that a constant $B>0$ and for every ϱ in $0<\varrho<1$ a function g_ϱ of class C^∞ on $\mathbf{R}$ are given such that*

1. *For $x\in\mathbf{R}$ and $0<\varrho<1$ is $|g_\varrho(x)|\leqslant 1$.*
2. *For $x\in\mathbf{R}$ and $0<\varrho<1$ is $|xg_\varrho'(x)|\leqslant B$.*
3. *For $x\leqslant\frac{1}{2}\varrho$ is $g_\varrho(x)=0$.*
4. *For $x\geqslant\varrho$ is $g_\varrho(x)=1$.*

For $\mathfrak{w}\in\mathbf{C}^p$ and $0<\varrho<1$, define $\lambda_\varrho(\mathfrak{w})=g_\varrho(|\mathfrak{w}|)$. Define

$$I(\varrho)=\int_H d^{\perp}\log|f|\wedge d(\lambda_\varrho\circ f)\wedge f^*(\omega_{p-1})\wedge\chi.$$

Then $$I(\varrho)\to\frac{2\pi^p}{(p-1)!}\int_{H\cap f^{-1}(0)}\nu_f\chi\quad\text{for }\varrho\to 0,$$

where, in the case $q=0$, χ is a function and the integral means a sum:

$$\int_{H\cap f^{-1}(0)}\nu_f\chi=\sum_{z\in H}\nu_f(z;0)\,\chi(z).$$

Proof. If $q>0$, then define J by

$$J(\mathfrak{w})=\int_{f^{-1}(\mathfrak{w})\cap H}\nu_f\chi.$$

According to II Theorem 3.9, J is continuous at $0\in\mathbf{C}^p$. If $q=0$, then define J by

$$J(\mathfrak{w})=\sum_{z\in H}\nu_f(z;\mathfrak{w})\,\chi(z).$$

According to I Proposition 3.2, J is continuous at $0\in\mathbf{C}^p$. For $0<\varrho<1$ is [1]

$$I(\varrho)=\int_{\mathbf{C}^p}J(\mathfrak{w})\,d^{\perp}\log|\mathfrak{w}|\wedge d\lambda_\varrho(\mathfrak{w})\wedge\omega_{p-1}(\mathfrak{w}).$$

Then $$d\lambda_\varrho=g_\varrho'(|\mathfrak{w}|)\,\frac{(d\mathfrak{w}|\mathfrak{w})+(\mathfrak{w}|d\mathfrak{w})}{2|\mathfrak{w}|}$$

[1] For $q>0$, see II, Proposition 2.9. For $q=0$, see II, Proposition 2.8.

Hence $$d^{\perp}\log|\mathfrak{w}|\wedge d\lambda_\varrho(\mathfrak{w})\wedge\omega_{p-1}(\mathfrak{w})=d^{\perp}\log|\mathfrak{w}|\wedge d\lambda_\varrho(\mathfrak{w})\wedge\frac{v_{p-1}(\mathfrak{w})}{|\mathfrak{w}|^{2p-2}}$$

$$=\frac{i}{4}\frac{1}{|\mathfrak{w}|^{2p+1}}g_\varrho'(|\mathfrak{w}|)\wedge((d\mathfrak{w}|\mathfrak{w})-(\mathfrak{w}|d\mathfrak{w}))\wedge((d\mathfrak{w}|\mathfrak{w}+(\mathfrak{w}|d\mathfrak{w}))\wedge v_{p-1}(\mathfrak{w})$$

$$=\frac{1}{|\mathfrak{w}|^{2p+1}}g_\varrho'(|\mathfrak{w}|)\frac{i}{2}(d\mathfrak{w}|\mathfrak{w})\wedge(\mathfrak{w}|d\mathfrak{w})\wedge v_{p-1}(\mathfrak{w}).$$

Define $$I_0(\varrho)=\int_{\mathbf{C}^p}(J(\mathfrak{w})-J(0))\,d^{\perp}\log|\mathfrak{w}|\wedge d\lambda_\varrho(\mathfrak{w})\wedge\omega_{p-1}(\mathfrak{w}),$$

$$I_1(\varrho)=\int_{\mathbf{C}^p}d^{\perp}\log|\mathfrak{w}|\wedge d\lambda_\varrho(\mathfrak{w})\wedge\omega_{p-1}(\omega).$$

Then $I(\varrho)=I_0(\varrho)+J(0)\,I_1(\varrho)$ for $0<\varrho<1$. It is

$$I_0(\varrho)=\int_{\varrho/2\leqslant|\mathfrak{w}|\leqslant\varrho}(J(\mathfrak{w})-J(0))\frac{1}{|\mathfrak{w}|^{2p+2}}(g_\varrho'(|\mathfrak{w}|)\,|\mathfrak{w}|)\,\frac{i}{2}(d\mathfrak{w}|\mathfrak{w})\wedge(\mathfrak{w}|d\mathfrak{w})\wedge v_{p-1}(\mathfrak{w})$$

$$=\int_{\frac{1}{2}\leqslant|\mathfrak{w}|\leqslant 1}(J(\varrho\mathfrak{w})-J(0))\frac{1}{|\mathfrak{w}|^{2p+2}}(g_\varrho'(|\mathfrak{w}|\,\varrho)\,|\mathfrak{w}|\,\varrho)\,\frac{i}{2}(d\mathfrak{w}|\mathfrak{w})\wedge(\mathfrak{w}|d\mathfrak{w})\wedge v_{p-1}.$$

Define $$D=\int_{\frac{1}{2}\leqslant|\mathfrak{w}|\leqslant 1}\frac{1}{|\mathfrak{w}|^{2p+1}}\frac{i}{2}\wedge(d\mathfrak{w}|\mathfrak{w})\wedge(\mathfrak{w}|d\mathfrak{w})\wedge v_{p-1}(\mathfrak{w}).$$

Then $0<D<\infty$. For every $\varepsilon>0$, a number $\varrho_0(\varepsilon)$ exists in $0<\varrho_0(\varepsilon)<1$ such that $|\mathfrak{w}|<\varrho_0(\varepsilon)$ implies $|J(\mathfrak{w})-J(0)|<\varepsilon$. Therefore $|J(\varrho\mathfrak{w})-J(0)|<\varepsilon$ if $|\mathfrak{w}|\leqslant 1$ and $0<\varrho<\varrho_0(\varepsilon)$. Hence $|I_0(\varrho)|\leqslant BD\varepsilon$ if $0<\varrho<\varrho_0(\varepsilon)$. Hence

$$I_0(\varrho)\to 0\quad\text{for }\varrho\to 0.$$

For I_1 is $$I_1(\varrho)=\int_{\varrho/2\leqslant|\mathfrak{w}|\leqslant\varrho}d^{\perp}\log|\mathfrak{w}|\wedge d\lambda_\varrho\wedge\omega_{p-1}(\mathfrak{w})$$

$$=-\int_{|\mathfrak{w}|\leqslant\varrho}d(\lambda_\varrho d^{\perp}\log|\mathfrak{w}|\wedge\omega_{p-1}(\mathfrak{w}))$$

because $$dd^{\perp}\log|\mathfrak{w}|\wedge\omega_{p-1}(\mathfrak{w})=-2\omega\wedge\omega_{p-1}=-2p\omega_p=0.$$

Let S_ϱ be the sphere of radius ϱ with center at $0\in\mathbf{C}^p$ oriented to the exterior of $\{\mathfrak{z}\mid|\mathfrak{z}|<\varrho\}$. Stokes' Theorem implies

$$\begin{aligned} I_1(\varrho) &= -\int_{S_\varrho} d^\perp \log |\mathfrak{w}| \wedge \omega_{p-1}(\mathfrak{w}) \\ &= -\int_{S_1} d^\intercal \log |\mathfrak{w}| \wedge \omega_{p-1}(\mathfrak{w}) \\ &= -\int_{S_1} d^\perp \log |\mathfrak{w}| \wedge v_{p-1}(\mathfrak{w}). \end{aligned}$$

If $j: S_1 \to \mathbf{C}^p$ is the inclusion map, the $j^*(d^\perp \log 1/|\mathfrak{w}| \wedge v_{p-1}(\mathfrak{w}))$ is the euclidean volume element of S_1. Hence

$$I_1(\varrho) = \frac{2\pi^p}{(p-1)!} \quad \text{if } 0 < \varrho < 1.$$

Hence
$$I(\varrho) = I_0(\varrho) + J(0)\, I_1(\varrho) \to \frac{2\pi^p}{(p-1)!} J(0)$$

for $\varrho \to 0$; q.e.d.

2. The intersection number

Let V be a complex vector space of dimension $n+1$. Let $\mathbf{P}(V)$ be the associated projective space and let $\varrho_V: V - \{0\} \to \mathbf{P}(V)$ be the associated projection such that

$$\varrho_V^{-1}(\varrho_V(\mathfrak{a})) = \{z\mathfrak{a} \mid 0 \neq z \in \mathbf{C}\}.$$

Let $V[p]$ be the p-folded exterior product of V. Define

$$\tilde{\mathfrak{G}}^{p+1}(V) = \{\mathfrak{c}_0 \wedge \dots \wedge \mathfrak{c}_p \mid \mathfrak{c}_\mu \in V\} \subseteq V[p+1].$$

Then the Grassmann-manifold

$$\mathfrak{G}^p(V) = \varrho_{V[p+1]}(\tilde{\mathfrak{G}}^{p+1}(V) - \{0\})$$

is a smooth, connected, compact submanifold of $\mathbf{P}(V[p+1])$,

If $\mathfrak{c}_0, \dots, \mathfrak{c}_p$ are linearly independent vectors of V, then

$$E(\mathfrak{c}_0, \dots, \mathfrak{c}_p) = \{\mathfrak{z} \mid \mathfrak{z} \wedge \mathfrak{c}_0 \wedge \dots \wedge \mathfrak{c}_p = 0\}$$

is the $(p+1)$-dimensional complex subspace of V which is spanned by $(\mathfrak{c}_0, \dots, \mathfrak{c}_p)$. If $\gamma \in \mathfrak{G}^p(V)$ then $E(\gamma) = E(\mathfrak{c}_0, \dots, \mathfrak{c}_p)$ is well-defined by $\gamma = \varrho_{V[p+1]}(\mathfrak{c}_0 \wedge \dots \wedge \mathfrak{c}_p)$. This map E of $\mathfrak{G}^p(V)$ onto the set of all $(p+1)$-dimensional complex subspaces of V is bijective. For $\gamma \in \mathfrak{G}^p(V)$ define $\ddot{E}(\gamma) = \ddot{E}(\mathfrak{c}_0, \dots, \mathfrak{c}_p) = \varrho_V(E(\gamma) - \{0\})$, where $\gamma = \varrho_{V[p+1]}(\mathfrak{c}_0 \wedge \dots \wedge \mathfrak{c}_p)$. Then $\ddot{E}$ defines a bijective map of $\mathfrak{G}^p(V)$ onto the set of p-dimensional complex planes of V.

$$GL(V) = \{(\mathfrak{c}_0, \dots, \mathfrak{c}_n) \mid \mathfrak{c}_0 \wedge \dots \wedge \mathfrak{c}_p \neq 0\}$$

is the set of all bases of V. It is the general linear group of V and is the complement of a thin analytic subset of $V \times \ldots \times V$ (n-times).

Now, a local coordinate system of $\mathfrak{G}^p(V)$ shall be introduced:

Lemma 2.1. *Let $\mathfrak{a} = (\mathfrak{a}_0, \ldots, \mathfrak{a}_n) \in GL(V)$ be a base of V. Let $0 \leqslant p < n$. Define*

$$\alpha = \varrho_{V[p+1]}(\mathfrak{a}_0 \wedge \ldots \wedge \mathfrak{a}_p) \in \mathfrak{G}^p(V).$$

Define

$$Z_0 = \{\mathfrak{w} \mid \mathfrak{w} \in \tilde{\mathfrak{G}}^{p+1}(V) \text{ with } \mathfrak{w} \wedge \mathfrak{a}_{p+1} \wedge \ldots \wedge \mathfrak{a}_n \neq 0\},$$

$$Z = \varrho_{V[p+1]}(Z_0).$$

Then Z is an open neighborhood of α. A holomorphic map

$$\zeta_0 : \mathbf{C}^{(n-p)(p+1)} \to \tilde{\mathfrak{G}}^{p+1}(V) - \{0\}$$

is defined in the following way: Take $\mathfrak{z} \in \mathbf{C}^{(n-p)(p+1)}$ then $\mathfrak{z}_\mu = (z_{\mu.p+1}, \ldots, z_{\mu n}) \in \mathbf{C}^{n-p}$ exists with $\mathfrak{z} = (\mathfrak{z}_0, \ldots, \mathfrak{z}_p)$.

Define
$$\mathfrak{c}_\mu = \sum_{\nu=p+1}^{\infty} z_{\mu\nu} \mathfrak{a}_\nu \quad \text{for } \mu = 0, \ldots, p.$$

Set
$$\zeta_0(\mathfrak{z}) = (\mathfrak{a}_0 + \mathfrak{c}_0) \wedge \ldots \wedge (\mathfrak{a}_p + \mathfrak{c}_p).$$

Then
$$\zeta = \varrho_{V[p+1]} \circ \zeta_0 : \mathbf{C}^{(n-p)(p+1)} \to Z$$

is a biholomorphic map onto the open neighborhood Z of $\alpha = \zeta(0)$ in $\mathfrak{G}^p(V)$.

Proof. 1. Denote $\varrho = \varrho_{V[p+1]}$. Clearly, $\mathfrak{a}_0 + \mathfrak{c}_0, \ldots, \mathfrak{a}_p + \mathfrak{c}_p$ are linearly independent. Hence ζ_0 and ζ (into $\mathfrak{G}^p(V)$) are well-defined and holomorphic. Obviously, Z_0 is an open neighborhood of $\mathfrak{a}_0 \wedge \ldots \wedge \mathfrak{a}_p$ in $\tilde{\mathfrak{G}}^{p+1}(V)$. Hence Z is an open neighborhood of $\alpha = \zeta(0)$ in $\mathfrak{G}^p(V)$.

2. If $\mathfrak{z} = (\mathfrak{z}_0, \ldots, \mathfrak{z}_p) \in \mathbf{C}^{(n-p)(p+1)}$ with $\mathfrak{z}_\mu \in \mathbf{C}^{n-p}$, then

$$\zeta_0(\mathfrak{z}) \wedge \mathfrak{a}_{p+1} \wedge \ldots \wedge \mathfrak{a}_n = (\mathfrak{a}_0 + \mathfrak{c}_0) \wedge \ldots \wedge (\mathfrak{a}_p + \mathfrak{c}_p) \wedge \mathfrak{a}_{p+1} \wedge \ldots \wedge \mathfrak{a}_n = \mathfrak{a}_0 \wedge \ldots \wedge \mathfrak{a}_n \neq 0$$

Therefore, ζ_0 maps into Z_0 and ζ maps into Z.

3. The map ζ is surjective: Take $\beta \in Z$. Then $\beta = \varrho(\mathfrak{b})$ with $\mathfrak{b} = \mathfrak{b}_0 \wedge \ldots \wedge \mathfrak{b}_p \neq 0$ and $\mathfrak{b}_\mu \in V$ and

$$\mathfrak{b}_0 \wedge \ldots \wedge \mathfrak{b}_p \wedge \mathfrak{a}_{p+1} \wedge \ldots \wedge \mathfrak{a}_n \neq 0.$$

Now, the following statement Sq shall be proved by induction for q in $0 \leqslant q \leqslant p+1$:

S_q: "*vectors $\mathfrak{c}_{\mu q} = \sum_{\nu=q}^{n} c_{\mu q \nu} \mathfrak{a}_\nu$ and a number $b_q \in \mathbf{C}$ exist such that*

$$\mathfrak{b} = b_q(\mathfrak{a}_0 + \mathfrak{c}_{0,q}) \wedge \ldots \wedge (\mathfrak{a}_{q-1} + \mathfrak{c}_{q-1,q}) \wedge \mathfrak{c}_{q,q} \wedge \ldots \wedge \mathfrak{c}_{p,q}\text{''}.$$

If $q=0$, choose $\mathfrak{c}_{\mu 0}=\mathfrak{b}_\mu$ for $\mu=0, ..., p$ and $b_0=1$. Then S_0 is true. Suppose that S_q is true. Then S_{q+1} shall be proved if $q+1 \leqslant p+1$. Then

$$0 \neq \mathfrak{b} \wedge \mathfrak{a}_{p+1} \wedge ... \wedge \mathfrak{a}_n = b_q \Delta \mathfrak{a}_0 \wedge ... \wedge \mathfrak{a}_n,$$

where Δ is computed that way: Let E_r be the unit matrix with r rows and columns. Let $\mathfrak{O}_{st}$ be the zero matrix with s rows and t columns. Define

$$C_{\mu\varrho}^{\nu\sigma} = \begin{pmatrix} c_{\nu q \sigma}, \dots, c_{\nu q \varrho} \\ \dots\dots\dots\dots\dots \\ c_{\mu q \sigma}, \dots, c_{\mu q \varrho} \end{pmatrix}.$$

Then
$$\Delta = \det \begin{pmatrix} E_q & C_{q-1,p}^{0,q} & C_{q-1,n}^{0,p+1} \\ \mathfrak{O}_{p-q+1,q} & C_{pp}^{qq} & C_{q,n}^{q,p+1} \\ \mathfrak{O}_{n-p,q} & \mathfrak{O}_{n-p,p-q+1} & E^{n-p} \end{pmatrix} = \det C_{pp}^{qq}.$$

Hence $c_{\mu,q,q} \neq 0$ for some μ in $q \leqslant \mu \leqslant p$. By changing the enumeration of $\mathfrak{c}_{q,q}, ..., \mathfrak{c}_{p,q}$, it can be assumed that $c_{q,q,q} \neq 0$. Hence $\mathfrak{c}_{q,q} = c_{q,q,q}(\mathfrak{a}_q + \mathfrak{c}_{q,q+1})$ where

$$\mathfrak{c}_{q,q+1} = \sum_{\nu=q+1}^{n} c_{q,q+1,\nu}\, \mathfrak{a}_\nu.$$

Define $b_{q+1} = c_{q,q,q} b_q$. Then

$$\mathfrak{b} = b_{q+1}(\mathfrak{a}_0 + \mathfrak{c}_{0,q}) \wedge ... \wedge (\mathfrak{a}_{q-1} + \mathfrak{c}_{q-1,q}) \wedge (\mathfrak{a}_q + \mathfrak{c}_{q,q+1}) \wedge \mathfrak{c}_{q+1,q} \wedge ... \wedge \mathfrak{c}_{pq}.$$

For $\mu \neq q$ define

$$\mathfrak{c}_{\mu,q+1} = \mathfrak{c}_{\mu,q} - c_{\mu,q,q}(\mathfrak{a}_q + \mathfrak{c}_{q,q+1}) = \sum_{\nu=q+1}^{n} c_{\mu,q+1,\nu}\, \mathfrak{a}_\nu.$$

Then
$$\mathfrak{b} = b_{q+1}(\mathfrak{a}_0 + \mathfrak{c}_{0,q+1}) \wedge ... \wedge (\mathfrak{a}_q + \mathfrak{c}_{q,q+1}) \wedge \mathfrak{c}_{q+1,q+1} \wedge ... \wedge \mathfrak{c}_{q+1,p}.$$

Hence S_{q+1} is proved. Especially S_{p+1} is true

$$\mathfrak{b} = b_{p+1}(\mathfrak{a}_0 + \mathfrak{c}_{0,p+1}) \wedge ... \wedge (\mathfrak{a}_p + \mathfrak{c}_{p,p+1})$$

with
$$\mathfrak{c}_{\mu,p+1} = \sum_{\nu=p+1}^{n} c_{\mu p+1 \nu}\, \mathfrak{a}_\nu.$$

Define $\mathfrak{z}_\mu = (c_{\mu,p+1,p+1}, ..., c_{\mu p+1,n})$ for $\mu=0, ..., p$ and $\mathfrak{z}=(\mathfrak{z}_0, ..., \mathfrak{z}_p) \in \mathbb{C}^{(n-p)(p+1)}$. Then $\mathfrak{b} = b_{p+1}\zeta_0(\mathfrak{z})$ and $\zeta(\mathfrak{z}) = \varrho(\zeta_0(\mathfrak{z})) = \varrho(\mathfrak{b}) = \beta$. Hence ζ is surjective.

4. The map ζ is injective: Let

for $\mu=0, ..., p$ and
$$\mathfrak{z}_\mu = (z_{\mu p+1}, ..., z_{\mu n}) \quad \text{and} \quad \mathfrak{v}_\mu = (v_{\mu p+1}, ..., v_{\mu n})$$

$$\mathfrak{z} = (\mathfrak{z}_0, \ldots, \mathfrak{z}_p), \quad \mathfrak{v} = (\mathfrak{v}_0, \ldots, \mathfrak{v}_p)$$

such that $\zeta(\mathfrak{z}) = \zeta(\mathfrak{v})$. Define

$$\mathfrak{c}_\mu = \sum_{\nu=p+1} z_{\mu\nu}\mathfrak{a} \quad \text{and} \quad \mathfrak{x}_\mu = \sum_{\nu=p+1} v_{\mu\nu}\mathfrak{a}_\nu$$

for $\mu = 0, \ldots, p$. A number $u \neq 0$ exists such that

$$\mathfrak{b} = (\mathfrak{a}_0 + \mathfrak{c}_0) \wedge \ldots \wedge (\mathfrak{a}_p + \mathfrak{c}_p) = u(\mathfrak{a}_0 + \mathfrak{x}_0) \wedge \ldots \wedge (\mathfrak{a}_p + \mathfrak{x}_p).$$

Then $$\mathfrak{b} \wedge \mathfrak{a}_{p+1} \wedge \ldots \wedge \mathfrak{a}_n = \mathfrak{a}_0 \wedge \ldots \wedge \mathfrak{a}_n = u\mathfrak{a}_0 \wedge \ldots \wedge \mathfrak{a}_n \neq 0.$$

Hence $u = 1$. Take μ with $0 \leqslant \mu \leqslant p$. Then

$$(\mathfrak{a}_\mu + \mathfrak{x}_\mu) \wedge (\mathfrak{a}_0 + \mathfrak{c}_0) \wedge \ldots \wedge (\mathfrak{a}_p + \mathfrak{c}_p) = 0.$$

Because $(\mathfrak{a}_0 + \mathfrak{c}_0) \wedge \ldots \wedge (\mathfrak{a}_p + \mathfrak{c}_p) \neq 0$, numbers $a_{\mu\varrho}$ exist such that

$$\mathfrak{a}_\mu + \mathfrak{x}_\mu = \sum_{\varrho=0}^{p} a_{\mu\varrho}(\mathfrak{a}_\varrho + \mathfrak{c}_\varrho).$$

Then
$$0 \neq \mathfrak{a}_0 \wedge \ldots \wedge \mathfrak{a}_n = \mathfrak{a}_0 \wedge \ldots \wedge \mathfrak{a}_{\mu-1} \wedge (\mathfrak{a}_\mu + \mathfrak{x}_\mu) \wedge \mathfrak{a}_{\mu+1} \wedge \ldots \wedge \mathfrak{a}_n = a_{\mu\mu}\mathfrak{a}_0 \wedge \ldots \wedge \mathfrak{a}_n.$$

Hence $a_{\mu\mu} = 1$ if $0 \leqslant \mu \leqslant p$. Take $\lambda \neq \mu$ in $0 \leqslant \lambda \leqslant p$. Then

$$\begin{aligned} 0 = \mathfrak{a}_\mu \wedge \mathfrak{a}_0 \wedge \ldots \wedge \mathfrak{a}_{\lambda-1} \wedge \mathfrak{a}_{\lambda+1} \wedge \ldots \wedge \mathfrak{a}_n &= (\mathfrak{a}_\mu + \mathfrak{x}_\mu) \wedge \mathfrak{a}_0 \wedge \ldots \wedge \mathfrak{a}_{\lambda-1} \wedge \mathfrak{a}_{\lambda+1} \wedge \ldots \wedge \mathfrak{a}_n \\ &= a_{\mu\lambda}\mathfrak{a}_\lambda \wedge \mathfrak{a}_0 \wedge \ldots \wedge \mathfrak{a}_{\lambda-1} \wedge \mathfrak{a}_{\lambda+1} \wedge \ldots \wedge \mathfrak{a}_n. \end{aligned}$$

Hence $a_{\mu\lambda} = 0$ if $\mu \neq \lambda$. Therefore $\mathfrak{a}_\mu + \mathfrak{x}_\mu = \mathfrak{a}_\mu + \mathfrak{c}_\mu$ or $\mathfrak{x}_\mu = \mathfrak{c}_\mu$ which implies $\mathfrak{v}_\mu = \mathfrak{z}_\mu$ for $\mu = 0, \ldots, p$. Hence $\mathfrak{v} = \mathfrak{z}$. Therefore ζ is injective; q.e.d.

Let M be a pure m-dimensional complex manifold. Let V be a complex vector space of dimension $n+1$. Suppose that $0 \leqslant p < n$. Let $f: M \to \mathbf{P}(V)$ be a holomorphic map. Then

$$F_p = F_p(f) = \{(z, \alpha) \mid f(z) \in \ddot{E}(\alpha), \quad \text{where} \quad (z, \alpha) \in M \times \mathfrak{G}^p(V)\}$$

is said to be the *graph of order p of f*. Obviously, $F_0(f)$ is the usual graph of f. Define

$$\pi_f: F_p(f) \to \mathfrak{G}^p(V) \quad \text{by } \pi_f(z, \alpha) = \alpha$$

$$\hat{\pi}_f: F_p(f) \to M \qquad \text{by } \hat{\pi}_f(z, \alpha) = z.$$

If $\alpha \in \mathfrak{G}^p(V)$, then

$$\pi_f^{-1}(\alpha) = \{(z, \alpha) \mid z \in M,\ f(z) \in \ddot{E}(\alpha)\} = f^{-1}(\ddot{E}(\alpha)) \times \{\alpha\}.$$

Hence

(1) $$\pi_f^{-1}(\alpha) = f^{-1}(\ddot{E}(\alpha)) \times \{\alpha\}$$

(2) $$\hat{\pi}_f(\pi_f^{-1}(\alpha)) = f^{-1}(\ddot{E}(\alpha)).$$

LEMMA 2.2. *Let M be a pure m-dimensional complex manifold. Let V be a complex vector space of dimension $n+1$. Suppose that $0 \leqslant p < n$. Let $f: M \to \mathbf{P}(V)$ be a holomorphic map. Then the graph $F_p(f)$ of order p of f is a smooth complex submanifold of $M \times \mathfrak{G}^p(V)$ with pure dimension $m + p(n-p)$.*

Remark: A local "coordinate system" at $(a, \alpha) \in F_p(f)$ can be obtained the following way:

1. *Step. Let*

$$\varrho: V - \{0\} \to \mathbf{P}(V) \qquad \underline{\varrho}: V[p+1] - \{0\} \to \mathbf{P}(V[p+1])$$

be the natural projections. Pick any base $(\mathfrak{a}_0, \ldots, \mathfrak{a}_n) \in GL(V)$ such that $\underline{\varrho}(\mathfrak{a}_0 \wedge \ldots \wedge \mathfrak{a}_p) = \alpha$ and $f(a) = \varrho(\mathfrak{a}_0)$.

2. *Step. Define*

$$V_0 = \mathfrak{a}_0 + E(\mathfrak{a}_1, \ldots, \mathfrak{a}_n) = \{\mathfrak{a}_0 + \sum_{\mu=1}^{n} z_\mu \mathfrak{a}_\mu \,|\, z_\mu \in \mathbf{C}\}$$

Then $\ddot{V}_0 = \varrho(V_0)$ is an open neighborhood of $f(a)$ in $\mathbf{P}(V)$. The map $\varrho_0 = \varrho \,|\, V_0 : V_0 \to \ddot{V}_0$ is biholomorphic.

3. *Step. Pick any open neighborhood A of a such that $f(A) \subseteq \ddot{V}_0$.*

4. *Step. Define a holomorphic map*

$$\xi: A \times \mathbf{C}^{p(n-p)} \to \mathfrak{G}^p(V)$$

by the following construction: Take $(z, \mathfrak{z}) \in A \times \mathbf{C}^{p(n-p)}$ then $\mathfrak{z} = (\mathfrak{z}_1, \ldots, \mathfrak{z}_p)$ with

$$\mathfrak{z}_\mu = (z_{\mu,p+1}, \ldots, z_{\mu,n}) \in \mathbf{C}^{n-p}.$$

Define

$$\varrho_0^{-1}(f(z)) = \mathfrak{a}_0 + \sum_{\mu=0}^{n} f_\mu(z)\, \mathfrak{a}_\mu,$$

$$\mathfrak{c}_\mu = \mathfrak{c}_\mu(\mathfrak{z}) = \sum_{\nu=p+1}^{n} z_{\mu\nu} \mathfrak{a}_\nu \quad \text{for } \mu = 1, \ldots, p,$$

$$\mathfrak{c}_0 = \mathfrak{c}_0(z, \mathfrak{z}) = \sum_{\nu=p+1}^{n} f_\nu(z)\, \mathfrak{a}_\nu - \sum_{\mu=1}^{p} f_\mu(z)\, \mathfrak{c}_\mu,$$

$$\xi(z, \mathfrak{z}) = \underline{\varrho}((\mathfrak{a}_0 + \mathfrak{c}_0) \wedge \ldots \wedge (\mathfrak{a}_p + \mathfrak{c}_p)).$$

5. *Step. For $(z, \mathfrak{z}) \in A \times \mathbf{C}^{p(n-p)}$, define*

$$\sigma(z, \mathfrak{z}) = (z, \xi(z, \mathfrak{z})) \in M \times \mathfrak{G}^p(V).$$

Then $B = \sigma(A \times \mathbf{C}^{p(n-p)})$ is an open neighborhood of $(a, \alpha) = \sigma(a, 0)$ in $F_p(f)$. Moreover, $\sigma: A \times \mathbf{C}^{p(n-p)} \to B$ is biholomorphic. Moreover, if Z is the neighborhood of α which was introduced in Lemma 2.1, then

$$B = (A \times Z) \cap F_p(f).$$

Proof. Clearly, if the statements of the Remark are proved, the Lemma 2.2 is also proved. Take $(a, \alpha) \in F_p$, then the Remark shall be proved. Now, $\mathfrak{a}_0 \in V - \{0\}$ with $\varrho(\mathfrak{a}_0) = f(a)$ exists where $f(a) \in \ddot{E}(\alpha)$ and $\ddot{E}(\alpha) = \varrho(E(\alpha) - \{0\})$. Hence $\mathfrak{a}_0 \in E(\alpha)$. Therefore, a base $\mathfrak{a} = (\mathfrak{a}_0, ..., \mathfrak{a}_n) \in GL(V)$ can be picked such that $\varrho(\mathfrak{a}_0 \wedge ... \wedge \mathfrak{a}_p) = \alpha$. This completes Step 1. Step 2, Step 3 and Step 4 are trivial and ξ is holomorphic.

For Step 5, define $B = (A \times Z) \cap F_p(f)$, where Z is the neighborhood of α which was defined in Lemma 2.1. Then B is an open neighborhood of (a, α) in $F_p(f)$. Holomorphic functions f_μ exist on A such that

$$\varrho_0^{-1}(f(z)) = \mathfrak{a}_0 + \sum_{\mu=1}^{n} f_\mu(z)\, \mathfrak{a}_\mu,$$

Here is $\varrho_0^{-1}(f(a)) = \varrho_0^{-1}(\alpha) + \mathfrak{a}_0$. Hence $f_\mu(a) = 0$ for $\mu = 1, ..., n$. Now, define $\beta: A \times \mathbf{C}^{p(n-p)} \to \mathbf{C}^{(p+1)(n-p)}$ by the following procedure: Take $(z, \mathfrak{z}) \in A \times \mathbf{C}^{p(n-p)}$. Then

$$\mathfrak{z} = (\mathfrak{z}_1, ..., \mathfrak{z}_p) \quad \text{with} \quad \mathfrak{z}_\mu = (z_{\mu p+1}, ..., z_{\mu n}) \in \mathbf{C}^{n-p}.$$

Define $\mathfrak{z}_0 = \mathfrak{z}_0(z, \mathfrak{z}) = (z_{0,p+1}, ..., z_{0,n})$ by

$$z_{0\nu} = z_{0\nu}(z, \mathfrak{z}) = f_\nu(z) - \sum_{\mu=1}^{n} f_m(z)\, z_{\mu\nu}.$$

Set $\beta(z, \mathfrak{z}) = (z, \mathfrak{z}_0, ..., \mathfrak{z}_p) \in A \times \mathbf{C}^{(p+1)(n-p)}$. Obviously β is a holomorphic, injective map with an Jacobian of rank $m + p(n-p)$. Hence $\beta(A \times \mathbf{C}^{p(n-p)}) = B'$ is a smooth complex submanifold of $A \times \mathbf{C}^{(p+1)(n-p)}$, with $\dim B' = m + p(n-p)$.

Now, $\zeta: \mathbf{C}^{(p+1)(n-p)} \to Z$ is biholomorphic. Hence

$$\bar{\zeta} = \mathrm{Id}_A \times \zeta: A \times \mathbf{C}^{(p+1)(n-p)} \to A \times Z$$

is a biholomorphic map onto an open subset of $M \times \mathfrak{G}^p(V)$. Hence $\bar{\zeta} \circ \beta: A \times \mathbf{C}^{p(n-p)} \to A \times Z$ is a holomorphic, injective map of rank $m + p(n-p)$ and its image $\bar{\zeta}(B') = B''$ is a smooth complex submanifold of $A \times Z$ with pure dimension $m + p(n-p)$. Now, $\bar{\zeta} \circ \beta = \sigma$ and $B'' = B$ is claimed:

For $(z, \mathfrak{z}) \in A \times \mathbf{C}^{p(n-p)}$ is $\mathfrak{z} = (\mathfrak{z}_1, \ldots, \mathfrak{z}_p)$ with $\mathfrak{z}_\mu = (z_{\mu, p+1}, \ldots, z_{\mu n})$. Define

$$\mathfrak{c}_\mu = \sum_{\nu=p+1}^{n} z_{\mu\nu}\, \mathfrak{a}_\nu \quad \text{for } \mu = 0, 1, \ldots, p.$$

Then $\mathfrak{z}_0 = (z_{0,p+1}, \ldots, z_{0,n})$ with $z_{0\nu} = f_\nu(z) - \sum_{\mu=1}^{p} f_\mu(z)\, z_{\mu\nu}$. Hence

$$\mathfrak{c}_0 = \sum_{\nu=p+1}^{n} f_\nu(z)\, \mathfrak{a}_\nu - \sum_{\mu=1}^{p} f_\mu(z) \sum_{\nu=p+1}^{n} z_{\mu\nu}\, \mathfrak{a}_\nu = \sum_{\nu=p+1}^{n} z_{0\nu}\, \mathfrak{a}_\nu.$$

Therefore $$\sigma(z, \mathfrak{z}) = (z, \xi(z, \mathfrak{z})) = (z, \underline{\varrho}((\mathfrak{a}_0 + \mathfrak{c}_0) \wedge \ldots \wedge (\mathfrak{a}_p + \mathfrak{c}_p)) = \tilde{\zeta}(z, \mathfrak{z}_0, \ldots, \mathfrak{z}_p) = \tilde{\zeta}(\beta(z, \mathfrak{z})).$$

Hence $\sigma = \tilde{\zeta} \circ \beta$.

Using the same notation, it is

$$\begin{aligned}
&\varrho_0^{-1}(f(z)) \wedge (\mathfrak{a}_0 + \mathfrak{c}_0) \wedge \ldots \wedge (\mathfrak{a}_p + \mathfrak{c}_p) \\
&\quad = (\mathfrak{a}_0 + \sum_{\mu=1}^{n} f_\mu(z)\, \mathfrak{a}_\mu) \wedge (\mathfrak{a}_0 + \sum_{\nu=p+1}^{n} f_\nu(z)\, \mathfrak{a}_\nu - \sum_{\mu=1}^{p} f_\mu(z)\, \mathfrak{c}_\mu) \wedge (\mathfrak{a}_1 + \mathfrak{c}_1) \wedge \ldots \wedge (\mathfrak{a}_p + \mathfrak{c}_p) \\
&\quad = \sum_{\mu=1}^{p} f_\mu(z)\, (\mathfrak{a}_\mu + \mathfrak{c}_\mu) \wedge (\mathfrak{a}_0 + \mathfrak{c}_0) \wedge (\mathfrak{a}_1 + \mathfrak{c}_1) \wedge \ldots \wedge (\mathfrak{a}_p + \mathfrak{c}_p) = 0.
\end{aligned}$$

Hence $f(z) \in \ddot{E}(\xi(z, \mathfrak{z}))$, which implies

$$\sigma(z, \mathfrak{z}) = (z, \xi(z, \mathfrak{z})) \in F_p.$$

Hence, $\sigma = \tilde{\zeta} \circ \beta$ is an injective, holomorphic map into B; hence $B'' \subseteq B$.

Take any $(z, \eta) \in B$. Then $\mathfrak{z} = (\mathfrak{z}_0, \ldots, \mathfrak{z}_p) \in \mathbf{C}^{(p+1)(n-p)}$ with $\mathfrak{z}_\mu = (z_{\mu p+1}, \ldots, z_{\mu n})$ exists such that $\zeta(\mathfrak{z}) = \eta$. Define $\mathfrak{c}_\mu = \sum_{\nu=p+1}^{n} z_{\mu\nu} \mathfrak{a}_\nu$ for $\mu = 0, \ldots, p$. Then

$$\underline{\varrho}((\mathfrak{a}_0 + \mathfrak{c}_0) \wedge \ldots \wedge (\mathfrak{a}_p + \mathfrak{c}_p)) = \eta.$$

Hence $$\varrho_0^{-1}(f(z)) \in E(\eta) = E(\mathfrak{a}_0 + \mathfrak{c}_0, \ldots, \mathfrak{a}_p + \mathfrak{c}_p).$$

Hence $$\mathfrak{a}_0 + \sum_{\mu=1}^{n} f_\mu(z) \mathfrak{a}_\mu = \sum_{\mu=0}^{p} g_\mu (\mathfrak{a}_\mu + \mathfrak{c}_\mu),$$

which implies $g_0 = 1$ and $g_\mu = f_\mu(z)$ for $\mu = 1, \ldots, p$ and

$$\sum_{\mu=p+1}^{n} f_\mu(z)\, \mathfrak{a}_\mu = \mathfrak{c}_0 + \sum_{\mu=0}^{p} f_\mu(z)\, \mathfrak{c}_\mu$$

or $$\mathfrak{c}_0 = \sum_{\nu=p+1}^{n} f_\nu(z)\, \mathfrak{a}_\nu - \sum_{\mu=0}^{p} f_\mu(z)\, \mathfrak{c}_\mu = \sum_{\nu=p+1}^{n} (f_\nu(z) - \sum_{\mu=0}^{p} f(z)\, z_{\mu\nu})\, \mathfrak{a}_\nu.$$

Hence $$z_{0\nu} = f_\nu(z) - \sum_{\mu=0}^{p} f_\mu(z)\, z_{\mu\nu}.$$

Therefore $$\beta(z, \mathfrak{z}_1, \ldots, \mathfrak{z}_p) = (z, \mathfrak{z}_0, \ldots, \mathfrak{z}_p) = (z, \mathfrak{z})$$

and $$\sigma(z, \mathfrak{z}_1, \ldots, \mathfrak{z}_p) = \tilde{\zeta}(\beta(z, \mathfrak{z}_1, \ldots, \mathfrak{z}_p) = (z, \zeta(\mathfrak{z})) = (z, \eta).$$

Therefore, σ is surjective. Hence $\sigma: A \times \mathbf{C}^{p(n-p)} \to B = B''$ is a biholomorphic map. Now, $f_\mu(a) = 0$ for $\mu = 1, \ldots, p$, implies

$$\sigma(a, 0) = (a, \xi(a, 0)) = (a, \underline{\varrho}(\mathfrak{a}_0 \wedge \ldots \wedge \mathfrak{a}_p)) = (a, \alpha),$$

q.e.d.

If M and N are complex manifolds and if $h:N\to M$ and $f:M\to \mathbf{P}(V)$ are holomorphic maps, define $g=h\circ f$. Then a holomorphic map $\hat{h}:F_p(g)\to F_p(f)$ is defined by $\hat{h}(z,\alpha)=(h(z),\alpha)$. Then $\hat{\pi}_f\circ\hat{h}=h\circ\hat{\pi}_g$. Especially, the identity map $j:\mathbf{P}(V)\to\mathbf{P}(V)$ is defined. Then $F_p(j)\to\mathbf{P}(V)$ is a fiber bundle over $\mathbf{P}(V)$ with general fiber $\mathfrak{G}^{p-1}(\mathbf{C}^n)$. Moreover, $F_p(f)$ is the bundle induced by f and $\hat{f}$ is the induced bundle map:

$$\begin{array}{ccc} F_p(f) & \xrightarrow{\hat{f}} & F_p(j) \\ \downarrow{\scriptstyle \hat{\pi}_f} & \circ & \downarrow{\scriptstyle \hat{\pi}_j} \\ M & \xrightarrow{f} & \mathbf{P}(V) \end{array}$$

However, this interpretation will not be needed now. Of importance is the local coordinate system introduced in Lemma 2.2.

DEFINITION 2.3. (*General maps of order r.*) *Let M be a pure m-dimensional complex manifold. Let V be a complex vector space of dimension $n+1$. Let r be an integer such that $0<r\leqslant n$ and $0<r\leqslant m$. Define*

$$p=n-r \quad \text{and} \quad q=m-r.$$

Let $f:M\to\mathbf{P}(V)$ be a holomorphic map. Then f is general of order r for $\alpha\in\mathfrak{G}^p(V)$ if and only if an open neighborhood U of α exists such that $f^{-1}(\ddot{E}(\zeta))$ is empty or an analytic set of pure dimension q whenever $\zeta\in U$. The map f is said to be general of order r if and only if f is general for every $\alpha\in\mathfrak{G}^p(V)$.

Obviously, if $r=n$, then f is general of order r if and only if f is q-fibering, i.e. if f has pure rank r.

LEMMA 2.4. *Let M be a pure m-dimensional complex manifold. Let V be a complex vector space of dimension $n+1$. Let $r\in\mathbf{N}$ with $p=n-r\geqslant 0$ and $q=m-r\geqslant 0$. Let $f:M\to\mathbf{P}(V)$ be holomorphic. Then the following statements are equivalent*

1. *f is general of order r.*
2. *For every $\alpha\in\mathfrak{G}^p(V)$, the set $f^{-1}(\ddot{E}(\alpha))$ is empty or has pure fiber dimension q.*
3. *The map $\pi_f:F_p(f)\to\mathfrak{G}^p(V)$ is q-fibering.*
4. *The map $\pi_f:F_p(f)\to\mathfrak{G}^p(V)$ is open.*

Proof. 2) is only a reformulation of the definition of 1). Because $\pi_f^{-1}(\alpha)=f^{-1}(\ddot{E}(\alpha))\times\{\alpha\}$, the statements 2 and 3 are equivalent. Because

$$\dim F_p(f)-\dim\mathfrak{G}^p(V)=m+p(n-p)-(p+1)(n-p)=q$$

conditions 3 and 4 are equivalent, q.e.d.

Obviously, f is general for $\alpha\in\mathfrak{G}^p(V)$, if and only if an open neighborhood U of α exists such that $f|f^{-1}(U)$ is general of order r. Suppose, that $f:M\to\mathbf{P}(V)$ is general of order r for $\alpha \in \mathfrak{G}^p(V)$ where m, n, p, q, r are as before. Then an open neighborhood U of α exists such that

$$\pi_f | \pi_f^{-1}(U) : \pi_f^{-1}(U) \to U$$

is a q-fibering map. If $z \in f^{-1}(\ddot{E}(\alpha))$, define(1) $\nu_f(z; \alpha) = \nu_{\pi_f}(z, \alpha)$ as the multiplicity of π_f at $(z, \alpha) \in F_p(f)$. If $z \notin f^{-1}(\ddot{E}(\alpha))$, define $\nu_f(z; \alpha) = 0$. The number $\nu_f(z; \alpha)$ does not depend on the choice of U and is called the *intersection number of f with α at z*. Moreover,(2)

$$F = f^{-1}(\ddot{E}(\alpha)) = \{z \mid \nu_f(z; \alpha) > 0\}$$

is called the intersection of f with α. According to *I* Theorem 5.6 the function $\nu_f(z; \alpha)$ of z is locally constant on the set $\dot{F}$ of simple points of $F = f^{-1}(\ddot{E}(\alpha))$. If B is branch of F, then $\nu_f(z; \alpha)$ is constant on $B \cap \dot{F}$ as a function of z.

Another representation of the intersection number will be obtained in Lemma 2.5 for later use.

LEMMA 2.5. *Let M be a pure m-dimensional complex manifold. Let V be a complex vector space of dimension $n+1>1$ with an Hermitian product $(\,|\,)$. Let $r \in \mathbf{N}$ with $q = m - r \geq 0$ and $p = n - r \geq 0$. Let ϱ: $V - \{0\} \to \mathbf{P}(V)$ and $\underline{\varrho}: V[p+1] - \{0\} \to \mathbf{P}(V[p+1])$ be the natural projections. Let $f: M \to \mathbf{P}(V)$ be a holomorphic map which is general of order r for $\alpha \in \mathfrak{G}^p(V)$. Let $a \in f^{-1}(\ddot{E}(\alpha))$. Then an orthonormal base $\mathfrak{a} = (\mathfrak{a}_1, \ldots, \mathfrak{a}_n)$ of V exists such that $\varrho(\mathfrak{a}_0) = f(a)$ and $\underline{\varrho}(\mathfrak{a}_0 \wedge \ldots \wedge \mathfrak{a}_p) = \alpha$. For any such a base $\mathfrak{a}$ of V an open neighborhood A of a and holomorphic functions $f_1, \ldots, f_n$ on A exist such that*

1. *For $z \in A$ is $f(z) = \varrho(\mathfrak{a}_0 + \sum_{\nu=1}^{n} f_\nu(z)\mathfrak{a}_\nu)$.*
2. *It is*

$$A \cap f^{-1}(\ddot{E}(\alpha)) = \bigcap_{\mu=p+1}^{n} \{z \mid f_\mu(z) = 0;\, z \in A\}.$$

3. *The map $\varphi = (f_{p+1}, \ldots, f_n): A \to \mathbf{C}^p$ is open and q-fibering. Moreover, if z is a simple point of $\varphi^{-1}(0) = A \cap f^{-1}(\ddot{E}(\alpha))$, then*

$$\nu_f(z; \alpha) = \nu_\varphi(z) = \nu_\varphi(z; 0).$$

Proof. An open neighborhood U of $\alpha \in \mathfrak{G}^p(V)$ exists such that for $\tilde{U} = \pi_f^{-1}(U)$ the map $\pi_f | \tilde{U}: \tilde{U} \to U$ is q-fibering.

Pick $\mathfrak{a}_0 \in \varrho^{-1}(f(a))$ with $|\mathfrak{a}_0| = 1$. Then $\mathfrak{a}_0 \in \varrho^{-1}(\ddot{E}(\alpha)) \subseteq E(\alpha)$. Hence an orthonormal base $(\mathfrak{a}_0, \ldots, \mathfrak{a}_n) = \mathfrak{a}$ of V exists such that $\mathfrak{a}_0, \ldots, \mathfrak{a}_p$ span $E(\alpha)$, i.e.: $\underline{\varrho}(\mathfrak{a}_0 \wedge \ldots \wedge \mathfrak{a}_p) = \alpha$. Let $\mathfrak{a}$ be any such a base. Define

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(1) Actually, $\nu_{\pi_f}(z, \alpha)$ ought to be written as $\nu_{\pi_f}((z, \alpha))$. Obviously, if $r = 0$, then $\nu_f(z; \alpha)$ is the α-multiplicity of f at z as defined in I § 4, because $\hat{\pi}_f$ is biholomorphic.

(2) Observe that $F \times \{\alpha\} = \pi_f^{-1}(\alpha)$.

$$V_0 = \{\mathfrak{a}_0 + \sum_{\mu=1}^{n} w_\mu \mathfrak{a}_\mu \mid w_\mu \in \mathbf{C}\}.$$

Then $\varrho(V_0) = \ddot{V}_0$ is open in $\mathbf{P}(V)$ and $\varrho_0 = \varrho \mid V_0 : V_0 \to \ddot{V}_0$ is biholomorphic. An open neighborhood A_0 of a exists such that $f(A_0) \subseteq \ddot{V}_0$. For $z \in A_0$ is

$$\varrho_0^{-1}(f(z)) = \mathfrak{a}_0 + \sum_{\mu=1}^{n} f_\mu(z)\, \mathfrak{a}_\mu,$$

where $f_1, \ldots, f_n$ are holomorphic on A_0. Define $\eta : A_0 \times \mathbf{C}^{pr} \to \mathbf{C}^r$ by

$$\eta(z, \mathfrak{z}) = (h_{p+1}(z, \mathfrak{z}), \ldots, h_n(z, \mathfrak{z})),$$

if $z \in A_0$ and $\mathfrak{z} = (\mathfrak{z}_1, \ldots, \mathfrak{z}_p) \in \mathbf{C}^{pr}$ with $\mathfrak{z}_\mu = (z_{\mu p+1}, \ldots, z_{\mu n}) \in \mathbf{C}^r$ and where

$$h_\nu(z, \mathfrak{z}) = f_\nu(z) - \sum_{\mu=1}^{p} f_\mu(z)\, z_{\mu\nu}.$$

Obviously, $h_{p+1}, \ldots, h_n$ and η are holomorphic on $A_0 \times \mathbf{C}^{pr}$. Define

$$\tilde{\eta} : A_0 \times \mathbf{C}^{pr} \to \mathbf{C}^r \times \mathbf{C}^{pr} = \mathbf{C}^{(p+1)r}$$

by setting $\tilde{\eta}(z, \mathfrak{z}) = (\eta(z, \mathfrak{z}), \mathfrak{z})$. Using this base $\mathfrak{a}$, construct $Z_0, Z, \zeta_0, \mathfrak{c}_\mu$ and ζ as in Lemma 2.1. Because $\varrho_0^{-1}(f(a)) = \mathfrak{a}_0$, it is $f_\mu(a) = 0$ for $\mu = 1, \ldots, n$. Therefore, $\eta(a, 0) = 0$ and $\tilde{\eta}(a, 0) = 0$. Hence

$$\zeta(\tilde{\eta}(a, 0)) = \zeta(0) = \alpha \in Z \cap U.$$

Open, connected neighborhoods A of a and Q of 0 exist such that $A \subseteq A_0$ and such that $\zeta \circ \tilde{\eta}(A \times Q) \subseteq Z \cap U$.

Because $f(A) \subseteq f(A_0) \subseteq V_0$, Lemma 2.2 can be applied using the same $\mathfrak{a}$, f_μ, A and obtaining ξ, σ, B. Then $\pi_f(B) \subseteq Z$. Define $\pi = \pi_f \mid B : B \to Z$. Hence, the following diagram is defined

$$\begin{array}{ccc} A \times \mathbf{C}^{pr} & \xrightarrow{\tilde{\eta}} & \mathbf{C}^r \times \mathbf{C}^{pr} \\ \downarrow \sigma & & \downarrow \zeta \\ B & \xrightarrow{\pi} & Z \end{array}$$

where σ and ζ are biholomorphic.

Now, it shall be proved, that the diagram is commutative. Take $(z, \mathfrak{z}) \in A \times \mathbf{C}^{pr}$. Then $\mathfrak{z} = (\mathfrak{z}_1, \ldots, \mathfrak{z}_p)$ with $\mathfrak{z}_\mu = (z_{\mu p+1}, \ldots, z_{\mu n})$. Define

$$\mathfrak{c}_\mu = \sum_{\nu=p+1}^{n} z_{\mu\nu}\, \mathfrak{a}_\nu \quad \text{for } \mu = 1, \ldots, p,$$

$$\mathfrak{c}_0 = \sum_{\nu=p+1}^{n} f_\nu(z)\,\mathfrak{a}_\nu - \sum_{\mu=1}^{p} f_\mu(z)\mathfrak{c}_\mu.$$

Then
$$\pi(\sigma(z, \mathfrak{z})) = \xi(z, \mathfrak{z}) = \underline{\varrho}((\mathfrak{a}_0 + \mathfrak{c}_0) \wedge \dots \wedge (\mathfrak{a}_p + \mathfrak{c}_p)).$$

Define
$$\mathfrak{z}_0 = \eta(z, \mathfrak{z}) = (h_{p+1}(z, \mathfrak{z}), \dots, h_n(z, \mathfrak{z})).$$

Then $\tilde{\eta}(z, \mathfrak{z}) = (\mathfrak{z}_0, \mathfrak{z}_1, \dots, \mathfrak{z}_p) \in \mathbf{C}^{(p+1)r}$ and

$$\mathfrak{c}_0 = \sum_{\nu=p+1}^{n} f_\nu(z)\,\mathfrak{a}_\nu - \sum_{\mu=1}^{p} f_\mu(z) \sum_{\nu=p+1}^{n} z_{\mu\nu}\mathfrak{a}_\nu = \sum_{\nu=p+1}^{n} \Big(f_\nu(z) - \sum_{\mu=1}^{p} f_\mu(z)\, z_{\mu\nu}\Big)\,\mathfrak{a}_\nu = \sum_{\nu=p+1}^{n} h_\nu(z, \mathfrak{z})\,\mathfrak{a}_\nu.$$

Hence
$$\zeta(\tilde{\eta}(z, \mathfrak{z})) = \underline{\varrho}((\mathfrak{a}_0 + \mathfrak{c}_0) \wedge \dots \wedge (\mathfrak{a}_p + \mathfrak{c}_p)) = \pi(\sigma(z, \mathfrak{z})).$$

Therefore, the diagram is commutative.

Now, $Z_1 = \zeta(\mathbf{C}^r \times Q)$ is an open neighborhood of $\alpha = \zeta(0)$. The map

$$\eta_1 = \eta \,|\, A \times Q : A \times Q \to \mathbf{C}^r$$

is holomorphic. Define $\tilde{\eta}_1 : A \times Q \to \mathbf{C}^r \times Q$ by setting $\tilde{\eta}_1(z, \mathfrak{z}) = (\eta_1(z, \mathfrak{z}), \mathfrak{z})$ if $(z, \mathfrak{z}) \in A \times Q$. Then $\tilde{\eta}_1 = \tilde{\eta} \,|\, A \times Q$. Moreover, $B_1 = \sigma(A \times Q)$ is an open neighborhood of $(a, \alpha) = \sigma(a, 0)$ with $B_1 \subseteq B$. Because the diagram is commutative, $\pi(B_1) \subseteq Z_1$. Define $\pi_1 = \pi \,|\, B_1$ as a map into Z_1. Define $\sigma_1 = \sigma \,|\, A \times Q$ and $\zeta_1 = \zeta \,|\, \mathbf{C}^r \times Q$. Then the restricted diagram

$$\begin{array}{ccc} A \times Q & \xrightarrow{\tilde{\eta}_1} & \mathbf{C}^r \times Q \\ \downarrow{\scriptstyle \sigma_1} & \circ & \downarrow{\scriptstyle \zeta_1} \\ B_1 & \xrightarrow{\pi_1} & Z_1 \end{array}$$

is commutative and σ_1 and ζ_1 are biholomorphic. Now $\pi_f(B_1) = \pi_1(B_1) = \zeta_1(\tilde{\eta}_1(A \times Q)) = \zeta(\tilde{\eta}(A \times Q)) \subseteq U$. Hence, $B_1 \subseteq \pi_f^{-1}(Y)$ and $\pi_1 = \pi_f \,|\, B_1$ is q-fibering. Moreover, the map η_1 is q-fibering and

$$\nu_f(z; \beta) = \nu_{\pi_f}(z, \beta) = \nu_{\pi_1}(z, \beta) = \nu_{\tilde{\eta}_1}(\sigma_1^{-1}(z, \beta))$$

if $(z, \beta) \in B_1$.

For $\mathfrak{z} \in Q$, define $\eta_{1\mathfrak{z}} : A \to \mathbf{C}^r$ by $\eta_{1\mathfrak{z}}(z) = \eta_1(z, \mathfrak{z})$ if $z \in A$. According to I Proposition 5.7 $\eta_{1\mathfrak{z}}$ is a q-fibering map; moreover, for every $\mathfrak{z} \in Q$ and every simple point z of $\eta_{1\mathfrak{z}}^{-1}(\eta_{1\mathfrak{z}}(z))$ is

$$\nu_{\tilde{\eta}_1}(z, \mathfrak{z}) = \nu_{\eta_{1\mathfrak{z}}}(z).$$

Now, take $\mathfrak{z} = 0 \in Q$. Then $h_\nu(z, 0) = f_\nu(z)$ for $z \in A$ and $\nu = p+1, \dots, n$. Hence

$$\eta_{10}(z) = (f_{p+1}(z), \dots, f_n(z)) = \varphi(z) \quad \text{if} \quad z \in A,$$

which means $\eta_{10} = \varphi : A \to \mathbf{C}^r$. Therefore, φ is a q-fibering map. Because $\dim A - \dim \mathbf{C}^r = m - r = q$, the map φ is open.

If $z \in A \cap f^{-1}(\ddot{E}(\alpha))$, then $f(z) \in \ddot{E}(\alpha)$ which implies $\varrho_0^{-1}(f(z)) \in V_0 \cap E(\mathfrak{a}_0, \ldots, \mathfrak{a}_p)$. Hence

$$f_\mu(z) = (\varrho_0^{-1}(f(z)) \,|\, \mathfrak{a}_\mu) = 0 \quad \text{for} \quad \mu = p+1, \ldots, n.$$

Hence $\varphi(z) = 0$. If $\varphi(z) = 0$, then $f_\mu(z) = 0$ for $\mu = p+1, \ldots, n$. Hence $\varrho_0^{-1}(f(z)) \in V_0 \cap E(\mathfrak{a}_0, \ldots, \mathfrak{a}_p)$ which implies $f(z) \in \ddot{E}(\alpha)$, or $z \in A \cap f^{-1}(\ddot{E}(\alpha))$. Therefore

$$A \cap f^{-1}(\ddot{E}(\alpha)) = \varphi^{-1}(0) = \eta_{10}^{-1}(\eta_{10}(a)) = \bigcap_{\mu=p+1}^{n} \{z \,|\, z \in A \text{ with } f_\mu(z) = 0\}.$$

If $z \in A \cap f^{-1}(\ddot{E}(\alpha))$, then $\sigma(z, 0) = (z, \xi(z, 0))$ where $\xi(z, 0) = \underline{\varrho}(\mathfrak{a}_0 + \mathfrak{c}_0) \wedge \ldots \wedge (\mathfrak{a}_p + \mathfrak{c}_p))$ with $\mathfrak{c}_\mu = 0$ for $\mu = 1, \ldots, p$ and

$$\mathfrak{c}_0 = \sum_{\nu=p+1}^{n} f_\nu(z)\,\mathfrak{a} - \sum_{\mu=1}^{p} f_\mu(z)\,\mathfrak{c}_\mu = 0.$$

Hence $\xi(z, 0) = \underline{\varrho}(\mathfrak{a}_0 \wedge \ldots \wedge \mathfrak{a}_p) = \alpha$, which implies $\sigma(z, 0) = (z, \alpha)$. If z is a simple point of $A \cap f^{-1}(\ddot{E}(\alpha))$, then

$$\nu_\varphi(z; 0) = \nu_\varphi(z) = \nu_{\eta_{10}}(z) = \nu_{\tilde{\eta}_1}(z, 0) = \nu_{\pi_1}(\sigma_1(z, 0))$$

$$= \nu_{\pi_f}(\sigma(z, 0)) = \nu_{\pi_f}(z, \alpha) = \nu_f(z; \alpha),$$

q.e.d.

The question, if this equality holds for all $z \in A \cap f^{-1}(\ddot{E}(\alpha))$, is open. However, for integration purposes, the Lemma 2.5 is sufficient.