HARMONIC ANALYSIS BASED ON CERTAIN COMMUTATIVE BANACH ALGEBRAS

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Introduction

In various papers (cf. the bibliography) A. Beurling has studied the harmonic analysis of functions on the real line. Using different approaches he has introduced the notion of the spectrum of a function as a set on the dual real line which, roughly speaking, consists of the frequencies of the characters of which the function can be regarded as composed,

The development of the theory of Banach algebras has made it possible to extend a wide sector of harmonic analysis into more abstract theories. It has therefore been natural to study spectral theory from a more general point of view. Thus Godement [8] gave a definition of the spectrum valid for bounded measurable functions on a locally compact Abelian group, and his approach was pursued by, among others, Kaplansky [10] and Helson [9].

Many problems in this field remain unsolved. It is by no means obvious to what extent the spectral theory depends on the metrical properties of the real line and on the structure of the function spaces which were considered by Beurling. The reason is perhaps that Beurling attained his results by means of a very large variety of methods. Algebraic arguments are sometimes used (as in [1]), but more often methods from the theory of analytic functions and potential theory are applied, and not all these methods are available in the general setting. Especially the generalization to groups has met many obstacles. More progress has been made when the theory has been restricted to the real line (Wermer [17]).

Beurling has given several more or less equivalent definitions of spectrum. The one which most easily lends itself to generalizations is the definition in [5], which defines the spectrum of a function in L^∞ as the set of frequencies of the characters which are included in the weak closure of the linear manifold spanned by the translations of the function. This definition was also used by Godement. In the theory which can be developed from this definition, it is of fundamental importance that L^1 be a commutative Banach algebra under convolution with the dual group as the regular maximal ideal space and with the property that every proper closed ideal is included in at least one regular maximal ideal. The last-mentioned property is closely connected with the general Tauberian theorem (Wiener [18]) as is shown e.g. in Loomis [11] §§ 25 D, 37 A.

A closer study of the possibilities offered by the above definition reveals that the concept of transformation is of a fundamental importance in the development of the theory. It is essential that L^1 can be considered as an algebra of transformations of

L^∞ into itself, if the transformation is defined as the ordinary convolution. In fact, even the definition of the spectrum can be expressed in terms of these transformations, and this opens the way to generalization in the following direction.

Let A be a normed linear space and F a commutative algebra with a representation onto an algebra of linear transformations of A into itself. For every $a \in A$ and $f \in F$ we denote by $f \circ a$ the corresponding transformed element in A. We assume that F is normed in such a way that

$$\| f \circ a \| \leq \| f \| \cdot \| a \|$$

and we suppose that F is a Banach algebra under this norm, with a space S of regular maximal ideals. We define, for every $a \in A$, the spectrum Λ_a as the subset of S consisting of all regular maximal ideals which contain the closed ideal of all $f \in F$ for which $f \circ a = \theta$. (θ denotes the null element in A.) We assume that every proper closed ideal is included in at least one regular maximal ideal, and then an empty Λ_a implies that $f \circ a = \theta$ for every $f \in F$. Let us finally assume that this is true only if $a = \theta$. Then Λ_a is empty only if $a = \theta$, and this fundamental uniqueness theorem gives us a solid basis for a general theory.

Of course very few of the problems in the Beurling spectral theory can be formulated in this abstract setting. The notion of translation has for instance disappeared as a main ingredient of the definition. The lost connections with the Fourier analysis can, however, partially be recovered if we assume that S is a locally compact Abelian group, and further specializations will of course lead us still closer to the field of study in the Beurling papers.

Our object in this paper is to study a class of algebras F of the above type, where S is a locally compact Abelian group, and then discuss the corresponding spectral definition. It will turn out that many of the essential results in the Beurling spectral theory can be approached in our rather general setting, e.g. the characterization of elements with one-point spectrum (originally studied by Beurling [2] and later by Kaplansky [10], Helson [9], Riss [15], Wermer [17] and others) and the spectral definition by means of the narrow closure (Beurling [2]).

As for the methods employed. we naturally have to utilize in a very essential way the general theory of commutative Banach algebras together with the special properties of the class of algebras which we discuss. The elementary Fourier analysis on groups is rather freely used, and results from the theory of analytic and quasi-analytic functions are applied at certain places where it has been possible to restrict the discussion to the real line. In paragraph 2.2 the structure theory of locally compact

Abelian groups is used in the discussion of a particular example of the algebras F, but apart from this, the theory does not depend on structure theory.

The first two chapters deal exclusively with the properties of the algebras F. Chapter III contains the definition of the spectrum and an account of certain of the most available spectral properties. In chapter IV the discussion centers around another definition of the spectrum. It is proved to be equivalent to the original definition and closely connected with the Beurling definition in [2].

Most of the results in chapter III are valid for more general classes of algebras F. The results in chapter IV, however, depend on the structural properties of F, and it is doubtful whether it is possible to prove similar results in greater generality.

It is assumed that the reader has a certain knowledge of the theory of commutative Banach algebras as in Gelfand [7] and parts of the theory of Fourier analysis on locally compact Abelian groups as in Pontrjagin [14], Weil [16] and Godement [8]. Whenever possible, however, we take the liberty of referring to the exposition in Loomis [11], and certain more or less standard arguments in harmonic analysis, such as convolutions, inversion theorems, etc., are used without reference.

Functions on the dual groups G and $\hat{G}$ are denoted by $f(x)$, $g(x), \ldots$, and $\hat{f}(\hat{x})$, $\hat{g}(\hat{x}), \ldots$, respectively. The only exceptions are the characters $(x, \hat{x})$, where the above notation is inconvenient. Whenever two functions, such as $f(x)$ and $\hat{f}(\hat{x})$, are mentioned in the same context, they indicate a pair of functions which in some sense are Fourier transforms. Addition is chosen as group operation.

CHAPTER I

A Class of Commutative Banach Algebras

1. Main assumptions and definitions

Let G be an Abelian locally compact group with the dual group $\hat{G}$. It will be convenient for our purposes to assume that the groups are Hausdorff spaces. This is no essential restriction as is shown in L. 28 D. (L. denotes here and in the following references to the corresponding paragraph in Loomis [11].)

We introduce a Banach space F of complex-valued functions $f(x)$, defined and finite everywhere on G. The addition of two elements in F is defined as the ordinary addition of the two functions and the multiplication of an element with a complex

constant is the ordinary multiplication of the function with the same constant. Different functions are supposed to be different elements, and for that reason $f(x)$ is the null-element if and only if $f(x) \equiv 0$.

Furthermore we suppose that if two functions $f_1(x)$ and $f_2(x)$ belong to F, then the same is true for the function

$$f(x) = f_1(x) \cdot f_2(x),$$

and the corresponding norms fulfill the relation

$$\|f\| \leq \|f_1\| \cdot \|f_2\|.$$

This implies that F is a commutative Banach algebra.

For any function $f(x) \in F$ we denote by Λ_f^0 the set in G where $f(x) \neq 0$, and by Λ_f the closure of Λ_f^0.

We shall introduce some further assumptions and notations:

I. *Suppose that for every neighborhood N of the identity in G there exists a non-negative, not identically vanishing function $f_N(x)$ in F with the following properties:*

A. $$\Lambda_{f_N} \subset N.$$

B. $$f_N(x) = \int\limits_{\hat{G}} \overline{(x, \hat{x})} \, \hat{f}_N(\hat{x}) \, d\hat{x},$$

where $\hat{f}_N(\hat{x})$ is continuous and $\in L^1(\hat{G})$.

C. *All continuous functions $\hat{g}(\hat{x})$ such that*

$$|\hat{g}(\hat{x})| \leq |\hat{f}_N(\hat{x})| \tag{1.11}$$

have the property that the functions

$$g(x) = \int\limits_{\hat{G}} \overline{(x, \hat{x})} \, \hat{g}(\hat{x}) \, d\hat{x} \tag{1.12}$$

belong to F, and their norms are uniformly bounded.

Before we can proceed with our assumptions we have to discuss a consequence of Assumption I.

Using the Pontrjagin duality theorem and the definition of the topologies of the dual groups we see (L. 34 C) that for every compact set $\hat{C} \subset \hat{G}$, the set of points $x \in G$, such that for every $\hat{x} \in \hat{C}$

$$|1 - (x, \hat{x})| < \tfrac{1}{2},$$

is an open set in G. Since it contains the identity o of G, it is a neighborhood N of o. The function $f_N(x)$ is non-negative and therefore, if $\hat{x} \in \hat{C}$,

$$\left| \hat{f}_N(\hat{x}) \right| = \left| \int\limits_G f_N(x)\,(x, \hat{x})\,dx \right| > \tfrac{1}{2} \int\limits_G f_N(x)\,dx.$$

(We assume here and in the following that the Haar measures on G and $\hat{G}$ are normed in such a way that the constant in the Fourier inversion formula has the value 1.)

This shows that it is possible to find, for every compact set $\hat{C}$, a function $f_N(x)$ such that $\left| \hat{f}_N(\hat{x}) \right|$ has a positive lower bound on $\hat{C}$. Using Assumption I C we see that this implies that *the class F_0 of functions*

$$g(x) = \int\limits_{\hat{G}} \overline{(x, \hat{x})}\,\hat{g}(\hat{x})\,d\hat{x},$$

where $\hat{g}(\hat{x})$ is continuous and vanishes outside a compact set, is a subclass of F. The formula

$$f_1(x) \cdot f_2(x) = \int\limits_{\hat{G}} \overline{(x, \hat{x})}\,d\hat{x} \int\limits_{\hat{G}} \hat{f}_1(\hat{x} - \hat{x}_0)\,\hat{f}_2(\hat{x}_0)\,d\hat{x}_0,$$

which is true if $\hat{f}_1(\hat{x})$ and $\hat{f}_2(\hat{x})$ belong to $L^1(\hat{G})$, shows that F_0 is moreover a sub-algebra (L. 28 A 4).

Our second assumption will be:

II. *Suppose that F_0 is dense in F.*

We shall introduce another subclass of F. Let us first form the class of all functions $g(x)$ of the type (1.12), for which $\hat{g}(\hat{x})$ is continuous and satisfies (1.11) for some N, and for which Λ_g is compact. *Then we denote by F' the class of functions of the type $\lambda \cdot g(x)$, where λ is an arbitrary constant.*

2. Some lemmas concerning the subclasses F_0 and F'

The classes F_0 and F' will play important rôles in the discussion of the Banach algebra F. For later use we shall collect in this section some lemmas on these subclasses.

If a function $g(x)$ belongs to F_0 or to F', then we shall use the term Fourier transform of $g(x)$ for the continuous function $\hat{g}(\hat{x})$, which in the sense of (1.12) is associated to $g(x)$.

LEMMA 1.21. *Consider for a given compact set $\hat{C}$ in $\hat{G}$ the subclass of all functions $g(x) \in F_0$, for which the Fourier transforms $\hat{g}(\hat{x})$ vanish outside $\hat{C}$. Then there exists a finite constant $d_{\hat{C}}$, such that for all these functions*

$$\|g(x)\| \le d_{\hat{C}} \|\hat{g}(\hat{x})\|_\infty,$$

where $\|\hat{g}(\hat{x})\|_\infty$ denotes the uniform norm of continuous functions on $\hat{G}$.

The proof follows at once from the discussion in 1.1.

LEMMA 1.22. *For every neighborhood $\hat{N}$ of the identity $\hat{o}$ in $\hat{G}$ there exists a function $f(x) \in F'$ such that the Fourier transform $\hat{f}(\hat{x})$ satisfies*

$$\left. \begin{aligned} 0 &\le \hat{f}(\hat{x}) \le 1, \\ \hat{f}(\hat{o}) &= 1, \\ \hat{f}(\hat{x}) &\le \tfrac{1}{2} \quad \text{outside } \hat{N}. \end{aligned} \right\} \tag{1.21}$$

PROOF. Let us start from a function $f_N(x)$ with compact N. $f_N(x)$ is non-negative, and for that reason

$$|\hat{f}_N(\hat{x})| \le \hat{f}_N(\hat{o}) \ne 0.$$

The function

$$\hat{h}(\hat{x}) = \frac{\hat{f}_N(\hat{x}) \overline{\hat{f}_N(\hat{x})}}{\{\hat{f}_N(\hat{o})\}^2}$$

is the Fourier transform of a function $\in F'$. It satisfies the first two of the conditions (1.21), and furthermore we know that

$$\hat{h}(\hat{x}) \le \tfrac{1}{2}$$

outside a certain compact set $\hat{C}$, since the Fourier transform of a function $\in L^1(G)$ vanishes at infinity.

Now let $\hat{N}$ be the given neighborhood. We may of course assume that it is open. The set $\hat{C}_0$ of all points in $\hat{C}$, which are not contained in $\hat{N}$, is then a compact set, not containing $\hat{o}$.

Let us for every point $x \in G$ denote by $\hat{O}_x$ the open set in $\hat{G}$ where

$$|1 + (x, \hat{x})|^2 < 2.$$

If $\hat{x} \ne \hat{o}$ there exists a point x_0 such that $(x_0, \hat{x}) \ne 1$. An elementary reasoning shows that for a suitable value of the integer n the number

$$(x_0, \hat{x})^n = (n\, x_0, \hat{x})$$

has to satisfy the inequality

$$|1 + (n\, x_0, \hat{x})| \le 1.$$

Therefore the sets $\hat{O}_x$ cover all points in $\hat{G}$ with the exception of $\hat{o}$, and as a result we may select a finite sub-sequence $\{\hat{O}_{x_\nu}\}_1^n$ which covers the compact set $\hat{C}_0$.

Let us now form the function

$$\hat{g}(\hat{x}) = \frac{1}{2^{2n}} \prod_1^n |1 + (x_\nu, \hat{x})|^2.$$

It satisfies the conditions (1.21), the third one, however, only inside $\hat{C}_0$. But the function

$$f(\hat{x}) = \hat{h}(\hat{x}) \cdot \hat{g}(\hat{x})$$

satisfies (1.21) in all details, and since $\hat{g}(\hat{x})$ is a linear combination of characters,

$$f(x) = \int\limits_{\hat{G}} \overline{(x, \hat{x})} f(\hat{x}) \, d\hat{x}$$

has compact Λ_f, and therefore it belongs to F'.

LEMMA 1.23. *For every neighborhood $\hat{N}$ of $\hat{o}$ and for every $\varepsilon > 0$ there exists a function $g(x) \in F'$ which has the representation*

$$g(x) = g_1(x) + \int\limits_{\hat{G}} \overline{(x, \hat{x})} \, \hat{g}_2(\hat{x}) \, d\hat{x},$$

where

$$\|g_1\| < \varepsilon,$$

and where $\hat{g}_2(\hat{x})$ is non-negative and continuous, vanishes outside $\hat{N}$ and satisfies

$$\int\limits_{\hat{N}} \hat{g}_2(\hat{x}) \, d\hat{x} = 1.$$

PROOF. Let us start from a function $f(x)$ in F' which satisfies the conditions in Lemma 1.22 with respect to $\hat{N}$. Let $\hat{k}(\hat{x})$ be a continuous function satisfying $0 \le \hat{k}(\hat{x}) \le 1$ and which vanishes outside $\hat{N}$ and assumes the value 1 on the set where

$$f(\hat{x}) \ge \tfrac{2}{3}.$$

Choose for every positive integer n the constant d_n such that

$$d_n \int\limits_{\hat{G}} \hat{k}(\hat{x}) \, [f(\hat{x})]^n \, d\hat{x} = 1.$$

Apparently

$$\lim_{n\to\infty} d_n^{1/n} = 1.$$

This relation and Assumption I C have as consequence that the norm of the function

$$d_n \int\limits_{\hat{G}} \overline{(x, \hat{x})} \, (1 - \hat{k}(\hat{x})) \, [f(\hat{x})]^n \, d\hat{x}$$

tends to 0, when $n \to \infty$. Let us assume that the norm is smaller than ε for $n = n_0$. Then the lemma follows by choosing

$$g(x) = d_{n_0} \int_{\hat{G}} \overline{(x, \hat{x})} \, [\hat{f}(\hat{x})]^{n_0} \, d\hat{x},$$

$$g_1(x) = d_{n_0} \int_{\hat{G}} \overline{(x, \hat{x})} \, (1 - \hat{k}(\hat{x})) \, [\hat{f}(\hat{x})]^{n_0} \, d\hat{x},$$

$$\hat{g}_2(\hat{x}) = d_{n_0} \hat{k}(\hat{x}) \, [\hat{f}(\hat{x})]^{n_0}.$$

LEMMA 1.24. *For every pair of sets C and O in G, where C is compact, O is open and $C \subset O$, there exists a function $f(x) \in F'$ such that*

$$0 \le f(x) \le 1 \quad \text{in } O,$$

$$f(x) = 1 \quad \text{in } C,$$

$$f(x) = 0 \quad \text{outside } O.$$

PROOF. We denote for every pair of sets E_1 and E_2 in G by $E_1 + E_2$ the set of all points $x = x_1 + x_2$, where $x_1 \in E_1$ and $x_2 \in E_2$. Then there exists (L. 5 F, L. 28 A 3) a compact symmetric neighborhood N of the identity in G such that

$$C + N + N \subset O.$$

Let us assume that the non-negative function $f_N(x)$ satisfies the relation

$$\int_G f_N(x) \, dx = 1.$$

If this is not the case, we may change the function by multiplying it with a suitable constant. Then let $f_1(x)$ be the characteristic function of the set $C + N$. The function

$$f(x) = \int_G f_N(x - x_0) \, f_1(x_0) \, dx_0 = \int_{\hat{G}} \overline{(x, \hat{x})} \, \hat{f}_N(\hat{x}) \, \hat{f}_1(\hat{x}) \, d\hat{x}$$

belongs to F', and it is very easy to verify that it has the required properties.

3. Linear functionals on F

We are going to show that we have a certain representation of the linear functionals on F as Borel measures on $\hat{G}$. Here the term Borel measure is used in the wide sense, i.e. it includes also complex set-functions.

Suppose that $f^*(f)$ is a linear functional on F. If we consider the functions $g(x) \in F_0$ for which the Fourier transforms $\hat{g}(\hat{x})$ vanish outside a fixed compact set $\hat{C}$, we get from Lemma 1.21

$$|f^*(g)| \le \|f^*\| \cdot \|g\| \le d_{\hat{C}} \|f^*\| \cdot \|\hat{g}(\hat{x})\|_\infty,$$

which shows that the functional is at the same time a linear functional on the class of functions $\hat{g}(\hat{x})$ under the uniform norm. Therefore

$$f^*(g) = \int_{\hat{C}} \hat{g}(\hat{x}) \, d\hat{\mu}_{\hat{C}}(-\hat{x}),$$

where $\hat{\mu}_{\hat{C}}$ is a Borel measure, uniquely defined on the interior of the compact set of points $\hat{x}$ such that $-\hat{x} \in \hat{C}$. Since $\hat{C}$ may be chosen arbitrarily we can extend this result to the following lemma:

LEMMA 1.31. *To each linear functional f^* on F there corresponds a unique Borel measure $\hat{\mu}$ on $\hat{G}$ such that if $f \in F_0$*

$$f^*(f) = \int_{\hat{G}} \hat{f}(\hat{x}) \, d\hat{\mu}(-\hat{x}).$$

Furthermore we have the following lemma, which is an immediate consequence of Assumption II.

LEMMA 1.32. *Two different functionals can not correspond to the same measure.*

Now let $f(x) \in F'$ and let f^* be a linear functional with the corresponding measure $\hat{\mu}$. If we let $g(x)$ run through all the elements in F_0 such that the Fourier transform $\hat{g}(\hat{x})$ satisfies

$$|\hat{g}(\hat{x})| \le |\hat{f}(\hat{x})|,$$

we get

$$\int_{\hat{G}} |\hat{f}(\hat{x})| \, |d\hat{\mu}(-\hat{x})| = \sup \left| \int_{\hat{G}} \hat{g}(\hat{x}) \, d\hat{\mu}(-\hat{x}) \right| = \sup |f^*(g)| \le \|f^*\| \cdot \sup \|g\| < \infty,$$

since the norms of the functions $g(x)$ are uniformly bounded (Assumption I C). Thus we get

LEMMA 1.33. *If $f(x) \in F'$ and if $\hat{\mu}$ is a measure which corresponds to a linear functional on F, then*

$$\int_{\hat{G}} |\hat{f}(\hat{x})| \, |d\hat{\mu}(-\hat{x})| < \infty.$$

4. Complex-valued homomorphisms of F

A homomorphism of F onto the complex numbers is a mapping

$$f(x) \to \lambda(f),$$

where $\lambda(f)$ for every $f(x) \in F$ is a finite complex number with the following properties:

A. $$\lambda\,(c_1\,f_1 + c_2\,f_2) = c_1\,\lambda\,(f_1) + c_2\,\lambda\,(f_2).$$

B. $$\lambda\,(f_1 \cdot f_2) = \lambda\,(f_1) \cdot \lambda\,(f_2)$$

for any two constants c_1 and c_2 and for any two elements $f_1(x)$ and $f_2(x)$.

C. $$\lambda\,(f) \doteq 0$$

for at least some $f\,(x)$.

Since F is a commutative algebra, $\lambda\,(f)$ is bounded, considered as a functional on F (L. 23 A). Therefore it is a linear functional on F, and in order to determine the complex-valued homomorphisms of F, we have only to find the not identically vanishing linear functionals which satisfy B.

Suppose that f^* is such a functional and suppose that it corresponds to the measure $\hat{\mu}$ in the sense of Lemma 1.31.

Since f^* is not identically vanishing and because of Assumption II, there exists a function $f_0(x) \in F_0$ such that

$$f^*\,(f_0) = \int\limits_{\hat{G}} f_0\,(\hat{x})\,d\hat{\mu}\,(-\hat{x}) = 1.$$

Put

$$\int\limits_{\hat{G}} f_0\,(\hat{x}_0 + \hat{x})\,d\hat{\mu}\,(-\hat{x}_0) = \hat{\alpha}\,(\hat{x}), \tag{1.41}$$

which is a continuous function, satisfying

$$\hat{\alpha}\,(\hat{o}) = 1. \tag{1.42}$$

Let $f_1\,(x)$ be a variable function in F_0. The relation

$$f^*\,(f_0\,f_1) = f^*\,(f_0) \cdot f^*\,(f_1) = f^*\,(f_1)$$

gives the formula

$$\int\limits_{\hat{G}} [\int\limits_{\hat{G}} f_1\,(\hat{x})\,f_0\,(\hat{x}_0 - \hat{x})\,d\hat{x}]\,d\hat{\mu}\,(-\hat{x}_0) = \int\limits_{\hat{G}} f_1\,(\hat{x})\,d\hat{\mu}\,(-\hat{x}),$$

and using (1.41) this may be written

$$\int\limits_{\hat{G}} f_1\,(\hat{x})\,\hat{\alpha}\,(-\hat{x})\,d\hat{x} = \int\limits_{\hat{G}} f_1\,(\hat{x})\,d\hat{\mu}\,(-\hat{x}).$$

Since this is true for every $f_1 \in F_0$, we obtain

$$\hat{\mu}\,(\hat{C}) = \int\limits_{\hat{C}} \hat{\alpha}\,(\hat{x})\,d\hat{x} \tag{1.43}$$

for every compact set $\hat{C}$ in $\hat{G}$.

As a consequence we have, if $f_1(x)$ and $f_2(x)$ are quite arbitrary functions in F_0,

$$\int_{\hat{G}} [\int_{\hat{G}} f_1(\hat{x}) f_2(\hat{x}_0 - \hat{x}) d\hat{x}] \, \hat{\alpha}(-\hat{x}_0) d\hat{x}_0 = \int_{\hat{G}} f_1(\hat{x}) \, \hat{\alpha}(-\hat{x}) d\hat{x} \cdot \int_{\hat{G}} f_2(\hat{x}) \, \hat{\alpha}(-\hat{x}) d\hat{x}.$$

And from this relation it is quite easy to see that

$$\hat{\alpha}(\hat{x}_1 + \hat{x}_2) = \hat{\alpha}(\hat{x}_1) \cdot \hat{\alpha}(\hat{x}_2) \tag{1.44}$$

for every pair of points in $\hat{G}$. Thus we have:

L E M M A 1.41. *A measure $\hat{\mu}$, corresponding to a linear functional which gives a complex-valued homomorphism, has to satisfy (1.43), where the continuous function $\hat{\alpha}(\hat{x})$ satisfies (1.42) and (1.44).*

We shall now proceed to prove the following more precise statement:

T H E O R E M 1.41. *The only functionals which give complex-valued homomorphisms, are the functionals $f^*(f) = f(x)$ for any $x \in G$.*

These functionals certainly give complex-valued homomorphisms. The only problem is to verify the condition C, which is, however, an immediate consequence of Lemma 1.24. We may mention that as a consequence these functionals are linear. They correspond in the sense of (1.43) to bounded functions $\hat{\alpha}(\hat{x})$, i.e. to the ordinary characters $(x, \hat{x})$.

Because of Lemma 1.32 no other linear functionals correspond to bounded functions $\hat{\alpha}(\hat{x})$. For that reason the only thing we have to prove is that no linear functional corresponds to a measure (1.43), where the continuous function $\hat{\alpha}(\hat{x})$ satisfies (1.44) and is unbounded. (Concerning the existence of such $\hat{\alpha}(\hat{x})$, see the remark after Theorem 2.31.) For the proof we need the following lemma:

L E M M A 1.42. *Let $\hat{x}_0$ be a fixed point in $\hat{G}$ and let c be a fixed real number such that*

$$0 < c < \pi.$$

Form the open set $O_c(\hat{x}_0)$ of all points $x \in G$ such that

$$-c < \arg (x, \hat{x}_0) < c \quad (\text{mod } 2\pi).$$

Suppose moreover that $\hat{f}(\hat{x})$ is the Fourier-Stieltjes transform of a bounded Borel measure μ, vanishing outside $O_c(\hat{x}_0)$ i.e.

$$\hat{f}(\hat{x}) = \int_{O_c(\hat{x}_0)} (x, \hat{x}) \, d\mu(x).$$

Then for every integer n

$$\hat{f}(n\hat{x}_0) = \int_{O_c(\hat{x}_0)} (x, \hat{x}_0)^n \, d\mu(x) = \int_{-c}^{c} e^{in\theta} \, db(\theta),$$

where $b(\theta)$ *is of bounded variation on* $(-c, c)$.

PROOF OF LEMMA 1.42. Let us denote by $g(e^{i\theta})$ an arbitrary continuous function on the unit circle. Consider the space of all these functions under the uniform norm. Then

$$\lambda(g) = \int_{O_c(\hat{x}_0)} g((x, \hat{x}_0)) \, d\mu(x)$$

is a linear functional and therefore it has the form

$$\lambda(g) = \int_{-\pi}^{\pi} g(e^{i\theta}) \, db(\theta),$$

where $b(\theta)$ is of bounded variation. By varying $g(e^{i\theta})$ it is easily seen that $b(\theta)$ is constant outside $(-c, c)$. And then the lemma follows by choosing $g(e^{i\theta}) = e^{in\theta}$.

PROOF OF THEOREM 1.41. Let us assume that a certain unbounded function $\hat{\alpha}(\hat{x})$ of the type described in Lemma 1.41 corresponds to a linear functional on F. We shall prove that this leads to a contradiction.

Let $\hat{x}_0$ be a point such that

$$|\hat{\alpha}(\hat{x}_0)| = d > 1.$$

Then we have for every integer n

$$|\hat{\alpha}(n\hat{x}_0)| = d^n.$$

Choose an arbitrary number c such that

$$0 < c < \pi.$$

The set $O_c(\hat{x}_0)$, defined in Lemma 1.42, is an open neighborhood of the identity in G. Therefore we can find a not identically vanishing function $f_1(x)$ in F' such that $\Lambda_{f_1} \subset O_c(\hat{x}_0)$. On account of Lemma 1.33 we have

$$\int_{\hat{G}} |\hat{f}_1(\hat{x})| \, |\hat{\alpha}(-\hat{x})| \, d\hat{x} < \infty.$$

Let us now choose a function $f_0(x) \in F_0$ such that

$$\hat{f}(\hat{x}) = \int_{\hat{G}} \hat{f}_1(\hat{x} - \hat{x}_0) \hat{f}_0(\hat{x}_0) \, d\hat{x}_0$$

satisfies

$$\hat{f}(\hat{o}) = 1.\tag{1.45}$$

The relation

$$|\hat{f}(\hat{x})| \leq \int_{\hat{G}} |\hat{f_1}(\hat{x} - \hat{x}_0)| \cdot |\hat{\alpha}(-\hat{x} + \hat{x}_0)| \cdot \frac{1}{|\hat{\alpha}(-\hat{x})| \cdot |\hat{\alpha}(\hat{x}_0)|} \cdot |\hat{f_0}(\hat{x}_0)| \, d\hat{x}_0$$

$$\leq \frac{1}{|\hat{\alpha}(-\hat{x})|} \cdot \int_{\hat{G}} |\hat{f_1}(\hat{x})| \, |\hat{\alpha}(-\hat{x})| \, d\hat{x} \cdot \sup_{\hat{x}_o \in \hat{G}} |\hat{\alpha}(-\hat{x}_0)| \cdot |\hat{f_0}(\hat{x}_0)| \quad (1.46)$$

shows that for some finite constant K

$$|\hat{f}(\hat{x})| \leq \frac{K}{|\hat{\alpha}(-\hat{x})|}$$

for every $\hat{x}$. In particular we have for every integer n

$$|\hat{f}(-n\,\hat{x}_0)| \leq \frac{K}{d^n}.\tag{1.47}$$

But $\hat{f}(\hat{x})$ is the Fourier transform of the function $f_1(x) \cdot f_0(x)$, and this function vanishes outside $O_c(\hat{x}_0)$. Therefore we may apply Lemma 1.42, and we then get for every integer n

$$\hat{f}(n\,\hat{x}_0) = \int_{-c}^{c} e^{-in\theta} \, db(\theta),\tag{1.48}$$

where $b(\theta)$ is a function of bounded variation.

(1.47) and (1.48) show that the analytic function

$$H(r\,e^{i\theta}) = H(z) = \sum_{n=-\infty}^{\infty} \hat{f}(-n\,\hat{x}_0)\,z^n$$

is regular in the region $1 < |z| < d$ and satisfies

$$\lim_{r \to 1+0} H(z) = 0$$

uniformly in any closed interval outside the interval $|\theta| \leq c$. Then it has to vanish identically which is contradictory to (1.45). This proves the theorem.

5. The space of regular maximal ideals

LEMMA 1.51. *The topology on G is the weakest topology in which all the functions $f(x) \in F$ are continuous.*

PROOF. Lemma 1.24 implies that the functions can not be continuous in any weaker topology. Thus it only remains to show that every $f(x) \in F$ is continuous in the topology on G.

All the functionals $f(x)$ have a norm ≤ 1, since this property is always true for a functional that gives a complex-valued homomorphism (L. 23 A). Therefore we have

$$\|f(x)\|_\infty \leq \|f\|. \tag{1.51}$$

However, because of Assumption II we can approximate every $f(x) \in F$ arbitrarily closely in the F-norm by means of functions in F_0. And thus (1.51) implies that every $f(x) \in F$ can be approximated arbitrarily closely in the uniform norm by means of continuous functions, and this has the consequence that every $f(x) \in F$ is continuous.

In the theory of commutative Banach algebras it is shown that there is a one-to-one correspondence between the regular maximal ideals and the complex-valued homomorphisms in the sense that every regular maximal ideal consists of the elements f, such that

$$f^*(f) = 0,$$

where f^* is the functional, which gives the corresponding homomorphism (L. 23 A). If M denotes a variable regular maximal ideal and f^* the corresponding functional in the above sense, then the function

$$f(M) = f^*(f)$$

is called the Gelfand representation of the element f on the space of regular maximal ideals. As topology on this space we choose the weakest topology in which all the functions $f(M)$ are continuous.

In our case the regular maximal ideals are in one-to-one correspondence to the points $x \in G$, since the functionals have the form $f(x)$. If we in this way identify the space of regular maximal ideals and G, Lemma 1.51 implies that the topology of the regular maximal ideal space is the original topology on G. *The topological space G is therefore the topological space of regular maximal ideals and every function is its own Gelfand representation.*

From the general theory of commutative Banach algebras we get the following theorem (L. 24 A Cor., L. 25 D). For the truth of B it is essential that the algebra is regular (L. 19 F), and this is the fact in our case because of Lemma 1.24.

THEOREM 1.51.   A.   *If $f(x) \in F$, then*

$$\lim_{n \to \infty} \|f^n\|^{1/n} = \|f(x)\|_\infty.$$

B. *Let E be a subset of G and suppose that we have an ideal in F with the property that, for any $x_0 \in E$, it contains a function $g(x) \in F$, such that $g(x_0) \neq 0$. Then the ideal contains every $f(x) \in F$, such that Λ_f is compact and included in E.*

Of fundamental importance is the following theorem:

T H E O R E M 1.52. *The elements $f(x)$ with compact Λ_f are dense in F.*

Before we start the proof, we shall introduce a new concept, using a terminology from L. 31 E.

D E F I N I T I O N 1.51. *A function $f(x) \in F$ is said to have an approximate identity if for every $\varepsilon > 0$ there exists a compact neighborhood $\hat{N}$ of $\hat{o}$ with the property that for every $f_0(x) \in F_0$ such that $\hat{f}_0(\hat{x})$ is non-negative, vanishes outside $\hat{N}$ and satisfies*

$$\int_{\hat{N}} \hat{f}_0(\hat{x}) \, d\hat{x} = 1,$$

we have

$$\| f(x) - f(x) \cdot f_0(x) \| \leq \varepsilon.$$

The proof of Theorem 1.52 will appear as an easy consequence of the following lemma:

L E M M A 1.52. *Every element $f(x) \in F$ with an approximate identity can be approximated arbitrarily closely by elements of the form $f(x) \cdot g(x)$, where $g(x) \in F$ and Λ_g is compact.*

P R O O F O F L E M M A 1.52. Choose an arbitrary $\varepsilon > 0$ and a set $\hat{N}$ which gives ε-approximations of $f(x)$ in the sense of Definition 1.51. Then we use the function $g(x)$, defined in Lemma 1.23. We then have

$$\| f - f \cdot g \| \leq \| f - f \cdot g_2 \| + \| f \| \cdot \| g_1 \| \leq \varepsilon + \varepsilon \cdot \| f \|,$$

which proves the lemma.

P R O O F O F T H E O R E M 1.52. The above lemma implies that the closure of the elements with compact Λ_f contains all functions with an approximate identity. However, Lemma 1.21 has the consequence that all elements in F_0 have an approximate identity, and therefore they are contained in the closure. And because of Assumption II every element in F is contained in the closure.

Theorem 1.51 B with $E = G$ and Theorem 1.52 have the following important consequence (the Wiener Tauberian theorem, cf. L. 25 D, Cor.).

THEOREM 1.53. *Suppose that a closed ideal in F has the property that it contains, for every $x_0 \in G$, a function $f(x)$, such that $f(x_0) \neq 0$. Then the ideal is the whole algebra F.*

Or, using algebraic terminology: *Every closed proper ideal is contained in at least one regular maximal ideal.*

Chapter II

Special Algebras and Special Elements

1. Various examples of Banach algebras F

A very simple example of an algebra F is the space of all continuous functions on G, vanishing at infinity, if we as norm choose the uniform norm. Assumption I is trivial to verify, and Assumption II is fulfilled as a consequence of the well-known fact that functions which are Fourier transforms of functions in $L^1(\hat{G})$ are dense in the class. This example shows that even if we can express the functions in F_0 and F' as Fourier transforms of functions on $\hat{G}$, this is in general not true for all the elements in F.

However, the particular cases, when this is possible, are of great interest. The classical example is the space of functions $f(x)$, which are Fourier transforms of functions $\hat{f}(\hat{x}) \in L^1(\hat{G})$ and with the norm

$$\|f\| = \int_G |\hat{f}(\hat{x})| \, d\hat{x}.$$

Beurling [1] and Wermer [17] have studied on the real line R more general Banach algebras of Fourier transforms of functions $\in L^1(\hat{R})$. (R denotes here as in the following the real line under the usual topology.) The Beurling algebras are said to be of non-quasianalytic type if for every neighborhood N of the identity in R they contain a not identically vanishing function which vanishes outside N. The corresponding subclasses of the Wermer algebras are algebras which satisfy a certain assumption (A), [17] p. 538. The Beurling non-quasianalytic algebras are apparently algebras of type F, and the same is true for those Wermer algebras, which satisfy (A), apart from an unessential difference in the definition of the norm [17] p. 537 (6).

For algebras of this kind, i.e. algebras which are defined as convolution algebras on the dual group, the verification of Assumption I A in 1.1 is often a very difficult problem. This matter was discussed in the cited papers and we shall illustrate it

further by discussing a natural generalization of the Beurling algebras to an arbitrary locally compact Abelian group G.

Let $\hat{p}(\hat{x})$ be a function on $\hat{G}$, measurable with respect to the Haar measure, bounded on every compact set and satisfying

$$\hat{p}(\hat{x}) \geq 1 \tag{2.11}$$

for every $\hat{x} \in \hat{G}$ and

$$\hat{p}(\hat{x}_1 + \hat{x}_2) \leq \hat{p}(\hat{x}_1) \cdot \hat{p}(\hat{x}_2) \tag{2.12}$$

for every pair of points $\hat{x}_1$ and $\hat{x}_2$ in $\hat{G}$.

Then consider the multiplicative Banach algebra of functions

$$f(x) = \int_G \overline{(x, \hat{x})} \hat{f}(\hat{x}) \, d\hat{x},$$

where $\hat{f}(\hat{x}) \in L^1(\hat{G})$, and where

$$\|f\| = \int_G |\hat{f}(\hat{x})| \, \hat{p}(\hat{x}) \, d\hat{x} < \infty.$$

DEFINITION 2.11. *We denote this Banach algebra* $F\{\hat{p}\}$.

It is easy to see that the inequality (2.11) is necessary in order to have (1.51) fulfilled, i.e. in order that the algebra is of type F.

The algebra $F\{\hat{p}\}$ *is an algebra* F *if and only if for every neighborhood* N *of the identity in* G *it contains a not identically vanishing function which vanishes outside* N. This will lead us to the following theorem:

THEOREM 2.11. $F\{\hat{p}\}$ *is an algebra* F *if and only if for every* $\hat{x}_0 \in \hat{G}$

$$\sum_1^\infty \frac{\log [\hat{p}(n\hat{x}_0)]}{n^2} < \infty. \tag{2.13}$$

The proof will be given in 2.2.

It may be pointed out that the condition (2.13) is well known on the real line. All questions of this kind are closely connected to notions of quasi-analyticity on the real line, and Theorem XII in Paley-Wiener [13] is a suitable tool in many similar cases.

An especially interesting case of the space $F\{\hat{p}\}$ is the following. Suppose that $G = R$, i.e. that $\hat{R}$ may be represented as the real line $-\infty < t < \infty$. We choose

$$\hat{p}(t) = \sum_0^\infty a_\nu |t|^\nu,$$

where $\{a_\nu\}_0^\infty$ is a sequence of non-negative numbers such that the least non-increasing majorant of

$$\sum_1^\infty a_\nu^{1/r}$$

is convergent, and such that for every m and n

$$a_{m+n}\,(m+n)\,!\leq a_m\,m\,!\cdot a_n\,n\,!.$$

The condition (2.12) is easy to verify, and (2.13) is fulfilled according to a theorem by Carleman [6] p. 50. Therefore the space is of type F.

The interest of this space lies in the fact that we may construct a very closely related space in the following way. Consider on R the space of all functions $f(x)$ such that

$$\|f\| = \sum_0^\infty a_\nu \sup_{|x|<\infty} |f^{(\nu)}(x)| < \infty,$$

while

$$\sum_0^\infty a_\nu \sup_{|x|>x_0} |f^{(\nu)}(x)| \to 0,$$

when $x_0 \to \infty$. This space is a multiplicative Banach algebra if $\|f\|$ is chosen as norm, and the fact that the functions in the algebra $F\{\hat{p}\}$ are dense in this new algebra can be used to show that it is of type F. We will not go into the details concerning the proof.

A particular case is when $a_\nu = 0$ if ν is greater than a certain index n. In this case we have to suppose that $f^{(n)}(x)$ is continuous and the algebra is then simply the multiplicative algebra of all functions with n continuous derivatives, vanishing together with the derivatives at infinity. It is then possible to use as norm

$$\|f\| = \sup_{|x|<\infty} \sum_0^n a_\nu |f^{(\nu)}(x)|.$$

Since we have introduced in Definition 1.51 the notion of elements in F with an approximate identity, it may be suitable to construct a space F where not all the elements are of that kind.

We form the function

$$\hat{p}(t) = \frac{1}{12\,\pi\,|t|^{\frac{1}{2}}\,(1+|t|)}$$

on $\hat{R}$. It is easy to show that if $t_0 \neq 0$

$$\int_{-\infty}^\infty \hat{p}(t)\,\hat{p}(t_0-t)\,dt \leq \hat{p}(t_0).$$

Let us then consider the class of functions

$$f(x) = \int\limits_{-\infty}^{\infty} e^{-itx} \hat{f}(t)\, dt,$$

where $\hat{f}(t)$ is continuous except possibly at $t = 0$, and where

$$\hat{f}(t) = o(\hat{p}(t))$$

at $t = 0$ and $t = \infty$.

Choosing the norm

$$\|f\| = \sup_{t \neq 0} \frac{\hat{f}(t)}{\hat{p}(t)},$$

we get an algebra F. However, an element such that $\hat{f}(t)$ is discontinuous at $t = 0$, can not have an approximate identity. For such an element $f(x)$ we have furthermore that, if $a \neq 0$, the function

$$f(x)\, e^{iax}$$

does not belong to the class. Therefore, in general we do not get new elements by multiplying an element with a character. This fact accounts for some of the complications in the discussions in chapters 3 and 4.

Finally we give the following space, which illustrates the fact that for a given value $x_0 \neq 0$, the functions $f(x)$ and $f(x + x_0)$ need not be elements of F at the same time.

We consider on R the class of functions

$$f(x) = f_1(x) + f_2(x) = \int\limits_{-\infty}^{\infty} e^{-itx} \hat{f}_1(t)\, dt + \int\limits_{-\infty}^{\infty} e^{-itx} \hat{f}_2(t)\, dt,$$

where $\hat{f}_1(t)(1 + |t|) \in L^1$, $\hat{f}_2(t) \in L^1$, and where $f_2(x)$ vanishes in the interval $0 \le x \le \infty$. Then we put

$$\|f\| = \inf \left\{ \int\limits_{-\infty}^{\infty} (1 + |t|)\, |\hat{f}_1(t)|\, dt + \int\limits_{-\infty}^{\infty} |\hat{f}_2(t)|\, dt \right\},$$

where we vary all the possible representations of $f(x)$. It is very simple to show that we get an algebra F.

2. Proof of Theorem 2.11

THE NECESSITY. Let us assume the opposite, namely that for some $\hat{x}_0 \in \hat{G}$

$$\sum_{1}^{\infty} \frac{\log [\hat{p}(n\hat{x}_0)]}{n^2} = \infty, \tag{2.21}$$

whereas for every neighborhood N of the identity in G we have a not identically vanishing function

$$f(x) = \int_{\hat{G}} \overline{(x, \hat{x})} f(\hat{x}) \, d\hat{x},$$

which vanishes outside N, and is such that

$$\int_{\hat{G}} \hat{p}(\hat{x}) |f(\hat{x})| \, d\hat{x} < \infty.$$

We may then proceed in exactly the same way as in the proof of theorem 1.41 with the only difference that $\hat{\alpha}(-\hat{x})$ all the time is exchanged to $\hat{p}(\hat{x})$. It is true that the inequality (1.46) uses the relation (1.44), which is not true for the function $\hat{p}(\hat{x})$, but (2.12) is apparently sufficient to guarantee that (1.46) is valid even if $\hat{\alpha}(-\hat{x})$ is exchanged to $\hat{p}(\hat{x})$. We therefore get for every c, such that $0 < c < \pi$, that there exists a function $b(\theta)$ of bounded variation such that Fourier coefficients

$$c_n = \int_{-c}^{c} e^{-in\theta} \, db(\theta)$$

satisfy

$$c_0 = 1 \tag{2.22}$$

and

$$|c_n| < \frac{K}{\hat{p}(n\hat{x}_0)} \tag{2.23}$$

for every integer n and for some finite constant K.

The reason why the proof of the contradiction in these relations causes us some trouble is that there exists no exact correspondence in the theory of Fourier series to the Theorem XII in Paley-Wiener [13] on Fourier integrals, which vanish on a half-line. The method which we shall use is to transfer the series into an integral with similar properties, and then use the Paley-Wiener theorem.

Standard arguments on the Fourier series in question show that there is no real restriction to assume that

$$\sum_{-\infty}^{\infty} \hat{p}(n\hat{x}_0) |c_n| < \infty. \tag{2.24}$$

Then we can prove the contradiction, starting from a value of c, such that $0 < c < \pi/2$.

For every real number y we define $a_n(y)$ as the Fourier coefficients of the function

$$e^{-iy\theta}$$

in $-\pi \leq \theta < \pi$, and put

$$d_n(y) = a_n(y) \cdot c_n.$$

Apparently

$$\sum_{-\infty}^{\infty} |d_n(y)| \cdot \hat{p}(n\hat{x}_0) \leq d \tag{2.25}$$

for some finite constant d, independent of y. In the interval $-\pi/2 \leq \theta \leq \pi/2$ we have

$$\sum_{-\infty}^{\infty} d_n(y) e^{in\theta} = 2\pi \int_{-c}^{c} e^{-iy(\theta-\psi)} \cdot db(\psi) = 2\pi e^{-iy\theta} B(-y), \tag{2.26}$$

where

$$B(t) = \int_{-c}^{c} e^{-it\theta} db(\theta).$$

Using the Parseval relation, (2.26) and (2.23) we obtain for any integer n

$$|B(n+y)| = \left| \int_{-c}^{c} e^{-in\theta - iy\theta} db(\theta) \right|$$

$$= \frac{1}{2\pi} \left| \sum_{m=-\infty}^{\infty} c_m d_{n-m}(y) \frac{1}{B(-y)} \right| \leq \frac{K}{2\pi |B(-y)|} \sum_{m=-\infty}^{\infty} \frac{|d_{n-m}(y)|}{\hat{p}(m\hat{x}_0)}.$$

However, by (2.12) and (2.25)

$$\sum_{m=-\infty}^{\infty} \frac{|d_{n-m}(y)|}{\hat{p}(m\hat{x}_0)} \leq \sum_{m=-\infty}^{\infty} \frac{\hat{p}(m\hat{x}_0)\, \hat{p}((n-m)\hat{x}_0) |d_{n-m}(y)|}{\hat{p}(n\hat{x}_0) \cdot \hat{p}(m\hat{x}_0)}$$

$$\leq \frac{1}{\hat{p}(n\hat{x}_0)} \sum_{m=-\infty}^{\infty} \hat{p}((n-m)\hat{x}_0) |d_{n-m}(y)| \leq \frac{d}{\hat{p}(n\hat{x}_0)}$$

and hence

$$|B(n+y)| \leq \frac{K \cdot d}{2\pi |B(-y)|} \cdot \frac{1}{\hat{p}(n\hat{x}_0)}.$$

It follows from (2.22) that

$$2\pi |B(-y)| \geq 1$$

for $|y| \leq \delta$, if δ is sufficiently small. Therefore

$$|B(n+y)| \leq \frac{K \cdot d}{\hat{p}(n\hat{x}_0)}$$

if n is an integer and $|y| \leq \delta$. By (2.24), $b(\theta)$ is absolutely continuous and $b'(\theta) \in L^2(-c, c)$. Hence $B(t) \in L^2(-\infty, \infty)$, and the inequality above, together with the assumption (2.21), gives

$$\int_{-\infty}^{\infty} \frac{\log |B(t)|}{1+t^2} dt = -\infty.$$

Using the cited theorem by Paley-Wiener we see that this implies that $B(t) \equiv 0$, which is contradictory to (2.22).

THE SUFFICIENCY. In the proof of the sufficiency it seems difficult to avoid structural considerations. We shall start by considering some cases when the group G has a very simple structure and then step by step extend the theorem to the general case.

The case when G is a discrete group D is trivial, since $\hat{G}$ is then compact (L. 38 A), i.e. $\hat{p}(\hat{x})$ is bounded. The cases when G is the real line R or the unit circle S, both under the usual topology, are obvious consequences of Theorem XII in Paley-Wiener [13].

Now let G be the direct product $G_1 \times G_2$ of two groups G_1 and G_2, for which the theorem is true. The points in G may be written in the form $x = x_1 + x_2$ where $x_1 \in G_1$, $x_2 \in G_2$. $\hat{G}$ is then the direct product of $\hat{G}_1$ and $\hat{G}_2$, and we may therefore put $\hat{x} = \hat{x}_1 + \hat{x}_2$, where $\hat{x}_1 \in \hat{G}_1$ and $x_2 \in \hat{G}_2$ (L. 35 A). This representation of G and $\hat{G}$ can be done such that always

$$(x, \hat{x}) = (x_1, \hat{x}_1) \cdot (x_2, \hat{x}_2).$$

We denote the identities in $G, G_1, G_2, \hat{G}_1$, and $\hat{G}_2$ by $o, o_1, o_2, \hat{o}_1$, and $\hat{o}_2$, and assume that the Haar measure on $\hat{G}$ is normed in such a way that it is the direct product of the Haar measures on $\hat{G}_1$ and $\hat{G}_2$.

Suppose that

$$\hat{p}(\hat{x}) = \hat{p}(\hat{x}_1 + \hat{x}_2)$$

fulfills the requirements of theorem 2.11. Let N be an arbitrary neighborhood of o. It contains a neighborhood of the form $N_1 \times N_2$ where $N_1 \subset G_1$ and $N_2 \subset G_2$ are neighborhoods of o_1 and o_2, because of the definition of the direct product topology. The functions

$$\hat{p}(\hat{x}_1) = \hat{p}(\hat{x}_1 + \hat{o}_2) \quad \text{and} \quad \hat{p}(\hat{x}_2) = \hat{p}(\hat{x}_2 + \hat{o}_1),$$

considered as functions on $\hat{G}_1$ and $\hat{G}_2$, satisfy the conditions in Theorem 2.11. And since the theorem was supposed to be true for these groups, we can for $\nu = 1, 2$ find a function

$$f_\nu(x_\nu) = \int_{\hat{G}_\nu} \overline{(x_\nu, \hat{x}_\nu)} f_\nu(\hat{x}_\nu) \, d\hat{x}_\nu,$$

vanishing outside N_ν and satisfying

$$0 < \int_{\hat{G}_\nu} |f_\nu(\hat{x}_\nu)| \, \hat{p}(\hat{x}_\nu) \, d\hat{x}_\nu < \infty.$$

Then let us form the function

$$f(x) = f(x_1 + x_2) = f_1(x_1) \cdot f_2(x_2)$$

$$= \int_{\hat{G}_1 \times \hat{G}_1} \overline{(x_1, \hat{x}_1)} \, \overline{(x_2, \hat{x}_2)} \, f_1(\hat{x}_1) \cdot f_2(\hat{x}_2) \, d\hat{x}_1 \, d\hat{x}_2 = \int_{\hat{G}} \overline{(x, \hat{x})} f_1(\hat{x}_1) f_2(\hat{x}_2) \, d\hat{x}.$$

It vanishes outside N, but not identically, and satisfies

$$\int_{\hat{G}} |\hat{f}_1(\hat{x}_1)| \cdot |\hat{f}_2(\hat{x}_2)| \, \hat{p}(\hat{x}) \, d\hat{x} \leq \int_{\hat{G}} |\hat{f}_1(\hat{x}_1)| \, |\hat{f}_2(\hat{x}_2)| \, \hat{p}(\hat{x}_1) \, \hat{p}(\hat{x}_2) \, d\hat{x}$$

$$= \int_{\hat{G}_1} |\hat{f}_1(\hat{x}_1)| \, \hat{p}(\hat{x}_1) \, d\hat{x}_1 \cdot \int_{\hat{G}_2} |\hat{f}_2(\hat{x}_2)| \, \hat{p}(\hat{x}_2) \, d\hat{x}_2 < \infty.$$

Therefore, the theorem is true for the group G.

As a consequence of this the theorem is true for all groups of the form

$$R^n \times S^m \times D. \tag{2.27}$$

We are now in a position to prove the theorem for an arbitrary locally compact Abelian group G.

Let N be an arbitrary neighborhood of the identity o in G. The multiplicative algebra of Fourier transforms of functions $\in L^1(G)$ under the usual norm is an algebra F. By applying Lemma 1.22 to this class of functions we see that there exists a function

$$g(x) = \int_{\hat{G}} \overline{(x, \hat{x})} \, \hat{g}(\hat{x}) \, d\hat{x},$$

where $\hat{g}(\hat{x}) \in L^1(\hat{G})$ vanishes outside a compact symmetric neighborhood $\hat{C}$ of $\hat{o}$, and such that

$$g(o) = 1, \tag{2.28}$$

while

$$|g(x)| \leq \tfrac{1}{2} \qquad \text{outside } N.$$

We denote by $\hat{G}_1$ the subgroup of $\hat{G}$, which consists of all points $\hat{x}_1$, included in some of the sets

$$n \cdot \hat{C} = \hat{C} + \hat{C} + \cdots + \hat{C}, \qquad n = 1, 2, \ldots,$$

i.e. the group, generated by $\hat{C}$. This is a new locally compact group under the induced topology, and we may as Haar measure on $\hat{G}_1$ choose the restriction to $\hat{G}_1$ of the Haar measure on $\hat{G}$. According to a theorem by A. Weil [16] p. 110, $\hat{G}_1$ is the dual group of a group G_1 of the type (2.27), and hence the theorem is true for the group G_1.

If the function $\hat{p}(\hat{x})$ satisfies the conditions in the theorem with respect to $\hat{G}$, then the same is true with respect to $\hat{G}_1$ for the function $\hat{p}(\hat{x}_1)$, defined as the restriction of $p(\hat{x})$ to G_1. For that reason we can find, for every neighborhood N_1 of the identity in the dual group G_1, a function $f(\hat{x}_1)$ on $\hat{G}_1$ such that

$$0 < \int_{\hat{G}_1} |f(\hat{x}_1)|\, \hat{p}(\hat{x}_1)\, d\,\hat{x}_1 < \infty,$$ (2.29)

while

$$\int_{\hat{G}_1} \overline{(x_1, \hat{x}_1)}\, f(\hat{x}_1)\, d\hat{x}_1$$

vanishes outside N_1. We may as N_1 choose the subset of G_1 where

$$\left| \int_{\hat{G}_1} \overline{(x_1, \hat{x}_1)}\, \hat{g}(\hat{x}_1)\, d\hat{x}_1 \right| > \tfrac{1}{2},$$

for this set is obviously open, and it contains the identity because of (2.28).

We extend the definition of $f(\hat{x})$ to the whole of $\hat{G}$ by defining $f(\hat{x}) = 0$ outside $\hat{G}_1$. And since for every $x \in G$ the restriction to $\hat{G}_1$ of the functions $(x, \hat{x})$ are characters on $\hat{G}_1$, we see that the above conditions imply that the function

$$\int_{\hat{G}} \overline{(x, \hat{x})}\, f(\hat{x})\, d\hat{x}$$

vanishes whenever $|g(x)| \leq \tfrac{1}{2}$, and hence it vanishes outside N. The condition (2.29) may be written

$$0 < \int_{\hat{G}} |f(\hat{x})|\, \hat{p}(\hat{x})\, d\hat{x} < \infty,$$

and since N was arbitrary, this proves the theorem in the general case.

3. The class Φ

Basic functions in the harmonic analysis on R are the functions $e^{i\lambda x}$, where λ is a complex number, and to some extent also the ordinary polynomials

$$P_n(x) = \sum_{m=0}^{n} a_m\, x^m.$$

These two classes of functions have correspondences on any locally compact Abelian group, and we shall now study to what extent these generalized exponentials and polynomials on G belong to F. To this end we shall make the following definition.

DEFINITION 2.31. *We denote by Φ the class of all functions on G, which coincide on any given compact set with some function in F.*

The following theorem shows that Φ contains all generalized exponentials.

THEOREM 2.31. *Suppose that $\alpha(x)$ is a continuous function on G, satisfying*

$$\alpha(x_1 + x_2) = \alpha(x_1) \cdot \alpha(x_2)$$

for every x_1 and x_2 in G. Then $\alpha(x) \in \Phi$.

PROOF. It is only necessary to consider the case when $\alpha(x) \not\equiv 0$. Lemma 1.24 then shows that it is possible to find a function $f(x) \in F'$, such that

$$\int_{\Lambda_f} f(x)\, \alpha(-x)\, dx = 1.$$

We denote by $g(x)$ the function which coincides with $\alpha(x)$ on the compact set, consisting of all points $x = x_1 - x_2$, where $x_1 \in C$ and $x_2 \in \Lambda_f$, while it vanishes outside the set. Then if $x \in C$

$$h(x) = \int_{\Lambda_f} f(x_0)\, g(x - x_0)\, dx_0 = \int_{\Lambda_f} f(x_0)\, \alpha(x - x_0)\, dx_0 = \alpha(x) \cdot \int_{\Lambda_f} f(x_0)\, \alpha(-x_0)\, dx_0 = \alpha(x),$$

and the Fourier transform $\hat{h}(\hat{x})$ satisfies

$$\hat{h}(\hat{x}) = \hat{f}(\hat{x}) \cdot \hat{g}(\hat{x}),$$

where $\hat{g}(\hat{x})$ is continuous and bounded. Therefore, it follows from Assumption I C in 1.1 that $h \in F$.

REMARK. The question concerning the existence of other functions $\alpha(x)$ than the bounded characters has been answered by Mackey [12]. He has found that *there exist unbounded functions $\alpha(x)$ if and only if there exist non-trivial continuous homomorphisms of R into $\hat{G}$, i.e. one-parameter subgroups of $\hat{G}$.*

DEFINITION 2.32. *A continuous function $P(x)$ on G is called a polynomial of degree n, if for every x and x_0 in G*

$$P(x + \nu x_0)$$

is a polynomial of degree $\leq n$, considered as a function of the variable non-negative integer ν, while at least one of these polynomials is exactly of degree n.

We may mention as an example that if $\alpha(x)$ is an unbounded function of the kind described in Theorem 2.31, then

$$\{\log |\alpha(x)|\}^n$$

is a polynomial of degree n.

THEOREM 2.32. *Every polynomial belongs to Φ.*

We need the following lemma:

LEMMA 2.31. *Let $P_n(x)$ be a polynomial of degree $\leq n$. Then for any given x_0*

$$\sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu} P_n(x + \nu x_0) = Q(x, x_0)$$

is independent of x.

PROOF OF LEMMA 2.31. The above definition of a polynomial of finite degree can be given on any semi-group, and if we omit the continuity assumption, the semi-group need not even have a topology. It will turn out from the proof that the lemma is still true in that general case if we make the extra assumption that the semi-group is commutative.

Apparently it is enough to prove that

$$Q(x, x_0) = Q(x + x_1, x_0) \tag{2.31}$$

for any given pair of points x and x_1 in G.

The expression

$$P_n(x + \mu x_1 + \nu x_0) = R(\mu, \nu)$$

has the property that for every choice of non-negative integers μ, ν, μ_0 and ν_0

$$R(\mu + \lambda \mu_0, \nu + \lambda \nu_0) \tag{2.32}$$

is a polynomial of degree $\leq n$ in the non-negative integer variable λ. If we choose $\mu = \nu_0 = 0$, $\mu_0 = 1$, and then $\nu = \mu_0 = 0$, $\nu_0 = 1$, it follows from the elementary theory of arithmetical series that

$$R(\mu, \nu) = \sum_{p,q=0}^{n} a_{p,q}\, \mu^p \nu^q.$$

By choosing suitable values of μ_0 and ν_0 in (2.32) it is obvious that the coefficients $a_{p,q}$ vanish whenever $p + q > n$. Then

$$\sum_{\nu=0}^{n} (-1)^{\nu} \binom{n}{\nu} R(\mu, \nu) = (-1)^n n!\, a_{0,n},$$

which is independent of μ. We obtain the two members of (2.31) by putting $\mu = 0$ and $\mu = 1$ in the above expression, and hence the relation (2.31) is true, which proves the lemma.

PROOF OF THEOREM 2.32. We shall prove the theorem by induction.

Lemma 1.24 shows that the theorem is true for polynomials of degree 0. Let us suppose that it is true for polynomials of degree $\leq n - 1$, and we have then only to prove that the construction is possible for an arbitrary polynomial $P_n(x)$ of degree n.

We choose a function $f(x) \in F'$ such that

$$\int_{\Lambda_f} f(x)\, dx = 1.$$

The function

$$P_{n-1}(x) = P_n(x) - \int\limits_{\Lambda_f} f(x_1) P_n(x - x_1)\, d x_1$$

satisfies because of lemma 2.31

$$\sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu} P_{n-1}(x + \nu x_0) = Q(x, x_0) - \int\limits_{\Lambda_f} f(x_1) Q(x - x_1, x_0)\, d x_1$$

$$= Q(x, x_0) - \int\limits_{\Lambda_f} f(x_1) Q(x, x_0)\, d x_1 = 0$$

for every x and x_0, and therefore it is a polynomial of degree $\leq n-1$.

Now let C be the given compact set, and let $g(x)$ coincide with $P_n(x)$ on the compact set, consisting of the points $x = x_1 - x_2$, where $x_1 \in C$ and $x_2 \in \Lambda_f$, while it vanishes outside this set. The function

$$h(x) = \int\limits_{\Lambda_f} f(x_0) g(x - x_0)\, d x_0$$

belongs to F, as can be shown in the same way as the similar statement in the proof of Theorem 2.31. And we have for $x \in C$

$$P_{n-1}(x) = P_n(x) - h(x).$$

By assumption we can find a function $k(x) \in F$, coinciding with the polynomial $P_{n-1}(x)$ on C. Hence we get if $x \in C$

$$P_n(x) = h(x) + k(x),$$

and this proves the theorem.

Our definition of polynomials is quite different from the definitions of polynomials and generalized polynomials in the theory of distributions on locally compact Abelian groups by J. Riss [15]. The connection of our concept and his is not obvious, and a study of this problem seems to require extensive structural considerations. The author hopes that he will be able to return to this subject.

CHAPTER III

The Spaces A and the Spectrum

1. The spaces A

Let F be a Banach algebra of the kind described in chapter I and let A be a normed linear space. We assume that to each $f \in F$ and each $a \in A$ there corresponds an element $f \circ a \in A$ and that this correspondence has the following properties:

$$f \circ (c_1 a_1 + c_2 a_2) = c_1 \cdot (f \circ a_1) + c_2 \cdot (f \circ a_2),$$
$$(c_1 f_1 + c_2 f_2) \circ a = c_1 \cdot (f_1 \circ a) + c_2 (f_2 \circ a)$$

and

$$f_1 f_2 \circ a = f_1 \circ (f_2 \circ a)$$

for any elements f, f_1, f_2 in F, any elements a, a_1, a_2 in A and any complex constants c_1 and c_2.

This implies that we have a homomorphism of F onto an algebra of linear transformations of A into itself, i.e. a representation of F. It is not necessary that different functions correspond to different transformations.

I. *We denote by $\|a\|$ the norm in A, and assume that we always have*

$$\| f \circ a \| \le \| f \| \cdot \| a \|. \tag{3.11}$$

II. *We denote by θ the null element in A, and assume that if for a given element* $a \in A$

$$f \circ a = \theta$$

for every $f \in F$, *then* $a = \theta$.

It is an easy consequence of (3.11) that *if $f \equiv 0$ or if $a = \theta$ then $f \circ a = \theta$.*

Starting from a given space F we can find a great number of spaces A. We shall mention some particularly important or interesting cases. It should be observed that A need not be complete (cf. example 4°).

1°. The space F itself if we for any pair of functions f and g in F put

$$f \circ g = f g.$$

Condition I is trivial and Condition II is an easy consequence of Lemma 1.24.

2°. The space F^* of all linear functionals on F with the usual norm. Here we define the functional $f \circ f^*$ as the functional which, operating on a function $g \in F$, gives the value $f^*(f g)$. Since

$$|f^*(f g)| \le \| f^* \| \cdot \| f \| \cdot \| g \|,$$

we have

$$\| f \circ f^* \| \le \| f^* \| \cdot \| f \|,$$

and hence I is true. II can be proved in the following way:

Suppose that for a given functional f^*

$$f \circ f^* = \theta$$

for every $f \in F$, i.e. that

$$f^* (f \cdot g) = 0$$

for every pair of functions f and g in F. The class of functions of the form $f \cdot g$ is according to Theorem 1.53 dense in F, and thus $f^* (f) = 0$ for every $f \in F$, i.e. $f^* = \theta$.

3°. Let F be a space $F\{\hat{p}\}$ (Definition 2.11), where $\hat{p}(\hat{x})$ then has to satisfy the condition in Theorem 2.11. Let p be a number such that $1 \le p < \infty$, and let $\hat{q}(\hat{x})$ be a positive measurable function such that for every pair of points $\hat{x}_1$ and $\hat{x}_2$ in $\hat{G}$

$$\hat{q}(\hat{x}_1 + \hat{x}_2) \le \hat{q}(\hat{x}_1) \cdot \hat{p}(\hat{x}_2). \tag{3.12}$$

Then let us form the Banach space of all measurable functions $\hat{a}(\hat{x})$ on $\hat{G}$ with the finite norm

$$\|\hat{a}\| = [\int_{\hat{G}} |\hat{a}(\hat{x})|^p \, \hat{q}(\hat{x})^p \, d\hat{x}]^{1/p}.$$

If $f \in F\{\hat{p}\}$ and

$$f(x) = \int_{\hat{G}} \overline{(x, \hat{x})} f(\hat{x}) \, d\hat{x},$$

we define $f \circ \hat{a}$ as the function

$$\int_{\hat{G}} f(\hat{x} - \hat{x}_0) \, \hat{a}(\hat{x}_0) \, d\hat{x}_0.$$

Using (3.12) and an inequality by Young, extended to groups by Weil [16] pp. 54–55, it is easy to prove that Condition I is satisfied. Condition II is also fulfilled, and hence the space is a space A with respect to the space $F\{\hat{p}\}$.

We shall present a method for constructing functions $\hat{q}(\hat{x})$ from given functions $\hat{p}(\hat{x})$. Let us assume that $\hat{p}(\hat{x}) = \hat{p}(-\hat{x})$ for every $\hat{x}$.

Suppose that $r(u)$ is a real-valued function of a real variable u such that for every u_1 and u_2

$$|r(u_1 - u_2)| \le |u_1 - u_2|.$$

Then

$$e^{r (\log \hat{p}(\hat{x}_1)) - r (\log \hat{p}(\hat{x}_2))} \le e^{|\log \hat{p}(\hat{x}_1) - \log \hat{p}(\hat{x}_2)|} \le e^{\log \hat{p}(\hat{x}_1 - \hat{x}_2)} \le \hat{p}(\hat{x}_1 - \hat{x}_2).$$

Hence

$$\hat{q}(\hat{x}) = e^{r (\log \hat{p}(\hat{x}))}$$

is an admissible function. In particular we may choose

$$r(u) = |u|^{1-c} \sin |u|^c,$$

where $0 < c < 1$, and then the corresponding function $\hat{q}(\hat{x})$ has the property, that if $\hat{p}(\hat{x})$ is unbounded, then $\hat{q}(\hat{x})$ assumes in general both arbitrarily large and arbitrarily small values.

$4°$. Let G be the real line R and F the class $F\{1\}$, i.e. the class of Fourier transforms of functions $\hat{f}(t) \in L^1(\hat{R})$.

Let A be the linear space of bounded measurable functions $\hat{a}(t)$ on $\hat{R}$.

We define $f \circ \hat{a}$ as the function

$$\int_{-\infty}^{\infty} \hat{f}(t - t_0)\, \hat{a}(t_0)\, dt_0,$$

and put

$$\|\hat{a}\| = \sup_{\|f\|=1} [\,\overline{\lim_{t \to +\infty}} \,|f \circ \hat{a}|\,],$$

where f varies in the class F.

The definition of the norm may look artificial, but it simplifies if we consider only the subclass of uniformly continuous functions $\hat{a}(t)$. Then

$$\|\hat{a}\| = \overline{\lim_{t \to +\infty}} \,|\hat{a}(t)|.$$

A null function is then every function $\hat{a}(t)$, which tends to 0, when $t \to +\infty$.

It is easy to see that the Conditions I and II are fulfilled for this class.

2. Definition and main properties of the spectrum

It is natural that the representation of F as an algebra of continuous functions on G will give us a certain characterization of the space A in terms of the space G. This can be effected by defining for every element $a \in A$ a subset $\Lambda_a \subset G$ which we call the spectrum of the element. The spectrum is defined in the following way:

DEFINITION 3.21. *For every fixed element $a \in A$ let us consider the class of all elements $g \in F$ such that $g \circ a = \theta$. Then Λ_a is defined as the complement of the set*

$$\bigcup \Lambda_g^0,$$

i.e. as the largest set, where all functions $g(x)$ vanish.

Apparently the spectrum is a closed subset of G. It follows immediately from Lemma 1.24 that in the case $1°$ in 3.1, i.e. when $A = F$, then the spectrum of an element f is exactly the set Λ_f, defined in 1.1 as the closure of Λ_f^0, and this justifies our notations.

In the case when $F = F\{1\}$ (Definition 2.11) and $A = F^*$, i.e. when A is equivalent to the space $L^\infty(\hat{G})$ of bounded measurable functions on $\hat{G}$, this definition of the spectrum coincides with the one introduced by Beurling in [5]. If, on the other

hand, A is a space of the kind described in 3.1 3° we get a definition of spectrum of the functions $\hat{a}(\hat{x})$ which is in its main features equivalent to the spectral definition for certain classes of functions on R given by Beurling in [3]. We may mention as an example that for functions $\hat{a}(\hat{x})$ in the classes $L^p(\hat{G})$, corresponding to $\hat{q}(\hat{x}) \equiv 1$, we have if $1 \leq p \leq 2$ that $\Lambda_{\hat{a}}$ is the smallest closed set, outside which the Fourier transform

$$\int_{\hat{G}} \overline{(x, \hat{x})}\, \hat{a}(\hat{x})\, d\hat{x}$$

vanishes almost everywhere.

The definition of spectrum in the classes of type 3.1 3° and 3.1 2° offers the possibility of defining a notion of spectrum for certain classes of functions (or more generally Borel measures) on $\hat{G}$. It can be proved, that these spectral sets are independent of the classes in which the functions or measures are considered as elements. This will not be proved here, since we have no use for the statement in the following, but we mention it in order to stress the quite different nature of the spectral definition in 3.1 4°. In that case we get a definition of spectrum for bounded measurable functions on $\hat{R}$, but this definition does not coincide with the Beurling definition. Thus by varying the topology in A it is possible to change the spectrum.

The following theorem together with its proof is an extension of mainly unpublished results in the Beurling spectral theory (cf. [3]).

THEOREM 3.21.   A. Λ_a *is empty if and only if* $a = \theta$.

B. $\Lambda_{ca} = \Lambda_a$ *if the constant* $c \neq 0$.

C. $\Lambda_{a_1 + a_2} \subset \Lambda_{a_1} \cup \Lambda_{a_2}$ *if* a_1 *and* a_2 *belong to* A.

D. $\Lambda_{f \circ a} \subset \Lambda_f \cap \Lambda_a$ *if* $f \in F$ *and* $a \in A$.

PROOF. A. $a = \theta$ implies that $f \circ a = \theta$ for every $f \in F$ and hence Λ_a is empty. If, on the other hand, Λ_a is empty, then the class of functions in F for which $f \circ a = \theta$ is a closed ideal, which is not contained in any regular maximal ideal. Because of Theorem 1.53 it has then to contain every function $f \in F$ and hence Assumption II in 3.1 shows that $a = \theta$.

B. The statement is obvious.

C. Let us choose an arbitrary point x_0 outside $\Lambda_{a_1} \cup \Lambda_{a_2}$. According to the definition of the spectrum there exist functions $f_1(x)$ and $f_2(x)$ in F such that $f_\nu(x_0) \neq 0$ while $f_\nu \circ a = \theta$ for $\nu = 1, 2$. Then

$$f_1 f_2 \circ (a_1 + a_2) = f_2 \circ (f_1 \circ a_1) + f_1 \circ (f_2 \circ a_2) = f_2 \circ \theta + f_1 \circ \theta = \theta.$$

But $f_1(x_0) \cdot f_2(x_0) \neq 0$, and therefore $x_0 \notin \Lambda_{a_1 + a_2}$.

D. Let x_0 be an arbitrary point in the complement of Λ_f. Since Λ_f is closed we can find a neighborhood N of x_0, included in the complement of Λ_f. And because of lemma 1.24 we can find a function $g(x) \in F$ such that $g(x_0) \neq 0$ while $\Lambda_g \subset N$. Then $g \cdot f \equiv 0$, and hence

$$g \circ (f \circ a) = g f \circ a = \theta,$$

so that $x_0 \notin \Lambda_{f \circ a}$.

Then let x_1 be an arbitrary point in the complement of Λ_a. As above we can find a function $g_1(x) \in F$ such that $g_1(x_1) \neq 0$ while Λ_{g_1} is contained in the complement of Λ_a. We can furthermore choose $g_1(x)$ so that Λ_{g_1} is compact.

By Theorem 1.51 B the ideal of all functions $g \in F$ such that $g \circ a = \theta$ contains the function g_1. Therefore

$$g_1 \circ (f \circ a) = f \circ (g_1 \circ a) = f \circ \theta = \theta,$$

and since $g_1(x_1) \neq 0$ this implies that $x_1 \notin \Lambda_{f \circ a}$.

We shall now make a comparison between our definition of the spectrum and the one used by Beurling in [3]. We shall restrict the discussion to the case when $A = F^*$, and we then need the following lemma.

LEMMA 3.21. *Let* $f^* \in F^*$. $x_0 \notin \Lambda_{f^*}$ *if and only if for some neigborhood* N *of* x_0

$$f^*(f) = 0$$

for every $f \in F$ *with* $\Lambda_f \subset N$.

PROOF. If there exists a neighborhood N such that $f^*(f) = 0$ if $f \in F$ and $\Lambda_f \subset N$, then obviously

$$f^*(f g) = 0$$

for every $g \in F$. Hence by definition $f \circ f^* = \theta$ for every such function f, and thus $x_0 \notin \Lambda_{f^*}$.

If on the other hand $x_0 \notin \Lambda_{f^*}$. then by Theorem 3.21 D $f \circ f^* = \theta$ for every $f \in F$ with Λ_f included in a certain compact neighborhood N of x_0. Hence $f^*(f g) = 0$ for every $g \in F$, and choosing $g \equiv 1$ on N, we obtain $f^*(f) = 0$.

Let us now assume that $G = R$ and that $F = F\{\hat{p}(t)\}$ (see Definition 2.11). It is easy to see that if $\sigma > 0$ and if λ is real, the function

$$\frac{2\sigma}{\sigma^2 + (x - \lambda)^2} = \int_{-\infty}^{\infty} e^{-it(x-\lambda) - \sigma|t|} dt$$

belongs to F. We shall then prove the following theorem.

THEOREM 3.22. *Let* $f^* \in F^*$. $x_0 \notin \Lambda_{f*}$ *if and only if*

$$U_{f*}(\sigma, \lambda) = f^* \left(\frac{2\sigma}{\sigma^2 + (x-\lambda)^2} \right) \to 0, \quad \text{when } \sigma \to +0,$$

uniformly for every λ *in an interval around* x_0.

PROOF. A measure $\hat{\mu}(t)$, which corresponds to a linear functional f^* is in this case absolutely continuous. Let us put $\hat{\mu}'(t) = \hat{a}(t)$. Apparently, for every $f \in F$

$$f^*(f) = \int_{-\infty}^{\infty} \hat{f}(t) \hat{a}(-t) dt$$

where the integral is absolutely convergent. Hence the Fourier transform of $U_{f*}(\sigma, \lambda)$, considered as a function of λ, is $e^{-\sigma|t|} \hat{a}(t)$.

Let us assume that $U_{f*}(\sigma, \lambda) \to 0$ uniformly in the interval

$$x_0 - \varepsilon \leq \lambda \leq x_0 + \varepsilon$$

where $\varepsilon > 0$. Let $f(x) \in F$ vanish outside the interval. Then

$$\frac{1}{2\pi} \int_{x_0-\varepsilon}^{x_0+\varepsilon} f(\lambda) U_{f*}(\sigma, \lambda) d\lambda = \int_{-\infty}^{\infty} \hat{f}(t) e^{-\sigma|t|} \hat{a}(-t) dt \to \int_{-\infty}^{\infty} \hat{f}(t) \hat{a}(-t) dt = f^*(f)$$

by the Lebesgue theorem on dominated convergence. Thus $f^*(f) = 0$ for every such function f, and by Lemma 3.21 $x_0 \notin \Lambda_{f*}$.

Before we start the proof of the opposite direction of the statement, let us observe the following. Using Lemma 1.24 we can, for every $\varepsilon > 0$, find a function $g_\varepsilon(x)$ with

$$\sup_{-\infty < x < \infty} |g_\varepsilon(x)| < 1$$

and which coincides with the function $1/(1+x^2)$ outside the interval $(-\varepsilon, \varepsilon)$. By Theorem 1.51 A

$$\sum_{n=1}^{\infty} \| g_\varepsilon(x)^n \| = B$$

is finite, and owing to the special choice of F

$$\sum_{n=1}^{\infty} \| g_\varepsilon(x-\lambda)^n \| = B$$

for every λ. If $\sigma^2 < 1$

$$\frac{1}{\sigma^2 + (x - \lambda)^2} = \sum_0^\infty \frac{(1 - \sigma^2)^n}{[1 + (x - \lambda)^2]^{n+1}}$$

and the right hand member may for $|x - \lambda| \geq \varepsilon$ be exchanged to

$$h(x - \lambda) = \sum_0^\infty (1 - \sigma^2)^n g_\varepsilon (x - \lambda)^{n+1}.$$

Hence if for a fixed x_0 $|\lambda - x_0| < \varepsilon$ and $|x - x_0| \geq 2\varepsilon$

$$\frac{1}{\sigma^2 + (x - \lambda)^2} = h(x - \lambda).$$

Let us now assume that $x_0 \notin \Lambda_{f*}$, and that ε is so small that $f^*(f) = 0$ if $\Lambda_f \subset (x_0 - 2\varepsilon, x_0 + 2\varepsilon)$ (Lemma 3.21). Then, if $|\lambda - x_0| < \varepsilon$ we obtain for $0 < \sigma < 1$

$$U_{f*}(\sigma, \lambda) = f^* \left(\frac{2\sigma}{\sigma^2 + (x - \lambda)^2} \right) = 2 \sigma f^* (h(x - \lambda))$$

and hence

$$|U_{f*}(\sigma, t)| \leq 2\sigma \|f^*\| \cdot \|h(x - \lambda)\| = 2\sigma B \cdot \|f^*\|$$

and this shows that $U_{f*}(\sigma, \lambda) \to 0$, when $\sigma \to +0$, uniformly if $|\lambda - x_0| < \varepsilon$.

REMARK. Theorem 3.22 shows that we may define the spectrum of the functionals, or, otherwise expressed, the spectrum of the functions $\hat{a}(t)$, by means of the function $U_{f*}(\sigma, \lambda)$. This function is harmonic, and therefore methods from potential theory and from the theory of analytic functions may be used in order to study the properties of this spectral definition. This has been done by Beurling [3] and Wermer [17]. It is true that they study more general functions than the functions $\hat{a}(t)$ in our theorem, but it is possible to modify theorem 3.22 to a more general theorem which shows the equivalence of the spectral definitions in still more general cases.

3. Theorems on iterated transformations

In Definition 2.31 we have introduced a certain class Φ of functions on G. Φ is apparently an algebra which contains F as a subalgebra. If $\varphi \in \Phi$ and if $f \in F$ has a compact Λ_f, then it follows at once that $\varphi(x) \cdot f(x) \in F$. We denote by Λ_φ the closure of the set where $\varphi(x) \neq 0$.

Let $\varphi \in \Phi$ and $a \in A$. Suppose that there exists an element $a_1 \in A$ with the property that for every $f \in F$ with compact Λ_f

$$f \circ a_1 = f \cdot \varphi \circ a.$$

This element is then unique, for if a_2 has the same property, then

$$f \circ (a_2 - a_1) = \theta$$

for every f with compact Λ_f. This implies that $\Lambda_{a_2 - a}$ is empty, and hence $a_2 - a_1 = \theta$.

Let us put $a_1 = \varphi \circ a$. We call this operation on the element a generalized transformation. In general it can not be defined for every element $a \in A$. We shall prove two properties of these generalized transformations.

THEOREM 3.31. A. $\varphi \circ a$ exists for every element a with compact Λ_a.

B. $\Lambda_{\varphi \circ a} \subset \Lambda_\varphi \cap \Lambda_a$, if $\varphi \circ a$ exists.

PROOF. A. There exists a function $g \in F$, coinciding with φ in an open set, including Λ_a. Then $\Lambda_{\varphi - g}$ and Λ_a have no point in common, and hence by Theorem 3.21 D

$$f \cdot (\varphi - g) \circ a = \theta$$

for every $f \in F$ with compact Λ_f. Thus we have

$$f \circ (g \circ a) = f \varphi \circ a$$

for every f of that kind, i.e. by definition

$$\varphi \circ a = g \circ a.$$

B. For every f with compact Λ_f we have by Theorem 3.21 D

$$\Lambda_{\varphi f \circ a} \subset \Lambda_a \cap \Lambda_{\varphi f} \subset \Lambda_a \cap \Lambda_f \cap \Lambda_\varphi.$$

Therefore $\Lambda_{\varphi f \circ a}$ is empty if Λ_f is included in the complement of $\Lambda_a \cap \Lambda_\varphi$. Thus under these conditions

$$f \circ (\varphi \circ a) = \theta.$$

And from this the lemma follows at once, using Lemma 1.24.

The reason why we have introduced the generalized transformations is that many functions on G of very simple nature belong to Φ but in general not to F. We may refer to 2.3 where we have shown that the functions $\alpha(x)$ and the polynomials belong to Φ. Apart from the identically vanishing function and from the case when G is compact, these functions do not belong to F, since the functions in F have to vanish at infinity (L. 19 B).

We shall now proceed to prove some theorems on what we may call iterated transformations. Starting from an element $a \in A$ and a function $\varphi \in \Phi$ we are going to show some connexions between Λ_a, $\varphi(x)$ and the norms of the elements $\varphi^n \circ a$.

THEOREM 3.32. *Suppose that C is a compact set and that $\varphi \in \Phi$. Then there exists a sequence of constants B_n, $n = 1, 2, \ldots$ with the property*

$$\lim_{n \to \infty} B_n^{1/n} = \sup_{x \in C} |\varphi(x)|,$$

so that for every n

$$\| \varphi^n \circ a \| \leq B_n \cdot \| a \|,$$

if $a \in A$ and $\Lambda_a \subset C$.

PROOF. We put

$$\sup_{x \in C} |\varphi(x)| = k$$

and choose an arbitrary positive number ε. Lemma 1.24 asserts that there exists a non-negative function $f_\varepsilon(x) \in F$ with the value 1 on an open set including C, vanishing outside the set where $|\varphi| < k + \varepsilon$ and with a value ≤ 1 elsewhere. Apparently we can choose f_ε in such a way that Λ_{f_ε} is compact.

Now

$$\sup_{x \in G} |\varphi(x) \cdot f_\varepsilon(x)| \leq k + \varepsilon,$$

and thus according to theorem 1.51 A the sequence

$$B_n^\varepsilon = \| [\varphi \cdot f_\varepsilon]^n \|$$

satisfies

$$\lim_{n \to \infty} [B_n^\varepsilon]^{1/n} \leq k + \varepsilon.$$

Arguing as in the proof of Theorem 3.31 A we get

$$\varphi^n \circ a = [\varphi \cdot f_\varepsilon]^n \circ a,$$

and thus

$$\| \varphi^n \circ a \| \leq \| [\varphi \cdot f_\varepsilon]^n \| \cdot \| a \| \leq B_n^\varepsilon \cdot \| a \|.$$

Then the theorem follows, since ε can be chosen arbitrarily small.

REMARK. It is easy to show that the theorem is still true if $\varphi \in F$ and if C instead of being compact is the complement of a compact set.

The following theorem is a rather strong converse of Theorem 3.32.

THEOREM 3.33. *If $a \in A$ and $\varphi \in \Phi$ and if $\varphi^{n_\nu} \circ a$ exists for an increasing sequence of positive integers n_ν, $\nu = 1, 2, \ldots$, then the relation*

$$\lim_{\nu \to \infty} \| \varphi^{n_\nu} \circ a \|^{1/n_\nu} = d$$

implies that

$$\sup_{x \in \Lambda_a} |\varphi(x)| \leq d.$$

PROOF. If at a point x_0, $|\varphi(x_0)| = d + \varepsilon$ for some $\varepsilon > 0$, then there exists because of Lemma 1.24 a function $g(x) \in F$, which assumes the value 1 in an open set O including x_0, while Λ_g is compact and included in the set where

$$|\varphi(x_0)| > d + \frac{\varepsilon}{2}.$$

According to Theorem 1.51 B this function is included in the ideal, consisting of functions in F of the form $g(x) \cdot \varphi(x)$, where $g(x) \in F$ and Λ_g is compact, and for that reason there exists a function $g(x) \in F$, such that in the set O

$$g(x) \cdot \varphi(x) = 1,$$

while

$$\sup_{x \in G} |g(x)| \leq \frac{1}{d + \dfrac{\varepsilon}{2}}.\tag{3.31}$$

For any $f \in F$ with Λ_f compact and $\subset O$

$$\|f \circ a\| = \|(f \cdot g^{n_\nu} \cdot \varphi^{n_\nu}) \circ a\| \leq \|f\| \cdot \|g^{n_\nu}\| \cdot \|\varphi^{n_\nu} \circ a\|.\tag{3.32}$$

By applying Theorem 1.51 A, (3.31) gives

$$\lim_{\nu \to \infty} \|g^{n_\nu}\|^{1/n_\nu} \leq \frac{1}{d + \dfrac{\varepsilon}{2}},$$

and hence the right hand member of (3.32) tends to 0 when $\nu \to \infty$. Thus $f \circ a = \theta$, which implies that $x_0 \notin \Lambda_a$.

As an application of the last two theorems we shall consider the spaces A of the type mentioned in 3.1 3°. Let us assume that $G = R$, i.e. that the elements in the space A are functions $\hat{a}(t)$ and the elements in the space $F\{\hat{p}(t)\}$ are functions

$$f(x) = \int_{-\infty}^{\infty} e^{-ixt} f(t)\, dt.$$

We are going to consider the generalized transformations obtained by the functions $(ix)^n$. These functions are polynomials and by Theorem 2.32 they belong to Φ. We shall show that these transformations are equivalent to derivations of the functions $\hat{a}(t)$. We say that the nth derivative $\hat{a}^{(n)}(t)$ of a function $\hat{a}(t)$ exists whenever there exists a function, equivalent to $\hat{a}$, absolutely continuous together with its $(n-1)$ first derivatives and with the nth derivative $\hat{a}^{(n)}(t)$.

THEOREM 3.34. $\hat{a}^{(n)}(t)$ exists and $\in A$ if and only if the element $(ix)^n \circ \hat{a}$ exists, and the elements are then identical.

PROOF. 1. *The sufficiency.* We denote by $g(x)$ an arbitrary function in F_0 such that the Fourier transform $\hat{g}(t)$ has n continuous derivatives. Then

$$g(x)(ix)^n = \int_{-\infty}^{\infty} e^{-ixt} \hat{g}^{(n)}(t)\, dt.$$

We suppose that $(ix)^n \circ \hat{a}$ exists, and call the corresponding function $\hat{a}_1(t)$. By definition

$$f \circ \hat{a}_1 = f(x) \cdot (ix)^n \circ \hat{a}$$

for every $f \in F$ with compact Λ_f, and hence

$$f \circ (g \circ \hat{a}_1) = f \circ [g(x) \cdot (ix)^n] \circ \hat{a}.$$

Since this is true for every f of the mentioned type, the spectrum of the element $g \circ \hat{a}_1 - [g(x) \cdot (ix)^n] \circ \hat{a}$ is empty, and hence

$$g \circ \hat{a}_1 = [g(x) \cdot (ix)^n] \circ \hat{a}.$$

Writing this as convolutions on $-\infty < t < \infty$ we get

$$\int_{-\infty}^{\infty} \hat{g}(t - t_0)\, \hat{a}_1(t_0)\, dt_0 = \int_{-\infty}^{\infty} \hat{g}^{(n)}(t - t_0)\, \hat{a}(t_0)\, dt_0,$$

or after n partial integrations

$$\int_{-\infty}^{\infty} \hat{g}^{(n)}(t - t_0)\, \{\int_0^{t_*} dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} \hat{a}_1(t_n)\, dt_n - \hat{a}(t_0)\}\, dt_0 = 0.$$

By varying g we see that this implies

$$\hat{a}_1(t) \equiv \hat{a}^{(n)}(t).$$

2. *The necessity.* On the other hand if we suppose that $\hat{a}^{(n)}(t)$ exists as an element in A, we can carry out the above argument in the opposite direction, and show that for every function g of the type mentioned above and for every function $h \in F$

$$(hg) \circ \hat{a}^{(n)} = (h(x) \cdot g(x) \cdot (ix)^n) \circ \hat{a}.$$

However, every function $f \in F$ with compact Λ_f belongs to the ideal which is formed by the elements $h \cdot g$ (Theorem 1.51 B). Therefore

$$f \circ \hat{a}^{(n)} = (f(x) \cdot (ix)^n) \circ \hat{a}$$

for every such function $f(x)$, i.e.

$$\hat{a}^{(n)} = (ix)^n \circ \hat{a}.$$

Let us now apply the Theorems 3.32 and 3.33 to this case. We assume that C is the closed interval $(-b, b)$. We then obtain:

THEOREM 3.35. *If* $\Lambda_{\hat{a}} \subset (-b, b)$, *then all the derivatives* $\hat{a}^{(n)}$ *exist and*

$$\overline{\lim_{n \to \infty}} \, \| \hat{a}^{(n)}(t) \|^{1/n} \leq b.$$

Conversely, if $\hat{a}(t)$ *is infinitely differentiable and*

$$\lim_{n \to \infty} \| \hat{a}^{(n)}(t) \|^{1/n} \leq b,$$

then $\Lambda_{\hat{a}} \subset (-b, b)$.

It is not difficult to realize that the functions $\hat{a}(t)$ in the theorem above have to coincide almost everywhere with the values on the real axis of analytic functions of exponential type. Interesting results may be obtained by studying the connexions between this class of functions and the generalized transformations obtained by the real functions $e^{\lambda x}$. We shall, however, not discuss this matter any further in this context.

4. Elements with one-point spectrum

The problem of characterizing the elements with a spectrum consisting only of one point, was first solved by Beurling [2] in the case when the space consists of all bounded, uniformly continuous functions on R. The spectrum was introduced by means of the closure properties of the translations of the function in a certain topology, the narrow topology. This definition gives the same spectrum as the one used in [5] (cf. [5] p. 225), which as we have mentioned earlier is related to our definition.

Godement [8] posed the problem for bounded measurable functions on an arbitrary locally compact Abelian group, using a definition of spectrum which corresponds to the Beurling definition in [5], and Kaplansky [10] and Helson [9] gave the solution. The problem has also been solved for more general classes of functions on R by Wermer [17] and for distributions on locally compact Abelian groups by Riss [15].

In our case we have to specialize in order to get results which are as simple as the ones obtained in the theories mentioned above. We are going to introduce a

notion of spaces F of polynomial growth, and for the corresponding spaces A we shall then get results which resemble the results by Wermer and Riss.

Let C be a compact set in G. We know that every character $(x, \hat{x}) \in \Phi$ by Theorem 2.31, and therefore there exist functions $f(x) \in F$, coinciding with $(x, \hat{x})$ on C. We form

$$\hat{p}_C(\hat{x}) = \inf \|f\|$$

taken over this class of functions, and make the following definition for every posititive integer q.

DEFINITION 3.41. *The space F is said to be of polynomial growth $<q$ with respect to $x \in G$ and $\hat{x} \in \hat{G}$ if for some compact neighborhood N around x the sequence*

$$\{\hat{p}_N(n\,\hat{x})\}_{-\infty}^{\infty} = \{p_n\}_{-\infty}^{\infty}$$

satisfies

$$p_n = O(|n|^q), \qquad |n| \to \infty, \tag{3.41}$$

and

$$\lim_{|n| \to \infty} \frac{p_n}{|n|^q} = 0. \tag{3.42}$$

In order to exemplify this definition, we may mention that the last of the spaces F, considered in 2.1, has the property that if $x \geq 0$, then F is of polynomial growth <2 with respect to x and every t, while if $x < 0$, it is of polynomial growth <1 with respect to x and every t.

It is easy to give examples of spaces which are not of polynomial growth with respect to all pairs x and $\hat{x}$. It is, however, always true that

$$\lim_{n \to \infty} [p_n]^{1/|n|} = 1$$

as a result of Theorem 3.32, and that

$$p_n \geq 1,$$

which is an easy consequence of (1.51).

We are going to prove the following general theorem:

THEOREM 3.41. *Suppose that F is of polynomial growth $<q$ with respect to x_0 and $\hat{x}_0$ and that $a \in A$ is such that Λ_a consists only of the point x_0. Then*

$$((x - x_0, \hat{x}_0) - 1)^q \circ a = 0.$$

We shall first prove two lemmas on Fourier series.

LEMMA 3.41. *Let* $\{p_n\}_{-\infty}^{\infty}$ *be a bounded sequence of positive numbers and let* $f(\theta)$ *be a function in* $(-\pi, \pi)$, *continuous and with absolutely convergent Fourier series.*

Then there exists for every $\varepsilon > 0$ *and every integer* m *a function*

$$g(\theta) = \sum_{-\infty}^{\infty} c_n \, e^{in\theta},$$

where

$$\sum_{-\infty}^{\infty} |c_n| < \infty$$

and

$$\sum_{-\infty}^{\infty} p_n \, |c_n| < |f(0)| \, p_m + \varepsilon,$$

such that $g(\theta)$ *and* $f(\theta)$ *coincide in an interval around* $\theta = 0$.

PROOF OF LEMMA 3.41. Let us put

$$p = \sup_{-\infty < n < \infty} p_n.$$

By the corollary in L. 37 C there exists for every function

$$f_1(\theta) = \sum_{-\infty}^{\infty} a_n \, e^{in\theta},$$

where

$$\sum_{-\infty}^{\infty} |a_n| < \infty$$

and $f_1(0) = 0$, a function

$$h(\theta) = \sum_{-\infty}^{\infty} d_n \, e^{in\theta},$$

such that

$$\sum_{-\infty}^{\infty} |d_n| < \frac{\varepsilon}{p},$$

and such that $h(\theta)$ and $f_1(\theta)$ coincide in an interval around $\theta = 0$. Let us apply this to the function

$$f_1(\theta) = f(\theta) - f(0) \, e^{im\theta}.$$

The function $h(\theta)$ thus obtained has the property that the function

$$g(\theta) = h(\theta) + f(0) \, e^{im\theta} = \sum_{-\infty}^{\infty} d_n \, e^{in\theta} + f(0) \, e^{im\theta}$$

satisfies the requirements in the lemma.

LEMMA 3.42. *Let* $\{p_n\}_{-\infty}^{\infty}$ *be a sequence of positive numbers such that for some positive integer* q *the relations* (3.41) *and* (3.42) *are fulfilled. Let the function* $f(\theta)$ *be continuous in* $(-\pi, \pi)$ *together with its first* n *derivatives and vanishing at* $\theta = 0$ *together with its first* $n-1$ *derivatives, and suppose that* $f^{(q)}(\theta)$ *has an absolutely convergent Fourier series.*

Then, for every $\varepsilon > 0$, *we can find a function*

$$g(\theta) = \sum_{-\infty}^{\infty} c_n e^{in\theta},$$

where

$$\sum_{-\infty}^{\infty} |c_n| < \infty$$

and

$$\sum_{-\infty}^{\infty} p_n |c_n| < \varepsilon,$$

such that $g(\theta)$ *and* $f(\theta)$ *coincide in an interval around* $\theta = 0$.

PROOF OF LEMMA 3.42. It is apparently enough to prove the lemma in the case when $p_0 = 1$ and

$$p_n \geq |n|^{q-1}$$

for every n.

By Lemma 3.41 we have for every $\delta > 0$ a function

$$h(\theta) = \sum_{-\infty}^{\infty} a_n e^{in\theta},$$

where the series is absolutely convergent and

$$|a_0| + \sum_{-\infty}^{-1} + \sum_{1}^{\infty} \frac{p_n}{|n|^q} |a_n| < \delta,$$

and which coincides with $f^{(q)}(\theta)$ in an interval around $\theta = 0$.

Then let us form the function

$$k(\theta) = \sum_{-\infty}^{-1} + \sum_{1}^{\infty} \frac{a_n}{(in)^q} e^{in\theta}.$$

We have in the above interval

$$f^{(q)}(\theta) - k^{(q)}(\theta) \equiv h(\theta) - k^{(q)}(\theta) \equiv a_0,$$

which shows that $f(\theta) - k(\theta)$ coincides in the interval with a polynomial $P_q(\theta)$ of degree $\leq q$. Now

$$| P_q^{(q)}(0) | = | a_0 | < \delta$$

and

$$| P_q^{(v)}(0) | = | f^{(v)}(0) - k^{(v)}(0) | = | k^{(v)}(0) | \leq \sum_{-\infty}^{-1} + \sum_{1}^{\infty} \frac{| a_n |}{| n |^{q-v}} \leq \sum_{-\infty}^{-1} + \sum_{1}^{\infty} \frac{p_n}{| n |^q} | a_n | < \delta$$

for $v = 0, 1, \ldots, q-1$.

Let ε be an arbitrary positive number. We may choose the number δ such that $\delta < \varepsilon/2$, and from the inequalities above it is trivial to conclude that if δ is sufficiently small there exists a function

$$l(\theta) = \sum_{-\infty}^{\infty} b_n e^{in\theta}$$

with

$$\sum_{-\infty}^{\infty} p_n | b_n | < \frac{\varepsilon}{2}$$

and coinciding with $P_q(\theta)$ in an interval around $\theta = 0$.

Hence the function

$$g(\theta) = k(\theta) + l(\theta)$$

coincides with $f(\theta)$ in an interval around $\theta = 0$ while its Fourier coefficients satisfy

$$\sum_{-\infty}^{\infty} p_n | c_n | \leq \sum_{-\infty}^{-1} + \sum_{1}^{\infty} \frac{p_n | a_n |}{| n^q |} + \sum_{-\infty}^{\infty} p_n | b_n | < \delta + \frac{\varepsilon}{2} < \varepsilon,$$

which was to be proved.

PROOF OF THEOREM 3.41. Let F be of polynomial growth $< q$ with respect to x_0 and $\hat{x}_0$, and let N be a neighborhood of x_0 with the property that

$$\{ \hat{p}_N(n\hat{x}_0) \}_{-\infty}^{\infty} = \{ p_n \}_{-\infty}^{\infty}$$

satisfies (3.41) and (3.42). Then there exists for every integer n a function $f_n(x) \in F$ which coincides with

$$(-x_0, \hat{x}_0)^n \cdot (x, \hat{x}_0)^n = (x - x_0, \hat{x}_0)^n$$

in an open set including N, and satisfies

$$\| f_n(x) \| < 2 p_n.$$

Now Lemma 3.42 shows that there exists for every $\varepsilon > 0$ a function

$$\sum_{-\infty}^{\infty} c_n e^{in\theta},$$

where

$$\sum_{-\infty}^{\infty} p_n \, |c_n| < \frac{\varepsilon}{2}$$

and which coincides with $(e^{i\theta}-1)^q$ in an interval around $\theta = 0$.

Hence

$$\sum_{-\infty}^{\infty} c_n \, \|f_n(x)\| < \varepsilon,$$

and

$$f(x) = \sum_{-\infty}^{\infty} c_n \, f_n(x)$$

defines an element in F which coincides with $((x-x_0, \hat{x}_0) - 1)^q$ in the intersection of N and an open set of type

$$-\delta < \ \arg \ (x - x_0, \hat{x}_0) < \delta \qquad (\text{mod } 2\pi),$$

i.e. in a neighborhood of $\hat{x}_0$. Obviously $\|f\| < \varepsilon$.

Hence if $a \in A$ and if Λ_a does not contain other points than x_0

$$a_1 = ((x-x_0, \hat{x}_0) - 1)^q \circ a = f(x) \circ a,$$

which implies that

$$\|a_1\| \le \|f\| \cdot \|a\| \le \varepsilon \cdot \|a\|.$$

And since ε was arbitrary, $a_1 = \theta$, which proves the theorem.

Theorem 3.41 has a particularly interesting interpretation if A is the space F^* of linear functionals on F in accordance to 3.1 2°. We shall first show a simple lemma, which is valid not only for this particular A.

LEMMA 3.43. *If* $a \in A$ *and if* Λ_a *is compact, then* a *can be written*

$$a = f_0 \circ a_0,$$

where $a_0 \in A$ *and* $f_0 \in F_0$.

PROOF. The space of the functions in F of the form $f_0(x) \cdot f_1(x)$ where $f_0 \in F_0$ and $f_1 \in F$ is an ideal, not contained in any regular maximal ideal. Hence by Theorem 1.51 B every function $f \in F$ with compact Λ_f can be written

$$f = f_0 \cdot f_1.$$

By Lemma 1.24 we can choose f in such a way that $f(x) \equiv 1$ in an open set, including Λ_a, and then we get

$$a = f \circ a = f_0 f_1 \circ a = f_0 \circ (f_1 \circ a),$$

which proves the lemma.

Thus if we choose $A = F^*$, every $f^* \in F^*$ with compact Λ_{f*} can be written

$$f^* = f_0 \circ f_0^*,$$

where $f_0 \in F_0$ and $f_0^* \in F^*$. Let us call the measures that in the sence of 1.3 correspond to f^* and f_0^*, $\hat{\mu}$ and $\hat{\mu}_0$. If we express in terms of the measures the equality

$$f^*(f) = f_0^*(f f_0),$$

which in particular is true for all $f \in F_0$, we get

$$\hat{\mu}(\hat{C}) = \int\limits_{\hat{C}} d\hat{x} \int\limits_{\hat{G}} f_0(\hat{x}_0 + \hat{x}) \, d\hat{\mu}_0(-\hat{x}_0)$$

for every compact $\hat{C}$. Hence

$$\hat{\mu}(\hat{C}) = \int\limits_{\hat{C}} \hat{F}(\hat{x}) \, d\hat{x},$$

where $\hat{F}(\hat{x})$ is a continuous function. Furthermore, Λ_{f*} being compact, $(x, \hat{x}_0) \circ f^*$ is welldefined by Theorem 3.31 A. If we put

$$(x, \hat{x}_0) \circ f^* = f_1^*,$$

we get

$$f_1^*(f) = f^*(f(x) \cdot (x, \hat{x}_0))$$

for every $f \in F_0$, and this shows that $(x, \hat{x}_0) \circ f^*$ corresponds in the above sense to the function $\hat{F}(\hat{x} + \hat{x}_0)$. Finally the functional

$$(1 - (x - x_0, \hat{x}_0))^q \circ f^* = (1 - (x, \hat{x}_0) \cdot (x_0, -\hat{x}_0))^q \circ f^*, \tag{3.43}$$

where x_0 and $\hat{x}_0$ are arbitrary fixed points, corresponds to the function

$$\sum_{m=0}^{q} (-1)^m \binom{q}{m} \hat{F}(\hat{x} + m\hat{x}_0)(x_0, -m\hat{x}_0). \tag{3.44}$$

This is true for every $f^* \in F^*$ with compact Λ_{f*}, then in particular for any f^* with Λ_{f*} consisting only of the point x_0. If moreover F is of polynomial growth $< q$ with respect to x_0 and $\hat{x}_0$, we see from Theorem 3.41 that the functional (3.43) is the null functional, which implies that (3.44) vanishes identically. If this then is true for every point $\hat{x}_0$, we see from definition 2.32 that the continuous function

$$\hat{F}(\hat{x})(x_0, -\hat{x})$$

is a polynomial of degree $\leq q - 1$. We formulate the result in the following theorem.

THEOREM 3.42. *If F is of polynomial growth $<q$ with respect to a fixed point x_0 and every $\hat{x}$, and if $f^* \in F^*$ has Λ_{f^*} consisting only of the point x_0, then the corresponding measure $\hat{\mu}$ satisfies for every compact set $\hat{C}$*

$$\hat{\mu}(\hat{C}) = \int_{\hat{C}} \hat{F}(\hat{x}) \, d\hat{x},$$

where $\hat{F}(\hat{x})$ is a function such that $\hat{F}(\hat{x}) \cdot (x_0, -\hat{x})$ is a polynomial of degree $\leq q-1$.

A similar interpretation is possible for the spaces of type 3.1 3°, namely that under the same conditions the element $\hat{a}(\hat{x})$ is equivalent to a polynomial of degree $\leq q-1$ multiplied by the character $(x_0, \hat{x})$. The condition that F is of polynomial growth may then be exchanged to certain restrictions in the growth of $\hat{p}(\hat{x})$.

In the case 3.1 4°, however, the corresponding theorem is not true. This would imply that if $\Lambda_{\hat{a}}$ consists of only one point, then $\hat{a}(t)$ is equivalent to a constant multiplied by a character. It is very easy to find examples which show that this is not the case.

CHAPTER IV

An equivalent definition of the spectrum

1. Elements in A with approximate identities. The subspace A_1

The following chapter deals with a different method of defining the spectrum, and the definition is essentially a generalization of the definition used by Beurling in [2]. Our definition will be expressed in terms of the generalized transformations obtained by the characters, and since they correspond to the translations in the Beurling theory, we shall use that terminology even in our case.

It is possible to prove special results for elements in a certain subspace A_1 of A, and we shall introduce this subspace by means of the following two definitions.

DEFINITION 4.11. *An element $a \in A$ is said to have an approximate identity if for every $\varepsilon > 0$ there exists a compact neighborhood $\hat{N}$ of $\hat{o}$ with the property that for every $f_0(x) \in F_0$ such that $\hat{f_0}(\hat{x})$ is non-negative, vanishes outside $\hat{N}$ and satisfies*

$$\int_{\hat{N}} \hat{f_0}(\hat{x}) \, d\hat{x} = 1,$$

we have

$$\| a - f_0 \circ a \| \leq \varepsilon.$$

We call such a neighborhood $\hat{N}$ an ε-neighborhood with respect to a.

DEFINITION 4.12. A_1 *is the subspace of* A *consisting of all elements* a *such that the translations* $(x, \hat{x}) \circ a$ *exist and such that* a *and all* $(x, \hat{x}) \circ a$ *have approximate identities.*

REMARK. It is quite obvious from the assumptions and results in chapter I that if the algebra F has the elements $f(x)$ and if x_0 denotes a fixed point in G, then the isomorphic algebra F_{x_0}, consisting of the functions $f(x + x_0)$ with the same norm as the corresponding functions $f(x)$ in F, is an algebra of type F. This new algebra may as well be used to define the spectrum of the elements in A, and we then obtain a spectrum which is the original spectrum translated by x_0.

Now it is quite easy to prove that the subspace A_1 is invariant under this translation of the spectrum. The only thing we need to check is whether the definition of elements with approximate identities changes when the functions $\hat{f}_0(\hat{x})$ are multiplied by $\overline{(x_0, \hat{x})}$. And this is not the case as is seen from the following argument.

Under the conditions in Definition 4.11 we have

$$\| f_0 \circ a \| \leq \| a \| + \varepsilon.$$

Hence, if $\hat{f}(\hat{x})$ is continuous and vanishes outside $\hat{N}$

$$\| f \circ a \| \leq 2 \, (\| a \| + \varepsilon) \int\limits_{\hat{N}} |\hat{f}(\hat{x})| \, d\hat{x}.$$

For that reason

$$\| a - \int\limits_{\hat{N}} \hat{f}_0(\hat{x}) \, \overline{(x_0, \hat{x})} \, d\hat{x} \circ a \| \leq \| a - f_0 \circ a \| + \| \int\limits_{\hat{N}} \hat{f}_0(\hat{x}) \, (1 - \overline{(x_0, \hat{x})}) \, d\hat{x} \circ a \|$$

$$\leq \varepsilon + 2 \, (\| a \| + \varepsilon) \int\limits_{\hat{N}} \hat{f}_0(\hat{x}) \, |1 - \overline{(x_0, \hat{x})}| \, d\hat{x}$$

and the right hand member is arbitrarily small if ε and $\hat{N}$ are sufficiently small.

This will be of great use for us in 4.3, where the new definition of the spectrum is introduced. By the above arguments, it has the same property as our first definition, i.e. the spectrum is translated in the above sense. Hence, in order to prove the equivalence of the definitions, it is enough to prove that the point o belongs to the spectrum in the sense of one definition if and only if it belongs to the spectrum in the sense of the other definition.

Before we proceed we shall prove a theorem on elements in A with approximate identities. The Definition 4.11 is nothing but an extension of Definition 1.51, and it is obvious that the proof of Lemma 1.52 can also be applied to the general case. Thus this lemma is true if the element $f(x)$ is exchanged to any element $a \in A$ with

an approximate identity, and if $f(x) \cdot g(x)$ is exchanged to $g \circ a$. We apply this new lemma to an element a with a discrete spectrum, and $g \circ a$ has then a finite spectrum. However, using Lemma 1.24, it is easy to show that every element with a finite spectrum may be expressed as a sum of elements with one-point spectrum. Hence we obtain the following theorem.

THEOREM 4.11. *Every $a \in A$ with an approximate identity and with discrete Λ_a can be approximated arbitrarily closely by finite sums of elements with one-point spectrum.*

The theorem should be compared with a result by Beurling [4].

The following theorem illustrates that the class A_1 can to a certain extent be characterized by continuity properties of the translations of its elements.

THEOREM 4.12. A. *If $a \in A_1$, then $\| (x, \hat{x}) \circ a - (x, \hat{x}_0) \circ a \|$ is a continuous function of $\hat{x}$ for every $\hat{x}_0$.*

B. *Suppose that A is complete, i.e. a Banach space. If all translations $(x, \hat{x}) \circ a$ of the element $a \in A$ exist, and if $\| (x, \hat{x}) \circ a - (x, \hat{x}_0) \circ a \|$ is continuous in $\hat{x}$ for every $\hat{x}_0$, then $a \in A_1$.*

PROOF. A. It is apparently enough to prove that the function corresponding to $\hat{x}_0 = \delta$ is continuous at $\hat{x} = \delta$.

Let us choose an arbitrary $\varepsilon > 0$ and let for every $\hat{x} \in \hat{G}$ $N_{\hat{x}}$ be an ε-neighborhood with respect to $(x, \hat{x}) \circ a$.

Then we choose $\hat{x}_1$ such that

$$-\hat{x}_1 \in \hat{N}_\delta, \tag{4.11}$$

and let $\hat{f}(\hat{x})$ be non-negative, continuous, satisfying

$$\int_{\hat{G}} \hat{f}(\hat{x}) \, d\hat{x} = 1,$$

and vanishing outside the set

$$\hat{N}_\delta \cap (\hat{N}_{\hat{x}_1} + (-\hat{x}_1)),$$

which is a neighborhood of $-\hat{x}_1$.

Let us put

$$f(x) = \int_{\hat{N}_\delta} \overline{(x, \hat{x})} \hat{f}(\hat{x}) \, d\hat{x}.$$

Then we get

$$f(x) \cdot \overline{(x, \hat{x}_1)} = \int_{\hat{N}_{\hat{x}_1}} \overline{(x, \hat{x})} \hat{f}(\hat{x} - \hat{x}_1) \, d\hat{x}.$$

Using the assumption that $\hat{N}_{\hat{o}}$ and $\hat{N}_{\hat{x}_1}$ are ε-neighborhoods we obtain

$$\| a - f(x) \circ a \| \leq \varepsilon$$

and

$$\| (x, \hat{x}_1) \circ a - [f(x) \cdot \overline{(x, \hat{x}_1)}] \circ [(x, \hat{x}_1) \circ a] \| = \| (x, \hat{x}_1) \circ a - f(x) \circ a \| \leq \varepsilon.$$

Hence we get

$$\| a - (x, \hat{x}_1) \circ a \| \leq 2\,\varepsilon. \tag{4.12}$$

This is true for every $\hat{x}_1$, which satisfies (4.11), and thus this part of the theorem is proved.

B. Let a be the given element and $\varepsilon > 0$ a given number. Let $\hat{N}$ be a compact neighborhood of $\hat{o}$ such that

$$\| a - (x, -\hat{x}) \circ a \| \leq \varepsilon$$

for every $\hat{x} \in \hat{N}$. We are going to show that this set is an ε-neighborhood with respect to a, and since this argument can be applied also to the elements $(x, \hat{x}) \circ a$, this will prove the theorem.

We choose a function $f \in F_0$ such that $\hat{f}(\hat{x})$ is non-negative, vanishes outside $\hat{N}$ and satisfies

$$\int\limits_{\hat{N}} \hat{f}(\hat{x}) \, d\hat{x} = 1,$$

and then we choose another arbitrary function $g \in F_0$.

For every positive integer n we are going to divide $\hat{N}$ into n distinct measurable subsets $\hat{E}_m^{(n)}$, such that every $\hat{E}_{m_0}^{(n)}$ is included in some of the sets $\hat{E}_m^{(n-1)}$, $n = 2, 3, \ldots$. Then we choose a point $\hat{x}_m^{(n)}$ in every $\hat{E}_m^{(n)}$.

We shall discuss the linear combinations of characters

$$f_n(x) = \sum_{m=1}^{n} \int\limits_{\hat{E}_m^{(n)}} \hat{f}(\hat{x}) \, d\hat{x} \cdot (x, -\hat{x}_m^{(n)}).$$

For every n we have $f_n \in \Phi$, and it is easy to see that

$$\| a - f_n \circ a \| \leq \varepsilon. \tag{4.13}$$

It is possible to show that we can choose the sets $\hat{E}_m^{(n)}$ in such a way that when $n \to \infty$,

$$\sup_{m} \int_{\hat{E}_m^{(n)}} \hat{f}(\hat{x}) \, d\hat{x} \to 0,$$

$$\sup_{m} \left[\sup_{\hat{x}_1, \hat{x}_2 \in \hat{E}_m^{(n)}} \| (x, -\hat{x}_0) \circ a - (x, -\hat{x}_1) \circ a \| \right] \to 0,$$

and

$$\sup_{m} \left[\sup_{\hat{x} \in \hat{G}, \hat{x}_1, \hat{x}_2 \in \hat{E}_m^{(n)}} | \hat{g}(\hat{x} - \hat{x}_1) - \hat{g}(\hat{x} - \hat{x}_2)| \right] \to 0.$$

Then $\{f_n \circ a\}_1^\infty$, is a Cauchy sequence, and since the space is complete, the sequence has a limit element a_1. By (4.13)

$$\| a - a_1 \| \leq \varepsilon. \tag{4.14}$$

It can also be proved that the Fourier transforms of the functions $f_n(x) \cdot g(x)$ converge uniformly to the Fourier transform of $f(x) \cdot g(x)$, and since all functions vanish outside some fixed compact set, Lemma 1.21 shows that we have at the same time convergence in norm. Thus

$$g \circ a_1 = \lim_{n \to \infty} g \circ (f_n \circ a) = \lim_{n \to \infty} (g \cdot f_n) \circ a = g \circ (f \circ a).$$

Now a_1 does not depend on the choice of g and hence by Assumption II in 1.1

$$a_1 = f \circ a.$$

Thus by (4.14)

$$\| a - f \circ a \| \leq \varepsilon,$$

and this proves that $\hat{N}$ is an ε-neighborhood with respect to a.

REMARK. The part B of the theorem is not true for all spaces A which are not complete. This is shown in the remark which follows after Theorem 4.31. In this remark it is shown that Theorem 4.31 fails to be true if the condition $a \in A_1$ is exchanged to the condition that the translations exist and $\| (x, \hat{x}) \circ a - (x, \hat{x}_0) \circ a \|$ is continuous.

2. Some lemmas

In order to prove Theorem 4.31 we need various lemmas, and we collect them in this section.

LEMMA 4.21. *Suppose that* $a \in A_1$. *Then there exists for every compact* $\hat{C}$ *and every* $\varepsilon > 0$ *a set* $\hat{N}$, *which is an* ε-*neighborhood with respect to all elements* $(x, \hat{x}_0) \circ a$, *when* $\hat{x}_0 \in \hat{C}$.

PROOF. Let $\hat{N}_{\delta}$ be an ε-neighborhood with respect to a. We can find a smaller symmetric neighborhood $\hat{N}'_{\delta}$ of δ such that

$$\hat{N}'_{\delta} + \hat{N}'_{\delta} \subset \hat{N}_{\delta}.$$

Then if the non-negative, continuous function $f(\hat{x})$ vanishes outside $\hat{N}'_{\delta}$ and satisfies

$$\int_{\hat{G}} f(\hat{x}) \, d\hat{x} = 1,$$

we have for every $\hat{x}_1$, such that $\hat{x}_1 \in \hat{N}'_{\delta}$

$$f(x) \cdot (x, \hat{x}_1) = \int_{\hat{N}_{\delta}} f(\hat{x} + \hat{x}_1) \, \overline{(x, \hat{x})} \, d\hat{x},$$

which implies that

$$\| a - f(x) \cdot (x, \hat{x}_1) \circ a \| < \varepsilon.$$

But (4.11) holds and hence (4.12) is true. Thus

$$\| (x, \hat{x}_1) \circ a - f(x) \cdot (x, \hat{x}_1) \circ a \| < 3\varepsilon,$$

and this relation shows that $\hat{N}_{\delta}$ is a 3ε-neighborhood with respect to every element $(x, \hat{x}_1) \circ a$ such that $\hat{x}_1 \in \hat{N}'_{\delta}$. By applying the same argument to the element $(x, \hat{x}_0) \circ a$ we see that there exists, for every $\hat{x}_0 \in \hat{G}$, a set $\hat{N}'_{\hat{x}_0}$ which is a 3ε-neighborhood with respect to all elements $(x, \hat{x}_1) \circ a$, such that

$$\hat{x}_1 - \hat{x}_0 \in \hat{N}'_{\hat{x}_0}.$$

The interiors of the sets $\hat{x}_0 + \hat{N}'_{\hat{x}_0}$ form an open covering of $\hat{C}$, and hence we can select a finite covering

$$\{\hat{x}_{\nu} + \hat{N}'_{\hat{x}_{\nu}}\}_1^n.$$

Thus every point $\hat{x} \in \hat{C}$ can be represented in the form

$$\hat{x}_{\nu} + \hat{x}'_{\nu}$$

for some ν, where $\hat{x}'_{\nu}$ denotes a point in $\hat{N}'_{\hat{x}_{\nu}}$. For that reason

$$\hat{x} - \hat{x}_{\nu} \in \hat{N}'_{\hat{x}_{\nu}},$$

which shows that $\hat{N}'_{\hat{x}_{\nu}}$ is a 3ε-neighborhood with respect to this particular element

$(x, \hat{x}) \circ a$. And in order to get a $3\,\varepsilon$-neighborhood with respect to all $(x, \hat{x}) \circ a$, where $\hat{x} \in \hat{C}$, we have only to form the set

$$\bigcap_1^n \hat{N}'_{\hat{x}_\nu}.$$

Since ε is arbitrary this proves the lemma.

LEMMA 4.22. *Suppose that* $a \in A_1$ *and* $f_0(x) \in F_0$, *let* $\varepsilon > 0$ *be an arbitrary number and* $\hat{C}$ *an arbitrary compact subset of* $\hat{G}$.

Then there exists a linear combination of translations of a

$$a' = \sum_1^n c_\nu\,(x, -\hat{x}_\nu) \circ a,$$

where the points $\hat{x}_\nu$ *belong to the closure of the set where the Fourier transform* $\hat{f}_0(\hat{x}) \neq 0$, *and with the properties*

$$\| (x, \hat{x}) \circ (f_0 \circ a - a') \| \leq \varepsilon \tag{4.21}$$

for every $\hat{x} \in \hat{C}$, $\sum_1^n c_\nu = f_0(o)$, *and*

$$\sum_1^n |c_\nu| \leq 4 \int_{\hat{G}} |\hat{f}_0(\hat{x})|\, d\hat{x}. \tag{4.22}$$

PROOF. It is obviously enough to prove the lemma in the case when $\hat{f}_0(\hat{x})$ is non-negative and

$$f_0(o) = \int_{\hat{G}} \hat{f}_0(\hat{x})\, d\hat{x} = 1$$

if we instead of (4.22) show the stronger relation

$$\sum_1^n |c_\nu| = 1. \tag{4.23}$$

We denote by $\hat{C}_0$ the compact closure of the set where $\hat{f}_0(\hat{x}) \neq 0$. Since the set $\hat{C} + (-\hat{C}_0)$ is compact we can, according to Lemma 4.21, find a set $\hat{N}$, which is an ε-neighborhood with respect to all elements $(x, \hat{x} - \hat{x}_0) \circ a$, where $\hat{x} \in \hat{C}$ and $\hat{x}_0 \in \hat{C}_0$.

The interiors of the sets $\hat{N} + \hat{x}_0$, where $\hat{x}_0 \in \hat{C}_0$, cover $\hat{C}_0$. Let us select a finite covering

$$\{\hat{N} + \hat{x}_\nu\}_1^n.$$

Obviously $\hat{f}_0(\hat{x})$ can be decomposed into a sum of non-negative, continuous functions $\hat{f}_\nu(\hat{x})$, such that $\hat{f}_\nu(\hat{x})$ vanishes outside $\hat{N} + \hat{x}_\nu$, $\nu = 1, 2, \ldots, n$. We put

$$\int_{\hat{N}+\hat{x}_\nu} f_\nu(\hat{x})\,d\hat{x} = c_\nu.$$

Apparently

$$\sum_1^n |c_\nu| = \sum_1^n c_\nu = 1,$$

which proves (4.23). Now

$$f_\nu(x)\cdot(x,\hat{x}_\nu) = \int_{\hat{N}} \overline{(x,\hat{x})}\, f_\nu(\hat{x}+\hat{x}_\nu)\,d\hat{x},$$

and since $\hat{N}$ is an ε-neighborhood with respect to the elements $(x,\hat{x}-\hat{x}_\nu)\circ a$ for every $\hat{x}\in\hat{C}$ we get

$$\|(x,\hat{x})\circ(f_0\circ a - a')\| \le \sum_1^n \|(x,\hat{x})\,f_\nu\circ a - c_\nu(x,\hat{x}-\hat{x}_\nu)\circ a\| =$$

$$= \sum_1^n \|f_\nu(x)\cdot(x,\hat{x}_\nu)\circ((x,\hat{x}-\hat{x}_\nu)\circ a) - c_\nu(x,\hat{x}-\hat{x}_\nu)\circ a\| \le \varepsilon \sum_1^n c_\nu = \varepsilon,$$

and this proves (4.21).

LEMMA 4.23. *Suppose that* $a\in A_1$ *and that* $f\in F$ *has compact* Λ_f. *Let* $\varepsilon>0$ *be an arbitrary number and* $\hat{C}_0$ *an arbitrary compact subset of* $\hat{G}$. *Then there exists a linear combination* a' *of translations*

$$a' = \sum_1^n c_\nu(x,\hat{x}_\nu)\circ a,$$

such that

$$\|(x,\hat{x}_0)\circ(f\circ a - a')\| < \varepsilon,$$

if $\hat{x}_0\in\hat{C}_0$.

PROOF. We can use Lemma 4.22, and hence the only thing we have to prove is that we can find an element $g(x)\in F_0$ such that

$$\|(x,\hat{x}_0)\cdot(f-g)\| < \varepsilon,$$

if $\hat{x}_0\in\hat{C}_0$.

Lemma 3.43 shows that $f(x)$ can be written

$$f(x) = f_0(x)\cdot f_1(x),$$

where $f_0\in F_0$ and $f_1\in F$. Let us put

$$\sup_{\hat{x}_0\in\hat{C}_0} \|f_0\cdot(x,\hat{x}_0)\| = B,$$

which is finite because of Lemma 1.21. Then we use Assumption II in 1.1, which shows that we can find a function $f_2(x)\in F_0$ such that

$$\left\| f_1(x) - f_2(x) \right\| < \frac{\varepsilon}{B} \cdot$$

Hence, if $\hat{x}_0 \in \hat{C}_0$

$$\left\| (x, \hat{x}_0) \cdot f - (x, \hat{x}_0) f_0 f_2 \right\| \leq \left\| f_0 \cdot (x, \hat{x}_0) \right\| \cdot \left\| f_1 - f_2 \right\| < \varepsilon.$$

We may therefore choose as $g(x)$ the function $f_0(x) f_2(x)$.

LEMMA 4.24. A. *Every $a \in A$ with compact Λ_a belongs to A_1.*

B. *Given any compact set $C \subset G$, there exist two continuous functions $\hat{p}_C(\hat{x})$ and $\hat{q}_C(\hat{x})$, where $\hat{q}_C(\hat{o}) = 0$, such that*

$$\left\| [(x, \hat{x} + \hat{x}_0) - (x, \hat{x})] \circ a \right\| \leq \|a\| \, \hat{p}_C(\hat{x}) \cdot \hat{q}_C(\hat{x}_0),$$

if $a \in A$ and $\Lambda_a \subset C$.

PROOF. Let us first prove the second part. We choose a function $f(x) \in F$ such that $f(x) = 1$ on an open set, including C, while Λ_f is compact. This choice is possible by Lemma 1.24.

From Lemma 3.43 and from the fact that all functions in F_0 have approximate identities (Definition 1.51) it is easily seen that the same is true for $f(x)$ and all functions $(x, \hat{x}) \cdot f(x)$. Therefore $f(x) \in F_1$ if the space F in the sense of 3.1 1° is considered as a space A. And by Theorem 4.12 A the functions

$$\left\| f(x) \cdot ((x, \hat{x}) - 1) \right\| = \hat{q}_C(\hat{x})$$

and

$$\left\| f(x) \cdot (x, \hat{x}) \right\| = \hat{p}_C(\hat{x})$$

are continuous and $\hat{q}_C(\hat{o}) = 0$. If $\Lambda_a \subset C$, we have

$$a = f \circ a = f^2 \circ a, \tag{4.24}$$

and thus

$$\left\| [(x, \hat{x})(x, \hat{x}_0) - (x, \hat{x})] \circ a \right\| = \left\| f \cdot [(x, \hat{x}_0) - 1] \cdot f \cdot (x, \hat{x}) \circ a \right\| \leq \|a\| \, \hat{p}_C(\hat{x}) \cdot \hat{q}_C(\hat{x}_0).$$

As for the first part, (4.24) shows that a has a representation of the form $f \circ a$, where $f \in F_1$. This implies at once that $a \in A_1$.

3. The spectral sets Λ'_a and Λ''_a

We are going to introduce two new definitions of spectrum of the elements in A. Only the first definition may be applied to every element in A, but the second one is often more convenient, since it is expressed only in terms of the translations.

DEFINITION 4.31. *For every* $a \in A$ *we define* Λ'_a *as the set of points* $x_0 \in G$, *with the property that for every* $\varepsilon > 0$ *and every compact set* $\hat{C} \subset \hat{G}$ *there exists an element* $f \in F$, *such that* $f \circ a$ *belongs to* A_1 *and satisfies*

$$\| [(x - x_0, \hat{x}) - 1] \cdot f \circ a \| < \varepsilon \| f \circ a \|$$

for every $\hat{x} \in \hat{C}$.

DEFINITION 4.32. *For every* $a \in A_1$ *we define* Λ''_a *as the set of points* $x_0 \in G$ *with the property that for every* $\varepsilon > 0$ *and every compact set* $\hat{C} \subset \hat{G}$ *there exists a linear combination of translations*

$$a' = \sum_1^n c_\nu (x, \hat{x}_\nu) \circ a,$$

such that

$$\| [(x - x_0, \hat{x}) - 1] \circ a' \| < \varepsilon \| a' \|$$

for every $\hat{x} \in \hat{C}$.

The following fundamental theorem connects these definitions and Definition 3.21.

THEOREM 4.31. $\Lambda_a = \Lambda'_a$ *for every* $a \in A$.

$\Lambda_a = \Lambda''_a$ *for every* $a \in A_1$.

REMARK. Of course the spectrum Λ''_a can be defined for a larger class of elements than A_1, e.g. for the class of elements $a \in A$ such that for every $\hat{x}_0$

$$\| (x, \hat{x}) \circ a - (x, \hat{x}_0) \circ a \|$$

exists and is a continuous function of $\hat{x}$. The following example, however, shows that we may then have $\Lambda_a \neq \Lambda''_a$. This shows at the same time that A_1 is a proper subclass of the mentioned class (cf. Theorem 4.12.).

Let $\hat{b}_1(t)$ be a bounded function $\in L^1(-\infty, \infty)$ such that if

$$\sum_1^n |c_\nu| \neq 0$$

none of the linear combinations of translations

$$\hat{d}_1(t) = \sum_1^n c_\nu \hat{b}_1(t + t_\nu)$$

is equivalent to a continuous function.

Let $\hat{b}_2(t)$ be another function with the same property. To each linear combination $\hat{d}_1(t)$ there corresponds in a unique way the function

$$\hat{d}_2(t) = \sum_1^n c_\nu \hat{b}_2(t + t_\nu).$$

We shall show that we can choose $\hat{b}_1(t)$ and $\hat{b}_2(t)$ in such a way that $\Lambda_{\hat{b}_1} \neq \Lambda_{\hat{b}_2}$, where these sets are defined as the closure of the sets where the corresponding Fourier transforms do not vanish.

Let us choose

$$\hat{b}_1(t) = e^{-|t|} \operatorname{sign} t.$$

It is easy to prove that $\Lambda_{\hat{b}_1} = (-\infty, \infty)$. Then we choose $\hat{c}(t)$ in $L^1(-\infty, \infty)$ and $L^\infty(-\infty, \infty)$ such that its Fourier transform is $\equiv 1$ in $(-1, 1)$. The function

$$\hat{b}_1 \star \hat{c} = \int_{-\infty}^{\infty} \hat{b}_1(t_0) \hat{c}(t - t_0) \, dt_0$$

is then bounded and continuous and its Fourier transform coincides with the Fourier transform of $\hat{b}_1$ in $(-1, 1)$. Therefore

$$\hat{b}_2 = \hat{b}_1 - \hat{b}_1 \star \hat{c}$$

fulfills $\Lambda_{\hat{b}_2} \neq \Lambda_{\hat{b}_1}$, and since each of the functions $\hat{b}_1$ and $\hat{b}_2$ has only one essential point of discontinuity, no non-trivial linear combination of translations is equivalent to a continuous function.

Let us now choose as F the space $F\{1\}$ of Fourier transforms of functions $\in L^1(-\infty, \infty)$, and as A the space A^1 of functions of the form

$$\hat{a}_1(t) = \hat{a}(t) + \hat{d}_1(t), \tag{4.31}$$

where $\hat{a}(t)$ is equivalent to a bounded continuous function $\in L^1$ and where $\hat{d}_1(t)$ is a linear combination of the kind defined above. We define the transformation $f \circ \hat{a}_1$ by means of the convolution

$$\int_{-\infty}^{\infty} \hat{a}_1(t - t_0) f(t_0) \, dt_0,$$

and since all functions $f \circ \hat{a}_1$ are equivalent to bounded continuous functions, they belong to the space A^1. We introduce the norm

$$\| \hat{a}_1 \| = \| \hat{a}(t) + \hat{d}_1(t) \| = \int_{-\infty}^{\infty} \left(|\hat{a}(t)| + |\hat{d}_1(t)| + |\hat{d}_2(t)| \right) dt,$$

which is uniquely determined by the unique decomposition (4.31). The spectrum of

an element $\hat{a}_1 \in A^1$ is apparently the set $\Lambda_{\hat{a}_1}$, defined above. The space A^1 fulfills every requirement which we demand of a space A.

For every element $\hat{a}_1(t)$ the translations $\hat{a}_1(t+t_\nu)$ exist and the function

$$\left\| \hat{a}_1(t+s) - \hat{a}_1(t+s_0) \right\|$$

is continuous.

If we exchange the rôle of the indices 1 and 2 in the above example, we get a space A^2 with analogous properties. Now the norms of the elements $\hat{d}_1(t) \in A^1$ are exactly the same as the norms of the corresponding functions $\hat{d}_2(t) \in A^2$. Hence by Definition 4.32 $\Lambda_{\hat{b}_1}'' = \Lambda_{\hat{b}_2}''$. But we have assumed $\Lambda_{\hat{b}_1} \neq \Lambda_{\hat{b}_2}$. Thus $\Lambda_{\hat{b}_1} \neq \Lambda_{\hat{b}_1}''$ or $\Lambda_{\hat{b}_2} \neq \Lambda_{\hat{b}_2}''$, i.e. the second part of Theorem 4.31 is false for at least one of the spaces A^ν.

PROOF OF THEOREM 4.31. The remark after Definition 4.12 makes it possible to restrict the discussion to the case when $x_0 = o$. The proof will be divided into three parts. In the first two parts, we are going to show that $o \in \Lambda_a$ implies $o \in \Lambda_a'$ and $o \in \Lambda_a''$, respectively, i.e. we have to prove that the Definitions 4.31 and 4.32 are fulfilled with $x_0 = o$, and for any prescribed ε and $\hat{C}$.

1. $\Lambda_a \subset \Lambda_{a'}$. We assume that $o \in \Lambda_a$, and have to construct an element $f \circ a \in A_1$, satisfying the relation in Definition 4.31.

Let C be an arbitrary open neighborhood of o. By Lemma 4.24 there exists an open neighborhood $\hat{N}$ of $\hat{o}$ such that

$$\left\| [(x, \hat{x} + \hat{x}_0) - (x, \hat{x})] \circ a_1 \right\| \le \frac{\varepsilon}{2} \left\| a_1 \right\|, \tag{4.32}$$

if $\hat{x} \in \hat{C}$, $\hat{x}_0 \in \hat{N}$, and if $a_1 \in A$ fulfills $\Lambda_{a_1} \subset C$.

The sets $\hat{x} + \hat{N}$, where $\hat{x} \in \hat{C}$, form an open covering of $\hat{C}$. Let us select a finite covering

$$\{\hat{x}_\nu + \hat{N}\}_1^m.$$

Now let C' be a compact neighborhood of o, included in C and in the open neighborhood of o where for every ν

$$\left| (x, \hat{x}_\nu) - 1 \right| < \frac{\varepsilon}{2}. \tag{4.33}$$

We choose a function $f_0 \in F$ with $\Lambda_{f_0} \subset C'$ and $f_0(o) \neq 0$. Since $o \in \Lambda_a$, we have $f_0 \circ a \neq \theta$. Let us then consider the class of elements

$$a_0 = \left[\prod_{\nu=1}^{m} ((x, \hat{x}_\nu) - 1)^{k_\nu} \right] \circ f_0 \circ a,$$

where the exponents k_ν denote arbitrary non-negative integers. Two cases may occur:

1°. Infinitely many of the elements a_0 are $\neq \theta$.

Let us apply Theorem 3.32 to the elements a_0. Since $\Lambda_{f_0} \subset C'$ and (4.33) holds in C' we obtain for the constants $B_n^{(\nu)}$ which correspond to $\varphi(x) = (x, \hat{x}_\nu) - 1$ in the theorem, the relation

$$\lim_{n \to \infty} [B_n^{(\nu)}]^{1/n} < \frac{\varepsilon}{2}.$$

This is true for all ν, and we have

$$\| a_0 \| \leq \| f_0 \circ a \| \prod_{\nu=1}^{m} B_{k_\nu}^{(\nu)}.$$

Hence the numbers

$$\sup_{\substack{m \\ \sum_{1} k_\nu = n}} \| a_0 \| = \delta_n,$$

which are all different from 0, have to satisfy

$$\overline{\lim_{n \to \infty}} \ \delta_n^{1/n} < \frac{\varepsilon}{2}.$$

This implies that at least one of them, let us say δ_{n_0}, satisfies

$$\delta_{n_0 + 1} < \frac{\varepsilon}{2} \cdot \delta_{n_0}.$$

The corresponding element, i.e. the element a_0 with $\sum_{1}^{m} k_\nu = n_0$ and for which

$$\| a_0 \| = \delta_{n_0}$$

then has to satisfy for every ν

$$\| [(x, \hat{x}_\nu) - 1] \circ a_0 \| \leq \delta_{n_0 + 1}.$$

Thus for every ν, this element, which we denote a_1, satisfies

$$\| [(x, \hat{x}_\nu) - 1] \circ a_1 \| < \frac{\varepsilon}{2} \| a_1 \|. \tag{4.34}$$

$2°$. Only a finite number of the elements a_0 are $\neq \theta$. Since in any case $f \circ a \neq \theta$, we can find an element a_1 among the elements a_0, such that $a_1 \neq \theta$ and for every ν

$$[(x, \hat{x}_\nu) - 1] \circ a_1 = 0,$$

then a fortiori (4.34).

The element a_1 has $\Lambda_{a_1} \subset C'$, and hence $a_1 \in A_1$ by Lemma 4.24, and it also satisfies (4.32). Combining (4.32) and (4.34) we get

$$\| [(x, \hat{x}_\nu + \hat{x}_0) - 1] \circ a_1 \| < \varepsilon \| a_1 \|$$

for every ν and if $\hat{x}_0 \in \hat{N}$. But every point in $\hat{C}$ may be written in the form $\hat{x}_\nu + \hat{x}_0$. And this concludes the proof.

2. $\Lambda_a \subset \Lambda_a''$. In the preceding part of the proof we saw that if $o \in \Lambda_a$, then $o \in \Lambda_a'$ and the function $f(x)$ can be chosen with compact Λ_f. Then, comparing Definition 4.31 with Definition 4.32, we see that the only thing we need to prove is that we can approximate $f \circ a$ arbitrarily closely by translations a' of a, in the sense that for every $\delta > 0$ we can find a' such that

$$\| f \circ a - a' \| < \delta$$

and

$$\| (x, \hat{x}) \circ [f \circ a - a'] \| < \delta,$$

if $\hat{x} \in \hat{C}$. And this approximation is possible, since we assume in this place that $a \in A_1$, and thus we can apply Lemma 4.23 with $\hat{C}_0$ consisting of the point $\hat{o}$ and the set $\hat{C}$.

3. $\Lambda_a' \subset \Lambda_a$ and $\Lambda_a'' \subset \Lambda_a$. Starting from a given element a, the two proofs will be exactly the same apart from the difference that the variable element a' should be interpreted as an element of the form $f \circ a$ in the first case, as a linear combination of translations in the second case. We suppose that $o \notin \Lambda_a$ and we are going to show that the relation $o \in \Lambda_a'$ or $o \in \Lambda_a''$, respectively, has to involve a contradiction.

It follows from the definition of Λ_a that there exists a function $f_1(x) \in F$ with $f_1(o) = 1$ and such that $f_1 \circ a = \theta$, i.e. such that for every a'

$$f_1 \circ a' = \theta.$$

We approximate f_1 by a function $f_0 \in F_0$ such that for a prescribed $\varepsilon > 0$

$$\| f_1 - f_0 \| < \varepsilon.$$

By (1.51)

$$|f_0(o)| \geq f_1(o) - |f_1(o) - f_0(o)| > 1 - \varepsilon, \tag{4.35}$$

and moreover we have

$$\|f_0 \circ a'\| \leq \|f_1 \circ a'\| + \|(f_1 - f_0) \circ a'\| \leq \varepsilon \|a'\|. \tag{4.36}$$

The Fourier transform $\hat{f}_0(\hat{x})$ vanishes outside a certain compact set $\hat{C}$. Let us put

$$\int_{\hat{C}} |\hat{f}_0(\hat{x})| \, d\hat{x} = B < \infty.$$

Since the Definitions 4.31 or 4.32, respectively, are assumed to be satisfied for $x_0 = o$, there exists an element $a' \in A_1$ such that if $\hat{x} \in \hat{C}$

$$\| [(x, -\hat{x}) - 1] \circ a' \| < \frac{\varepsilon}{4 B} \|a'\|. \tag{4.37}$$

Then, by Lemma 4.22 there exists a linear combination of translations

$$\sum_1^n c_\nu (x, -\hat{x}_\nu) \circ a',$$

where $\hat{x}_\nu \in \hat{C}$, satisfying $\sum_1^n c_\nu = f_0(o)$,

$$\left\| f_0 \circ a' - \sum_1^n c_\nu (x, -\hat{x}_\nu) \circ a' \right\| < \varepsilon \|a'\| \tag{4.38}$$

and

$$\sum_1^n |c_\nu| \leq 4 \int_{\hat{G}} |\hat{f}_0(\hat{x})| \, d\hat{x} = 4 B. \tag{4.39}$$

The formulae (4.38) and (4.37) give

$$\left\| f_0 \circ a' - \sum_1^n c_\nu a' \right\| < \varepsilon \|a'\| + \frac{\varepsilon \cdot \sum_1^n |c_\nu|}{4 B} \cdot \|a'\|,$$

i.e. together with (4.39)

$$\| f_0 \circ a' - f_0(o) \cdot a' \| < 2 \varepsilon \|a'\|.$$

And combining this inequality and (4.36) we obtain

$$|f_0(o)| \cdot \|a'\| < 3 \varepsilon \|a'\|,$$

which is contradictory to (4.35) since we can assume $\varepsilon < \frac{1}{4}$.

4. The narrow topology

The relation between our spectral definitions in this chapter and our original definition is essentially the same as the relation between the Beurling definition in [2] and the definition which he used in [5]. This will appear from the following discussion.

Let $\hat{p}(\hat{x})$ be a function of the kind used in Definition 2.11. $\hat{p}(-\hat{x})$ is a function with the same properties. We form the space $F\{\hat{p}(-\hat{x})\}$ and let A be the corresponding space of linear functionals in the sense of 3.1 2°. Then A is isomorphic to the following space of functions $\hat{a}(\hat{x})$ on $\hat{G}$.

$\hat{a}(\hat{x})$ is defined, finite and measurable a. e. on $\hat{G}$ and

$$\|\hat{a}(\hat{x})\| = \sup. \text{ ess. } \frac{\hat{a}(\hat{x})}{\hat{p}(\hat{x})}.$$

The linear transformation $f \circ \hat{a}$ of the element $\hat{a}(\hat{x})$ is the ordinary convolution

$$f \circ \hat{a} = \int_{\hat{G}} \hat{a}(\hat{x} - \hat{x}_0) \hat{f}(\hat{x}_0) d\hat{x}_0,$$

and the translation by $(x, \hat{x}_0)$ is the ordinary translation

$$(x, \hat{x}_0) \circ \hat{a} = \hat{a}(\hat{x} + \hat{x}_0),$$

which exists for every element.

Given $f \in F\{\hat{p}(-\hat{x})\}$, we can for every $\varepsilon > 0$ and every compact set $\hat{C} \subset \hat{G}$ find a function $f_0 \in F_0$, such that

$$\|(x, \hat{x}) \cdot (f - f_0)\| < \varepsilon$$

for every $\hat{x} \in \hat{C}$. The functions $f_0(x)$ have, however, the property that

$$\|[(x, \hat{x}) - (x, \hat{x}_0)] \cdot f_0\|$$

is a continuous function of $\hat{x}$ for every $\hat{x}_0$. Thus the same is true for the function $f(x)$. Hence we may conclude that for every element of the form $f \circ \hat{a}$ the function

$$\|[(x, \hat{x}) - (x, \hat{x}_0)] \circ (f \circ \hat{a})\|$$

is continuous, and thus by Theorem 4.12 B, using the fact that A is complete, we conclude that $f \circ \hat{a} \in A_1$.

By definition every element in A_1 can be approximated arbitrarily closely by functions $f_0 \circ \hat{a}$, where $f_0 \in F_0$. Thus it is a limit of continuous functions and hence

it is a continuous function itself. Therefore, A_1 consists by Theorem 4.12 of all functions, equivalent to continuous functions $\hat{a}(\hat{x})$, for which

$$\sup_{\hat{x} \in \hat{G}} \frac{|\hat{a}(\hat{x} + \hat{x}_0) - \hat{a}(\hat{x})|}{\hat{p}(\hat{x})}$$

is continuous at $\hat{x}_0 = \hat{o}$.

In the case when $\hat{p}(\hat{x})$ is bounded, the space is the space of all measurable and bounded functions, and A_1 is then the subspace, consisting of all functions which are equivalent to uniformly continuous functions.

Let us now make the extra assumption that $\hat{p}(\hat{x})$ is continuous and satisfies $\hat{p}(\hat{o}) = 1$. The characters $(x, \hat{x})$ belong to the space and since we assume $\hat{p}(\hat{x}) \geq 1$, we have

$$\| (x, \hat{x}) \| = 1. \tag{4.41}$$

Now we introduce a new topology in the space, the *narrow topology*, by choosing as a neighborhood base of an element $\hat{a}_0$ the subspaces of elements $\hat{a}$ for which

$$\sup_{\hat{x} \in \hat{C}} |\hat{a}(\hat{x}) - \hat{a}_0(\hat{x})| + | \, \|\hat{a}\| - \|\hat{a}_0\| \, | < \varepsilon,$$

where ε is an arbitrary positive number and $\hat{C}$ an arbitrary compact subset of $\hat{G}$. We can then prove the following two theorems.

THEOREM 4.41. $\Lambda_{\hat{a}}$ consists of the points x_0 for which $(x_0, \hat{x})$ is included in the narrow closure of the class of functions of the form $f \circ \hat{a}$.

THEOREM 4.42. If $\hat{a} \in A_1$, then $\Lambda_{\hat{a}}$ consists of the points x_0 for which $(x_0, \hat{x})$ is included in the narrow closure of the class of linear combinations of translations

$$\sum_{1}^{n} c_\nu \, \hat{a}(\hat{x} + \hat{x}_\nu).$$

REMARK. The narrow topology was introduced by Beurling [2] in the space of bounded, uniformly continuous functions on R, and he proved that the narrow closure of the class of linear combinations of translations of a given function always contains a character. His arguments may as well be used to prove that the definitions of spectrum in [2] and [5] coincide (cf. [5] p. 225), which is the same as the truth of Theorem 4.42 for that particular case.

The sets of points x_0 for which the corresponding characters $(x_0, \hat{x})$ are included in the narrow closure of the spaces mentioned in the theorems, are sets which are very closely related to the spectral sets $\Lambda_{\hat{a}}'$ and $\Lambda_{\hat{a}}''$. The fact that these sets by Theorem 4.31 coincide with $\Lambda_{\hat{a}}$ makes it possible for us to prove the above theorems.

PROOF OF THEOREM 4.41 AND THEOREM 4.42. The two proofs will only differ in the respect that $\hat{a}_1$ in the first case denotes a function of the form $f \circ \hat{a}$, in the second case a linear combination of the translations of $\hat{a}$, which then is supposed to belong A_1. In any case $\hat{a}_1 \in A_1$, and we may suppose that the function is continuous.

If $x_0 \notin \Lambda_{\hat{a}}$ there exists a function f with $f(x_0) \neq 0$ such that $f \circ \hat{a} = \theta$. Hence $f \circ \hat{a}_1 = \theta$, i.e.

$$\int_{-\infty}^{\infty} \hat{a}_1 (-\hat{x}) f(\hat{x}) \, d\hat{x} = 0$$

for every $\hat{a}_1$. If $(x_0, \hat{x})$ were included in the narrow closure of the functions $\hat{a}_1$ we would have

$$f(x_0) = \int_{-\infty}^{\infty} (x_0, -\hat{x}) f(\hat{x}) \, d\hat{x} = 0,$$

i.e. a contradiction.

Thus we have only to show that if $x_0 \in \Lambda_{\hat{a}}$, then $(x_0, \hat{x})$ is included in the narrow closure of the elements $\hat{a}_1$. We apply Theorem 4.31 which shows that there exists for every $\varepsilon > 0$ and every compact set $\hat{C}$ a function $\hat{a}_1$ such that if $\hat{x}_0 \in \hat{C}$

$$\| \hat{a}_1 (\hat{x} + \hat{x}_0) - (x_0, \hat{x}_0) \cdot \hat{a}_1 (\hat{x}) \| < \varepsilon \| \hat{a}_1 \|. \tag{4.42}$$

We shall modify this statement. There exists of course a point $\hat{x}_1$ such that

$$| \hat{a}_1 (\hat{x}_1) | > \hat{p} (\hat{x}_1) \frac{1}{1 + \varepsilon} \| \hat{a}_1 \|. \tag{4.43}$$

The function

$$1 \leq \hat{a}_2 (\hat{x}) = \frac{\hat{a}_1 (\hat{x} + \hat{x}_1)}{\hat{a}_1 (\hat{x}_1)}$$

is also of the type $\hat{a}_1$. It satisfies

$$\hat{a}_2 (\hat{o}) = 1, \tag{4.44}$$

and using (2.12) and (4.43) it is easily seen that

$$1 \leq \| \hat{a}_2 (\hat{x}) \| \leq 1 + \varepsilon. \tag{4.45}$$

Furthermore we get from (2.12), (4.42) and (4.43)

$$\| \hat{a}_2 (\hat{x} + \hat{x}_0) - (x_0, \hat{x}_0) \hat{a}_2 (\hat{x}) \| < \varepsilon (1 + \varepsilon)$$

for $\hat{x}_0 \in \hat{C}$, i.e. by (4.44)

$$| \hat{a}_2 (\hat{x}_0) - (x_0, \hat{x}_0) | < \varepsilon (1 + \varepsilon). \tag{4.46}$$

If we finally combine (4.41), (4.45) and (4.46) we obtain

$$\sup_{\hat{x}_0 \in \hat{C}} \left| \hat{a}_2(\hat{x}_0) - (x_0, \hat{x}_0) \right| + \left| \|\hat{a}_2\| - \|(x_0, \hat{x})\| \right| < \varepsilon \, (2 + \varepsilon).$$

Since ε was chosen arbitrarily and $\hat{p}(\hat{x})$ is bounded in the compact set $\hat{C}$, this shows that we may find a function $\hat{a}_1$ in any neighborhood of $(x_0, \hat{x})$, i.e. $(x_0, \hat{x})$ belongs to the narrow closure of the space of the functions $\hat{a}_1$.

We shall illustrate the results in this chapter by means of another example.

Let A be a Banach space and let us have a strongly continuous homomorphism (i.e. a representation) of $\hat{G}$ onto a group of linear bounded transformations of A into itself. (Cf. L. 32 A.) We let $T_{\hat{x}}$ denote the transformation which corresponds to $\hat{x}$ and assume that for every $\hat{x}$

$$\sum_{1}^{\infty} \frac{\log \|T_{n\hat{x}}\|}{n^2} < \infty.$$

Since apparently

$$\|T_{\hat{x}_1 + \hat{x}_2}\| = \|T_{\hat{x}_1} T_{\hat{x}_2}\| \leq \|T_{\hat{x}_1}\| \cdot \|T_{\hat{x}_2}\|$$

the space $F\{\|T_{\hat{x}}\|\}$ (Definition 2.11) is of type F. Using the same technique as in the proof of Theorem 4.12 B (cf. also L. 32 B) it is possible to prove that the group of transformations can be extended to an algebra of the following kind:

There is an algebra of transformations of A of the kind described in 3.1, with $F = F\{\|T_{\hat{x}}\|\}$ and such that for every $\hat{x}$ and a $T_{\hat{x}}$ a coincides with the translation $(x, \hat{x}) \circ a$.

We have moreover $A_1 = A$, and this is by the way important in the proof of the above statement, since it shows that every a can be approximated arbitrarily closely by elements $f \circ a$, $f \in F\{\|T_{\hat{x}}\|\}$, and this has as a result that Assumption II is fulfilled.

Therefore the whole spectral theory may be applied and owing to Theorem 4.31 we have now a possibility to express the definition of the spectrum in terms of the "translations" $T_{\hat{x}}$. In order to get a briefer formulation let us say that a subclass A_0 of A contains approximate eigenelements corresponding to x_0 if for every $\varepsilon > 0$ and every compact set $\hat{C} \subset \hat{G}$ it contains an element a' such that

$$\|(T_{\hat{x}} - (x_0, \hat{x}) \cdot I) a'\| < \varepsilon \|a'\|$$

for every $\hat{x} \in \hat{C}$, where I denotes the identical transformation. Then Definition 4.32 yields:

$x_0 \in \Lambda_a$ *if and only if the linear manifold spanned by the translations of a contains*
approximate eigenelements corresponding to x_0.

And the fundamental Theorem 3.21 A shows that *if* $a \neq \theta$, *then we can always*
find a point x_0 *with this property.*

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