THE RATIONAL HOMOTOPY THEORY OF CERTAIN PATH SPACES WITH APPLICATIONS TO GEODESICS

BY

KARSTEN GROVE(1), STEPHEN HALPERIN and MICHELINE VIGUÉ–POIRRIER

University of Copenhagen, Denmark *University of Toronto, Canada*
Université de Paris-Sud, Orsay, France

It is well known that the topology of various path spaces on a complete riemannian manifold M is closely related to the existence of various kinds of geodesics on M. Classical Morse theory and the theory of closed geodesics are beautiful examples of this sort.

The motivation for the present paper is the study of geodesics satisfying a very general boundary condition of which the above examples and the example of isometry-invariant geodesics are particular cases. In particular, we generalize a result of Sullivan–Vigué [16].

Let $N \subset M \times M$ be a submanifold of the riemannian product $M \times M$. An *N-geodesic* on M is a geodesic c: $[0, 1] \to M$ which satisfies the boundary condition

$$(N) \qquad (c(0), c(1)) \in N \quad \text{and} \quad (\dot{c}(0), -\dot{c}(1)) \in TN^{\perp},$$

where $TN^{\perp}$ is the normal bundle of N in $M \times M$. If $N = V_1 \times V_2$, where $V_i \subset M$, $i = 1, 2$ are submanifolds of M then an N-geodesic is simply a $V_1 - V_2$ *connecting geodesic* (orthogonal to each V_i). If N is the graph of an isometry, A, of M then an N-geodesic is a geodesic which extends uniquely to an *A-invariant geodesic* c: $\mathbf{R} \to M$; i.e.

$$c(t+1) = A(c(t)), \quad t \in \mathbf{R}.$$

When A has finite order ($A^k = \mathrm{id}$) then c is in fact closed ($c(t+k) = c(t)$, $t \in \mathbf{R}$).

The study of N-geodesics on M proceeds via critical point theory for the energy integral on a suitable Hilbert manifold of curves with endpoints in N. This Hilbert manifold is homotopy equivalent to the space M_N^I of continuous curves f: $[0, 1] \to M$ satisfying $(f(0), f(1)) \in N$, with the compact open topology (cf. Grove [4], [6]).

(1) Part of this work was done while the first named author visited the IHES at Bures-sur-Yvette during May 1976.

In this paper we apply Sullivan's theory of minimal models to study the rational homotopy type of M_N^I, and hence to obtain information about N-geodesics.

Sullivan's theory (cf. [14], [15] and [8]) associates with each path connected space S a certain differential algebra $(\Lambda X_S, d_S)$ over $\mathbf{Q}$ which describes its rational homotopy type. $(\Lambda X_S, d_S)$ is called the minimal model of S and $H(\Lambda X_S)$ is the rational (singular) cohomology of S. As an algebra ΛX_S is the free graded commutative algebra over the graded space X_S. If S is nilpotent and its rational cohomology has finite type then X_S is the (rational) dual of the graded space $\pi_*(S)\otimes\mathbf{Q}$. (See section 1 for more details.)

Our main result is an explicit construction of the *minimal* model for the space $M_{G(g)}^I$, where $G(g)$ is the graph of a so called 1-rigid map and M is any 1-connected topological space whose rational cohomology has finite type (Theorem 3.17). This gives in particular a new proof of Sullivan's theorem for the space of closed curves M^{S^1} [14]. Surprisingly enough the minimal model for $M_{G(g)}^I$ has exactly the same form as the minimal model for the space of closed curves on a space M'. This space, however, is not obviously related to M and it can be much bigger than M. For this reason the results of Sullivan–Vigué [16] do not carry over to our more general case in a completely satisfactory manner although some of the methods from [16] are important for us.

The minimal model for $M_{G(g)}^I$ contains all information about the rational homotopy theory of $M_{G(g)}^I$, in particular about the cohomology. An immediate consequence of the model is the following (Theorem 4.1).

THEOREM. *If the rational cohomology of $M_{G(g)}^I$ is non trivial and g is rigid at 1 then $M_{G(g)}^I$ has non-zero cohomology in an infinite arithmetic sequence of dimensions.*

The main application of the model is however (cf. Theorem 4.5).

THEOREM. *If M is 1-connected, $H^*(M)$ finite dimensional and $g\colon M\to M$ rigid at 1, then $M_{G(g)}^I$ has a bounded sequence of Betti numbers if and only if*

$$\dim \pi_*^{\text{even}}(M)^{g_\#}\otimes\mathbf{Q} \leqslant \dim \pi_*^{\text{odd}}(M)^{g_\#}\otimes\mathbf{Q}\leqslant 1$$

where $\pi_*(M)^{g_\#}$ is the homotopy of M fixed by the induced map $g_\#$.

When $g=\text{id}$ this specializes to the main theorem of Sullivan–Vigué [16]. If we combine this result with the main theorem of Grove–Tanaka [7] we obtain (generalizing the application by Sullivan–Vigué of Gromoll–Meyer [3]).

THEOREM. *Let M be a compact 1-connected riemannian manifold and let g be a finite order isometry of M. If g has at most finitely many invariant geodesics then*

$$\dim \pi_*^{\text{even}}(M)^{g\#} \otimes \mathbf{Q} \leqslant \dim \pi_*^{\text{odd}}(M)^{g\#} \otimes \mathbf{Q} \leqslant 1.$$

As a consequence we obtain (cf. Cor. 4.10).

COROLLARY. *Let M be a 1-connected, compact riemannian manifold whose cohomology is spherically generated (e.g. M formal) and let g be a finite order isometry of M. If the induced map g^* on cohomology fixes at least two generators then g has infinitely many invariant geodesics.*

The paper is divided into 4 sections. In section 1 we recall briefly the main results in the theory of (minimal) models and explain how they generalize when an action of a finite group is involved. Besides being of interest in itself we use these results in section 3. In section 2 we translate the fibration

$$\Omega M \longrightarrow M_N^I \xrightarrow{\pi_N} N,$$

to models. Here M is any 1-connected space, and N a path connected subspace of $M \times M$. Furthermore, $\pi_N(f) = (f(0), f(1))$, ΩM is the ordinary loop space of M and M_N^I is defined as above. We exhibit a (not necessarily minimal) model for M_N^I (Theorem 2.8). In particular (Cor. 2.11) we obtain explicitly the space of generators for the minimal model of M_N^I. We also apply results from the theory of models to our model of M_N^I (Theorem 2.15 and Cor. 2.16).

In particular, suppose N is a closed submanifold of $M \times M$ and M is a complete riemannian manifold. Let p_i: $N \to M$, $i=0, 1$ be the left and right projections and assume that either $p_0(N)$ or $p_1(N)$ is compact and that $V = N \cap \triangle(M)$ is a closed submanifold of N. Then according to Grove [5] if there are no N-geodesics on M the inclusion $V \to M_N^I$ is a homotopy equivalence. Thus Theorem 2.15 yields:

THEOREM. *Suppose in addition to the above conditions N is 1-connected and let*

$$(p_i)_\# : \pi_*(N) \otimes \mathbf{Q} \to \pi_*(M) \otimes \mathbf{Q}, \quad i = 0, 1$$

be the linear maps induced by p_i, $i=0$, 1. If for some complete metric on M there are no N-geodesics, then coker $((p_0)_\# - (p_1)_\#)$ *is spanned by elements of even degree and*

$$\dim \operatorname{coker}((p_0)_\# - (p_1)_\#) \leqslant \dim V.$$

As a second application we get from Example 2.21 the

THEOREM. *Let Σ, Σ_1 and Σ_2 be spheres (possibly exotic) and suppose Σ_1 and Σ_2 are imbedded in Σ so that $\Sigma_1 \cap \Sigma_2$ is a (collection of) closed submanifold(s) of Σ. Then for any riemannian metric on Σ there are $\Sigma_1 - \Sigma_2$ connecting geodesics.*

Finally in section 3 and section 4 we specialize to the case $N=G(g)$ and get the results on isometry invariant geodesics.

1. Equivariant minimal models

Throughout the paper all vector spaces are defined over the rationals $\mathbf{Q}$ unless otherwise said. We begin by recalling some facts from Sullivan's theory of minimal models (see Sullivan [14], [15] and Halperin [8]).

A *commutative graded differential algebra (c.g.d.a.)* is a pair (A, d_A) where $A=\oplus_{p=0}^{\infty} A^p$ is a non-negatively graded algebra (over $\mathbf{Q}$) with identity, such that $ab=(-1)^{pq}ba$ for $a\in A^p$, $b\in A^q$ and d_A: $A\to A$ is a derivation of degree 1 with $d_A^2=0$.

$\wedge X$ will denote the *free graded commutative algebra* over a graded space X i.e.

$$\wedge X = \text{exterior } (X^{\text{odd}}) \otimes \text{symmetric } (X^{\text{even}}).$$

$\wedge^+ X$ is the ideal of polynomials with no constant term i.e. $\wedge^+ X=\sum_{j\geq 1}\wedge^j X$.

A *KS-complex* is a c.g.d.a. $(\wedge X, d)$ which satisfies:

(ks_1) There is a homogeneous basis $\{x_\alpha\}_{\alpha\in J}$ for X indexed by a well ordered set J such that dx_α is a polynomial in the x_β with $\beta<\alpha$.

If $(\wedge X, d)$ in addition to (ks_1) satisfies

(ks_2) $dX\subset \wedge^+ X\cdot\wedge^+ X$

then $(\wedge X, d)$ is said to be *minimal*.

In the rest of the paper $(\wedge X, d)$ is always assumed to be a connected KS-complex. Let $Q(\wedge X)=\wedge^+X/\wedge^+X\cdot\wedge^+X$ be the indecomposables of $\wedge X$ and ζ: $\wedge^+X\to Q(\wedge X)$ the projection. Define a differential $Q(d)$ on $Q(\wedge X)$ by $Q(d)\zeta=\zeta d$. Then $(\wedge X, d)$ is minimal if and only if $Q(d)=0$. If ψ: $(\wedge X, d)\to(\wedge X', d')$ is a c.g.d.a. map, we define $Q(\psi)$: $Q(\wedge X)\to Q(\wedge X')$ by $Q(\psi)\zeta=\zeta'\psi$. Note that ζ restricts to an isomorphism $X\to Q(\wedge X)$ which allows us to identify these spaces.

We shall now recall the notation of homotopy due to Sullivan [15, § 3] (see also [8; chap. 5]). Let $(\wedge X, d)$ be a KS-complex with X strictly positively graded (i.e. $\wedge X$ is connected.)

$(\wedge X^I, D)$ is the c.g.d.a. obtained by tensoring $(\wedge X, d)$ with the "contractible" c.g.d.a. $(\wedge\bar{X}\otimes\wedge D\bar{X}, D)$, where

(c_1) $\bar{X}$ is the suspension of X i.e. $\bar{X}^p=X^{p+1}$

and

(c_2) D: $\bar{X}\to D\bar{X}$ is an isomorphism.

The degree -1 isomorphism $X = \bar{X}$ is written $x \mapsto \bar{x}$.

A derivation i of degree -1 and a derivation θ of degree zero in $\wedge X^I$ are defined by

$$ix = \bar{x},\ i\bar{x} = iD\bar{x} = 0 \quad \text{for all } x \in X$$

and

$$\theta = Di + iD.$$

Let λ_0: $\wedge X \to \wedge X^I$ denote the standard inclusion and set $\lambda_1 = e^\theta \circ \lambda_0$. Here e^θ is well defined because for any $\Phi \in \wedge X^I$ there is an integer n such that $\theta^n \Phi = 0$ [8]. Note that if Π: $\wedge X^I \to \wedge X$ is the projection defined by

$$\Pi\, x = x,\ \Pi\, \bar{x} = \Pi\, D\bar{x} = 0 \quad \text{for all } x \in X$$

then λ_0 and Π induce inverse cohomology isomorphisms because $(\wedge \bar{X} \otimes \wedge D\bar{X}, D)$ is acyclic.

Definition 1.1. Two homomorphisms γ_0, γ_1: $(\wedge X, d) \to (A, d_A)$ of c.g.d.a.'s are called *homotopic* (written $\gamma_0 \sim \gamma_1$) if there is a c.g.d.a. map Γ: $(\wedge X^I, D) \to (A, d_A)$ such that $\Gamma \circ \lambda_i = \gamma_i$ $i = 0, 1$.

If the c.g.d.a. (A, d_A) is homology connected i.e. $H^0(A) = \mathbf{Q}$ a *model* for (A, D_A) is a KS-complex $(\wedge X, d)$ together with a homomorphism of c.g.d.a.'s

$$\varphi\colon (\wedge X, d) \to (A, d_A)$$

which satisfies

(m) φ induces an isomorphism φ^* on cohomology.

If the KS-complex $(\wedge X, d)$ is minimal we speak of the *minimal model* φ: $(\wedge X, d) \to (A, d_A)$.

We can now state the following important result (see [15, § 5] and [8, chap. 6]).

THEOREM 1.2. *Let (A, d_A) be a c.g.d.a. with $H^0(A) = \mathbf{Q}$. Then there is a minimal model*

$$\varphi\colon (\wedge X, d) \to (A, d_A).$$

If φ': $(\wedge X', d') \to (A, d_A)$ is another minimal model, then there is an isomorphism of c.g.d.a.'s α: $(\wedge X, d) \to (\wedge X', d')$ such that $\varphi \sim \varphi' \circ \alpha$. Finally, α is unique up to homotopy.

A number of choices are involved in the construction of φ: $(\wedge X, d) \to (A, d_A)$. If a finite group G acts on (A, d_A), the flexibility in the construction enables us to obtain an induced action of G on $(\wedge X, d)$ and to make φ equivariant. In fact, one can carry out Sullivan's proof of Theorem 1.2 equivariantly using that any G-invariant subspace of a vector space has a G-invariant complement. Hence

THEOREM 1.3. *Let (A, d_A) be a c.g.d.a. with $H^0(A)=\mathbf{Q}$ and let G be a finite group acting on A by c.g.d.a. maps. Then there is a minimal model*

$$\varphi: (\Lambda X, d) \to (A, d_A)$$

such that G acts on $(\Lambda X, d)$ and φ is equivariant. If φ': $(\Lambda X', d') \to (A, d_A)$ is another G-equivariant minimal model, then there is a G-isomorphism α: $(\Lambda X, d) \to (\Lambda X', d')$ such that $\varphi \sim \varphi' \circ \alpha$ and α is unique up to homotopy.

There is also an equivariant theorem for maps which again can be proved by making the corresponding non-equivariant proof (cf. e.g. [8, Theorem 5.19]) equivariant.

THEOREM 1.4. *Let (A, d_A) and $(A', d_{A'})$ be a c.g.d.a.'s with $H^0(A)=H^0(A')=\mathbf{Q}$ and with actions of a finite group G. Furthermore, let*

$$\varphi: (\Lambda X, d) \to (A, d_A) \quad \textit{and } \varphi': (\Lambda X', d') \to (A', d_A)$$

be equivariant minimal models as in Theorem 1.3. Then for any equivariant c.g.d.a. map Ω: $(A, d_A) \to (A', d_{A'})$ there is an equivariant c.g.d.a. map ω: $(\Lambda X, d) \to (\Lambda X', d')$ such that $\varphi' \circ \omega \sim \Omega \circ \varphi$.

Now suppose M is a topological space. Denote by $(A(M), d)$ the c.g.d.a. of *rational differential (PL) forms* on M.

A *rational p-form* $\Phi \in A^p(M)$ on M is a function which assigns to each singular q-simplex σ: $\Delta^q \to M$ a C^∞ differential p-form Φ_σ on the standard q-simplex Δ^q such that

(d_1) Φ_σ is in the c.g.d.a. generated (over $\mathbf{Q}$) by the barycentric coordinate functions.

and

(d_2) The map $\sigma \mapsto \Phi_\sigma$ is compatible with face and degeneracy operations.

Multiplication and differentiation are defined in $A(M)$ by $(\Phi \wedge \Psi)_\sigma = \Phi_\sigma \wedge \Psi_\sigma$ and $(d\Phi)_\sigma = d(\Phi_\sigma)$.

If g: $M \to M'$ is a continuous map, there is an induced map $A(g)$: $A(M') \to A(M)$ of c.g.d.a.'s given by $(A(g)\Phi)_\sigma = \Phi_{g\circ\sigma}$. One has the following important result.

THEOREM 1.5. (Sullivan–Whitney–Thom). *Integration yields a natural isomorphism of graded algebras*

$$\int^* : H^*(A(M)) \to H^*(M)$$

where $H^*(M)$ denotes singular cohomology with coefficients in $\mathbf{Q}$.

When M is path connected a (minimal) model for $(A(M), d)$ is called simply a (*minimal*) *model for* M. The minimal model for M will be denoted by

$$\varphi_M\colon (\Lambda X_M, d_M) \to (A(M), d).$$

The space of indecomposable elements:

$$\pi_\psi^*(M) = Q(\Lambda X_M) \cong X_M$$

is called the *pseudo dual homotopy of* M. If $H^*(M)$ has finite type (i.e. finite dimensional in each degree) and M is nilpotent then there is a natural isomorphism

$$\pi_\psi^*(M) \to \operatorname{Hom}_{\mathbf{Z}}(\pi_*(M), \mathbf{Q})$$

(cf. [15] and [8]).

2. A model for the space M_N^I

Let M be a simply connected space whose rational cohomology has finite type, and fix a path connected subspace $N \subset M \times M$.

Let M^I be the space of continuous maps $f\colon [0, 1] \to M$ with the compact open topology. In this section we shall determine a model for the subspace $M_N^I \subset M^I$ given by

$$M_N^I = \{f \in M^I \mid (f(0), f(1)) \in N\}.$$

We have the commutative diagram

$$\begin{array}{ccccc} M \times M & \xleftarrow{\ \pi\ } & M^I & \xleftarrow{\ j\ } & \Omega M \\ {\scriptstyle i_N}\uparrow & & \uparrow & & \| \ \text{incl.} \\ N & \xleftarrow[\ \pi_N\ ]{} & M_N^I & \xleftarrow[\ j_N\ ]{} & \Omega M \end{array} \tag{2.1}$$

where $\pi(f) = (f(0), f(1))$, π_N is the restriction of π and $\Omega M = \pi_N^{-1}(m_0, m_1) = \{f \in M^I \mid f(0) = m_0$ and $f(1) = m_1\}$ for a chosen base point $(m_0, m_1) \in N$.

Both rows in (2.1) are Hurewicz fibrations which we denote respectively by $\mathfrak{F}$ and $\mathfrak{F}_N$. Note that $\mathfrak{F}_N = i_N^*(\mathfrak{F})$.

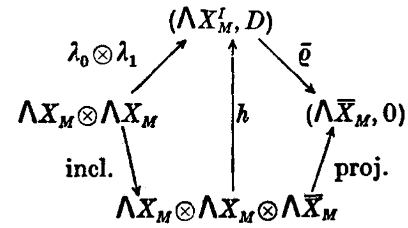
We also have a homotopy equivalence $\eta\colon M \to M^I$ given by: $\eta(m)$ is the constant map $I \to m$. Clearly

$$\pi \circ \eta = \Delta\colon M \to M \times M \tag{2.2}$$

is the diagonal of M.

Now we begin the translation of (2.1) to models. Since M is 1-connected and $H^*(M)$ has finite type it follows that ΛX_M is 1-connected; i.e. $X_M^0 = X_M^1 = 0$, and has finite type (see [8; Cor. 3.11 and Cor. 3.15]).

Consider the diagram



(2.3)

where λ_0 and λ_1 are defined on page 281 and

$$\lambda_0 \otimes \lambda_1(\Phi \otimes \Psi) = \lambda_0 \Phi \cdot \lambda_1 \Psi$$

$$\bar{\varrho} x = \bar{\varrho} D\bar{x} = 0 \quad \text{and } \bar{\varrho}\bar{x} = \bar{x}$$

and

$$h(\Phi \otimes \Psi \otimes \bar{x}) = \lambda_0 \Phi \cdot \lambda_1 \Psi \cdot (1 \otimes \bar{x} \otimes 1).$$

By [8, Lemma 5.28] h is an isomorphism of graded algebras (because ΛX_M is minimal.) Since ΛX_M is 1-connected, $d_M X_M^p \subset \Lambda\,(\oplus_{j=2}^{p-1} X_M^j)$. Hence (5.5) and (5.6) of [8] yield

$$\lambda_1 x - \lambda_0 x = D\bar{x} + \Omega(x), \quad x \in X_M^p \tag{2.4}$$

where
$$\Omega(x) = \sum_{n=1}^{\infty} \frac{(iD)^n}{n!} x \in \{\Lambda(X_M^{<p}) \otimes \Lambda(\bar{X}_M^{<p-1}) \otimes \Lambda(D\bar{X}_M^{<p})\} \cap \ker \Pi$$

and Π is defined on p. 281.

An easy calculation shows that $\bar{\varrho} D = \bar{\varrho} i D = 0$, and it follows from (2.4) that (2.3) is commutative. Thus (cf. [8, chapers 1 and 5]) (2.3) exhibits $\Lambda X_M \otimes \Lambda X_M \to \Lambda X_M^I \to \Lambda \bar{X}_M$ as a minimal KS-extension.

We shall now define a commutative diagram of c.g.d.a.'s

$$\begin{array}{ccccc} \Lambda X_M \otimes \Lambda X_M & \xrightarrow{\lambda_0 \otimes \lambda_1} & \Lambda X_M^I & \xrightarrow{\bar{\varrho}} & \Lambda X_M \\ \downarrow{\scriptstyle \varphi_{M\times M}} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \varphi_\Omega} \\ A(M \times M) & \xrightarrow[A(\pi)]{} & A(M^I) & \xrightarrow[A(j)]{} & A(\Omega M) \end{array} \tag{2.5}$$

in which all the vertical maps induce isomorphisms on cohomology.

First let P_L, P_R: $M \times M \to M$ be the left and right projections, and define

$$\varphi_{M\times M}(\Phi \otimes \Psi) = A(P_L) \circ \varphi_M \Phi \cdot A(P_R) \circ \varphi_M \Psi.$$

Since $H^*(M)$ has finite type, the Künneth theorem holds and $\varphi_{M\times M}$ induces an isomorphism $\varphi^*_{M\times M}$ on cohomology. In particular $\varphi_{M\times M}$: $\wedge X_M \otimes \wedge X_M \to A(M\times M)$ is a minimal model for $M\times M$.

Next, note that the projection Π: $\wedge X_M^I \to \wedge X_M$ satisfies $\Pi\circ\lambda_0 = \Pi\circ\lambda_1 = \mathrm{id}$. Hence $\Pi\circ(\lambda_0\otimes\lambda_1) = \mu$ is the multiplication homomorphism

$$\mu\colon \wedge X_M \otimes \wedge X_M \to \wedge X_M.$$

From this and (2.2) we see that the following diagram is commutative.

$$\begin{array}{ccc} A(M^I) & \xrightarrow{A(\eta)} & A(M) \\ {\scriptstyle A(\pi)\circ\varphi_{M\times M}}\uparrow & & \uparrow{\scriptstyle \varphi_M\circ\pi} \\ \wedge X_M\otimes\wedge X_M & \xrightarrow[\lambda_0\otimes\lambda_1]{} & \wedge X_M^I \end{array}$$

Since η is a homotopy equivalence it induces an isomorphism $A(\eta)^*$ on cohomology. Therefore by Sullivan [15, § 3] or Theorem 5.19 of [8] there is a homomorphism of c.g.d.a.'s

$$\psi\colon (\wedge X_M^I, D) \to (A(M^I), d)$$

such that $\psi\circ(\lambda_0\otimes\lambda_1) = A(\pi)\circ\varphi_{M\times M}$ and $A(\eta)\circ\psi \sim \varphi_M\circ\Pi$. Because $A(\eta)^*$, φ_M^* and Π^* are all cohomology isomorphisms, so is ψ^*.

Finally (2.3) shows that ker $\bar{\varrho}$ is generated by $\lambda_0\otimes\lambda_1(X_M\oplus X_M)$ and hence ψ (ker $\bar{\varrho}$) is generated by $A(\pi)\circ\varphi_{M\times M}(X_M\oplus X_M)$. Since $A(j)\circ A(\pi)=0$ on elements of degree >0 it follows that ψ factors to give a c.g.d.a. homomorphism

$$\varphi_\Omega\colon (\wedge \bar{X}_M, 0) \to (A(\Omega M), d)$$

such that (2.5) commutes.

Now since $\mathfrak{J}$ is a Hurewicz fibration, M is 1-connected and $H^*(M)$ has finite type, a theorem of Grivel [2] or [8, Th. 20.3] asserts that because $\varphi^*_{M\times M}$ and ψ^* are isomorphisms so is φ^*_Ω. In particular φ_Ω: $(\wedge\bar{X}_M, 0) \to A(\Omega M)$ is a minimal model for the loop space of M.

We now turn our attention to the fibration $\mathfrak{J}_N$. Recall that φ_N: $(\wedge X_N, d_N) \to (A(N), d)$ is a minimal model for the path connected space N.

Use (2.1) to obtain from (2.5) the commutative diagram

$$\begin{array}{ccccc} \wedge X_M\otimes\wedge X_M & \xrightarrow{\lambda_0\otimes\lambda_1} & \wedge X_M^I & \xrightarrow{\bar{\varrho}} & \wedge\bar{X}_M \\ {\scriptstyle A(i_N)\circ\varphi_{M\times M}}\downarrow & & \downarrow{\scriptstyle A(\mathrm{incl.})\circ\psi} & & \downarrow{\scriptstyle \varphi_\Omega} \\ A(N) & \xrightarrow[A(\pi_N)]{} & A(M_N^I) & \xrightarrow[A(j_N)]{} & A(\Omega M) \end{array} \tag{2.6}$$

Using again Sullivan [15, § 5] or [8, Th. 5.19] we obtain unique (up to homotopy) c.g.d.a. maps

$$\varphi_0\colon (\wedge X_M, d_M) \to (\wedge X_N, d_N)$$

and

$$\varphi_1\colon (\wedge X_M, d_M) \to (\wedge X_N, d_N)$$

such that $\varphi_N \circ \varphi_0 \sim A(P_L \circ i_N) \circ \varphi_M$ and $\varphi_N \circ \varphi_1 \sim A(P_R \circ i_N) \circ \varphi_M$. Define a homomorphism of c.g.d.a.'s

$$\mu_N\colon \wedge X_M \otimes \wedge X_M \to \wedge X_N$$

by

$$\mu_N(\Phi \otimes \Psi) = \varphi_0(\Phi) \cdot \varphi_1(\Psi).$$

Then

$$\varphi_N \circ \mu_N \sim A(i_N) \circ \varphi_{M \times M}.$$

Therefore we can apply (9.15.4) of [8] to obtain from (2.6) another commutative diagram of c.g.d.a.'s

$$\begin{array}{ccccc}
\wedge X_M \otimes \wedge X_M & \xrightarrow{\lambda_0 \otimes \lambda_1} & \wedge X_M^I & \xrightarrow{\bar{\varrho}} & \wedge \bar{X}_M \\
\downarrow{\scriptstyle \varphi_N \circ \mu_N} & & \downarrow{\scriptstyle \psi_N} & & \downarrow{\scriptstyle \varphi'_\Omega} \\
A(N) & \xrightarrow[A(\pi_N)]{} & A(M_N^I) & \xrightarrow[A(j_N)]{} & A(\Omega M)
\end{array}$$

in which $\varphi'_\Omega \sim \varphi_\Omega$. In particular φ'^*_Ω is an isomorphism.

Finally, write $\wedge X_M^I = \wedge X_M \otimes \wedge X_M \otimes \wedge \bar{X}_M$ using the isomorphism h of (2.3). The ideal $\ker \mu_N \otimes \wedge \bar{X}_M$ is D-stable, and so a c.g.d.a.

$$(\wedge X_N \otimes \wedge \bar{X}_M, D_N)$$

is defined by

$$D_N(\Phi \otimes 1) = d_N \Phi \otimes 1 \quad \text{and} \quad D_N \circ (\mu_N \otimes \mathrm{id}) = (\mu_N \otimes \mathrm{id}) \circ D.$$

Clearly ψ_N factors through $(\wedge X_N \otimes \wedge \bar{X}_M, D_N)$ to produce the commutative diagram of c.g.d.a.'s

$$\begin{array}{ccccc}
\wedge X_N & \xrightarrow{\text{incl.}} & (\wedge X_N \otimes \wedge \bar{X}_M, D_N) & \xrightarrow{\text{proj.}} & \wedge \bar{X}_M \\
\downarrow{\scriptstyle \varphi_N} & & \downarrow{\scriptstyle \psi'_N} & & \downarrow{\scriptstyle \varphi'_\Omega} \\
A(N) & \xrightarrow[A(\pi_N)]{} & A(M_N^I) & \xrightarrow[A(j_N)]{} & A(\Omega M)
\end{array} \tag{2.7}$$

Because φ_N^* and φ'^*_Ω are isomorphisms the comparison theorem, applied to the spectral sequence of Grivel [2] or [8, Th. 20.5] for the fibration $\mathfrak{F}_N$, shows that $\psi_N'^*$ is an isomorphism. Thus we have established

THEOREM 2.8. *A model for the space M_N^I is given by*

$$\psi_N': (\wedge X_N \otimes \wedge \bar{X}_M, D_N) \to (A(M_N^I), d).$$

In particular (cf. Sullivan [15] or [8, Cor. 2.4]) the minimal model of M_N^I is generated by $H(X_N \oplus \bar{X}_M, Q(D_N))$, i.e.

$$\pi_\psi^*(M_N^I) = H(X_N \oplus \bar{X}_M, Q(D_N)).$$

Next recall that $\wedge X_N$ is minimal and (cf. sec. 1) project the top row of (2.7) to the short exact sequence

$$0 \to (X_N, 0) \to (X_N \oplus \bar{X}_M, Q(D_N)) \to (\bar{X}_M, 0) \to 0.$$

This leads to a long exact sequence

$$\dots \xrightarrow{\partial^*} X_N^p \longrightarrow H^p(X_N \oplus \bar{X}_M, Q(D_N)) \longrightarrow \bar{X}_N^p \xrightarrow{\partial^*} X_N^{p+1} \longrightarrow \dots \tag{2.9}$$

in which clearly $\partial^* = Q(D_N)$.

A straightforward calculation using (2.4) shows that

$$D_N(1 \otimes \bar{x}) = (\varphi_1 - \varphi_0)x - (\mu_N \otimes \mathrm{id})\Omega(x), \quad x \in X_M.$$

Since $\Omega(x)$ is decomposable we conclude

$$\partial^* \bar{x} = (Q(\varphi_1) - Q(\varphi_0))x.$$

If $\partial_M^*: \bar{X}_M \to X_M$ is the canonical isomorphism we can write this as

$$\partial^* = [Q(\varphi_1) - Q(\varphi_0)] \circ \partial_M^*. \tag{2.10}$$

Now the sequence (2.9) allows us to identify $H(X_N \oplus \bar{X}_M, Q(D_N))$ with coker $\partial^* \oplus \overline{\ker \partial^*}$, and so Theorem 2.8 has the following

COROLLARY 2.11. *The space of generators for the minimal model of M_N^I is given by*

$$\pi_\psi^*(M_N^I) = H(X_N \oplus \bar{X}_M, Q(D_N)) = \mathrm{coker}\,(Q(\varphi_1) - Q(\varphi_0)) \oplus \overline{\ker\,(Q(\varphi_1) - Q(\varphi_0))}.$$

Next recall that we identify $X_N = \pi_\psi^*(N)$ etc. Since φ_0 and φ_1 correspond respectively to $p_0 = P_L \circ i_N: N \to M$ and $p_1 = P_R \circ i_N: N \to M$ we have $Q(\varphi_i) = p_i^\#$, and (2.9) can be written in the form (cf. [10, sec. 4])

$$\dots \longrightarrow \pi_\psi^p(N) \xrightarrow{\pi_N^\#} \pi_\psi^p(M_N^I) \xrightarrow{j_N^\#} \pi_\psi^p(\Omega M) \xrightarrow{(p_1^\# - p_0^\#)\partial_M^*} \pi_\psi^{p+1}(N) \longrightarrow \dots \tag{2.12}$$

Observe that (2.10) is analogous to a result of Grove [6] and that (2.12) is the ψ-analogue of a sequence in [6, Theorem 1.3]. However, unless N is assumed nilpotent (2.12) cannot be obtained from [6] by dualizing; it may be a different sequence entirely!

Now let $V = N \cap \Delta(M)$ and let σ: $V \to M_N^I$ be the inclusion defined by

$$\sigma(x, x): I \to x, \quad (x, x) \in N \cap \Delta(M).$$

Because of applications to geodesics we consider the following conditions:

$$\sigma \text{ is a homotopy equivalence} \tag{2.13}$$

$$H^p(V) = 0, \quad p > r. \tag{2.14}$$

Note that (2.13) implies that V is path connected, and that σ induces an isomorphism $\pi_\psi^*(M_N^I) \to \pi_\psi^*(V)$. Moreover if γ: $V \to N$ is the inclusion then $\pi_N \circ \sigma = \gamma$, and so we can identify $\pi_N^\#$ with $\gamma^\#$.

THEOREM 2.15. *Suppose* (2.13) *and* (2.14) *hold. Then*

(i) $\ker(p_1^\# - p_0^\#)$ *has finite dimension* $\leq r$, *and is spanned by elements of even degree.*

(ii) *The sequence*

$$\begin{array}{c} 0 \longrightarrow \pi_\psi^{\mathrm{odd}}(M) \xrightarrow{p_1^\# - p_0^\#} \pi_\psi^{\mathrm{odd}}(N) \xrightarrow{\gamma^\#} \pi_\psi^{\mathrm{odd}}(V) \\ \swarrow \partial_M^* \circ j_N^\# \circ (\sigma^\#)^{-1} \\ \pi_\psi^{\mathrm{even}}(M) \xrightarrow[p_1^\# - p_0^\#]{} \pi_\psi^{\mathrm{even}}(N) \xrightarrow[\gamma^\#]{} \pi_\psi^{\mathrm{even}}(V) \longrightarrow 0 \end{array}$$

is exact.

Proof. (i) follows from Lemma 2.18 below, applied to $(\Lambda X_N \otimes \Lambda \bar{X}_M, D_N)$. (ii) follows from (i) and the exactness of (2.12).

COROLLARY 2.16. *The following are equivalent when* (2.13) *and* (2.14) *hold*

(i) $\dim \pi_\psi^*(N) < \infty$

and

(ii) $\dim \pi_\psi^*(V) < \infty$ *and* $\dim \pi_\psi^*(M) < \infty$.

Furthermore, if (i) and (ii) *hold then*

$$\chi_\pi(N) = \chi_\pi(M) + \chi_\pi(V),$$

where $\chi_\pi = \dim \pi_\psi^{\mathrm{even}} - \dim \pi_\psi^{\mathrm{odd}}$ *is the homotopy Euler characteristic.*

Proof. If (i) holds then $\dim \pi_\psi^{\mathrm{odd}}(M) < \infty$; then $\pi_\psi^{2p-1}(M) = 0$, if $2p-1 \geqslant m$, some m. Apply Theorem 5.9 of [10] to the projection $(\wedge X_M, d) \to \wedge(\sum_{j>m} X_M^j), 0)$ to obtain $X_M^j = 0$, $j > m$. Hence $\dim \pi_\psi^*(M) < \infty$ and so (i) implies (ii).

Consider in general (cf. top row of (2.7)) a sequence of connected KS complexes of the form

$$(\wedge Y, d) \xrightarrow{i} (\wedge Y \otimes \wedge X, D) \xrightarrow{\varrho} (\wedge X, 0)$$

in which $(\wedge Y, d)$ is minimal. As above we obtain a long exact sequence

$$\dots \longrightarrow Y^p \xrightarrow{Q(i)^*} H^p(Y \oplus X, Q(D)) \xrightarrow{Q(\varrho)^*} X^p \xrightarrow{\partial^*} Y^{p+1} \longrightarrow \dots \tag{2.17}$$

LEMMA 2.18. *If* $H^i(\wedge Y \otimes \wedge X, D) = 0$ *for* $i > r$ *then every homogeneous element in* $\ker \partial^*$ *has odd degree and* $\dim \ker \partial^* \leqslant r$.

Proof. Choose a graded subspace $X_1 \subset X$ so that

$$X = X_1 \oplus \ker \partial^*.$$

This decomposition defines a linear projection $X \to \ker \partial^*$ which extends to a homorphism

$$\varrho_1\colon \wedge X \to \wedge \ker \partial^*.$$

Composing with ϱ we obtain

$$\varrho_2 = \varrho_1 \circ \varrho\colon (\wedge Y \otimes \wedge X, D) \to (\wedge \ker \partial^*, 0).$$

Moreover, by exactness $\ker \partial^* = \operatorname{im} Q(\varrho)^*$ and since $Q(\varrho_1)$ is the identity in $\ker \partial^*$ we obtain that $Q(\varrho_2)^*$ is surjective. Thus Theorem 5.9 of [10] applies and shows that the product of any $r+1$ elements of positive degree in $H(\wedge \ker \partial^*)$ is zero. Since $H(\wedge \ker \partial^*) = \wedge \ker \partial^*$ this implies the lemma.

We close this section with two examples in which $N = V_0 \times V_1$ and $V_i \subset M$, $i = 0, 1$. Note by the way that it would be no real restriction to consider only the case $N = V_0 \times V_1$ since in fact $M_N^I = M \times M_{N \times \Delta(M)}^I$.

If $N = V_0 \times V_1$ and $i_j\colon V_j \to M$, $j = 0, 1$ are the inclusions then $p_1^\# - p_0^\#\colon \pi_\psi^*(M) \to \pi_\psi^*(N)$ can be written as

$$i_1^\# - i_0^\#\colon \pi_\psi^*(M) \to \pi_\psi^*(V_0) \oplus \pi_\psi^*(V_1) \tag{2.19}$$

and if (2.13) and (2.14) hold this can be substituted in the sequence of Theorem 2.15 (ii).

Example 2.20. Suppose V_0 and V_1 are even spheres of dimensions $2l$ and $2m$, and $V = V_0 \cap V_1$ is properly contained in each. Assume (2.13) and (2.14) hold and $\dim H^*(M) < \infty$. Then

$$H^*(V) = H^*(pt)$$

and

$$\sum_p \dim H^p(M)t^p = (1+t^{2l})(1+t^{2m}). \tag{2.21}$$

Indeed, since V is contractible in each of V_0 and V_1, $\gamma^* = 0$. From (ii) of Theorem 2.15 we then deduce that

$$i_1^\# - i_0^\# : \pi_\psi^{\text{odd}}(M) \to \pi_\psi^{\text{odd}}(V_0 \times V_1)$$

is an isomorphism and

$$i_1^\# - i_0^\# : \pi_\psi^{\text{even}}(M) \to \pi_\psi^{\text{even}}(V_0 \times V_1)$$

is surjective. Since $\dim \pi_\psi^{\text{odd}}(V_0 \times V_1) = \dim \pi_\psi^{\text{even}}(V_0 \times V_1) = 2$ on the one hand, and since by Theorem 1′ of [9]

$$\dim \pi_\psi^{\text{odd}}(M) \geqslant \dim \pi_\psi^{\text{even}}(M)$$

on the other, we must have equality above and hence

$$i_1^\# - i_0^\# : \pi_\psi^*(M) \to \pi_\psi^*(V_0 \times V_1)$$

is an isomorphism. Again by Theorem 2.15 (ii), this implies $\pi_\psi^*(V) = 0$ and so $H^*(V) = H^*(pt)$. It also allows us to apply Corollary 2 to Theorem 5 of [9] which gives (2.21).

Example 2.22. Let M, V_0 and V_1 all be spheres and suppose $V_0 \cap V_1$ is properly contained in each V_i, $i = 0, 1$. Then (2.13) and (2.14) cannot hold. Otherwise as in the above example

$$i_1^\# - i_0^\# : \pi_\psi^{\text{odd}}(M) \to \pi_\psi^{\text{odd}}(V_0 \times V_1)$$

would be an isomorphism, but $\dim \pi_\psi^{\text{odd}}(M) = 1$ and $\dim \pi_\psi^{\text{odd}}(V_0 \times V_1) = 2$.

3. The minimal model for the space of g-invariant curves

Let M continue to denote a 1-connected space whose rational cohomology has finite type, and fix a continuous map $g: M \to M$. We shall apply the results of section 2 to the case N is the graph of g:

$$N = G(g) = \{(x, g(x)) \mid x \in M\}.$$

When g satisfies a condition we call *rigidity at* 1 (this is always true if $g^k = \text{id}$, some k) then we give an explicit form of the minimal model of $M^I_{G(g)}$.

Since $M^I_{G(g)}$ consists of paths $f: I \to M$ such that $f(1) = g(f(0))$ we can identify it with the space of paths

$$f: \mathbf{R} \to M \quad \text{satisfying } f(t+1) = g(f(t)),$$

i.e. the space of *g-invariant curves*. Similarly if $g^k = \mathrm{id}$ we can identify $M^I_{G(g)}$ with the space of continuous maps

$$f: S^1 \to M \quad \text{such that } f(e^{2\pi i/k}e^{i\theta}) = g(f(e^{i\theta})),$$

i.e. $M^I_{G(g)}$ is then the space of *g-invariant circles* on M.

For the moment let $g: M \to M$ be any continuous map. We translate from section 2 with $N = G(g)$. Note that $p_0: G(g) \to M$ is a homeomorphism, and so φ_0 (which represents it) is an isomorphism. Moreover if

$$\psi_g: (\Lambda X_M, d_M) \to (\Lambda X_M, d_M)$$

represents g ($\varphi_M \circ \psi_g \sim A(g) \circ \varphi_M$) then p_1 is represented by $\varphi_1 = \varphi_0 \circ \psi_g$.

Next recall (Theorem 2.8) the model $(\Lambda X_{G(g)} \otimes \Lambda \bar{X}_M, D_{G(g)})$ for $M^I_{G(g)}$. Define a c.g.d.a. $(\Lambda X_M \otimes \Lambda \bar{X}_M, D_g)$ by requiring that

$$\varphi_0 \otimes \mathrm{id}: (\Lambda X_M \otimes \Lambda \bar{X}_M, D_g) \to (\Lambda X_{G(g)} \otimes \Lambda \bar{X}_M, D_{G(g)})$$

be an isomorphism. Set $\varphi'_g = \psi'_{G(g)} \circ (\varphi_0 \otimes \mathrm{id})$, then Theorem 2.8 reads:

COROLLARY 3.1. *A model for* $M^I_{G(g)}$ *is given by*

$$\varphi'_g: (\Lambda X_M \otimes \Lambda \bar{X}_M, D_g) \to (A(M^I_{G(g)}), d),$$

where D_g *is determined by*

$$D_g \circ (\mu_g \otimes \mathrm{id}) = (\mu_g \otimes \mathrm{id}) \circ D,$$

and $\mu_g: \Lambda X_M \otimes \Lambda X_M \to \Lambda X_M$ *is given by*

$$\mu_g(\Phi \otimes \Psi) = \Phi \cdot \psi_g(\Psi).$$

For the induced differential $Q(D_g)$ *we have*

$$Q(D_g) X_M = 0 \quad \textit{and via } (2.10)$$

$$Q(D_g)\bar{x} = (Q(\psi_g) - id)x, \quad \bar{x} \in \bar{X}_M \tag{3.2}$$

which translates Lemma 1.5 of [*6*].

Remark 3.3. In view of our hypotheses on M there is a canonical isomorphism as mentioned at the end of section 1,

$$Q(\Lambda X_M) \xrightarrow{\cong} \mathrm{Hom}_{\mathbf{Z}}(\pi_*(M); \mathbf{Q}).$$

Because M is simply connected g induces a well defined homomorphism of homotopy groups

$$g_{\#}: \pi_*(M) \to \pi_*(M)$$

even though g may not preserve base points. Moreover if

$$g^{\#}: \text{Hom}\,(\pi_*(M); \mathbf{Q}) \to \text{Hom}\,(\pi_*(M); \mathbf{Q})$$

is the dual of $g_{\#}$, then the isomorphism above identifies $Q(\psi_g)$ with $g^{\#}$. In particular the generators for the minimal model of $M^I_{G(g)}$ are determined by $g_{\#}$.

Now let $(\Lambda X_M)_0$ be the subalgebra of ΛX_M of elements Φ satisfying

$$\psi_g \Phi = \Phi,$$

and let $Q(\Lambda X_M)_0$ be the subspace of elements $a \in Q(\Lambda X_M)$ satisfying

$$Q(\psi_g) a = a.$$

Definition 3.4. A map g: $M \to M$ will be called *rigid at* 1 if

$$Q(\Lambda X_M) = Q(\Lambda X_M)_0 \oplus \text{im}\,(Q(\psi_g) - \text{id}) \tag{3.5}$$

and if for a suitable choice of ψ_g the projection

$$\zeta: (\Lambda^+ X_M)_0 \to Q(\Lambda X_M)_0 \tag{3.6}$$

is surjective.

Remark 3.7. Since $Q(\Lambda X_M) \cong X_M$ is a graded space of finite type, condition (3.5) simply says that if $(Q(\psi_g) - \text{id})^n a = 0$ then $Q(\psi_g) a = a$. Equivalently, $Q(\psi_g) - \text{id}$ restricts to an isomorphism of the subspace im $(Q(\psi_g) - \text{id})$.

Condition (3.6) says that any $Q(\psi_g)$-invariant vector can be represented by a ψ_g-invariant element in ΛX_M.

Thus while (3.5) can be interpreted as a condition on $g_{\#}$, (3.6) is more subtle. *Note that if ψ_g and X_M can be chosen so that X_M is stable under ψ_g then* (3.6) *is automatic.*

Example 3.8. Suppose g: $M \to M$ is a continuous map such that $g^k = \text{id}$ for some $k \in \mathbf{Z}$. Thus g makes M into a G-space, where $G = \mathbf{Z}_k$. In this case by Theorem 1.3 we can choose ψ_g so that $\psi_g^k = \text{id}$, which allows us to choose X_M to be stable under ψ_g. (In fact the constructions in the proof of 1.3 already make ψ_g act on X_M with order k.) According to the remark above g is rigid at 1.

Using another approach we have more generally

THEOREM 3.9. *Let M be 1-connected and suppose g: $M \to M$ satisfies*

$$g^k \sim \text{id}.$$

Then g is rigid at 1.

Proof. Let $\varphi_M\colon \Lambda X\to A(M)$ be the minimal model and choose $\psi_1\colon \Lambda X\to\Lambda X$ so that

$$\varphi_M\psi_1\sim A(g)\varphi_M.$$

Then $\psi_1^k\sim\mathrm{id}$.

By a result of Sullivan [15; Prop. 6.5] or [8, Th. 11.21], this implies

$$\psi_1^k=e^\theta=\sum_0^\infty\frac{\theta^m}{m!}$$

where $\theta=sd+ds$ and s is a derivation of degree -1 in ΛX. Moreover

$$\theta=\ln(\psi_1^k)=\sum_{n\geqslant 1}(-1)^{n-1}\frac{(\psi_1^k-\mathrm{id})^n}{n}.$$

In particular

$$\theta\psi_1=\psi_1\theta$$

Set $\theta_1=-\theta/k=-\left(\frac{s}{k}d+d\frac{s}{k}\right)$; then $e^{\theta_1}\sim\mathrm{id}$ (cf. Sullivan [15, Prop. A.3]) or [8, Th. 11.21]. Also $\theta_1\psi_1=\psi_1\theta_1$, whence

$$e^{\theta_1}\psi_1=\psi_1e^{\theta_1}.$$

Hence
$$(e^{\theta_1}\psi_1)^k=e^{k\theta_1}\psi_1^k=e^{-\theta}\psi_1^k=\mathrm{id}$$

and
$$e^{\theta_1}\psi_1\sim\psi_1.$$

Put $\psi=e^{\theta_1}\psi_1$. Then

$$\psi\sim\psi_1\Rightarrow\varphi_M\psi\sim A(g)\varphi_M$$

and so ψ represents g. On the other hand

$$\psi^k=\mathrm{id}\quad\text{in }\Lambda X$$

and so by the argument above ψ is rigid at 1.

Remark 3.10. Without proof we mention that there are many more 1-rigid maps e.g. retractions and more generally maps g satisfying $g^{k+s}=g^k$ for some k and s.

Henceforth we assume g to be rigid at 1 and determine the minimal model of $M^I_{G(g)}$.

It is immediate from definition 3.4 that we can choose X_M and ψ_g so that $X_M=Y\oplus U$, where

$$\psi_g y=y,\quad y\in Y$$

and

$$U\subset\mathrm{im}\,(\psi_g-\mathrm{id}).$$

LEMMA 3.11. *With the choices above*

(i) im $(\psi_g - \mathrm{id}) \subset \wedge Y \otimes \wedge^+ U$, *and*

(ii) $\wedge Y \otimes \wedge^+ U$ *is* d_M*-stable.*

Proof. (i): Choose a graded subspace $V \subset \wedge^+ X_M$ so that $\zeta(V) \subset U$ and $(\psi_g - \mathrm{id})$: $V \to U$ is an isomorphism. If we regard U as a subspace of $Q(\wedge X_M)$, then clearly

$$(\psi_g - \mathrm{id}) = (Q(\psi_g) - \mathrm{id}) \circ \zeta \colon V \to U.$$

Since $\psi_g - \mathrm{id}$: $V \to U$ is an isomorphism it follows that ζ: $V \to U$ is an isomorphism. Therefore

$$\wedge^+ X_M = \wedge^+ X_M \cdot \wedge^+ X_M \oplus Y \oplus V$$

and so

$$(\psi_g - \mathrm{id}) \wedge^+ X_M = (\psi_g - \mathrm{id})(\wedge^+ X_M \cdot \wedge^+ X_M) + U \subset [(\psi_g - \mathrm{id}) \wedge^+ X_M] \cdot \wedge^+ X_M + \wedge Y \otimes \wedge^+ U.$$

An easy degree argument completes the proof.

(ii): Since $\wedge Y \otimes \wedge^+ U$ is the ideal generated by U, (ii) follows from the relation

$$d_M U \subset d_M \,\mathrm{im}\, (\psi_g - \mathrm{id}) \subset \mathrm{im}\, (\psi_g - \mathrm{id}) \subset \wedge Y \otimes \wedge^+ U.$$

Since the ideal $\wedge Y \otimes \wedge^+ U$ is d_M-stable we may divide out by it to obtain a c.g.d.a. $(\wedge Y, \delta)$ such that the projection

$$P \colon \wedge X_M \to \wedge Y \tag{3.12}$$

is a homomorphism of c.g.d.a.'s.

We now associate to $(\wedge Y, \delta)$ the corresponding c.g.d.a. $(\wedge Y^I, D)$ (p. 280), with $\wedge Y^I = \wedge Y \otimes \wedge \bar{Y} \otimes \wedge D\bar{Y}$, and derivations i and θ in $\wedge Y^I$, and c.g.d.a. maps λ_0, λ_1: $\wedge Y \to \wedge Y^I$. Moreover λ_0 and λ_1 determine an isomorphism

$$\lambda_0 \otimes \lambda_1 \otimes \mathrm{id} \colon \wedge Y \otimes \wedge Y \otimes \wedge \bar{Y} \to \wedge Y^I$$

(compare (2.3)). Thus a homomorphism of graded algebras

$$\mu \otimes \mathrm{id} \colon \wedge Y^I \to \wedge Y \otimes \wedge \bar{Y}$$

is defined by

$$(\mu \otimes \mathrm{id}) \lambda_0 \Phi = (\mu \otimes \mathrm{id}) \lambda_1 \Phi = \Phi \quad \text{and } (\mu \otimes \mathrm{id}) \bar{y} = \bar{y}$$

for all $\Phi \in \wedge Y$ and $\bar{y} \in \bar{Y}$. As in section 2 a differential $\bar{D}$ in $\wedge Y \otimes \wedge \bar{Y}$ is defined by requiring $\mu \otimes \mathrm{id}$ to be a map of c.g.d.a.'s.

In order to identify $\bar{D}$, we define a degree -1 derivation i_Y in $\wedge Y \otimes \wedge \bar{Y}$ by

$$i_Y y = \bar{y} \quad \text{and} \quad i_Y \bar{y} = 0,$$

and a degree $+1$ derivation d_g in $\wedge Y \otimes \wedge \bar{Y}$ by

$$d_g y = \delta y \quad \text{and} \quad d_g \bar{y} = -i_Y \delta y, \quad y \in Y.$$

Since obviously $i_Y^2 = 0$ we get

$$d_g \circ i_Y + i_Y \circ d_g = 0 \tag{3.13}$$

and therefore $d_g^2 = 0$; i.e. $(\wedge Y \otimes \wedge \bar{Y}, d_g)$ is a c.g.d.a.

Remark. $(\wedge Y \otimes \wedge \bar{Y}, d_g)$ is obviously a minimal KS complex. If Y is the minimal model for a space S, then $(\wedge Y \otimes \wedge \bar{Y}, d_g)$ is Sullivan's model for the space of maps $S^1 \to S$ ([14], [16]).

LEMMA 3.14. *The differentials $\bar{D}$ and d_g agree, i.e.*

$$\mu \otimes \mathrm{id}\colon (\wedge Y^I, D) \to (\wedge Y \otimes \wedge \bar{Y}, d_g)$$

is a homomorphism of c.g.d.a.'s.

Proof. Note that $\bar{D} = \delta$ in $\wedge Y$. Hence we need only show

$$\bar{D}\bar{y} = -i_Y \delta y, \quad y \in Y.$$

which we do by induction on the degree of y.

First recall that the derivation i in $\wedge Y^I$ (p. 281) satisfies $i^2 = 0$, whence by (2.4) $i(\lambda_1 y) = i(\lambda_0 y) = \bar{y}$ for all $y \in Y$. If follows that

$$(\mu \otimes \mathrm{id}) \circ i = i_Y \circ (\mu \otimes \mathrm{id})$$

and using (2.4) we conclude

$$\bar{D}\bar{y} = -\sum_{n=1}^{\infty} \frac{(i_Y \bar{D})^n}{n!} y = -\sum_{n=0}^{\infty} \frac{(i_Y \bar{D})^n}{(n+1)!} i_Y \delta y$$

$$= -i_Y \delta y - \sum_{n=1}^{\infty} \frac{(i_Y \bar{D})^n}{(n+1)!} i_Y \delta y$$

If $\deg y = p$ then δy is a polynomial in the y_j's with $\deg y_j < p$ ($(\wedge Y, \delta)$ is a 1-connected KS-complex) and it follows from (3.13) and our induction hypothesis that

$$\bar{D} i_Y \delta y = d_g i_Y \delta y = i_Y \delta^2 y = 0.$$

Hence the equation above reads $\bar{D}\bar{y} = -i_Y \delta y$ and we are done.

Now extend the c.g.d.a. map P of (3.12) to a c.g.d.a. map $P^I\colon (\wedge X_M^I, D) \to (\wedge Y^I, D)$ by setting

$$P^I \bar{x} = \overline{Px} \quad \text{and} \quad P^I D\bar{x} = D\,\overline{Px}, \quad x \in Y$$

and

$$P^I \bar{x} = P^I D\bar{x} = 0, \quad x \in U.$$

Then P^I commutes with i and θ so that

$$P^I \circ \lambda_0 = \lambda_0 \circ P \quad \text{and} \quad P^I \circ \lambda_1 = \lambda_1 \circ P. \tag{3.15}$$

Also, extend P to an algebra homomorphism

$$P_g\colon \wedge X_M \otimes \wedge \bar{X}_M \to \wedge Y \otimes \wedge \bar{Y}$$

by setting $P_g \bar{x} = \overline{Px}$ for all $x \in X_M$ (i.e. $P_g \bar{x} = 0$, $x \in U$).

For these extensions we have

LEMMA 3.16. *The diagram*

$$\begin{array}{ccc} \wedge X_M^I & \xrightarrow{\;P^I\;} & \wedge Y \\ {\scriptstyle \mu_g \otimes \mathrm{id}}\downarrow & & \downarrow {\scriptstyle \mu \otimes \mathrm{id}} \\ \wedge X_M \otimes \wedge \bar{X}_M & \xrightarrow[\;P_g\;]{} & \wedge Y \otimes \wedge \bar{Y} \end{array}$$

commutes. In particular $P_g \circ D_g = d_g \circ P_g$, i.e. P_g is a homomorphism of c.g.d.a.'s.

Proof. If $x \in X_M$ then $(\mu \otimes \mathrm{id}) \circ P^I \bar{x} = P_g \circ (\mu_g \otimes \mathrm{id}) \bar{x}$ is immediate from the definitions. Moreover by (3.15)

$$(\mu \otimes \mathrm{id}) \circ P^I \lambda_0 x = (\mu \otimes \mathrm{id}) \circ \lambda_0 \circ Px = Px = P_g \circ (\mu_g \otimes \mathrm{id})\, \lambda_0 x.$$

Finally recall that $\operatorname{im}(\psi_g - \mathrm{id}) \subset \wedge Y \otimes \wedge^+ U$ by Lemma 3.11. It follows that

$$P \circ \psi_g = P$$

and hence by (3.15)

$$(\mu \otimes \mathrm{id}) \circ P^I \circ \lambda_1 x = (\mu \otimes \mathrm{id}) \circ \lambda_1 \circ Px = Px = P \circ \psi_g x = P_g \circ \psi_g x = P_g \circ (\mu_g \otimes \mathrm{id}) \circ \lambda_1 x$$

i.e. the diagram commutes. Since $\mu_g \otimes \mathrm{id}$, P^I and $\mu \otimes \mathrm{id}$ are all morphisms of c.g.d.a.'s and $\mu_g \otimes \mathrm{id}$ is surjective, it follows that P_g is also a c.g.d.a. homomorphism.

THEOREM 3.17. *The homomorphism P_g induces an isomorphism*

$$H(\wedge X_M \otimes \wedge \bar{X}_M, D_g) \to H(\wedge Y \otimes \wedge \bar{Y}, d_g)$$

of cohomology. In particular $(\wedge Y \otimes \wedge \bar{Y}, d_g)$ is the minimal model of $M^I_{G(g)}$.

Proof. According to Theorem 7.1 in [8] we need only check that

$$Q(P_g)^*\colon H(X_M \oplus \bar{X}_M, Q(D_g)) \to Y \oplus \bar{Y}$$

is an isomorphism. But it follows from 3.2 that $Q(D_g)$ is zero on X_M and on $\bar{Y}$ and restricts to an isomorphism $\bar{U} \to U$. Hence $Q(P_g)^*$ identifies $H(X_M \oplus \bar{X}_M, Q(D_g))$ with $Y \oplus \bar{Y}$.

Finally, consider the commutative diagram

$$\begin{array}{ccc} \Lambda X_M \otimes \Lambda \bar{X}_M & \xrightarrow{P_g} & \Lambda Y \otimes \Lambda \bar{Y} \\ \Big\downarrow & & \Big\downarrow \mathrm{id} \\ \mathbf{Q} & \longrightarrow & \Lambda Y \otimes \Lambda \bar{Y}. \end{array}$$

Since P_g^* is an isomorphism Sullivan [15] or Theorem 5.19 of [8] implies there is a homomorphism φ: $(\Lambda Y \otimes \Lambda \bar{Y}, d_g) \to (\Lambda X_M \otimes \Lambda \bar{X}_M, D_g)$ of c.g.d.a.'s such that φ^* is the isomorphism inverse to P_g.

Thus

$$\varphi_g: (\Lambda Y \otimes \Lambda \bar{Y}, d_g) \to (A(M^I_{G(g)}), d)$$

is a minimal model for $A(M^I_{G(g)})$, where $\varphi_g = \varphi'_g \circ \varphi$.

Remark. As mentioned earlier the c.g.d.a. $(\Lambda Y \otimes \Lambda \bar{Y}, d_g)$ is exactly Sullivan's construction applied to $(\Lambda Y, \delta)$. Moreover if $g = \mathrm{id}_M$ then $\psi_g = \mathrm{id}$, $X_M = Y$ and $P_g = \mathrm{id}$. Hence we recover Sullivan's theorem [14] (with a different proof) as a special case of Theorem 3.15.

Remark 3.18. The fact that the minimal model of $M^I_{G(g)}$ appears to be the minimal model for a space of closed curves can be explained as follows:

Let $A(p)$ be the rational c.g.d.a.$\subset A(\Delta^p)$ generated by the barycentric coordinate functions. In [15, § 8] Sullivan constructs the function adjoint to "differential forms" which associates with each c.g.d.a. (R, d_R) the simplicial set $\langle R \rangle$ given by

$$\langle R \rangle_p = \{\text{all homomorphisms } (R, d_R) \to (A(p), d))\}.$$

Now suppose g is rigid at 1. The homomorphism ψ_g yields a map of simplicial sets

$$\langle \psi_g \rangle: \langle \Lambda X_M \rangle \to \langle \Lambda X_M \rangle.$$

The fixed point set of $\langle \psi_g \rangle$ is the sub-simplicial set $\langle \Lambda X_M \rangle^g$ defined by

$$\langle \Lambda X_M \rangle^g_p = \{\text{all homomorphisms } (\Lambda X_M, d_M) \xrightarrow{\eta} (A(p), d) \quad \text{such that } \eta \circ \psi_g = \eta\}.$$

On the other hand, since g is rigid at 1 we have that the ideal generated by im $(\psi_g - \mathrm{id})$ is exactly $\Lambda Y \otimes \Lambda^+ U$. Hence we obtain $\langle \Lambda X_M \rangle^g_p = \langle \Lambda Y \rangle_p$ i.e.

$$\langle \Lambda X_M \rangle^g = \langle \Lambda Y \rangle.$$

Let $|\langle \Lambda X_M \rangle|$ and $|\langle \Lambda Y \rangle|$ be the geometric realizations (cf. Milnor [13]). $\langle \psi_g \rangle$ defines a continuous map $\bar{g}$ of $|\langle \Lambda X_M \rangle|$ and we have that the fixed point set of $\bar{g}$ is given by

$$|\langle \Lambda X_M \rangle|^{\bar{g}} = |\langle \Lambda Y \rangle|.$$

Finally note that $|\langle \Lambda X_M \rangle|$ is the "rationalization of M" and $\bar{g}$ is the rationalization of g; thus ΛY is the minimal model of the fixed point set of the rationalization of g. Moreover the model of the g-invariant paths on M coincides with the model of the space of all closed paths in the fixed point set of the rationalization of g, $\bar{g}$.

Remark 3.19. Note that if $g: M \to M$ is periodic i.e. $g^k = \mathrm{id}_M$ then we can prove Theorem 3.17 directly via Sullivan's theorem by studying the inclusion of $M^I_{G(g)}$ into the space of all circles on M (cf. the beginning of sec. 3) and using (3.3) and the remarks concluding section 1.

4. On the cohomology of $M^I_{G(g)}$

Throughout this section M is a 1-connected space whose rational cohomology has finite type and $g: M \to M$ is a 1-rigid map. In particular we have a minimal model for the space $M^I_{G(g)}$ of g-invariant curves as in Theorem 3.17.

We show how one can use the minimal model for $M^I_{G(g)}$ in order to obtain information about the cohomology $H^*(M^I_{G(g)})$. In particular we are interested in the Betti-numbers of $M^I_{G(g)}$, because of their significance in applications to geodesics.

As a first application we have the following immediate generalization of a theorem due to Sullivan [14].

THEOREM 4.1. *If the rational cohomology of $M^I_{G(g)}$ is not trivial, then $M^I_{G(g)}$ has non-zero Betti numbers in an infinite arithmetic sequence of dimensions.*

Proof. First suppose $(\Lambda Y, \delta)$ ((3.12)) has no odd dimensional generators; i.e. ΛY is a polynomial algebra in even dimensional generators (which exist for otherwise $Y = \bar{Y} = \{0\}$ and consequently $H^*(M^I_{G(g)})$ would be trivial) and $\delta = 0$. Then $d_g = 0$ and the d_g-closed elements $\{x^j\}_{j \in \mathbf{N}}$ in $\Lambda Y \otimes \Lambda \bar{Y}$ provides us with an infinite sequence of non-zero cohomology classes.

Secondly, if ΛY has odd dimensional generators we proceed exactly as in Sullivan [14, p. 46].

We are now interested in finding necessary and sufficient conditions in order for $M^I_{G(g)}$ to have an unbounded sequence of Betti numbers. Note that as a consequence of Theorem 4.1 we have

COROLLARY 4.2. *Suppose the rational cohomology of the spaces $(M_i)^I_{G(g_i)}$, $i=1, 2$ is non-trivial. Then $(M_1 \times M_2)^I_{G(g_1 \times g_2)}$ has an unbounded sequence Betti numbers.*

We return to the general case corresponding to the direct sum decomposition $Y = Y^{\text{odd}} \oplus Y^{\text{even}}$

$$\chi_0 = \dim\ Y^{\text{odd}}$$

and

$$\chi_e = \dim\ Y^{\text{even}}$$

if both χ_0 and χ_e are finite

$$\chi_\pi = \chi_e - \chi_0$$

is the homotopy Euler characteristic of $(\Lambda Y, \delta)$.

PROPOSITION 4.3. *The sequence of Betti numbers for $M^I_{G(g)}$ is unbounded if and only if one of the following conditions is fulfilled:*

(i) $\chi_0 \geqslant 2$

(ii) $\chi_0 = 0$ and $\chi_e \geqslant 2$

(iii) $\chi_0 = 1$, $\delta Y^{\text{odd}} = \{0\}$ *and* $\chi_e \geqslant 1$

(iv) $\chi_0 = 1$, $\delta Y^{\text{odd}} \neq \{0\}$ *and* $\chi_e \geqslant 3$

(v) $\chi_0 = 1$, $\delta Y^{\text{odd}} \neq \{0\}$, $\chi_e = 2$ *and* $\dim \mathbf{Q}[x_1, x_2]/(\partial P/\partial x_1, \partial P/\partial x_2) = \infty$, *where* $Y^{\text{even}} = \operatorname{span}\{x_1, x_2\}$ *and* $\delta y = P(x_1, x_2)$, $y \in Y^{\text{odd}}$.

Proof. In [16] it has in particular been proved that $\chi_0 \geqslant 2$ implies that $H(\Lambda Y \otimes \Lambda \overline{Y}, d_g)$ has an unbounded sequence $\{b_i\}_{i \in \mathbf{N}}$ of Betti numbers.

If $\chi_0 = 0$ then $d_g = 0$ and $\{b_i\}_{i \in \mathbf{N}}$ is clearly unbounded if and only if $\chi_e \geqslant 2$.

Assume in the following that $\chi_0 = 1$. First let $\delta Y^{\text{odd}} = \{0\}$. If $\chi_e = 0$ then $\Lambda Y = \mathbf{Q}(y, \bar{y})$ and $d_g = 0$. Thus $\{b_i\}$ is bounded. Suppose now on the other hand that $\chi_e \geqslant 1$. Then clearly the ideal im d_g in ker d_g is contained in the ideal generated by y and $\bar{y}$, where $y \in Y^{\text{odd}}$. Hence $\dim \ker \delta \cap Y^{\text{even}} \geqslant 2$ implies that $\{b_i\}_{i \in N}$ is unbounded. If there are not two even closed generators of Y we range the generators of Y^{even} by increasing degrees $x_1, x_2, \ldots, x_n, \ldots$ so that $\delta x_1 = 0$, $\delta x_2 = x_1^\alpha y, \ldots, \delta x_n = P_n(x_1, \ldots, x_{n-1}) y, \ldots$ and P_n, $n \geqslant 3$, belongs to the ideal generated by $x_2, \ldots, x_{n-1}$. Then we have

$$d_g \bar{x}_2 = \alpha x_1^{\alpha-1} \bar{x}_1 y + x_1^\alpha \bar{y}$$

and

$$d_g \bar{x}_n = \sum_{k=1}^{n-1} \frac{\partial P_n}{\partial x_k} \bar{x}_k y + P_n \cdot \bar{y}$$

for $n \geqslant 3$. Hence in $\Lambda Y \otimes \Lambda \overline{Y}$, im d_g is contained in the ideal

$$(d_g x_2, d_g \bar{x}_2, x_2 \bar{y}, \ldots, x_n \bar{y}, \ldots, \bar{x}_2 y, \ldots, \bar{x}_n y, \ldots, x_2 y, \ldots, x_n y, \ldots)$$

so the family of closed elements $\{x_1^a\bar{y}^b\}$, $(a, b) \in \mathbf{N}\times\mathbf{N}$ are homologically independent, in particular $\{b_i\}_{i\in\mathbf{N}}$ is unbounded.

In the rest of the proof we assume besides $\chi_0=1$ that $\delta Y^{\mathrm{odd}} \neq \{0\}$. Then $\delta Y^{\mathrm{even}}=0$ since $\delta^2=0$.

If $\chi_e=1$ we have $\Lambda Y=\mathbf{Q}(x, y)$ with $\delta x=0$ and $\delta y=x^h$. It is then easy to prove that $\{b_i\}_{i\in\mathbf{N}}$ are bounded (see Addendum in [16]). If $\chi_e=\infty$ we obviously have $\{b_i\}_{i\in\mathbf{N}}$ unbounded.

We shall now show that $3\leqslant\chi_e<\infty$ implies $\{b_i\}_{i\in\mathbf{N}}$ unbounded. Let $x_1, ..., x_p$, $p\geqslant 3$, be a basis for Y^{even}. An element of the polynomial ring $\mathbf{Q}[x_1, ..., x_p]$ is easily seen to be a boundary in $(\Lambda Y\otimes\Lambda\bar{Y}, d_g)$ if and only if it is in the ideal generated by $d_g y$, $y\in Y^{\mathrm{odd}}$. Now, consider the graded ring $A=\mathbf{Q}[x_1, ..., x_p]/(d_g y)$ of Krull dimension $q=p-1\geqslant 2$. By lemme 1 of [12] there are positive integers N and α and a polynomial P with $\deg P = q-1\geqslant 1$, such that for all $n\geqslant N$ and $n\equiv 0 \pmod{\alpha}$ we have $\dim A_n=P(n)$, where A_n is the subspace of A of elements of degree n.

Finally assume $\chi_e=2$ and let x_1, x_2 be a basis for Y^{even}. If $y\in Y^{\mathrm{odd}}$ $\delta y=P(x_1, x_2)$ and hence $\operatorname{im} d_g$ is contained in the ideal generated by $\partial P/\partial x_1$ and $\partial P/\partial x_2$. If $A=\mathbf{Q}[x_1, x_2]/(\partial P/\partial x_1, \partial P/\partial x_2)$ is not finite dimensional, then A has Krull dimension $\geqslant 1$ and the ring $B=A\otimes\mathbf{Q}(\bar{y})$ has therefore Krull dimension $\geqslant 2$. Again by Lemma 1 of [12] we conclude that $\{\dim B_n\}_{n\in\mathbf{N}}$ is unbounded. But for any non-zero element $\bar{\beta}\in B$ the element $\bar{x}_1\bar{x}_2\bar{\beta}$ is a cocycle in $(\Lambda Y\otimes\Lambda\bar{Y}, d_g)$ and not a boundary i.e. $\{b_i\}_{i\in\mathbf{N}}$ is unbounded. If $\dim A<\infty$ a direct but lengthy computation of $H(\Lambda Y\otimes\Lambda\bar{Y}, d_g)$ in even and odd degrees shows that $\{b_i\}_{i\in\mathbf{N}}$ is bounded.

From Proposition 1 in [16] and the above proposition we get

COROLLARY 4.4. *The sequence of Betti numbers for $M^I_{G(g)}$ is bounded if and only if the cohomology ring $H(\Lambda Y, \delta)$ has one of the following types:*

(i) $H(\Lambda Y, \delta)=\mathbf{Q}$

(ii) *$H(\Lambda Y, \delta)$ is generated by one element*

(iii) *$H(\Lambda Y, \delta)$ is a polynomial algebra in two variables x_1, x_2 truncated by an ideal generated by one element P such that* $\dim \mathbf{Q}[x_1, x_2]/(\partial P/\partial x_1, \partial P/\partial x_2)<\infty$.

In Proposition 4.3 and Corollary 4.4 the cohomology of M was only supposed to be of finite type. If we assume $H^*(M)$ to be finite dimensional (e.g. M a finite complex) we can apply some recent results of Halperin [9] and [10] to obtain:

THEOREM 4.5. *Let M be a 1-connected space with finite dimensional cohomology $H^*(M)$ and let $g\colon M\to M$ be a 1-rigid map. Then exactly one of the following holds:*

(I) $\chi_0=\chi_e=0$. *In this case* $\Lambda Y=\mathbf{Q}$ *and* $H^*(M^I_G(g))=\mathbf{Q}$.

(II) $\chi_0=1$, $\chi_e=0$. *In this case* $\Lambda Y=\Lambda(y)$ *and* $H^*(M^I_{G(g)})=\Lambda(y,\bar{y})$ *is the exterior algebra on* y *tensor the polynomial algebra on* $\bar{y}$.

(III) $\chi_0=\chi_e=1$. *In this case* $\Lambda Y=\Lambda(y,x)$ *with* $\delta x=0$, $\delta y=x^{n+1}$ *and* $H^*(M^I_{G(g)})=\Lambda^+(x,\bar{x})/(x^{n+1},x^n\bar{x})\otimes\Lambda(\bar{y})$. *In particular* $\{b_i(M^I_{G(g)})\}$ *is bounded.*

(IV) $\{b_i(M^I_{G(g)})\}_{i\in\mathbf{N}}$ *is unbounded.*

In particular $\{b_i\}$ *is bounded if and only if* $\chi_e\leqslant\chi_0\leqslant 1$.

Proof. If dim $Y=\infty$ we see from Proposition 4.3 that $\{b_i\}_{i\in\mathbf{N}}$ is unbounded.

Suppose now that dim $Y<\infty$. Since $\dim H^*(M)=\dim H(\Lambda X_M,d_m)<\infty$ Corollary 5.13 of Halperin [10] implies that $\dim H(\Lambda Y,\delta)<\infty$. We can therefore apply the finiteness results of Halperin [9]. In particular $\chi_\pi=\chi_e-\chi_0\leqslant 0$ by Theorem 1 in [9].

If $\chi_0\geqslant 2$ we know from Proposition 4.3 that $\{b_i\}_{i\in\mathbf{N}}$ is unbounded.

If $\chi_0=1$ we must have $\chi_e\leqslant 1$. Suppose $\chi_e=1$. Then $\delta x=0$ and $\delta y=x^{n+1}$ for some n because $H(\Lambda Y,\delta)$ is finite dimensional. The actual computation of $H^*(M^I_{G(g)})$ is then contained in the Addendum of [16].

The case $\chi_0=1$ and $\chi_e=0$ is clear.

Finally $\chi_0=\chi_e=0$ if and only if $H^*(M^I_{G(g)})$ is trivial.

Note that if $\dim H^*(M)<\infty$ then (iii) in Corollary 4.4 is impossible. If $g=\mathrm{id}_M$ then $Y=X_M$; i.e. (i) is also impossible and Corollary 4.4 is nothing but the main theorem of Sullivan and Vigué [16].

Theorem 4.5 gives a necessary and sufficient condition on the action of g on $\pi_*(M)\otimes\mathbf{Q}$ in order for $H^*(M^I_{G(g)})$ to have an unbounded sequence of Betti numbers. As in the case $g=\mathrm{id}_M$ it would be interesting also to have a (necessary and sufficient) condition on the action of g on $H^*(M)$ in order for $H^*(M^I_{G(g)})$ to have an unbounded sequence of Betti numbers. We can illustrate the subtlety of this problem with the following examples.

Example 4.6. Let $M=S^{2p}\times S^{2q}$ with $p\neq q$ and $p,q\geqslant 1$. Then $\Lambda X_{S^{2p}}=\Lambda(x_1,y_1)$ with $\deg x_1=2p$, $\deg y_1=4p-1$, $dx_1=0$ and $dy_1=x_1^2$ and similarly for $\Lambda X_{S^{2q}}=\Lambda(x_2,y_2)$. Thus any 1-rigid homotopy equivalence g of M will fix at least the generators y_i, $i=1,2$ and by Theorem 4.5 $M^I_{G(g)}$ will have an unbounded sequence of Betti numbers. However, g may map x_i to $-x_i$, $i=1,2$ and hence not fix any generators in the cohomology $H^*(M)$.

Example 4.7. Take $M=\mathbf{C}P^{2p+1}\times\mathbf{C}P^{2q+1}$ with $p\neq q$ and $p,q\geqslant 0$. Then $\Lambda X_{\mathbf{C}P^{2p+1}}=\Lambda(x_1,y_1)$ with $\deg x_1=2$, $\deg y_1=2(2p+1)+1$, $dx_1=0$ and $dy_1=x_1^{2p+2}$ and similarly for $\Lambda X_{\mathbf{C}P^{2q+1}}=\Lambda(x_2,y_2)$. We can therefore draw exactly the same conclusions as above.

Example 4.8. Endow S^{2p} and $\mathbf{C}P^{2q}$ with their standard riemannian metrics and $S^{2p} \times \mathbf{C}P^{2q}$ with the product metric. Let $q_1 = -\mathrm{id}_{S^{2p}}$ be the antipodal map on S^{2p} and g_2 the conjugate map on $\mathbf{C}P^{2q}$ i.e. in homogeneous coordinates $g_2(z_1, \ldots, z_{2q+1}) = (\bar{z}_1, \ldots, \bar{z}_{2q+1})$. If $M = T_1(S^{2p} \times \mathbf{C}P^{2q})$ is the unit tangent bundle of $S^{2p} \times \mathbf{C}P^{2q}$ then the differential of the involutive isometry $g_1 \times g_2$ restricts to an involution g on M.

Note that M is the total space of the fibre bundle $M \to S^{2p} \times \mathbf{C}P^{2q}$ with fiber $S^{2p+4q-1}$. Therefore $\wedge X_M = \wedge X_{S^{2p}} \otimes \wedge X_{\mathbf{C}P^{2q}} \otimes \wedge X_{S^{2p+4q-1}} = \wedge(x_1, x_2, y_1, y_2, y_3)$ with $\deg x_1 = 2p$, $\deg x_2 = 2$, $\deg y_1 = 4p-1$, $\deg y_2 = 4q+1$, $\deg y_3 = 2p+4q-1$ and $dx_1 = dx_2 = 0$, $dy_1 = x_1^2$, $dy_2 = x_2^{2q+1}$ and $dy_3 = (4q+2)x_1 x_2^{2q}$ ($x_1 x_2^{2q}$ = orientation class of $S^{2p} \times \mathbf{C}P^{2q}$ and Euler class of bundle $= (4q+2) \cdot$ orientation class). Furthermore g induces an involution on $\wedge X_M$ which is given on generators by $x_1 \to -x_1$, $x_2 \to -x_2$ and hence $y_1 \to y_1$, $y_2 \to -y_2$ and $y_3 \to -y_3$; i.e. $\chi_0 = 1$ and $\chi_e = 0$. According to Theorem 4.5 the Betti numbers for $M^I_{G(g)}$ are uniformly bounded, in fact $H^*(M^I_{G(g)}) = \wedge(y_1, \bar{y}_1)$.

On the other hand, let $u_1 = (4q+2)x_2^{2q} y_1 - x_1 y_3$ and $u_2 = (4q+2)x_1 y_2 - x_2 y_3$. Then a family of generators for $H(\wedge X_M, d)$ contains x_1, x_2, u_1 and u_2 (or linear combinations of these), and on cohomology $g^*(u_i) = u_i$, $i = 1, 2$ i.e. g fixes two generators of $H^*(M)$ but the sequence of Betti numbers for $M^I_{G(g)}$ is bounded.

We finally restrict our attention to spaces whose cohomology (over $\mathbf{Q}$) is spherically generated.

Definition 4.9. Let M be a 1-connected space whose cohomology is of finite type. We say that $H^*(M)$ is spherically generated if

$$\ker \zeta^* = H^+(\wedge X_M) \cdot H^+(\wedge X_M)$$

where ζ^* is the induced map on cohomology by the projection $\zeta: \wedge^+ X_M \to Q(\wedge X_M)$ (p. 280).

Note that ζ^* is the dual of the Hurewicz map. The above definition is therefore equivalent to saying that ζ^* imbeds the generators of $H^*(M)$ into $\mathrm{Hom}\,(\pi^*(M), \mathbf{Q})$.

COROLLARY 4.10. *Let M be a 1-connected space whose cohomology is finite dimensional and spherically generated, and let g be a 1-rigid map of M. Then $M^I_{G(g)}$ has an unbounded sequence of Betti numbers if the induced map g^* on cohomology $H^*(M)$ fixes at least two generators.* (1)

Proof. By hypothesis, $H^*(M)$ is spherically generated, so ζ^* induces an embedding

$$H^+(M)/H^+(M) \cdot H^+(M) \to Q(\wedge X_M)$$

(1) i.e. the subspace fixed by the linear map induced by g^* on $H^+(M)/H^+(M) \cdot H^+(M)$ has dimension $\geqslant 2$.

commuting with the induced actions by g. Hence we can choose the generators of ΛX_M so that we have two closed generators fixed by ψ_g. They give two closed generators of ΛY, and we conclude using Theorem 4.5.

Remark 4.11. According to example 8.13 of [11] any formal space (its minimal model is a formal consequence of its cohomology) has spherically generated cohomology. Thus Corollary 4.10 applies in particular to formal spaces. Among formal spaces are riemannian symmetric spaces [14] and Kähler manifolds [1] (and [11, Cor. 6.9]).

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