THE INHOMOGENEOUS MINIMA OF BINARY QUADRATIC FORMS (I).

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Table of Contents.

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1. Let $f(x, y) = ax^2 + bxy + cy^2$ be an indefinite binary quadratic form with real coefficients and discriminant $D = b^2 - 4ac > 0$. For any real numbers x_0, y_0 we define $M(f; x_0, y_0)$ to be the lower bound of $|f(x + x_0, y + y_0)|$ taken over all integer sets x, y. It is clear that if

$$x_0' \equiv x_0,\ y_0' \equiv y_0 \pmod 1 \tag{1.1}$$

then

$$M(f; x_0', y_0') = M(f; x_0, y_0). \tag{1.2}$$

We identify pairs of real numbers (x, y) with the points P of the Cartesian plane, and say that any two points are congruent if their coordinates differ by integers. Writing $M(f; P)$ for $M(f; x_0, y_0)$, we now define the inhomogeneous minimum of $f(x, y)$ to be

$$M(f) = \text{upper bound } M(f; P). \tag{1.3}$$

The upper bound is taken over all points of the plane, but in virtue of (1.2) may merely be taken over any complete set of incongruent points.

It follows from these definitions that corresponding to any point $P = (x_0, y_0)$ and any $\varepsilon > 0$ we can find an integer point (x, y) such that

$$|f(x + x_0, y + y_0)| < M(f) + \varepsilon. \tag{1.4}$$

If we can in fact satisfy

$$|f(x + x_0, y + y_0)| \leq M(f) \tag{1.5}$$

for every (x_0, y_0) and a corresponding integer point (x, y), we shall say that $M(f)$ is an *attained* minimum.

We use the customary definition of equivalent forms, without however distinguishing between proper and improper equivalence. Thus two forms $f(x, y)$, $f'(x, y)$ are equivalent if there exists a transformation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{1.6}$$

with integral coefficients and determinant $\alpha\delta - \beta\gamma = \pm 1$ such that $f'(x', y') = f(x, y)$. Writing (1.6) as $P' = T(P)$, it is clear that for any point P we have

$$M(f; P) = M(f'; T(P)). \tag{1.7}$$

Since the points $T(P_1)$, $T(P_2)$ are congruent if and only if P_1 and P_2 are congruent, it follows that

$$M(f) = M(f'). \tag{1.8}$$

Thus equivalent forms have the same minimum. We also note that, trivially,

$$M(\lambda f) = |\lambda| M(f) \tag{1.9}$$

for any real λ.

Let C be the set of points P for which $M(f; P) = M(f)$, and define $M_2(f)$ as the upper bound of $M(f; P)$ taken over all P not belonging to C. Obviously

$$M_2(f) \leq M(f). \tag{1.10}$$

If strict inequality holds in (1.10), we say that $M(f)$ is an *isolated minimum*, and we call $M_2(f)$ a *second minimum*.[1] Similarly we define successive minima $M_3(f), M_4(f), \ldots$ the sequence being strictly decreasing until a non-isolated minimum is reached.

2. The first result on the inhomogeneous minimum $M(f)$ was found by Minkowski [1]. He showed that

$$M(f) \leq \frac{1}{4}\sqrt{D}, \tag{2.1}$$

the equality sign being necessary if and only if $f(x, y) \sim bxy$. Contributions have since been made by many authors[2], who have found upper bounds for $M(f)$ in terms of a value or values assumed by $f(x, y)$ for integral x, y. In many cases, the bound determined has been precise, and more recently some particular forms have been examined in detail by Davenport [1], Varnavides [2, 3], Bambah [1], and Inkeri [3, 4, 5]. Explicit acknowledgements of these results will be made below.

In the opposite direction, Davenport [4] has proved the remarkable result that

$$M(f) > \frac{1}{128}\sqrt{D} \tag{2.2}$$

for forms not representing zero; and Prasad has sharpened this to about

$$M(f) > \frac{1}{36}\sqrt{D}. \tag{2.3}$$

A particular impetus has been given to research on this problem by its close association with the Euclidean algorithm in real quadratic number fields. An algebraic number field is said to be Euclidean if for any number $\varkappa$ of the field there exists an integer ϱ of the field such that

$$|\operatorname{norm}(\varrho + \varkappa)| < 1. \tag{2.4}$$

The elements of a real quadratic field $k(\sqrt{m})$, where m is a square-free positive integer, are of the form $x + \omega y$ with rational x, y, the integers of the field corresponding to rational integers x, y. Then norm $(x + \omega y)$ is an indefinite quadratic form $f_m(x, y)$, where

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[1] This definition is perhaps a departure from convention, since we do not exclude the possibility of $\mathfrak{C}$ containing an infinity of incongruent points. The reader should note that Inkeri [3] uses M_1 and M_2 to distinguish between what we call attained and unattained first minima.

[2] See for example Heinhold [1, 2], Davenport [1], Varnavides [1], Cassels [1], Barnes [1], Inkeri [3].

$$f_m(x, y) = x^2 - m y^2, \ \omega = \sqrt{m} \text{ if } m \equiv 2, 3 \pmod 4; \tag{2.5}$$

$$f_m(x, y) = x^2 + xy - \frac{1}{4}(m-1) y^2, \ \omega = \frac{1}{2}(1 + \sqrt{m}) \text{ if } m \equiv 1 \pmod 4. \tag{2.6}$$

If we write $\varkappa = x_0 + \omega y_0$, $\varrho = x + \omega y$, (2.4) becomes

$$|f_m(x + x_0, \ y + y_0)| < 1. \tag{2.7}$$

Thus $k(\sqrt{m})$ is Euclidean if and only if

$$M(f_m; P) < 1 \tag{2.8}$$

for all *rational* points P. A sufficient condition for this is clearly that

$$M(f_m) < 1. \tag{2.9}$$

The main interest in the problem of inhomogeneous minima has therefore been in proving (2.9), or disproving (2.8) for some particular rational P. In addition to (2.2) Davenport has shown that if $f(x, y)$ has rational coefficients,

$$M(f; P) > \frac{1}{128}\sqrt{D} \tag{2.10}$$

for some rational P. Thus there are only a finite number of Euclidean fields $k(\sqrt{m})$, and the set of such fields has now been completely determined.[1]

3. In this paper we shall be concerned with forms $f(x, y)$ with rational coefficients which do not represent zero for integral x, y not both zero. Such forms have an infinity of automorphs; in this section we discuss these automorphs and their relation to the problem of inhomogeneous minima.

In virtue of (1.9) we may take $f(x, y)$ in the form

$$f(x, y) = a x^2 + b x y + c y^2$$

where a, b, c are integers with no common factor. We shall call an integral unimodular transformation

$$T = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{3.1}$$

an *automorph* of $f(x, y)$ if $f(x, y)$ is equivalent under T to either[2] $\pm f(x, y)$. It is known that these automorphs fall into three classes:

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[1] Note, however, that Rédei is in error in stating ([1], p. 607) that $k(\sqrt{97})$ is Euclidean, as we shall show in theorem 15 below.

[2] It is convenient to make this extension of the term "automorph" to include all transformations of $|f(x, y)|$ into itself.

I. Proper transformations of $f(x, y)$ into itself, given by

$$T = \begin{pmatrix} \frac{1}{2}t - \frac{1}{2}bu & -cu \\ au & \frac{1}{2}t + \frac{1}{2}bu \end{pmatrix} \tag{3.2}$$

where t, u is any integral solution of the Pellian equation

$$t^2 - Du^2 = 4. \tag{3.3}$$

II. Improper transformations of $f(x, y)$ into $-f(x, y)$ given by (3.2), where now t, u is any integral solution of

$$t^2 - Du^2 = -4. \tag{3.4}$$

III. Improper transformations of $f(x, y)$ into itself, and proper transformations of $f(x, y)$ into $-f(x, y)$.

Automorphs of type I always exist, and there are an infinity of them, given by

$$T = \pm T_0^n \quad (n = 0, \pm 1, \pm 2, \ldots) \tag{3.5}$$

where T_0 is given by (3.2) with t, u the least positive pair satisfying (3.3).

Automorphs of type II may or may not exist. If in fact there are any solutions of (3.4), then all automorphs of types I and II are expressible in the form (3.5), where t, u is now the least positive pair satisfying (3.4). We call T_0 the *fundamental automorph* of $f(x, y)$ in each of these two cases.

Criteria for the existence of an automorph of type III are not so simple, and we shall merely note here that the "ambiguous" forms $f_m(x, y)$ of (2.5), (2.6) have respectively the automorphs

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}.$$

The properties of automorphs which are of primary interest to us are:

Theorem A. *If T is any automorph of $f(x, y)$, and P is any point, then*

$$M(f; P) = M(f; T(P)). \tag{3.6}$$

Proof: The result follows at once from (1.7).

Theorem B. *Let T be an automorph of $f(x, y) = ax^2 + bxy + cy^2$ given by* (3.2), *where $t^2 - Du^2 = \pm 4$, $t > 0$, $u > 0$. Let $\{P_i\}$ $(i = 1, 2, \ldots, N)$ be a finite set of incongruent points such that the set $\{T(P_i)\}$ is a permutation of $\{P_i\}$ modulo* 1. *Suppose that*

$$M(f; P_i) < K \quad (i = 1, 2, \ldots, N). \tag{3.7}$$

Then there exists a point (x, y) *with the properties:*

(i) $(x, y) \equiv P_i$ *for some* i,

(ii) $|f(x, y)| < K$,

(iii) $D y^2 < K|a|(t+2)$ *if* $t^2 - D u^2 = 4$,

$t y^2 < K|a| u^2$ *if* $t^2 - D u^2 = -4$.

Proof:[1] Consider the set of points satisfying (i) and (ii), which is not null by (3.7). Since there are only a finite number of points P_i, we can choose a pair (x, y) in this set for which $|y|$ has its least possible value. The theorem will be established if we show that this value of $|y|$ satisfies (iii). Suppose it does not. Then, taking first the case when $t^2 - D u^2 = 4$, we have

$$D y^2 \geq K|a|(t+2). \tag{3.8}$$

We may suppose for simplicity that $a > 0$, since we may replace $f(x, y)$ by $-f(x, y)$. Also, since $f(-x, -y) = f(x, y)$, we may take $y > 0$ in (3.8). Then we have

$$|f(x, y)| < K \leq \frac{D y^2}{a(t+2)},$$

i.e.

$$\left|\left(x + \frac{b}{2a} y\right)^2 - \frac{D}{4a^2} y^2\right| < \frac{D y^2}{a^2(t+2)},$$

or

$$\frac{D y^2}{4a^2}\left(1 - \frac{4}{t+2}\right) < \left(x + \frac{b}{2a} y\right)^2 < \frac{D y^2}{4a^2}\left(1 + \frac{4}{t+2}\right).$$

On replacing D by $(t^2 - 4)/u^2$ and simplifying, this gives

$$\frac{y^2(t-2)^2}{4a^2u^2} < \left(x + \frac{b}{2a} y\right)^2 < \frac{y^2(t-2)(t+6)}{4a^2u^2} < \frac{y^2(t+2)^2}{4a^2u^2},$$

or

$$\frac{y(t-2)}{2au} < \left|x + \frac{b}{2a} y\right| < \frac{y(t+2)}{2au}. \tag{3.9}$$

Thus *either*

$$\frac{y(t-2)}{2au} < x + \frac{b}{2a} y < \frac{y(t+2)}{2au},$$

i.e.

$$\left|aux - \frac{1}{2}(t - bu) y\right| < y; \tag{3.10}$$

or

[1] Infinite descent, based on the method of Davenport [1], Lemma 3.

$$\frac{y(t-2)}{2au} < -x - \frac{b}{2a}y < \frac{y(t+2)}{2au},$$

i.e.

$$\left| aux + \frac{1}{2}(t+bu)y \right| < y. \tag{3.11}$$

But the y-coordinates of $T(x, y)$, $T^{-1}(x, y)$ are respectively $aux + \frac{1}{2}(t+bu)y$, $-aux + \frac{1}{2}(t-bu)y$; and by hypothesis each of the points $T(x, y)$, $T^{-1}(x, y)$ is congruent to some P_i. Hence, since either (3.10) or (3.11) holds, we have a point satisfying (i) and (ii) with a smaller value of $|y|$, which contradicts the initial choice of y.

Consider now the case when $t^2 - Du^2 = -4$. Then we have

$$ty^2 \geq K|a|u^2,$$

where as above we may suppose that $a > 0$, $y > 0$. Then

$$|f(x, y)| < K \leq \frac{ty^2}{au^2},$$

i.e.

$$\frac{y^2}{4a^2}\left(D - \frac{4t}{u^2}\right) < \left(x + \frac{b}{2a}y\right)^2 < \frac{y^2}{4a^2}\left(D + \frac{4t}{u^2}\right).$$

Replacing D by $(t^2+4)/u^2$ and taking the square root, this yields precisely the inequality (3.9). The proof may now be completed as above.

This theorem (which is best possible) is designed to give the least number of possible values of y to test. Results of this sort are, of course, not original; but we have not been able to find an explicit statement of this theorem.

4. It is clear from Theorems A and B that, for any given automorph T of $f(x, y)$, the set of points F for which $T(F) \equiv F$ will be of special interest. Such points will be called *fixed points* of T. For any integral point A, there is a fixed point F satisfying

$$T(F) = F + A,$$

and if the matrix $(T - I)$ is non-singular, this equation defines a unique, rational F. If $(T - I)$ is singular, it is easily seen, since $|T| = \pm 1$, that T is of finite order (in fact $T^2 = \pm I$); we shall exclude such transformations from the following discussion.

We now prove some important results on transformations of infinite order and their fixed points. For an integral unimodular T,

$$T = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \tag{4.1}$$

we set

$$\varepsilon = \alpha\delta - \beta\gamma = \pm 1, \tag{4.2}$$

$$\varkappa = \begin{cases} \dfrac{1}{2\beta}[\delta - \alpha - \sqrt{\{(\alpha+\delta)^2 - 4\varepsilon\}}] & \text{if } \alpha + \delta > 0, \\ \dfrac{1}{2\beta}[\delta - \alpha + \sqrt{\{(\alpha+\delta)^2 - 4\varepsilon\}}] & \text{if } \alpha + \delta < 0. \end{cases} \tag{4.3}$$

We shall always suppose that $\varkappa$ is *real.*

It is convenient to define the expression

$$P \in \mathcal{R} \pmod 1,$$

where P is a point and $\mathcal{R}$ a point set, as meaning that there exists a point Q congruent to P modulo 1 lying in $\mathcal{R}$.

Theorem C.[1] *Let T, defined by (4.1), be of infinite order, and let $\mathcal{R}$ be a bounded point set such that, for some given set $\mathcal{R}^*$ and some given integer point A, any point $P \in \mathcal{R}$ has the property that either $T(P) \in \mathcal{R}^*$ (mod 1) or $T(P) - A \in \mathcal{R}$. Let $F = (x_0, y_0)$ be the fixed point of T defined by $T(F) = F + A$. Then if $P = (x, y) \in \mathcal{R}$, and $T^n(P)$ is not congruent to a point of $\mathcal{R}^*$ for any $n = 1, 2, \ldots$, P lies on the line*

$$y - y_0 = \varkappa(x - x_0)$$

through F, where $\varkappa$ is defined by (4.3). Moreover, F belongs to the closure $\overline{\mathcal{R}}$ of $\mathcal{R}$, and for each n there exists a point $Q_n \equiv T^n(P)$ such that $Q_n \to F$ as $n \to +\infty$.

For the proof of Theorem C we need two preliminary lemmas.

Lemma 1. *Let S be the transformation*

$$(x', y') = (tx, \pm y/t),$$

where t is real and $|t| > 1$, and let $\mathcal{R}$ be a bounded point set. Suppose that $S^n(P) \in \mathcal{R}$ for all $n \geq 0$. Then P lies on the line $x = 0$, the origin O belongs to the closure $\overline{\mathcal{R}}$ of $\mathcal{R}$, and $S^n(P) \to 0$ as $n \to +\infty$.

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[1] The reader will find the purpose of this theorem clearer if he refers to the first part of § 8, where there is a general account of the methods used.

Theorem C is basically the well-known result: if the vector $\overline{FP}$ remains bounded under all positive powers of the transformation T, then it is an eigenvector of T. In fact the region $\mathcal{R}^*$ and the point A of Theorem C arise naturally in the applications, whereas an explicit boundedness condition does not; it is therefore more convenient to have the theorem stated in this more complex form.

Proof: Let P be the point (x, y). Then $S^n(P) = (t^n x, \pm t^{-n} y) \in \mathfrak{R}$, and so, since $\mathfrak{R}$ is bounded, $t^n x$ is bounded as $n \to +\infty$. Since $|t| > 1$, it follows that $x = 0$. Finally, $S^n(P) = (0, \pm t^{-n} y) \to 0$ as $n \to +\infty$, so that $O \in \overline{\mathfrak{R}}$.

Lemma 2. *Let T be defined by (4.1), and let $\mathfrak{R}$ be a bounded point set. Suppose that $P = (x, y)$ is a point such that $T^n(P) \in \mathfrak{R}$ for all $n \geq 0$. Then P lies on the line*

$$y = \varkappa x,$$

where $\varkappa$ is defined by (4.3); $O \in \overline{\mathfrak{R}}$; and $T^n(P) \to O$ as $n \to +\infty$.

Proof: Let λ_1, λ_2 be the roots of the equation

$$\begin{vmatrix} \alpha - \lambda & \beta \\ \gamma & \delta - \lambda \end{vmatrix} = \lambda^2 - (\alpha + \delta)\lambda + \varepsilon = 0.$$

Then it is well known that the linear substitution

$$X = (\delta - \lambda_1)x - \beta y,$$

$$Y = (\delta - \lambda_2)x - \beta y$$

reduces T to the form

$$(X', Y') = (\lambda_1 X, \lambda_2 Y).$$

Since $\lambda_1 \lambda_2 = \varepsilon = \pm 1$, and λ_1, λ_2 are real and not both ± 1, we may choose $|\lambda_1| > 1$. Lemma 1 now shows that $O \in \overline{\mathfrak{R}}$, that $T^n(P) \to O$ as $n \to +\infty$, and that P lies on the line

$$X = (\delta - \lambda_1)x - \beta y = 0.$$

Now $a + \delta \neq 0$, since T is of infinite order; and

$$\lambda_1 = \frac{1}{2}[\alpha + \delta + \sqrt{\{(\alpha + \delta)^2 - 4\varepsilon\}}] \text{ if } \alpha + \delta > 0,$$

$$\lambda_1 = \frac{1}{2}[\alpha + \delta - \sqrt{\{(\alpha + \delta)^2 - 4\varepsilon\}}] \text{ if } \alpha + \delta < 0,$$

so that the above line is $y = \varkappa x$, as required.

Proof of Theorem C: Suppose that $P_0 \in \mathfrak{R}$ and that $T^n(P_0)$ is not congruent to a point of $\mathfrak{R}^*$ for any $n \geq 0$.

Let A be the point (a, b), and define an inhomogeneous transformation U by $U(P) = T(P) - A$. Then, since $U(P) \equiv T(P)$ for all P, $U^n(P)$ is not congruent to a point of $\mathfrak{R}^*$ for any $n \geq 0$.

We now change the origin to F, so that the new coordinates (x', y') are given by

$$(x', y') = (x - x_0, y - y_0),$$

or $P' = P - F$. The transformation U then becomes

$$(x', y') \to (\alpha x' + \beta y', \gamma x' + \delta y').$$

Now by hypothesis, $P_0 \in \mathcal{R}$,

$$U(P_0) = T(P_0) - A = Q_1 \in \mathcal{R},$$

$$U^2(P_0) = U(Q_1) = T(Q_1) - A = Q_2 \in \mathcal{R},$$

and so on, so that $Q_n = U^n(P_0) \in \mathcal{R}$ for all $n \geq 0$. It follows at once from Lemma 2 that $F \in \overline{\mathcal{R}}$, $Q_n \to F$ as $n \to +\infty$, and that P_0 lies on the line $y' = \varkappa x'$, i.e. on the line $y - y_0 = \varkappa(x - x_0)$.

We note that there is a simple and obvious generalization of Theorem C to the case of a finite number of bounded point sets:

Theorem C'. *Let T, defined by* (4.1), *be of infinite order, and let $\mathcal{R}_0, \mathcal{R}_1, \ldots, \mathcal{R}_{k-1}$ be a finite number of bounded point sets. Suppose that for some $\mathcal{R}^*$ and some integer points $A_1, \ldots, A_k$, every point $P_i \in \mathcal{R}_i$ $(i = 0, 1, \ldots, k-1)$ has the property that either $T(P_i) \in \mathcal{R}^*$* (mod 1) *or $T(P_i) - A_{i+1} \in \mathcal{R}_{i+1}$ (where $\mathcal{R}_k$ is interpreted to be $\mathcal{R}_0$). Let*

$$A = A_k + T(A_{k-1}) + T^2(A_{k-2}) + \cdots + T^{k-1}(A_1),$$

and let $F = (x_0, y_0)$ be the fixed point of T^k defined by $T^k(F) = F + A$. Then if $P = (x, y) \in \mathcal{R}_0$ and $T^n(P)$ is not congruent to a point of $\mathcal{R}^*$ for any $n \geq 0$, P must lie on the line

$$y - y_0 = \varkappa(x - x_0).$$

Moreover, $F \in \overline{\mathcal{R}}_0$, and for each n there exists a point $Q_{nk} \equiv T^{nk}(P)$ such that $Q_{nk} \to F$ as $n \to +\infty$.

Proof: Suppose that $P \in \mathcal{R}_0$ and that $T^n(P)$ is not congruent to a point of $\mathcal{R}^*$ for any $n \geq 0$. Then by hypothesis,

$$T(P) = P_1 + A_1, \quad P_1 \in \mathcal{R}_1,$$

$$T(P_1) = P_2 + A_2, \quad P_2 \in \mathcal{R}_2,$$

$$\cdots\cdots\cdots\cdots$$

$$T(P_{k-1}) = P_k + A_k, \quad P_k \in \mathcal{R}_k = \mathcal{R}_0.$$

Hence

$$T^k(P) = P_k + A_k + T(A_{k-1}) + \cdots + T^{k-1}(A_1) = P_k + A,$$

so that $T^k(P) - A \in \mathcal{R}_0$.

The required results now follow from Theorem C, on replacing T by T^k. The constant $\varkappa$ is unchanged, since T and T^k have the same eigenvectors.

The essential point of Theorem C is that we consider the transformations T^n only for *positive* n. If we consider also T^n for $n < 0$, we have the following simpler result, due to Cassels[1]:

Theorem D. *Suppose that the hypotheses of Theorem C are satisfied, and further that for any point* $P \in \mathcal{R}$, $T^{-1}(P)$ *is congruent to a point of either* $\mathcal{R}^*$ *or of* $\mathcal{R}$. *Then if* $P \in \mathcal{R}$ *and* $T^n(P)$ *is not congruent to a point of* $\mathcal{R}^*$ *for any* $n \gtrless 0$, P *is the fixed point* F.

Proof: The result is most simply proved by extending Lemma 1 above to show that $S^n(P)$ can belong to $\mathcal{R}$ for all $n \gtreqqless 0$ only if P is the origin, and by making the corresponding extension in Lemma 2. The introduction of the "asymptotic line" $y - y_0 = \varkappa(x - x_0)$ is then unnecessary. However, we may deduce the result directly from Theorem C. We first note that there exists an integer point B such that for any $P \in \mathcal{R}$, either $T^{-1}(P) \in \mathcal{R}^*$ (mod 1), or $T^{-1}(P) - B \in \mathcal{R}$. For if $P \in \mathcal{R}$, $T^{-1}(P) \in \mathcal{R}$ (mod 1), then

$$T^{-1}(P) = B + Q, \quad Q \in \mathcal{R}, \quad B \text{ integral},$$

and so

$$P = T(B) + T(Q).$$

The hypothesis of Theorem C now show that $T(B) = -A$, so that $B = -T^{-1}(A)$, which is independent of P.

Suppose now that $P \in \mathcal{R}$ and that $T^n(P)$ is not congruent to a point of $\mathcal{R}^*$ for any n. Theorem C shows that $P = (x, y)$ lies on the line

$$y - y_0 = \varkappa(x - x_0). \tag{4.4}$$

Also, applying Theorem C with T replaced by T^{-1}, P must also lie on the line

$$y - y_0 = \varkappa'(x - x_0), \tag{4.5}$$

where $\varkappa'$ is obtained from T^{-1} in the same way as $\varkappa$ is obtained from T. It is easily seen that $\varkappa' \neq \varkappa$ ($\varkappa'$ being in fact derived from $\varkappa$ by changing the sign of the radical). Hence P must be F, the point of intersection of the lines (4.4), (4.5).

There is an obvious generalization of Theorem D, corresponding to the above generalization of Theorem C.

5. We are now in a position to prove some general results about the inhomogeneous minima of *rational* forms $f(x, y)$. We shall abbreviate $M(f)$, $M(f; P)$ to

[1] Quoted by Bambah [1].

M, $M(P)$ respectively; and we shall write $f(P)$ for $f(x, y)$. The first result, which we shall quote without proof, is given by Heinhold ([1], p. 660):

Theorem E. *If M is an attained minimum, there is at least one point P for which $M(P) = M$, and an integral point Q corresponding to P such that $f(P+Q) = \pm M$.*

The proof is based on a simple application of the Heine-Borel covering theorem. If the minimum is unattained, the first part of the Theorem is still (trivially) true; the second part is true also, but we do not prove this here.

Theorem F. *$M(P)$ is upper semi-continuous; i.e. for any point P and any $\varepsilon > 0$, we can find a $\delta > 0$ such that $M(P') < M(P) + \varepsilon$ whenever $|P' - P| < \delta$.*

The proof is immediate, since $f(P)$ is a continuous function of P.

Our next result is new, and gives a criterion for the existence of isolated minima. We use the (permanent) notation S for the closed unit square: $|x| \leq \frac{1}{2}$, $|y| \leq \frac{1}{2}$; clearly any point P of the plane is congruent to a point of S.

Theorem G. *Suppose that $M(P) < k$ for all but a finite set of points of S. Then there exists a number $k' < k$ such that $M(P) \leq k'$ for all points $P \in S$ except the given finite set.*

Proof: Let $\{P_i\}$ $(i = 1, 2, \ldots, N)$ be the set of points $P \in S$ for which $M(P) \geq k$, and let T_0 be the fundamental automorph of $f(P)$. From Theorem A we see that T_0 permutes the set $\{P_i\}$ (mod 1), and so there is an integer r, $1 \leq r \leq N!$, such that P_i is a fixed point of $T = T_0^r$ for each i. Let $T(P_i) = P_i + A_i$.

Let $\mathcal{R}_i^{(1)}$ be the region

$$|P - P_i| < \varepsilon_1 \quad (i = 1, 2, \ldots, N),$$

where ε_1 is so small that no two of the sets $\overline{\mathcal{R}}_i^{(1)}$ have a common point. Now choose $\varepsilon_2 \leq \varepsilon_1$ so that, if $\mathcal{R}_i^{(2)}$ is the region

$$|P - P_i| < \varepsilon_2 \quad (i = 1, 2, \ldots, N),$$

we have $T(\mathcal{R}_i^{(2)}) < \mathcal{R}_i^{(1)} + A_i$, $T^{-1}(\mathcal{R}_i^{(2)}) < \mathcal{R}_i^{(1)} - T^{-1}(A_i)$. This is always possible since $T(P_i) = P_i + A_i$, $T^{-1}(P_i) = P_i - T^{-1}(A_i)$, and a sufficiently small neighbourhood of P_i transforms continuously with P_i under T and T^{-1}. Now let

$$\mathcal{R}^* = S - \sum_i \mathcal{R}_i^{(2)}.$$

Then $\mathcal{R}^*$ is a closed bounded set, and since $\mathcal{R}^*$ contains none of the points P_i, $M(P) < k$ for all $P \in \mathcal{R}^*$. Thus for each $P \in \mathcal{R}^*$ there exists an integer point L such that P is an interior point of

$$|f(P + L)| \leq k.$$

It follows from the Heine-Borel theorem that there exist a finite number of such sets, defined by $L_1, L_2, \ldots, L_n$, say, such that any $P \in \mathcal{R}^*$ is an interior point of one of them. Since $\min\limits_{1 \leq i \leq n} |f(P + L_i)|$ is a continuous function of P and is strictly less than k in $\mathcal{R}^*$, it attains its upper bound $k' < k$. Hence $M(P) \leq k' < k$ for $P \in \mathcal{R}^*$.

It remains to show that $M(P) \leq k'$ if $P \in \mathcal{R}_i^{(2)}$, $P \neq P_i$, for any $i = 1, 2, \ldots, N$. Now by Theorem A, $M(P) = M(T^n(P)) \leq k'$ if $T^n(P) \in \mathcal{R}^*$ (mod 1); also from the construction of the sets $\mathcal{R}_i^{(1)}$, $\mathcal{R}_i^{(2)}$ we see that if $P \in \mathcal{R}_i^{(2)}$,

$$\textit{either } T(P) \in \mathcal{R}^* \pmod 1 \textit{ or } T(P) - A_i \in \mathcal{R}_i^{(2)},$$

and

$$\textit{either } T^{-1}(P) \in \mathcal{R}^* \pmod 1 \textit{ or } T^{-1}(P) + T^{-1}(A_i) \in \mathcal{R}_i^{(2)}.$$

Theorem D now shows that either $T^n(P) \in \mathcal{R}^*$ (mod 1) for some n, or $P = P_i$, which completes the proof.

6. Before proceeding to the evaluation of $M(f)$ for particular forms $f(x, y)$, we shall establish the following arithmetical result.

For any real number α, we write

$$\phi(\alpha) = \underset{\lambda}{\text{u.b.}} \ \underset{x}{\text{l.b.}} \ |(x + \lambda)^2 - \alpha|, \tag{6.1}$$

where the upper bound is taken over all real λ and the lower bound over all *integers* x. Thus $\phi(\alpha)$ is the lower bound of all numbers $m(\alpha)$ such that the inequality

$$|(x + \lambda)^2 - \alpha| \leq m(\alpha)$$

can be satisfied for every real λ and some corresponding integer x.

Partial results on the value of $\phi(\alpha)$ have been known for many years, and have been used[1] to deduce upper bounds for $M(f)$. Thus it has been shown that $\phi(\alpha) < 1$ if $0 \leq \alpha \leq 2$, $\alpha \neq \frac{5}{4}$, a result which has an immediate application to the problem of the Euclidean algorithm outlined in § 2. The strongest result has been given by Davenport.[2] It may be stated as:

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[1] See for example MORDELL [1], PERRON [1], DAVENPORT [1], VARNAVIDES [1].

[2] DAVENPORT [2], Lemma 5.

Theorem H. *If* $B \geq \frac{1}{4}$ *and*

$$0 \leq \alpha \leq B + \frac{1}{4}[2B]^2, \tag{6.2}$$

then

$$\phi(\alpha) \leq B. \tag{6.3}$$

This form of statement is particularly useful in the applications, since we usually wish to determine the α for which $\phi(\alpha)$ is less than some preassigned number B.

It is, however, quite easy to find an explicit formula for $\phi(\alpha)$:

Theorem J.[1] (i) *If* $\alpha < \frac{1}{8}$, *then*

$$\phi(\alpha) = \frac{1}{4} - \alpha, \tag{6.4}$$

the upper bound being attained only when $\lambda \equiv \frac{1}{2} \pmod 1$.

(ii) *If* $\alpha \geq \frac{1}{8}$, *let* n *be the non-negative integer determined by*

$$\frac{1}{4}n^2 + \frac{1}{4}n + \frac{1}{8} \leq \alpha < \frac{1}{4}(n+1)^2 + \frac{1}{4}(n+1) + \frac{1}{8} = \frac{1}{4}n^2 + \frac{3}{4}n + \frac{5}{8}. \tag{6.5}$$

Then

$$\phi(\alpha) = \alpha - \frac{1}{4}n^2 \quad \text{if} \quad \frac{1}{4}n^2 + \frac{1}{4}n + \frac{1}{8} \leq \alpha \leq \frac{1}{4}n^2 + \frac{1}{2}n + \frac{1}{2}, \tag{6.6}$$

$$\phi(\alpha) = \left(\frac{1}{2}n + 1\right)^2 - \alpha \quad \text{if} \quad \frac{1}{4}n^2 + \frac{1}{2}n + \frac{1}{2} < \alpha < \frac{1}{4}n^2 + \frac{3}{4}n + \frac{5}{8}. \tag{6.7}$$

Moreover, the upper bound $\phi(\alpha)$ *is attained only when* $\lambda \equiv \frac{1}{2}n \pmod 1$, *or when* $\alpha = \frac{1}{4}n^2 + \frac{1}{4}n + \frac{1}{8}$ *and* $2\lambda \equiv 0 \pmod 1$.

Proof: (i) If $\alpha < \frac{1}{8}$ we choose, for any λ, an integer x_0 to satisfy $|x_0 + \lambda| \leq \frac{1}{2}$. Then

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[1] The graph of $\phi(\alpha)$ for $\alpha \geq 0$ is easily seen to be a zig-zag line with peaks at the points $\alpha = \frac{1}{4}n^2 + \frac{1}{2}n + \frac{1}{2}$ $(n = 0, 1, \ldots)$. To determine the values of α for which $\phi(\alpha)$ does not exceed some given number K, the reader may find it more convenient to draw the graph of $\phi(\alpha)$ in terms of α and consider its intersections with the line $\phi(\alpha) = K$.

$$(x_0 + \lambda)^2 - \alpha \leq \frac{1}{4} - \alpha,$$

$$(x_0 + \lambda)^2 - \alpha \geq -\alpha > \alpha - \frac{1}{4},$$

so that

$$|(x_0 + \lambda)^2 - \alpha| \leq \frac{1}{4} - \alpha, \quad \phi(\alpha) \leq \frac{1}{4} - \alpha. \tag{6.8}$$

Clearly equality can occur in (6.8) only if $|x_0 + \lambda| = \frac{1}{2}$, so that $\lambda \equiv \frac{1}{2}$ (mod 1). If in fact we have $\lambda \equiv \frac{1}{2}$ (mod 1), then for any integer x

$$\textit{either } |x + \lambda| = \frac{1}{2}, \quad |(x + \lambda)^2 - \alpha| = \frac{1}{4} - \alpha,$$

$$\textit{or} \quad |x + \lambda| \geq \frac{3}{2}, \quad |(x + \lambda)^2 - \alpha| \geq \frac{9}{4} - \alpha > \frac{1}{4} - \alpha.$$

Hence $\phi(\alpha) \geq \frac{1}{4} - \alpha$, and part (i) of the theorem is proved.

(ii) Suppose next that for some $n \geq 0$,

$$\frac{1}{4}n^2 + \frac{1}{4}n + \frac{1}{8} \leq \alpha \leq \frac{1}{4}n^2 + \frac{1}{2}n + \frac{1}{2}. \tag{6.9}$$

Then for any λ we can choose an integer x_0 to satisfy

$$\frac{1}{2}n \leq |x_0 + \lambda| \leq \frac{1}{2}(n + 1).$$

Then

$$(x_0 + \lambda)^2 - \alpha \leq \frac{1}{4}(n + 1)^2 - \alpha \leq \alpha - \frac{1}{4}n^2, \quad \text{using (6.9);}$$

$$(x_0 + \lambda)^2 - \alpha \geq \frac{1}{4}n^2 - \alpha.$$

Hence

$$|(x_0 + \lambda)^2 - \alpha| \leq \alpha - \frac{1}{4}n^2, \quad \phi(\alpha) \leq \alpha - \frac{1}{4}n^2. \tag{6.10}$$

Clearly equality can arise in (6.10) only if $|x_0 + \lambda| = \frac{1}{2}n$, $\lambda \equiv \frac{1}{2}n$ (mod 1); or if $\alpha = \frac{1}{4}n^2 + \frac{1}{4}n + \frac{1}{8}$ and $|x_0 + \lambda| = \frac{1}{2}(n + 1)$, $\lambda \equiv \frac{1}{2}(n + 1)$ (mod 1).

If now in fact we have $\lambda \equiv \frac{1}{2}n$ (mod 1), then for any integer x, *either* $|x+\lambda| = \frac{1}{2}n, \quad |(x+\lambda)^2-\alpha| = \alpha - \frac{1}{4}n^2,$

or $|x+\lambda| \geq \frac{1}{2}(n+2), \quad |(x+\lambda)^2-\alpha| \geq \frac{1}{4}(n+2)^2 - \alpha \geq \alpha - \frac{1}{4}n^2,$ by (6.9),

or (when $n \geq 2$) $|x+\lambda| \leq \frac{1}{2}(n-2), \quad |(x+\lambda)^2-\alpha| \geq \alpha - \frac{1}{4}(n-2)^2 > \alpha - \frac{1}{4}n^2.$

Thus in all cases,

$$\operatorname*{l.b.}_{x} |(x+\lambda)^2-\alpha| \geq \alpha - \frac{1}{4}n^2, \tag{6.11}$$

so that $\phi(\alpha) \geq \alpha - \frac{1}{4}n^2$, and therefore $\phi(\alpha) = \alpha - \frac{1}{4}n^2$.

It may be similarly verified that (6.11) also holds when $\alpha = \frac{1}{4}n^2 + \frac{1}{4}n + \frac{1}{8}$ and $\lambda \equiv \frac{1}{2}(n+1) \pmod 1$.

(iii) Suppose finally that

$$\frac{1}{4}n^2 + \frac{1}{2}n + \frac{1}{2} < \alpha < \frac{1}{4}n^2 + \frac{3}{4}n + \frac{5}{8}. \tag{6.12}$$

For any λ we can choose an integer x_0 to satisfy

$$\frac{1}{2}(n+1) \leq |x_0+\lambda| \leq \frac{1}{2}(n+2).$$

Then

$$(x_0+\lambda)^2 - \alpha \geq \frac{1}{4}(n+1)^2 - \alpha > \alpha - \frac{1}{4}(n+2)^2, \text{ using (6.12)};$$

$$(x_0+\lambda)^2 - \alpha \leq \frac{1}{4}(n+2)^2 - \alpha.$$

Hence

$$|(x_0+\lambda)^2-\alpha| \leq \frac{1}{4}(n+2)^2 - \alpha, \quad \phi(\alpha) \leq \frac{1}{4}(n+2)^2 - \alpha. \tag{6.13}$$

Equality can arise in (6.13) only if $|x_0+\lambda| = \frac{1}{2}(n+2)$, i.e. if $\lambda \equiv \frac{1}{2}n \pmod 1$. If in fact we have $\lambda \equiv \frac{1}{2}n \pmod 1$, then for any integer x,

either $|x+\lambda| = \frac{1}{2}(n+2), \quad |(x+\lambda)^2-\alpha| = \frac{1}{4}(n+2)^2 - \alpha,$

or $|x+\lambda| \geq \frac{1}{2}(n+4), \quad |(x+\lambda)^2-\alpha| \geq \frac{1}{4}(n+4)^2 - \alpha > \frac{1}{4}(n+2)^2 - \alpha,$

or $|x+\lambda| \leq \frac{1}{2}n, \quad |(x+\lambda)^2-\alpha| \geq \alpha - \frac{1}{4}n^2 > \frac{1}{4}(n+2)^2 - \alpha,$ by (6.12).

Thus in all cases

$$\underset{x}{\text{l.b.}}\ |(x+\lambda)^2-\alpha| \geq \frac{1}{4}(n+2)^2-\alpha,$$

so that $\phi(\alpha) \geq \frac{1}{4}(n+2)^2-\alpha$, and therefore $\phi(\alpha) = \frac{1}{4}(n+2)^2-\alpha$.

We note that Theorem H may be simply deduced from Theorem J, though we do not give the deduction here.

7. A weaker form of Theorem H was used by Davenport [1] to obtain an estimate for $M(f)$ in terms of a value a assumed by $f(x, y)$ for coprime integers x, y; and Theorem H itself was applied by Varnavides [1] in the same way to deduce a correspondingly stronger estimate. We shall not go into details of these results, but we shall make frequent applications of Davenport's method.

Let $f(x, y) = ax^2 + bxy + cy^2$ be a real non-zero form, so that $a \neq 0$. Then

$$|f(x, y)| = |a| \left| \left(x + \frac{b}{2a}y\right)^2 - \frac{D}{4a^2}y^2 \right|.$$

If $P = (x_0, y_0)$ is any point, we have

$$|f(u + x_0, y_0)| = |a| \left| \left(u + x_0 + \frac{b}{2a}y_0\right)^2 - \frac{D}{4a^2}y_0^2 \right|,$$

from which we deduce at once

$$M(f; P) \leq |a|\, \phi\left(\frac{D}{4a^2}y_0^2\right). \tag{7.1}$$

As a corollary, we have

Theorem K. *Suppose that for some $K > 0$ the inequality*

$$\phi\left(\frac{D}{4a^2}y^2\right) \leq \frac{K}{|a|} \tag{7.2}$$

holds for a complete set of incongruent values of y (mod 1). *Then $M(f) \leq K$, and moreover $M(f; P) < K$ except possibly for those points (if any) for which there is equality in* (7.2) *and also in the relevant equality clause of Theorem J, where*

$$\lambda = x + \frac{b}{2a}y. \tag{7.3}$$

It is sometimes possible to obtain the precise value of $M(f)$ by using Theorem K. As Varnavides [1] has pointed out, Heinhold's results, [1], for norm-forms $f_m(x, y)$ may be deduced very simply from Theorem K and the case $B = \frac{1}{4}$ of Theorem H;

and these estimates are known[1] to be best possible for many general classes of forms $f_m(x, y)$. By applying the stronger Theorem J, we have been able to find simple proofs for the minima of some further classes of norm-forms.

Theorem 1. *Let*

$$f_m(x, y) = x^2 + xy - \frac{m-1}{4} y^2,$$

where

$$m = (2n+1)^2 - 4, \quad n \geq 3.$$

Then

$$M(f) = \frac{n^2 + n - 1}{2n + 3},$$

and is attained only for points

$$P \equiv \pm \left(\frac{n}{2} + \frac{n+2}{2(2n+3)}, \frac{n+1}{2n+3}\right).$$

Proof: (i) Take $f(x, y) = f_m(x, y)$, and let $K = (n^2 + n - 1)/(2n + 3)$. Since $[2K] = n - 1$, we have

$$\phi\left(\frac{m}{4} y^2\right) \leq K \tag{7.4}$$

by Theorem H, provided that

$$\frac{m}{4} y^2 \leq K + \frac{1}{4}(n-1)^2, \tag{7.5}$$

and, by Theorem J, also, provided that

$$\frac{1}{4}(n+1)^2 - K \leq \frac{1}{4} m y^2 \leq K + \frac{1}{4} n^2. \tag{7.6}$$

Substituting for K and m, (7.5) becomes

$$|y| \leq \frac{n+1}{2n+3}, \tag{7.7}$$

and (7.6) becomes

$$k_n < |y| \leq \frac{n+2}{2n+3}, \tag{7.8}$$

where

$$k_n^2 = \frac{2n^3 + 3n^2 + 4n + 7}{(2n+3)^2(2n-1)} < \frac{1}{4}, \text{ since } n \geq 3.$$

We may therefore replace (7.8) by the smaller range

[1] See Inkeri [3], pp. 17–26, where an omission in Varnavides' proof is also rectified.

$$\frac{1}{2} \leq |y| \leq \frac{n+2}{2n+3}. \tag{7.9}$$

The intervals (7.7), (7.9) clearly include a complete set of incongruent values of y, so that by Theorem K, $M(f) \leq K$, and $M(f; P) < K$ except possibly when

$$y \equiv \pm \frac{n+1}{2n+3}, \quad x + \frac{1}{2} y \equiv \frac{1}{2}(n-1) \pmod{1},$$

which gives the points stated in the Theorem.

(ii) It remains to show that $M(f; P) = K$ where

$$P = \left(\frac{n}{2} + \frac{n+2}{2(2n+3)}, \frac{n+1}{2n+3}\right).$$

For this we use Theorem B.

The fundamental solution of the Pellian equation $t^2 - m u^2 = 4$ is clearly given by $t = 2n+1$, $u = 1$, giving the fundamental automorph

$$T = \begin{pmatrix} n & m \\ 1 & n+1 \end{pmatrix}$$

of $f(x, y)$. Now it is easily verified that $T(P) \equiv -P$. Hence by Theorem B, if $M(f; P) < K$, there exists a point $(x, y) \equiv \pm P$ with $|f(x, y)| < K$ and

$$y^2 < \frac{K(t+2)}{D} = \frac{n^2+n-1}{(2n+1)^2-4} < \frac{1}{4},$$

i.e.

$$|y| < \frac{1}{2}.$$

Since $f(x, y) = f(-x, -y)$, it follows that there is an integer u for which

$$\left| f\left(u + \frac{n}{2} + \frac{n+2}{4n+6}, \frac{n+1}{2n+3}\right)\right| < K,$$

i.e.

$$g(u) = \left| u^2 + (n+1)u - \frac{n^2+n-1}{2n+3} \right| < K = \frac{n^2+n-1}{2n+3}. \tag{7.10}$$

But it is easily seen that $g(0) = g(-n-1) = K$, and that $g(u) > K$ for $u \neq 0$, $-n-1$. Hence (7.10) cannot be satisfied, and so $M(f; P) \geq K$. In view of (i) we now have $M(f; P) = K$, as required.

Theorem 2. *Let*

$$f_m(x, y) = x^2 - m y^2,$$

where

$$m = (2n+1)^2 - 2, \quad n \geq 2.$$

Then

$$M(f_m) = \frac{8n^3 + 6n^2 - 6n + 1}{2m} = \left(n - \frac{1}{4}\right)\left(1 - \frac{1}{m}\right),$$

and is attained only for points

$$P \equiv \pm \left(\frac{1}{2}, \frac{1}{2} - \frac{n}{m}\right).$$

Proof: (i) Take $f(x, y) = 2x^2 + 2(2n+1)xy + y^2$, which is equivalent to $f_m(x, y)$ under the transformation $(x, y) \to ((2n+1)x + y, x)$. Write

$$K = \frac{8n^3 + 6n^2 - 6n + 1}{2m} = \left(n - \frac{1}{4}\right)\left(1 - \frac{1}{m}\right).$$

Since $[K] = n - 1$, we have

$$\phi\left(\frac{1}{4}my^2\right) \leq \frac{1}{2}K \tag{7.11}$$

by Theorem H, provided that

$$\frac{1}{4}my^2 \leq \frac{1}{4}(n-1)^2 + \frac{1}{2}K, \tag{7.12}$$

and also, by Theorem J, provided that

$$\frac{1}{4}(n+1)^2 - \frac{1}{2}K \leq \frac{1}{4}my^2 \leq \frac{1}{4}n^2 + \frac{1}{2}K. \tag{7.13}$$

Substituting for K and m, (7.12) reduces to

$$|y| \leq \frac{n(2n+1)}{m}, \tag{7.14}$$

and (7.13) reduces to

$$k_n \leq |y| \leq \frac{2n^2 + 3n - 1}{m}, \tag{7.15}$$

where

$$k_n^2 = \frac{1}{m}\{(n+1)^2 - 2K\} < \frac{1}{4}, \quad \text{when } n \geq 2.$$

Thus (7.15) may be replaced by

$$\frac{1}{2} \leq |y| \leq \frac{2n^2 + 3n - 1}{m} = 1 - \frac{n(2n+1)}{m}. \tag{7.16}$$

The intervals (7.14), (7.16) clearly include a complete set of incongruent values of y, so that by Theorem K, $M(f) \leq K$; and $M(f; P) < K$ except possibly when

$$y \equiv \pm \frac{n(2n+1)}{m}, \quad x + \frac{1}{2}(2n+1)y \equiv \frac{1}{2}(n-1) \pmod 1,$$

which gives the points stated in the theorem.

(ii) Returning to the form $f_m(x, y)$, we have to show that $M(f_m; P) = K$ when

$$P = \left(\frac{1}{2}, \frac{1}{2} - \frac{n}{m}\right).$$

For this we use Theorem B, as in the proof of Theorem 1. The analysis is rather more complicated, since the bound provided by Theorem B is not an absolute constant, but depends on n.

The fundamental solution of the Pellian equation $t^2 - 4mu^2 = 4$ is given by $t = 8n(n+1)$, $u = 2n+1$, yielding the fundamental automorph

$$T = \begin{pmatrix} 4n(n+1) & m(2n+1) \\ 2n+1 & 4n(n+1) \end{pmatrix}$$

of $f_m(x, y)$. Now it is easily verified that $T(P) \equiv P$. Hence by Theorem B, if $M(f_m; P) < K$, there exists a point $(x, y) \equiv P$ with $|f_m(x, y)| < K$ and

$$y^2 < \frac{K(t+2)}{D} = \frac{1}{2}\left(n - \frac{1}{4}\right)\left(1 - \frac{1}{m}\right)\frac{4n^2 + 4n + 1}{4n^2 + 4n - 1},$$

so that crudely

$$y^2 < \frac{1}{2} n. \tag{7.17}$$

Since $(x, y) \equiv P$, we may set

$$x = u + \frac{1}{2}, \quad y = v + \frac{1}{2} - \frac{n}{m},$$

where u, v are integral, and then

$$|f_m(x, y)| = \left|\left(u + \frac{1}{2}\right)^2 - m\left(v + \frac{1}{2} - \frac{n}{m}\right)^2\right| < K.$$

We subdivide the argument into two main cases, according as $v \geq 0$ or $v < 0$.

Suppose first that $v \geq 0$, and that

$$\left|u + \frac{1}{2}\right| \geq (2n+1)\left(v + \frac{1}{2}\right).$$

Then

$$f_m(x, y) \geq (2n+1)^2\left(v + \frac{1}{2}\right)^2 - \{(2n+1)^2 - 2\}\left(v + \frac{1}{2} - \frac{n}{m}\right)^2$$

$$= 2\left(v + \frac{1}{2}\right)^2 + n(2v+1) - \frac{n^2}{m} \geq \frac{1}{2} + n - \frac{n^2}{m} > n > K.$$

If, however,

$$\left|u + \frac{1}{2}\right| \leq (2n+1)\left(v + \frac{1}{2}\right) - 1,$$

then

$$-f_m(x, y) \geq \{(2n+1)^2 - 2\}\left(v + \frac{1}{2} - \frac{n}{m}\right)^2 - \left\{(2n+1)\left(v+\frac{1}{2}\right) - 1\right\}^2$$

$$= (n+1)(2v+1) - 2\left(v + \frac{1}{2}\right)^2 - 1 + \frac{n^2}{m}. \tag{7.18}$$

For $v = 0$, (7.18) gives

$$-f_m(x, y) \geq n - \frac{1}{2} + \frac{n^2}{m} = K;$$

if $v \geq 1$, we write (7.18) in the form

$$-f_m(x, y) \geq n(2v+1) + v - \frac{1}{2} - 2v^2 + \frac{n^2}{m},$$

where by (7.17)

$$v^2 < \left(v + \frac{1}{2} - \frac{n}{m}\right)^2 = y^2 < \frac{1}{2}n.$$

Then

$$-f_m(x, y) \geq n(2v+1) + v - \frac{1}{2} - n + \frac{n^2}{m} \geq 2n + 1 + \frac{n^2}{m} > K.$$

Suppose next that $v \leq -1$. Write $v = -w - 1$, so that $w \geq 0$ and

$$f_m(x, y) = \left(u + \frac{1}{2}\right)^2 - m\left(w + \frac{1}{2} + \frac{n}{m}\right)^2 = \left(u + \frac{1}{2}\right)^2 - m\left(w + \frac{1}{2}\right)^2 - n(2w+1) - \frac{n^2}{m}.$$

If now

$$\left|u + \frac{1}{2}\right| \geq (2n+1)\left(w + \frac{1}{2}\right) + 1,$$

we have

$$f_m(x, y) \geq \left\{(2n+1)\left(w + \frac{1}{2}\right) + 1\right\}^2 - \{(2n+1)^2 - 2\}\left(w + \frac{1}{2}\right)^2 - n(2w+1) - \frac{n^2}{m}$$

$$= (n+1)(2w+1) + 1 + 2\left(w + \frac{1}{2}\right)^2 - \frac{n^2}{m} \geq n + \frac{5}{2} - \frac{n^2}{m} > K.$$

If, however,

$$\left|u + \frac{1}{2}\right| \leq (2n+1)\left(w + \frac{1}{2}\right),$$

then we have

$$-f_m(x, y) \geq -(2n+1)^2\left(w + \frac{1}{2}\right)^2 + \{(2n+1)^2 - 2\}\left(w + \frac{1}{2}\right)^2 + n(2w+1) + \frac{n^2}{m}$$

$$= n(2w+1) + \frac{n^2}{m} - 2\left(w + \frac{1}{2}\right)^2. \tag{7.19}$$

For $w = 0$, (7.19) gives

$$-f_m(x, y) \geq n - \frac{1}{2} + \frac{n^2}{m} = K.$$

For $w \geq 1$, we have by (7.17)

$$\left(w + \frac{1}{2}\right)^2 < \left(w + \frac{1}{2} + \frac{n}{m}\right)^2 = y^2 < \frac{1}{2}n,$$

and so, from (7.19),

$$-f_m(x, y) > n(2w + 1) + \frac{n^2}{m} - n \geq 2n + \frac{n^2}{m} > K.$$

In all cases, therefore, we have proved that the inequality $|f_m(x, y)| < K$ cannot be satisfied with $(x, y) \equiv P$, $y^2 < \frac{1}{2}n$. This contradiction shows that $M(f_m; P) \geq K$. It now follows from (i) that $M(f_m; P) = K$, as required.

For the positive values of n excluded in Theorems 1 and 2, we note that

(i) Theorem 1 is false when $n = 1$, $m = 5$; Heinhold [1] gives in fact $M(f_5) = \frac{1}{4}$. The results are, however, valid for $n = 2$, $m = 21$, as we shall show in our next theorem.

(ii) Theorem 2 is true for $n = 1$, $m = 7$, but it is more convenient to deduce the result as a particular case of Theorem 5 below.

Theorem 3. *If*

$$f_{21}(x, y) = x^2 + xy - 5y^2,$$

then

$$M(f_{21}) = \frac{5}{7},$$

and is attained only for points

$$P \equiv \pm\left(\frac{2}{7}, \frac{3}{7}\right).$$

Proof: Part (ii) of the proof of Theorem 2, together with the fact that

$$f_{21}\left(\frac{2}{7}, \frac{3}{7}\right) = -\frac{5}{7},$$

shows that $M(f_{21}; P) = \frac{5}{7}$ for the points P quoted in the enunciation.

Making the transformation $(x, y) \to (x + y, y)$, we have therefore to show that for the form

$$f(x, y) = x^2 + 3xy - 3y^2$$

we have

$$M(f; P) < \frac{5}{7} \tag{7.20}$$

whenever $P \not\equiv \pm\left(-\frac{1}{7}, \frac{3}{7}\right)$.

Now

$$|f(x, y)| = \left|\left(x + \frac{3}{2}y\right)^2 - \frac{21}{4}y^2\right| = 3\left|\left(y - \frac{1}{2}x\right)^2 - \frac{7}{12}x^2\right|,$$

and so, applying Theorem J, we see that (7.20) holds if $P = (x, y)$ satisfies

$$\frac{21}{4}y^2 < \frac{1}{4} + \frac{5}{7}, \quad \text{or} \quad \frac{1}{4} - \frac{5}{21} < \frac{7}{12}x^2 < \frac{5}{21},$$

and therefore if

$$|y| < \frac{3}{7}, \quad \text{or} \quad \frac{1}{7} < |x| \leq \frac{1}{2}.$$

Any exceptional point P (i.e. a point for which (7.20) does not hold) must therefore lie (mod 1) in the region

$$-\frac{1}{7} \leq x \leq \frac{1}{7}, \quad \frac{3}{7} \leq y \leq \frac{4}{7}. \tag{7.21}$$

Now the fundamental automorph of $f(x, y)$ is

$$T = \begin{pmatrix} 1 & 3 \\ 1 & 4 \end{pmatrix}.$$

Since $M(f; P) = M(f; T(P))$, it follows that any exceptional point must also satisfy

$$-\frac{1}{7} \leq x + 3y \leq \frac{1}{7} \pmod 1. \tag{7.22}$$

But the only points satisfying (7.21) and (7.22) are

$$(x, y) = \left(-\frac{1}{7}, \frac{3}{7}\right), \quad \left(\frac{1}{7}, \frac{4}{7}\right),$$

which are congruent to the excluded points $\pm\left(-\frac{1}{7}, \frac{3}{7}\right)$.

8. The proofs of Theorems 1, 2 and 3 above may serve as a model for many of the proofs which will be given in this and later sections. Although there may be considerable variation in detail, the basic pattern is as follows:

(i) Taking K as the supposed value of $M(f)$, we apply Theorem J to $f(x, y)$ (or suitably chosen equivalent forms) and so find a set $\mathfrak{R}^*$ of points P for which $M(f; P) < K$.

(ii) We call P an *exceptional point* if it satisfies $M(f; P) \geq K$. Since $M(f; T(P)) = M(f; P)$, it follows that an exceptional point cannot transform (mod 1) into the set $\mathcal{R}^*$ under any automorph T of $f(x, y)$.

Occasionally (as in Theorem 3) we need use this principle for a finite number only of transformations T in order to determine all the exceptional points. More often, however, we must use Theorems C or D, the transformation T of these theorems usually being taken as the fundamental automorph T_0 of $f(x, y)$.

(iii) When the set C of incongruent exceptional points has been found, it remains only to establish the value of $M(f; P)$ for these points. If now C is finite, it may clearly be divided into subsets, each of which consists of a point and its transforms under powers of T_0; Theorem B may then be applied to each subset, precisely as in Theorems 1 and 2.

The method fails if C is infinite. We shall meet this case only in Theorem 7, where we have to deal with a set $\{T^n(P)\}$ $(n = 0, 1, 2, \ldots)$ arising from an application of Theorem C. A modification of Theorem B proves to be sufficient, together with the fact that the set has a fixed point F of T as its limiting point.

It is clear that the applications of Theorem J in (i) yield a set $\mathcal{R}^*$ which is strictly contained in a finite number of hyperbolic regions

$$|f(u + x, v + y)| < K \quad (u, v \text{ integral}). \tag{8.1}$$

Two obvious objections can therefore be made to the method outlined above. First, would not the use of the hyperbolic regions themselves, combined with (ii) if necessary, give a simpler proof of the result? Secondly, what guarantee is there that a set $\mathcal{R}^*$, obtained either from Theorem J or from consideration of a finite number of regions (8.1), is large enough to enable one to carry through part (ii) of the method?

The answer to the second objection is provided by Theorem B. For suppose that Theorem B (iii) gives the bound $|y| < C$. Then if $\mathcal{R}^*$ is the part of the unit square S contained in the *finite* number of regions (8.1) with $|v| < C + \frac{1}{2}$, it follows that any point P either is exceptional or transforms into $\mathcal{R}^*$ (mod 1) under some power of the fundamental automorph T_0. In some cases, in fact, the set $S - \mathcal{R}^*$ obtained in this way consists of a finite number of isolated points only, and then the use of automorphs is unnecessary; as examples, we may quote the proofs of $M(f_7)$ and $M(f_{11})$ given by Inkeri [5].

However, it may be very difficult to specify the set $\mathcal{R}^*$ obtained from hyperbolic regions, particularly if the number of these is large. The use of Theorem J, which defines a set $\mathfrak{R}^*$ by means of *linear* inequalities, is therefore preferable, in general, from the point of view of numerical detail.

Thus the essential question is whether, by using Theorem J instead of considering the regions (8.1), we are left with too large a set $S - \mathfrak{R}^*$ to eliminate by applying the automorphs of $f(x, y)$. The answer to this question seems to depend mainly on the magnitude of the fundamental solution of the Pellian equation $t^2 - D u^2 = \pm 4$. If this solution is large (in comparison with D), the set $S - \mathfrak{R}^*$ must be correspondingly small before the hypotheses of Theorems C or D can be satisfied, and the loss involved in using Theorem J may be too great. Clearly, also, part (ii) of the method will not normally succeed if the set $S - \mathfrak{R}^*$ contains fixed points of T_0 or $-T_0$ which are not exceptional points; and the number of such fixed points is very large when t, u are large.

In the proofs given below, in which the method outlined above is successful, either the solution of the Pellian equation is fairly small (t being always less than $2D$), or (as in Theorem 8) it is not necessary to apply the fundamental automorph.

Finally it may be noted that, with an appropriate choice of the constant K, the above method may clearly be used to isolate the minimum $M(f)$ or to establish the value of $M_2(f)$, $M_3(f)$, ... However, the arithmetical details become very complicated if too small a value of K is chosen, and for this reason we investigate only first minima, except when the second minimum can be obtained without a great deal more trouble.

Theorem 4. *Let*

$$f_m(x, y) = x^2 - m y^2$$

where

$$m = (2n + 1)^2 + 2, \quad n \geq 1.$$

Then

$$M(f_m) = \frac{8n^3 + 6n^2 + 6n - 1}{2m} = n - \frac{2n^2 + 1}{2m},$$

and is attained only for points

$$P \equiv \pm \left(\frac{1}{2}, \frac{1}{2} + \frac{n+1}{m}\right).$$

Proof: (i) Take $f(x, y) = x^2 + 2(2n + 1)xy - 2y^2$, which is equivalent to $f_m(x, y)$ under the transformation $(x, y) \to (x + (2n + 1)y, y)$. Write

$$K = \frac{8n^3 + 6n^2 + 6n - 1}{2m} = n - \frac{2n^2 + 1}{2m}.$$

Since

$$|f(x, y)| = |\{x + (2n+1)y\}^2 - my^2|,$$

we have

$$M(f; P) < K \tag{8.2}$$

provided that

$$\phi(my^2) < K. \tag{8.3}$$

Clearly $[2K] = 2n - 1$, so that (8.3) holds if

$$my^2 < K + \frac{1}{4}(2n-1)^2 \tag{8.4}$$

or if

$$\frac{1}{4}(2n+1)^2 - K < my^2 < K + \frac{1}{4}(2n)^2. \tag{8.5}$$

On substituting the value of K, (8.4) becomes

$$|y| < \frac{1}{2} - \frac{n+1}{m}. \tag{8.6}$$

Also, using the inequalities

$$\frac{1}{4}(2n+1)^2 - K < m\left(\frac{1}{2} - \frac{n}{m}\right)^2, \quad n^2 + K > m\left(\frac{1}{2} - \frac{1}{m}\right)^2,$$

which may easily be verified for $n \geq 1$, we may replace (8.5) by the smaller region

$$\frac{1}{2} - \frac{n}{m} \leq |y| \leq \frac{1}{2} - \frac{1}{m}. \tag{8.7}$$

Next, since

$$\frac{1}{2}|f(x, y)| = \left|\left\{y - \frac{1}{2}(2n+1)x\right\}^2 - \frac{1}{4}mx^2\right|,$$

(8.2) also holds if

$$\phi\left(\frac{1}{4}mx^2\right) < \frac{1}{2}K. \tag{8.8}$$

Since $[K] = n - 1$, (8.8) is true if

$$\frac{1}{4}mx^2 < \frac{1}{2}K + \frac{1}{4}(n-1)^2 \tag{8.9}$$

or if

$$\frac{1}{4}(n+1)^2 - \frac{1}{2}K < \frac{1}{4}mx^2 < \frac{1}{2}K + \frac{1}{4}n^2. \tag{8.10}$$

Using the inequality

$$m\{2K + (n-1)^2\} > \left(2n^2 + n + 1 + \frac{1}{2n+1}\right)^2,$$

which can be easily verified for $n \geq 1$, we may replace (8.9) by the smaller region

$$|x| \leq \frac{1}{m}\left(2n^2 + n + 1 + \frac{1}{2n+1}\right) = \frac{1}{2} - \frac{2n+1}{2m} + \frac{1}{m(2n+1)}. \tag{8.11}$$

Also (8.10) may be written as

$$(2n^2 + n + 2)^2 < m^2 x^2 < m(n^2 + 2n) - 2n^2 - 1 \tag{8.12}$$

where the expression on the right is greater than $\frac{1}{4}m^2$ for $n \geq 2$. Thus for $n \geq 2$ we may replace (8.12) by

$$\frac{2n^2 + n + 2}{m} < |x| \leq \frac{1}{2}. \tag{8.13}$$

(ii) If $n = 1$, $m = 11$, the results of (i) hold as far as (8.12), but this latter inequality gives only

$$\frac{2n^2 + n + 2}{m} < |x| < \frac{\sqrt{30}}{11}. \tag{8.13$'$}$$

However, using the automorph

$$U = \begin{pmatrix} 1 & 0 \\ 3 & -1 \end{pmatrix}$$

of $f(x, y) = x^2 + 6xy - 2y^2$, it is easily shown that (8.2) still holds for all points P satisfying (8.13). For, by (8.13)$'$ and (8.6), we need consider only points satisfying

$$\frac{\sqrt{30}}{11} \leq x \leq 1 - \frac{\sqrt{30}}{11}, \quad \frac{7}{22} \leq y \leq \frac{15}{22}.$$

These inequalities give at once

$$|3x - y - 1| < \frac{4 \cdot 13}{22},$$

whence, by (8.6), $U(P) = (x, 3x - y)$ is not exceptional.

(iii) We have therefore shown that, for any $n \geq 1$, (8.1) holds if $P = (x, y)$ satisfies (8.6), (8.7), (8.11) or (8.13). Any exceptional point must therefore lie (mod 1) in one of the six regions:

$$\mathfrak{R}_1: \begin{cases} \frac{1}{2} - \frac{2n+1}{m} + \frac{1}{m(2n+1)} < x \leq \frac{1}{2} - \frac{2n-1}{m} \\ \frac{1}{2} - \frac{n+1}{m} \leq y < \frac{1}{2} - \frac{n}{m} \end{cases}$$

$$\mathfrak{R}_2: \begin{cases} \frac{1}{2} - \frac{2n+1}{m} + \frac{1}{m(2n+1)} < x \leq \frac{1}{2} - \frac{2n-1}{m} \\ \frac{1}{2} - \frac{1}{m} < y < \frac{1}{2} + \frac{1}{m} \end{cases}$$

$$\mathcal{R}_3: \begin{cases} \dfrac{1}{2} - \dfrac{2n+1}{m} + \dfrac{1}{m(2n+1)} < x \leq \dfrac{1}{2} - \dfrac{2n-1}{m} \\ \qquad\qquad\qquad \dfrac{1}{2} + \dfrac{n}{m} < y \leq \dfrac{1}{2} + \dfrac{n+1}{m} \end{cases}$$

and the images $\mathcal{R}_1'$, $\mathcal{R}_2'$, $\mathcal{R}_3'$ of these in the origin.

We now use the transformations

$$U = \begin{pmatrix} 1 & 0 \\ 2n+1 & -1 \end{pmatrix}, \quad V = \begin{pmatrix} -1 & -2(2n+1) \\ 0 & 1 \end{pmatrix},$$

which are automorphs of $f(x, y)$ of finite order.

(a) Suppose that $P \in \mathcal{R}_1$. Then we have

$$(n-1) + \frac{1}{2} + \frac{n+2}{m} < (2n+1)x - y \leq (n-1) + \frac{1}{2} + \frac{3n+3}{m},$$

and so $U(P) = (x, (2n+1)x - y)$ cannot be congruent to a point of any of the above six regions. Hence no point of $\mathcal{R}_1$ is exceptional; by symmetry, no point of $\mathcal{R}_1'$ is exceptional.

(b) Suppose next that $P \in \mathcal{R}_3$. Then we have

$$-(2n+3) + \frac{1}{2} - \frac{2n-1}{m} \leq -x - 2(2n+1)y < -(2n+3) + \frac{1}{2} + \frac{6n+7}{2m} - \frac{1}{(2n+1)m},$$

with equality on the left only if

$$(x, y) = \left(\frac{1}{2} - \frac{2n-1}{2m}, \frac{1}{2} + \frac{n+1}{m}\right) = P_0. \tag{8.14}$$

This inequality shows that $V(P) = (-x - 2(2n+1)y, y)$ cannot be congruent to a point of $\mathcal{R}_2$, $\mathcal{R}_2'$, $\mathcal{R}_3$ or $\mathcal{R}_3'$ unless $P = P_0$. Hence the only exceptional points of $\mathcal{R}_3$ and $\mathcal{R}_3'$ are $\pm P_0$.

(c) Suppose finally that $P \in \mathcal{R}_2$. Then we have

$$n - \frac{1}{2} + \frac{1}{m} < (2n+1)x - y < n - \frac{1}{2} + \frac{2n+3}{m},$$

so that $U(P) = (x, (2n+1)x - y)$ cannot be congruent to a point of $\mathcal{R}_2$ or $\mathcal{R}_2'$. Also, since $U(P_0) \equiv P_0$, $U(P)$ cannot be congruent to $\pm P_0$. Hence no point of $\mathcal{R}_2$ or $\mathcal{R}_2'$ is exceptional.

(iii) We have now shown that (8.2) holds for all P, except possibly when $P \equiv \pm P_0$. To complete the proof of the theorem, we have therefore only to show that

$$M(f; P_0) = K. \tag{8.15}$$

Now the fundamental automorph of $f(x, y)$ is

$$T = \begin{pmatrix} 1 & 2(2n+1) \\ 2n+1 & 2(2n+1)^2+1 \end{pmatrix} = -UV,$$

corresponding to the solution $t = 2(2n+1)^2 + 2$, $u = 2n+1$ of $t^2 - 4mu^2 = 4$. It is easily verified that $T(P_0) \equiv -P_0$.

If now $M(f; P_0) < K$, Theorem B shows that there exists a solution of $|f(x, y)| < K$ with $(x, y) \equiv \pm P_0$ and

$$y^2 < \frac{K(t+2)}{D} = \frac{1}{2}K < \frac{1}{2}n;$$

and, since $f(-x, -y) = f(x, y)$, we need consider only solutions with $(x, y) \equiv P_0$. But it may easily be shown that $|f(x, y)| \geq K$ for all such points (x, y); we omit the details, since they are exactly parallel to those of Theorem 2 (ii).

Thus $M(f; P_0) \geq K$. Since

$$f\left(\frac{1}{2} - \frac{2n-1}{2m}, -\frac{1}{2} + \frac{n+1}{m}\right) = -K,$$

it follows that $M(f; P_0) = K$, as required.

Theorem 5. *Let*

$$f(x, y) = nx^2 + nxy - (2n-1)y^2, \quad \textit{where } n \geq 2.$$

Then

$$M(f) = \frac{(2n-1)^2}{9n-4},$$

and is attained only for points

$$P \equiv \pm\left(\frac{2n-1}{9n-4}, -\frac{4n-2}{9n-4}\right).$$

Proof: Set $K = \dfrac{(2n-1)^2}{9n-4}$.

(i) We have

$$|f(x, y)| = n\left|\left(x + \frac{1}{2}y\right)^2 - \frac{9n-4}{4n}y^2\right| = (2n-1)\left|\left(y - \frac{n}{4n-2}x\right)^2 - \frac{n(9n-4)}{4(2n-1)^2}x^2\right|.$$

Hence

$$M(f; P) < K \tag{8.16}$$

either if

$$\phi\left(\frac{9n-4}{4n}y^2\right) < \frac{K}{n} = \frac{(2n-1)^2}{n(9n-4)}, \tag{8.17}$$

or if

$$\phi\left(\frac{n(9n-4)}{4(2n-1)^2}x^2\right) < \frac{K}{2n-1} = \frac{2n-1}{9n-4}. \tag{8.18}$$

The r.h.s. of (8.17) lies between $\frac{1}{4}$ and $\frac{1}{2}$, since $n \geq 2$, and the r.h.s. of (8.18) is less than $\frac{1}{4}$. Theorem J now shows that these inequalities hold if

$$\frac{9n-4}{4n}y^2 < \frac{(2n-1)^2}{n(9n-4)}$$

or if

$$\frac{1}{4} - \frac{2n-1}{9n-4} < \frac{n(9n-4)}{4(2n-1)^2}x^2 < \frac{2n-1}{9n-4},$$

and so certainly if

$$|y| < \frac{4n-2}{9n-4}, \quad \text{or} \quad \frac{2n-1}{9n-4} < |x| \leq \frac{1}{2}.$$

(ii) Any exceptional point P must therefore lie (mod 1) in the region $\mathcal{R}$ defined by

$$|x| < \frac{2n-1}{9n-4}, \; \frac{4n-2}{9n-4} \leq y \leq \frac{5n-2}{9n-4}.$$

Now

$$U = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$$

is an automorph of $f(x, y)$, and if $P = (x, y) \in \mathcal{R}$ we have

$$\frac{2n-1}{9n-4} \leq x + y \leq 1 - \frac{2n-1}{9n-4}. \tag{8.19}$$

Hence $U(P)$ can be congruent to a point of $\mathcal{R}$ only if there is equality on either side of (8.19), i.e. only if

$$(x, y) = \left(-\frac{2n-1}{9n-4}, \frac{4n-2}{9n-4}\right) \quad \text{or} \quad \left(\frac{2n-1}{9n-4}, \frac{5n-2}{9n-4}\right).$$

(iii) The only possible exceptional points are therefore those congruent to $\pm P_0$, where

$$P_0 = \left(\frac{2n-1}{9n-4}, -\frac{4n-2}{9n-4}\right).$$

We complete the proof of the theorem by showing that $M(f; P_0) = K$.

It is convenient to make the transformation: $X = x$, $Y = x - y$, so that

$$f(x, y) = X^2 + (3n-2)XY - (2n-1)Y^2 = F(X, Y), \text{ say,}$$

and P_0 becomes

$$Q_0 = (X_0, Y_0) = \left(\frac{2n-1}{9n-4}, \frac{6n-3}{9n-4}\right).$$

The fundamental automorph of $F(X, Y)$ is

$$T = \begin{pmatrix} 2 & 6n-3 \\ 3 & 9n-4 \end{pmatrix},$$

corresponding to the solution $t = 9n-2$, $u = 3$ of $t^2 - n(9n-4)u^2 = 4$. It is easily verified that $T(Q_0) \equiv Q_0$.

Suppose if possible that $M(F; Q_0) < K$. Then, by Theorem B, there exists a solution of $|F(X, Y)| < K$ with $(X, Y) \equiv Q_0$ and

$$Y^2 < \frac{K(t+2)}{D} = \frac{(2n-1)^2}{9n-4} \cdot \frac{9n}{n(9n-4)},$$

i.e.

$$|Y| < \frac{6n-3}{9n-4}.$$

We need therefore consider only the values

$$(X, Y) = \left(u + \frac{2n-1}{9n-4}, -\frac{3n-1}{9n-4}\right) \quad (u \text{ integral}).$$

Then

$$F(X, Y) = \left(u - \frac{n-1}{2}\right)^2 - \frac{n(3n-1)^2}{4(9n-4)},$$

and it is easily seen that

$$\min |F(X, Y)| = \frac{n(3n-1)^2}{4(9n-4)} - \frac{1}{4}(n-1)^2 = K. \tag{8.20}$$

This contradiction shows that $M(F; Q_0) \geq K$; and from (8.20) we see that therefore $M(F; Q_0) = K$.

The form $f(x, y) = nx^2 + nxy - (2n+1)y^2$ $(n \geq 2)$ may be treated by exactly the same argument as that used in Theorem 5, taking now $K = \dfrac{(2n+1)^2}{9n+4}$. Parts (i) and (ii) of the proof of Theorem 5 go through, with only changes of sign required, to show that $M(f; P) < K$ except possibly for $P \equiv \pm P_0$, where

$$P_0 = \left(\frac{2n+1}{9n+4}, -\frac{4n+2}{9n+4}\right).$$

However, the proof that $M(f; P_0) = K$ breaks down if $n \leq 4$; the result is in fact not true if $n = 2$, 3 or 4.

We therefore state without proof:

Theorem 6. *Let*

$$f(x, y) = n x^2 + n x y - (2n + 1) y^2, \quad n \geq 5.$$

Then

$$M(f) = \frac{(2n + 1)^2}{9n + 4},$$

and is attained only for points

$$P \equiv \pm \left(\frac{2n + 1}{9n + 4}, \ -\frac{4n + 2}{9n + 4}\right).$$

We may note that the values $n = 2, 3$ give $f(x, y)$ equivalent to the norm-forms

$$f_{11}(x, y) = x^2 - 11 y^2,$$

$$f_{93}(x, y) = x^2 + x y - 23 y^2,$$

respectively. The first of these has been dealt with in Theorem 4 (the particular case $n = 1$), and the second will be considered in Theorem 10 below. If $n = 4$, $f(x, y) = 4x^2 + 4xy - 9y^2$, and it may be proved that

$$M(f) = \frac{79}{40} < \frac{(2n + 1)^2}{9n + 4} = \frac{81}{49}.$$

In each of these cases, though the value of the minimum is less than that given by Theorem 6, it is in fact taken at the points quoted in the theorem and at no others.

9. As a final example of a general class of norm-forms $f_m(x, y)$, we shall consider the forms

$$f_m(x, y) = x^2 + x y - \frac{1}{4}(m - 1) y^2 \tag{9.1}$$

with

$$m = (2n + 1)^2 + 4, \quad n \geq 1. \tag{9.2}$$

These are the only non-zero forms known to us for which the inhomogeneous minimum $M(f)$ is unattained. The particular case $n = 1$, $m = 13$ has been investigated by Inkeri[1] [4], who proves, using the technique of Davenport [1], that $M(f_{13}) = \frac{1}{3}$ and is unattained. He also states that $M(f_{13}; P) \leq \frac{4}{13}$ for all points P at which $M(f_{13}; P) \neq \frac{1}{3}$; so that in our notation $M_2(f_{13}) \leq \frac{4}{13}$.

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[1] We are grateful to Professor L. J. Mordell for bringing Inkeri's results to our notice. An announcement that $M(f_{13}) = \frac{1}{3}$, and is unattained, is made in the final footnote of Inkeri [3].

We shall show that in fact $M_2(f_{13}) = \frac{4}{13}$, and that entirely analogous results hold for the general form (9.1), (9.2). The first part of the proof (Lemma 3), in which Theorem C plays a fundamental role, goes through generally for $n \geq 3$, and a slight modification deals with the cases $n = 1$ and 2. The final investigation of the minima at the exceptional points (Lemmas 4, 5, 6) is valid for all $n \geq 1$.

The complete theorem is most conveniently stated in terms of the form $f(x, y) = x^2 + (2n + 1)xy - y^2$, which is equivalent to $f_m(x, y)$ and has an obvious symmetry.

Theorem 7. *Let*

$$f(x, y) = x^2 + (2n + 1)xy - y^2, \quad n \geq 1. \tag{9.3}$$

Then

$$\text{(i)} \qquad M(f; P) < \frac{2n^3 + n^2 + 2n - 1}{4n^2 + 4n + 5} = \frac{2n^3 + n^2 + 2n - 1}{m}, \tag{9.4}$$

except when P is congruent to a point of one of the following three sets:

$$C_1: \quad \left(\pm \frac{n}{2n+1}, \pm \frac{n}{2n+1}\right);$$

$$C_2: \quad \pm\left(\frac{2n^2+n+1}{m}, \frac{2n^2+n+3}{m}\right), \quad \pm\left(\frac{2n^2+n+3}{m}, -\frac{2n^2+n+1}{m}\right);$$

$$C_3: \quad \pm\begin{pmatrix} 0 & 1 \\ 1 & 2n+1 \end{pmatrix}^k P_0, \quad \pm\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 2n+1 \end{pmatrix}^k P_0,$$

where k is any integer and

$$P_0 = \left(\frac{1}{2} - \frac{1}{2\sqrt{m}}, \frac{n}{2n+1} + \frac{1}{(2n+1)\sqrt{m}}\right); \tag{9.5}$$

(ii) *For these exceptional points we have*

$$M(f; P) = \frac{n^2}{2n+1} = M(f) \text{ if } P \in C_1 \text{ or } P \in C_3,$$

$$M(f; P) = \frac{2n^3 + n^2 + 2n - 1}{m} = M_2(f) \text{ if } P \in C_2;$$

and if $P \in C_3$ the minimum $M(f; P) = M(f)$ is unattained.

Lemma 3. *Apart from the exceptional points quoted,* (9.4) *holds for all $n \geq 1$.*

Proof: Set $K = \frac{2n^3 + n^2 + 2n - 1}{m}$.

(i) We have

$$|f(x, y)| = \left|\left\{x + \frac{1}{2}(2n + 1)y\right\}^2 - \frac{1}{4}my^2\right|,$$

so that (9.4) certainly holds if

$$\phi\left(\frac{1}{4}my^2\right) < K. \tag{9.6}$$

Clearly $[2K] = n - 1$, so that by Theorems H and J (9.6) holds if

$$\frac{1}{4}my^2 < K + \frac{1}{4}(n-1)^2$$

or if

$$\frac{1}{4}(n+1)^2 - K < \frac{1}{4}my^2 < \frac{1}{4}n^2 + K,$$

and these inequalities simplify respectively to

$$|y| < \frac{2n^2 + n + 1}{m} \tag{9.7}$$

and

$$\frac{2n^2 + n + 3}{m} < |y| < \frac{1}{m}\sqrt{(4n^4 + 12n^3 + 9n^2 + 8n - 4)}. \tag{9.8}$$

(ii) If now $n \geq 3$, the r.h.s. of (9.8) is greater than $\frac{1}{2}$. Then (9.7) and (9.8) show that any exceptional point must satisfy

$$\frac{2n^2 + n + 1}{m} \leq |y| \leq \frac{2n^2 + n + 3}{m} \pmod 1; \tag{9.9}$$

and since $f(x, y) = f(-y, x)$, it follows that also

$$\frac{2n^2 + n + 1}{m} \leq |x| \leq \frac{2n^2 + n + 3}{m} \pmod 1. \tag{9.10}$$

This result is also true for $n = 1, 2$ as we now show.

(a) If $n = 1$, $m = 13$, (9.7) is

$$|y| < \frac{4}{13},$$

and the interval (9.8) no longer exists. Thus exceptional points must satisfy

$$\frac{4}{13} \leq |x|, |y| \leq \frac{1}{2} \pmod 1. \tag{9.11}$$

We now use the fundamental automorph

$$T = \begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix}$$

of $f(x, y)$ for $n = 1$. For points $P = (x, y)$ satisfying

$$\frac{4}{13} \leq x \leq \frac{9}{13}, \quad \frac{6}{13} < y < \frac{7}{13}, \tag{9.12}$$

we have

$$\frac{22}{13} < x + 3y < \frac{30}{13},$$

i.e.

$$|x + 3y - 2| < \frac{4}{13}. \tag{9.13}$$

Since $T(P) = (y, x + 3y)$, (9.13) shows that $T(P)$ does not satisfy (9.11) and so is not exceptional. Hence no exceptional point lies in the region defined by (9.12). By symmetry, it follows that no exceptional point lies in the region

$$\frac{6}{13} < x < \frac{7}{13}, \quad \frac{4}{13} \leq y \leq \frac{9}{13}. \tag{9.14}$$

Thus (9.11) may be strengthened to

$$\frac{4}{13} \leq |x|, |y| \leq \frac{6}{13} \pmod 1,$$

which is (9.9) and (9.10) exactly.

(b) If $n = 2$, $m = 29$, (9.7) and (9.8) are

$$|y| < \frac{11}{29}$$

and

$$\frac{13}{29} < |y| < \frac{\sqrt{208}}{29} < \frac{1}{2}.$$

We now use the fundamental automorph

$$T = \begin{pmatrix} 0 & 1 \\ 1 & 5 \end{pmatrix}$$

of $f(x, y)$ for $n = 2$. For points $P = (x, y)$ satisfying

$$\frac{11}{29} \leq x \leq \frac{18}{29}, \quad \frac{\sqrt{208}}{29} \leq y \leq 1 - \frac{\sqrt{208}}{29}, \tag{9.15}$$

we find as in (a) above

$$|x + 5y - 3| \leq \frac{76 - 5\sqrt{208}}{29} < \frac{4}{29},$$

so that $T(P) = (y, x + 5y)$ is not exceptional; and by symmetry no point of the region obtained from (9.15) by interchanging x, y can be exceptional either. Thus all exceptional points must satisfy

$$\frac{11}{29} \leq |x|, |y| \leq \frac{13}{29} \pmod 1,$$

which is precisely (9.9) and (9.10).

(iii) We have now shown that for $n \geq 1$ any exceptional point must be congruent to a point of one of the four regions

$$\mathcal{R}_1: \frac{2n^2+n+1}{m} \leq x \leq \frac{2n^2+n+3}{m}, \quad \frac{2n^2+n+1}{m} \leq y \leq \frac{2n^2+n+3}{m};$$

$$\mathcal{R}_2: \frac{2n^2+n+1}{m} \leq -x \leq \frac{2n^2+n+3}{m}, \quad \frac{2n^2+n+1}{m} \leq y \leq \frac{2n^2+n+3}{m};$$

$$\mathcal{R}_1': \frac{2n^2+n+1}{m} \leq -x \leq \frac{2n^2+n+3}{m}, \quad \frac{2n^2+n+1}{m} \leq -y \leq \frac{2n^2+n+3}{m};$$

$$\mathcal{R}_2': \frac{2n^2+n+1}{m} \leq x \leq \frac{2n^2+n+3}{m}, \quad \frac{2n^2+n+1}{m} \leq -y \leq \frac{2n^2+n+3}{m}.$$

We now use the automorphs

$$U = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 \\ 1 & 2n+1 \end{pmatrix}$$

of $f(x, y)$. U is of finite order, while T is the fundamental automorph. It is clear that

$$\mathcal{R}_2 = U(\mathcal{R}_1), \ \mathcal{R}_1' = U^2(\mathcal{R}_1), \ \mathcal{R}_2' = U^3(\mathcal{R}_1), \ U^4 = I. \tag{9.16}$$

(a) Let now $P = (x, y)$ be any exceptional point of $\mathcal{R}_1$. Then

$$\frac{2n^2 - n + 2}{m} \leq x + (2n+1)y - n \leq \frac{2n^2 + 3n + 6}{m}.$$

Since $T(P) = (y, x + (2n+1)y)$ is congruent to a point of one of the four regions, we see that *either*

$$T(P) + (0, -n) \in \mathcal{R}_1, \tag{9.17}$$

or

$$T(P) + (0, -n-1) \in \mathcal{R}_2'. \tag{9.18}$$

(b) Consider now $T^{-1}(P) = (-(2n+1)x + y, x)$. Since

$$\frac{2n^2 - n - 2}{m} \leq -(2n+1)x + y + n \leq 1 - \frac{2n^2 + n + 3}{m}, \tag{9.19}$$

where we have equality on the right only for the one point

$$(x, y) = \left(\frac{2n^2 + n + 1}{m}, \frac{2n^2 + n + 3}{m}\right), \tag{9.20}$$

it follows from (9.19) that except for the point (9.20) we have

$$T^{-1}(P) + (n, 0) \in \mathcal{R}_1. \tag{9.21}$$

(c) The point (9.20) and the points obtained from it by powers of U are precisely the set C_2. We exclude them from consideration for the rest of this proof, so that (9.21) holds for all exceptional points in $\mathcal{R}_1$. The fixed point which satisfies

$$T^{-1}(F) + (n, 0) = F$$

is

$$F = \left(\frac{n}{2n+1}, \frac{n}{2n+1}\right). \tag{9.22}$$

Theorem C now shows that any exceptional point of $\mathcal{R}_1$ lies on the line

$$y - \frac{n}{2n+1} = \varkappa\left(x - \frac{n}{2n+1}\right) \tag{9.23}$$

through F, where

$$\varkappa = \frac{1}{2}[2n + 1 + \sqrt{\{(2n+1)^2 + 4\}}] = \frac{1}{2}(2n + 1 + \sqrt{m}). \tag{9.24}$$

By applying powers of U to (9.24) we obtain the lines on which exceptional points of $\mathcal{R}_2$, $\mathcal{R}_1'$, $\mathcal{R}_2'$ must lie. In particular, the exceptional points of $\mathcal{R}_2$ lie on

$$x - \frac{n}{2n+1} = -\varkappa\left(y + \frac{n}{2n+1}\right). \tag{9.25}$$

(d) We now go back to the results found in (a). If of the two alternatives (9.17) and (9.18), (9.17) holds for P, $T(P)$, $T^2(P)$, ..., then it follows from (9.17), (9.21) and Theorem D that P is the fixed point F of (9.22). The points obtained from F under powers of U are precisely the set C_1.

If, however, (9.18) holds for some exceptional point $P \in \mathcal{R}_1$, then from (c), $P = (x, y)$ lies on the line (9.23) and $T(P) + (0, -n-1) = (y, x + (2n+1)y - n - 1)$ lies on the line (9.25). This gives two equations for P, whose solution is easily found to be the point P_0 of (9.5).

This proves the Lemma, and we have only to establish the values of $M(f; P)$ at the exceptional points. This we do by Theorem B, or a modification of it.

Lemma 4. *If* $P \in C_1$, *then*

$$M(f; P) = \frac{n^2}{2n+1}. \tag{9.26}$$

Proof: As appeared in the proof of Lemma 3, the points of C_1 are all fixed points of T, and are equivalent under powers of U, so that it is sufficient to prove (9.26) for the point F of (9.22). Now

$$f\left(\frac{n}{2n+1}, \frac{n}{2n+1}\right) = \frac{n^2}{2n+1},$$

so that

$$M(f; F) \leq \frac{n^2}{2n+1}. \tag{9.27}$$

Suppose if possible that there is strict inequality in (9.27). Then by Theorem B there is a solution of

$$|f(x, y)| < \frac{n^2}{2n+1}$$

with $(x, y) \equiv F$ and

$$y^2 < \frac{n^2}{2n+1} \cdot \frac{u^2}{t} = \frac{n^2}{2n+1} \cdot \frac{1}{2n+1},$$

i.e.

$$|y| < \frac{n}{2n+1};$$

and this is clearly impossible.

Lemma 5. *If* $P \in C_2$, *then*

$$M(f; P) = K = \frac{2n^3 + n^2 + 2n - 1}{m}.$$

Proof: The points of C_2 are permuted (mod 1) by T, and therefore give the same value of $M(f; P)$. Since

$$f\left(\frac{2n^2+n+1}{m}, \frac{2n^2+n+3}{m}\right) = K,$$

it follows that

$$M(f; P) \leq K. \tag{9.28}$$

Suppose that there is inequality in (9.28). Then by Theorem B there is a solution of $|f(x, y)| < K$ with (x, y) congruent to a point of C_2 and

$$y^2 < \frac{K u^2}{t} = \frac{K}{2n+1}.$$

It is easily verified that this gives *a fortiori*

$$|y| < \frac{2n^2+n+3}{m}.$$

Since $f(x, y) = f(-x, -y)$, it follows that there is a solution of

$$\left| f\left(u - \frac{2n^2+n+3}{m}, \frac{2n^2+n+1}{m}\right)\right| < K$$

with integral u. But

$$f\left(u - \frac{2n^2+n+3}{m}, \frac{2n^2+n+1}{m}\right) = u(u+n-1) - K,$$

and since $K < \frac{1}{2}n$, the least value of $|f|$ is given by $u = 0, 1-n$, $|f| = K$, which contradicts the assumption above and so proves the Lemma.

Lemma 6. *If* $P \in C_3$, *then*

$$M(f; P) = \frac{n^2}{2n+1}. \tag{9.29}$$

However, this lower bound $M(f; P)$ *is not attained for any point* P *of the set.*

Proof: From the construction of the set C_3 in terms of automorphs of $f(x, y)$, all points of C_3 give the same value of $M(f; P)$. Further, from the final clause of Theorem C we see that the fixed points C_1 are limiting points of C_3. Hence from Lemma 4 and Theorem F,

$$M(f; P) \leq M(f; C_1) = \frac{n^2}{2n+1}, \quad \text{for } P \in C_3. \tag{9.30}$$

We next prove that the inequality

$$f(x, y) \geq \frac{n^2}{2n+1} \tag{9.31}$$

holds for all (x, y) congruent to a point of C_3. For suppose not. Then for some number $C < \frac{n^2}{2n+1}$ and some (x, y) congruent to a point of C_3, we have

$$|f(x, y)| \leq C < \frac{n^2}{2n+1}. \tag{9.32}$$

Since T permutes the points of C_3 (mod 1), the argument of Theorem B shows that if Y is the lower bound of values of $|y|$ such that, for some x, (x, y) is congruent to a point of C_3 and satisfies (9.32), then

$$Y^2 \leq \frac{C u^2}{t} = \frac{C}{2n+1} < \frac{n^2}{(2n+1)^2},$$

whence

$$Y < \frac{n}{2n+1}. \tag{9.33}$$

But this is impossible, since it is easily verified that

$$|y| > \frac{n}{2n+1}$$

for all points of C_3; thus, for example, each point $T^k(P_0)$ $(k \leq 0)$ is congruent to a point on the segment $F P_0$ of the line (9.23).

This establishes (9.31) and therefore (9.29). It remains only to show that there cannot be equality in (9.31). From the formation of C_3, if this were possible, it would be possible with $(x, y) \equiv P_0$. But if we write

$$x = u + \frac{1}{2} - \frac{1}{2\sqrt{m}}, \quad y = v + \frac{n}{2n+1} + \frac{2}{(2n+1)\sqrt{m}},$$

then

$$f(x, y) = f\left(u + \frac{1}{2},\ v + \frac{n}{2n+1}\right) + \frac{1}{m} f\left(-\frac{1}{2},\ \frac{1}{2n+1}\right) - \frac{\sqrt{m}}{4n+2}\left(v + \frac{n}{2n+1}\right),$$

which is clearly irrational for integral values of u, v.

Theorem 7 now follows at once from Lemmas 3–6, on noting that

$$\frac{n^2}{2n+1} > K = \frac{2n^3 + n^2 + 2n - 1}{m}.$$

10. The general classes of norm-forms $f_m(x, y)$ discussed in the preceding three sections include many forms of small discriminant for which the value of $M(f_m)$ was not previously known.

As we remarked above, general estimates for $M(f)$ have been given by various authors, and these have been shown to be precise for many forms $f_m(x, y)$; while certain other values of m have been examined in detail. If we combine the results above with those of Heinhold [1][1], we find that the value of $M(f_m)$ is now known for all $m \leq 101$ except the following:

[1] Note however that there are some omissions and inaccuracies in Heinhold's table (namely for $m = 23$, 34, 43, 55 and 82) as has been pointed out by Inkeri [3].

$$f_m(x, y) = x^2 - m y^2, \quad m = 19, 22, 31, 43, 46, 58, 59, 67, 70, 71, 86, 94; \tag{10.1}$$

$$f_m(x, y) = x^2 + xy - \frac{1}{4}(m-1) y^2, \quad m = 33, 41, 57, 61, 73, 89, 93, 97. \tag{10.2}$$

We have also obtained the values of $M(f_m)$ for most — though not all — of the forms (10.1), (10.2). They can be proved by the technique developed above (described at the beginning of § 8), combined in some of the more difficult cases with a direct consideration of hyperbolic regions $|f(x + x_0, y + y_0)| < K$. We give as specimens the cases $m = 31, 41, 93$ and (in the next section) 61.

Theorem 8. *If* $f_m(x, y) = f_{31}(x, y) = x^2 - 31 y^2$, *then* $M(f_{31}) = \frac{45}{31}$ *and is attained only for points* $P \equiv \left(0, \pm \frac{13}{31}\right)$.

Proof: We use the equivalent form

$$f(x, y) = 5x^2 + 2xy - 6y^2$$

which is obtained from $f_{31}(x, y)$ by the transformation $(x, y) \to (6x - 5y, x - y)$. The exceptional points of the Theorem now become $\pm\left(\frac{3}{31}, -\frac{15}{31}\right)$.

(i) $$|f(x, y)| = 5\left|\left(x + \frac{1}{5}y\right)^2 - \frac{31}{25}y^2\right| = 6\left|\left(y - \frac{1}{6}x\right)^2 - \frac{31}{36}x^2\right|,$$

so that $M(f; P) < \frac{45}{31}$ if either

$$\phi\left(\frac{31}{36}x^2\right) < \frac{15}{62} \quad \text{or} \quad \phi\left(\frac{31}{25}y^2\right) < \frac{9}{31}.$$

By Theorem J, these inequalities are equivalent to

$$\frac{1}{4} - \frac{15}{62} < \frac{31}{36}x^2 < \frac{15}{62} \quad \text{and} \quad \frac{31}{25}y^2 < \frac{9}{31},$$

which are implied by the stronger

$$\frac{3}{31} < |x| \leq \frac{1}{2}, \quad |y| < \frac{15}{31}.$$

(ii) Any exceptional point is therefore congruent to a point of the region

$$\mathcal{R}: \quad -\frac{3}{31} \leq x \leq 0, \quad \frac{15}{31} \leq y \leq \frac{16}{31},$$

or to a point of $\mathcal{R}'$, its image in the origin.

It is easily verified by partial differentiation that $f(x+2,\ y-2)$ is a strictly increasing function of x and y in $\mathcal{R}$, so that

$$-\frac{45}{31} = f\left(\frac{59}{31},\ -\frac{47}{31}\right) \le f(x+2,\ y-2) \le f\left(2,\ -\frac{46}{31}\right) < \frac{45}{31},$$

with equality at the lower end only for $(x, y) = \left(-\frac{3}{31},\ \frac{15}{31}\right)\cdot$ Thus this point is the only possible exceptional point of $\mathcal{R}$.

(iii) It only remains to prove that

$$M\left(f;\frac{3}{31},\ -\frac{15}{31}\right) = \frac{45}{31}\cdot \tag{10.3}$$

Now

$$f\left(u+\frac{3}{31},\ v-\frac{15}{31}\right) = 5\,u^2 + 2\,u\,v - 6\,v^2 + 6\,v - \frac{45}{31},$$

so that (10.3) holds unless we can find integers u, v for which

$$5\,u^2 + 2\,u\,v - 6\,v^2 + 6\,v = 1 \text{ or } 2.$$

The second alternative is clearly impossible, by considering congruences modulo 4. The first gives

$$31\,(5\,u + v)^2 - (31\,v - 15)^2 = -70,$$

which is also impossible since 31 is a quadratic non-residue of 7.

Theorem 9. *If* $f_m(x, y) = f_{41}(x, y) = x^2 + x\,y - 10\,y^2$, *then* $M(f_{41}) = \frac{23}{32}$ *and is attained only for points congruent to* $\pm\left(\frac{5}{16},\ \frac{5}{16}\right),\ \pm\left(\frac{3}{8},\ \frac{5}{16}\right)\cdot$

Proof: We use the equivalent form

$$f(x, y) = 2\,x^2 + 5\,x\,y - 2\,y^2$$

which is obtained from $f_{41}(x, y)$ by the transformation $(x, y) \to (3\,x + 8\,y,\ x + 3\,y)$.

(i) We have

$$|f(x, y)| = 2\left|\left(x + \frac{5}{4}y\right)^2 - \frac{41}{16}y^2\right|,$$

and so

$$M(f;\ P) < \frac{23}{32} \tag{10.4}$$

if

$$\phi\left(\frac{41}{16}y^2\right) < \frac{23}{64}\cdot$$

By Theorem J, this holds if

$$\frac{41}{16}y^2 < \frac{23}{64},$$

and so certainly if

$$|y| \leq 0{\cdot}3744. \tag{10.5}$$

By the obvious symmetry properties of $f(x, y)$, (10.4) also holds if

$$|x| \leq 0{\cdot}3744. \tag{10.6}$$

In the same way, since

$$|f(x, y)| = \left|\left(\frac{7}{2}x + \frac{19}{2}y\right)^2 - \frac{41}{4}(x + 3y)^2\right|,$$

(10.4) holds if

$$\phi\left(\frac{41}{4}(x + 3y)^2\right) < \frac{23}{32},$$

and hence by Theorem J if

$$\frac{41}{4}(x + 3y)^2 < \frac{1}{4} + \frac{23}{32} = \frac{31}{32},$$

and so certainly if

$$|x + 3y| \leq 0{\cdot}3074. \tag{10.7}$$

By symmetry, (10.4) also holds if

$$|3x - y| \leq 0{\cdot}3074. \tag{10.8}$$

(ii) By (10.5) and (10.6), any exceptional points must be congruent to a point of the region

$$\mathcal{R}:\ 0{\cdot}3744 < x,\ y < 0{\cdot}6256.$$

For points of $\mathcal{R}$ we have

$$1{\cdot}4976 < x + 3y < 2{\cdot}5024, \quad 0{\cdot}4976 < 3x - y < 1{\cdot}5024.$$

Thus in view of (10.7) and (10.8), any exceptional points in $\mathcal{R}$ must satisfy

$$x + 3y < 1{\cdot}6926 \quad \text{or} \quad x + 3y > 2{\cdot}3074,$$

and

$$3x - y < 0{\cdot}6926 \quad \text{or} \quad 3x - y > 1{\cdot}3074.$$

This leaves us with four regions $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R}_3$, $\mathcal{R}_4$, in one of which any exceptional point of $\mathcal{R}$ must lie. $\mathcal{R}_1$ is defined by

$$x > 0{\cdot}3744, \quad y > 0{\cdot}3744, \quad x + 3y < 1{\cdot}6926, \quad 3x - y < 0{\cdot}6926. \tag{10.9}$$

$\mathcal{R}_2$, $\mathcal{R}_3$ and $\mathcal{R}_4$ may be obtained from $\mathcal{R}_1$ by applying powers of the trivial automorph $U(x, y) = (y, -x)$.

The inequalities (10.9) give easily, in $\mathfrak{R}_1$:

$$0{\cdot}3744 < x < 0{\cdot}3771, \quad 0{\cdot}4306 < y < 0{\cdot}4394. \tag{10.10}$$

Applying the transformation U, we have also

$$\text{in } \mathfrak{R}_2: \quad 0{\cdot}4306 < x < 0{\cdot}4394, \quad 0{\cdot}6229 < y < 0{\cdot}6256; \tag{10.11}$$

$$\text{in } \mathfrak{R}_3: \quad 0{\cdot}6229 < x < 0{\cdot}6256, \quad 0{\cdot}5606 < y < 0{\cdot}5694; \tag{10.12}$$

$$\text{and in } \mathfrak{R}_4: \quad 0{\cdot}5606 < x < 0{\cdot}5694, \quad 0{\cdot}3744 < y < 0{\cdot}3771. \tag{10.13}$$

(iii) We now use the fundamental automorph

$$T = \begin{pmatrix} 7 & 20 \\ 20 & 57 \end{pmatrix}$$

of $f(x, y)$, obtained from the fundamental solution $t = 64$, $u = 10$ of the Pellian equation $t^2 - 41\,u^2 = -4$.

If $P = (x, y)$ is an exceptional point of $\mathfrak{R}_1$, then from (10.10),

$$11{\cdot}2328 < 7\,x + 20\,y < 11{\cdot}4277,$$

$$32{\cdot}0322 < 20\,x + 57\,y < 32{\cdot}5878,$$

$$4{\cdot}4122 < 20\,x - 7\,y < 4{\cdot}5278.$$

Since $T(P)$ and $T^{-1}(P)$ are exceptional points, the above inequalities show in view of (10.10)–(10.13):

$$T(P) - (11, 32) \in \mathfrak{R}_1,$$

and

$$T^{-1}(P) \in \mathfrak{R}_1 \pmod 1.$$

From Theorem D, the only exceptional point of $\mathfrak{R}_1$ is therefore given by

$$T(F) - (11, 32) = F,$$

i.e.

$$F = \left(\frac{3}{8}, \frac{7}{16}\right).$$

Applying powers of the transformation U, we now see that the only possible exceptional points of $\mathfrak{R}$ are

$$\pm\left(\frac{3}{8}, \frac{7}{16}\right) \quad \text{and} \quad \pm\left(\frac{7}{16}, -\frac{3}{8}\right),$$

which correspond to the points cited in the statement of the Theorem.

(iv) It remains only to establish the value of the minimum at the exceptional points. Since they can all be obtained by applying powers of U to any one of them, we need only consider one. Reverting to the original form, we need thus only show

$$M\left(f_{41};\frac{5}{16},\frac{5}{16}\right)=\frac{23}{32}. \tag{10.14}$$

Now

$$f_{41}\left(\frac{5}{16}-1,\frac{5}{16}\right)=-\frac{23}{32},$$

so that if (10.14) is false, by Theorem B there is a solution of

$$|f_{41}(x,y)|<\frac{23}{32}$$

with $(x, y) \equiv \pm\left(\frac{5}{16},\frac{5}{16}\right)$ and

$$y^2<\frac{23}{32}\cdot\frac{u^2}{t}=\frac{575}{512}.$$

This last inequality shows that we need only consider the values

$$x=u+\frac{5}{16},\quad y=\frac{5}{16}, \tag{10.15}$$

and

$$x=u+\frac{5}{16},\quad y=-\frac{11}{16}. \tag{10.16}$$

If (10.15) holds,

$$f_{41}(x,y)=u^2+\frac{15}{16}u-\frac{25}{32},$$

and the least value of $|f_{41}(x,y)|$ is $\frac{23}{32}$, attained at $u=-1$. If (10.16) holds,

$$f_{41}(x,y)=u^2-\frac{1}{16}u-\frac{155}{32},$$

and the least value of $|f_{41}(x,y)|$ is $\frac{23}{32}$, attained at $u=-2$.

These results together establish (10.14) and hence the Theorem.

Theorem 10. *If $f_m(x,y)=f_{93}(x,y)=x^2+xy-23y^2$, then $M(f_{93})=\frac{44}{31}$ and is attained only at points congruent to $\pm\left(\frac{5}{31},-\frac{10}{31}\right)$.*

Proof: We use the equivalent form

$$f(x,y)=x^2+9xy-3y^2,$$

which is equivalent to $f_{93}(x, y)$ under the transformation $(x, y) \to (x + 4y, y)$. The exceptional points become $\pm\left(\frac{14}{31}, -\frac{10}{31}\right)$.

(i) We have

$$|f(x, y)| = \left|\left(x + \frac{9}{2}y\right)^2 - \frac{93}{4}y^2\right| = 3\left|\left(y - \frac{3}{2}x\right)^2 - \frac{31}{12}x^2\right|,$$

so that

$$M(f; P) < \frac{44}{31} \tag{10.17}$$

if

$$\phi\left(\frac{31}{12}x^2\right) < \frac{44}{93} \quad \text{or} \quad \phi\left(\frac{93}{4}y^2\right) < \frac{44}{31}.$$

By Theorem J, these hold respectively if

$$\frac{31}{12}x^2 < \frac{44}{93} \quad \text{or} \quad \frac{49}{93} = 1 - \frac{44}{93} < \frac{31}{12}x^2 < \frac{1}{4} + \frac{44}{93} = \frac{269}{372},$$

and

$$\frac{93}{4}y^2 < 1 + \frac{44}{31} = \frac{75}{31} \quad \text{or} \quad \frac{80}{31} = 4 - \frac{44}{31} < \frac{93}{4}y^2 < \frac{9}{4} + \frac{44}{31} = \frac{455}{124};$$

and so *a fortiori* (10.17) holds if

$$|x| \le \frac{13 \cdot 26}{13} \quad \text{or} \quad \frac{14}{31} < |x| \le \frac{1}{2}, \tag{10.18}$$

or if

$$|y| < \frac{10}{31} \quad \text{or} \quad \frac{10 \cdot 33}{31} \le |y| \le \frac{12 \cdot 31}{31}. \tag{10.19}$$

(ii) Automorphs (of finite order) of $f(x, y)$ are

$$U = \begin{pmatrix} -1 & -9 \\ 0 & 1 \end{pmatrix}, \quad V = \begin{pmatrix} 1 & 0 \\ 3 & -1 \end{pmatrix}.$$

Suppose now that $P = (x, y)$ satisfies

$$\frac{13 \cdot 26}{31} < x \le \frac{14}{31}, \quad \frac{10}{31} \le y < \frac{18 \cdot 69}{31} = 1 - \frac{12 \cdot 31}{31}. \tag{10.20}$$

Then

$$\frac{21 \cdot 09}{31} < 3x - y \le \frac{32}{31},$$

so that in view of (10.19), $V(P)$ and therefore P cannot be exceptional. Combining this, and its image, with (10.18) and (10.19), we see that any exceptional point must be congruent to a point of

$$\mathcal{R}:\ \frac{13\cdot 26}{31} < x \leq \frac{14}{31},\quad -\frac{10\cdot 33}{31} < y \leq -\frac{10}{31}, \tag{10.21}$$

or of $\mathcal{R}'$, its image in the origin.

Suppose now that $P = (x, y)$ is an exceptional point of $\mathcal{R}$. Then from (10.21),

$$\frac{14}{31} \leq -x - 9y - 2 < \frac{17\cdot 71}{31}. \tag{10.22}$$

By consideration of the y-coordinate, it is impossible to have $U(P) \in \mathcal{R}'$ (mod 1); and from (10.22) and (10.21), we can only have $U(P) \in \mathcal{R}$ (mod 1) if there is equality on the left in (10.22), i.e. for the point $\left(\frac{14}{31}, -\frac{10}{31}\right)$. Thus the only possible exceptional points are those named above.

(iii) It remains only to show that

$$M(f; P) = \frac{44}{31} \tag{10.23}$$

when $P = \left(\frac{14}{31}, -\frac{10}{31}\right)$. Now the fundamental automorph of $f(x, y)$ is

$$T = -UV = \begin{pmatrix} 1 & 9 \\ 3 & 28 \end{pmatrix},$$

and $T(P) \equiv -P$. Since

$$f\left(\frac{14}{31}, -\frac{10}{31}\right) = -\frac{44}{31},$$

Theorem B now shows that (10.23) holds unless there exists a solution of $|f(x, y)| < \frac{44}{31}$ with $(x, y) \equiv \pm P$ and

$$y^2 < \frac{44}{31}\cdot\frac{t+2}{D} = \frac{44}{93}.$$

Since $f(x, y) = f(-x, -y)$, we need only examine the sets

$$(x, y) = \left(u + \frac{14}{31}, -\frac{10}{31}\right), \tag{10.24}$$

$$(x, y) = \left(u + \frac{14}{31}, \frac{21}{31}\right). \tag{10.25}$$

But if (10.24) holds,

$$|f(x, y)| = \left|(u-1)^2 - \frac{75}{31}\right| \geq \frac{44}{31},$$

and if (10.25) holds,

$$|f(x, y)| = \left|\left(u + \frac{7}{2}\right)^2 - \frac{1323}{124}\right| \geq \left|\frac{49}{4} - \frac{1323}{124}\right| = \frac{49}{31}.$$

This contradiction proves the Theorem.

11. We next consider the form

$$f_{61}(x, y) = x^2 + xy - 15y^2.$$

This form has occupied the attention of several writers[1], mainly in connection with the problem of the Euclidean Algorithm (cf. § 2). Rédei [1] was the first to prove that $k(\sqrt{61})$ is not Euclidean, by finding (in our notation) a rational point P for which

$$M(f_{61}; P) = \frac{41}{39} > 1. \tag{11.1}$$

He also stated ([1], p. 601, footnote) that for every rational point either (11.1) holds or else $M(f_{61}; P) < 1$; and on this basis Inkeri [3] conjectured that $M(f_{61}) = \frac{41}{39}$.

Both the statement and the conjecture are false, as we shall now show; the constant $\frac{41}{39}$ is in fact the second minimum.

Theorem 11. *If*

$$f_m(x, y) = f_{61}(x, y) = x^2 + xy - 15y^2, \tag{11.2}$$

then

$$M(f_{61}) = \frac{1611}{1525}, \quad M_2(f_{61}) = \frac{41}{39}.$$

Moreover, $M(f_{61})$ *is attained only at points congruent to*

$$\pm\left(\frac{66}{305}, -\frac{132}{305}\right) \quad \textit{or} \quad \pm\left(\frac{67}{305}, -\frac{134}{305}\right); \tag{11.3}$$

and the only rational points at which $M_2(f_{61})$ *is attained are those congruent to*

$$\pm\left(\frac{8}{39}, -\frac{17}{39}\right) \quad \textit{or} \quad \pm\left(\frac{9}{39}, -\frac{17}{39}\right). \tag{11.4}$$

Proof: We use the equivalent form

$$f(x, y) = 3x^2 + 5xy - 3y^2,$$

which is obtained from $f_{61}(x, y)$ by the transformation $(x, y) \to (7x - 3y, 2x - y)$. The points (11.3), (11.4) become respectively

$$\pm\left(\frac{148}{305}, -\frac{141}{305}\right) \quad \text{and} \quad \pm\left(\frac{141}{305}, \frac{148}{305}\right), \tag{11.5}$$

and

$$\pm\left(\frac{18}{39}, \frac{19}{39}\right) \quad \text{and} \quad \pm\left(\frac{19}{39}, -\frac{18}{39}\right). \tag{11.6}$$

[1] See Rédei [1], Hua and Shih [1], Inkeri [2].

(i) We have

$$|f(x, y)| = 3\left|\left(x + \frac{5}{6}y\right)^2 - \frac{61}{36}y^2\right|,$$

so that

$$M(f; P) < \frac{41}{39} \tag{11.7}$$

if

$$\phi\left(\frac{61}{36}y^2\right) < \frac{41}{117}. \tag{11.8}$$

By Theorem J, this is equivalent to

$$\frac{61}{36}y^2 < \frac{41}{117},$$

so that (11.7) holds *a fortiori* if

$$|y| \leq 0{\cdot}4547,$$

or by symmetry if

$$|x| \leq 0{\cdot}4547.$$

Thus any exceptional point is congruent to a point of the region

$$\mathcal{R}:\ 0{\cdot}4547 < x, y < 0{\cdot}5453. \tag{11.9}$$

(ii) We now use the automorphs

$$T = \begin{pmatrix} 7 & 15 \\ 15 & 32 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

where T is the fundamental automorph corresponding to the solution $t = 39$, $u = 5$ of $t^2 - 61u^2 = -4$.

Let $P = (x, y)$ be any exceptional point of $\mathcal{R}$. Using (11.9), we find that

$$10{\cdot}0034 < 7x + 15y < 11{\cdot}9966, \tag{11.10}$$

$$3{\cdot}0034 < 15x - 7y < 4{\cdot}9966. \tag{11.11}$$

Since $T(P)$ is exceptional, we must have either

$$10{\cdot}4547 < 7x + 15y < 10{\cdot}5453, \tag{11.12}$$

or

$$11{\cdot}4547 < 7x + 15y < 11{\cdot}5453, \tag{11.13}$$

by (11.10) and (11.9); and since $T^{-1}(P)$ is exceptional, by (11.11) and (11.9) we must have either

$$3{\cdot}4547 < 15x - 7y < 3{\cdot}5453, \tag{11.14}$$

or

$$4{\cdot}4547 < 15x - 7y < 4{\cdot}5453. \tag{11.15}$$

These last four inequalities enable us to specify four incongruent regions $\mathcal{R}_i$ ($i = 1, 2, 3, 4$), to one of which any exceptional point of $\mathcal{R}$ must belong. We choose the notation so that $\mathcal{R}_1$ is defined by (11.12) and (11.14), $\mathcal{R}_2$ by (11.13) and (11.14), $\mathcal{R}_3$ by (11.13) and (11.15), and $\mathcal{R}_4$ by (11.12) and (11.15). It is easily seen that the regions $\mathcal{R}_i$ are permuted cyclicly by the transformation U (mod 1).

The inequalities (11.12) and (11.14) give the bounds

$$0{\cdot}4562 < x < 0{\cdot}4635, \quad 0{\cdot}4817 < y < 0{\cdot}4891 \tag{11.16}$$

for points of $\mathcal{R}_1$. Applying U repeatedly, we therefore have

$$\text{in } \mathcal{R}_2 \qquad 0{\cdot}4817 < x < 0{\cdot}4891, \quad 0{\cdot}5365 < y < 0{\cdot}5438; \tag{11.17}$$

$$\text{in } \mathcal{R}_3 \qquad 0{\cdot}5365 < x < 0{\cdot}5438, \quad 0{\cdot}5109 < y < 0{\cdot}5183; \tag{11.18}$$

$$\text{and in } \mathcal{R}_4 \quad 0{\cdot}5109 < x < 0{\cdot}5183, \quad 0{\cdot}5462 < y < 0{\cdot}4635. \tag{11.19}$$

Since $T(P) \in \mathcal{R}$ (mod 1) for any exceptional point $P \in \mathcal{R}_1$, we see from (11.12) and (11.16) that

$$T(P) - (10, 22) \in \mathcal{R}, \tag{11.20}$$

and from (11.14) and (11.16) that

$$T^{-1}(P) - (-8, 3) \in \mathcal{R}. \tag{11.21}$$

Similar results hold, of course, for the other regions $\mathcal{R}_i$.

(iii) We now consider which of the regions $\mathcal{R}_i$ an exceptional point $P \in \mathcal{R}_1$ can transform into (mod 1) under T and T^{-1}. We show that in fact $T(P)$ must lie (mod 1) in $\mathcal{R}_3$ or $\mathcal{R}_4$, and that $T^{-1}(P)$ must lie (mod 1) in $\mathcal{R}_2$ or $\mathcal{R}_3$.

For suppose $T(P)$ is congruent to a point of $\mathcal{R}_1$ or $\mathcal{R}_2$. Then from (11.16), (11.17) and (11.20) we have

$$7x + 15y < 10{\cdot}4891, \quad 15x + 32y > 22{\cdot}4817,$$

and so

$$y = 15(7x + 15y) - 7(15x + 22y)$$
$$< 15 \times 10{\cdot}4891 - 7 \times 22{\cdot}4817 = -0{\cdot}0354,$$

which is impossible.

For convenience, we shall assume through the rest of this proof that all suffixes are taken modulo 4. Then applying U repeatedly to the above result, we see that if P_i is any exceptional point of $\mathcal{R}_i$ then

$$T(P_i) \in \mathcal{R}_{i+2} \text{ or } \mathcal{R}_{i+3} \text{ (mod 1)}. \tag{11.22}$$

An obvious formal argument from the four results (11 22) now shows that

$$T^{-1}(P_i) \in \mathcal{R}_{i+1} \quad \text{or} \quad \mathcal{R}_{i+2} \ (\text{mod } 1). \tag{11.23}$$

We now define the sets $S^{(i)}_{j,k}$ $(i, j, k = 1, 2, 3, 4)$ by the rule: $P \in S^{(i)}_{j,k}$ if $P \in \mathcal{R}_i$, $T(P) \in \mathcal{R}_j$ (mod 1), $T^{-1}(P) \in \mathcal{R}_k$ (mod 1). Thus $S^{(i)}_{j,k}$ is a subset of $\mathcal{R}_i$.

In view of (11.22) and (11.23), many of these sets are empty. Thus for $i = 1$ only four of these sets need be considered, namely

$$S^{(1)}_{3,2}, \ S^{(1)}_{3,3}, \ S^{(1)}_{4,2} \text{ and } S^{(1)}_{4,3}.$$

We next show that $S^{(1)}_{4,3}$ can contain no exceptional points, since it is entirely contained in the hyperbolic region $|f(x, y)| < \frac{41}{39}$. For let $P = (x, y)$ be any point in $S^{(1)}_{4,3}$. Then from (11.19) and (11.20),

$$10{\cdot}5109 < \ 7x + 15y < 10{\cdot}5183, \tag{11.24}$$

$$22{\cdot}4562 < 15x + 32y < 22{\cdot}4635; \tag{11.25}$$

while from (11.18) and (11.21),

$$-7{\cdot}4635 < -32x + 15y < -7{\cdot}4562, \tag{11.26}$$

$$3{\cdot}5109 < \ 15x - 7y < \ 3{\cdot}5183. \tag{11.27}$$

Combining (11.24) with (11.26) and (11.25) with (11.27) we have

$$17{\cdot}9671 < 39x < 17{\cdot}9818, \tag{11.28}$$

$$18{\cdot}9379 < 39y < 18{\cdot}9526. \tag{11.29}$$

We now change the coordinates, writing

$$x = \frac{18}{39} - \xi, \quad y = \frac{19}{39} - \eta,$$

so that (11.28) and (11.29) become

$$0{\cdot}0182 < 39\,\xi < 0{\cdot}0329, \tag{11.30}$$

$$0{\cdot}0474 < 39\,\eta < 0{\cdot}0621. \tag{11.31}$$

Since

$$f(x, y) = 3\,\xi^2 + 5\,\xi\,\eta - 3\,\eta^2 - \frac{203}{39}\,\xi + \frac{24}{39}\,\eta + \frac{41}{39},$$

the inequalities (11.30) and (11.31) show at once that

$$\frac{41}{39} > f(x, y) > -\frac{41}{39}.$$

(iv) We are now in a position to apply the transformation theory of Theorems C and D. By the definition of the $S_{j,k}^{(i)}$, if $P \in S_{j,k}^{(i)}$ is an exceptional point, then

$$T(P) \in S_{m,i}^{(j)} \pmod 1, \quad T^{-1}(P) \in S_{i,n}^{(k)} \pmod 1$$

for some m, n. Thus

$$P \in S_{4,2}^{(1)} \text{ implies } T(P) \in S_{3,1}^{(4)} \text{ or } S_{2,1}^{(4)} \pmod 1,\ T^{-1}(P) \in S_{1,3}^{(2)} \pmod 1; \tag{11.32}$$

$$P \in S_{3,3}^{(1)} \text{ implies } T(P) \in S_{1,1}^{(3)} \pmod 1,\ T^{-1}(P) \in S_{1,1}^{(3)} \text{ or } S_{1,4}^{(3)} \pmod 1; \tag{11.33}$$

$$P \in S_{3,2}^{(1)} \text{ implies } T(P) \in S_{1,1}^{(3)} \pmod 1,\ T^{-1}(P) \in S_{1,3}^{(2)} \pmod 1; \tag{11.34}$$

with the corresponding results obtained from these by cyclic permutation of the indices; and the shifts are in every case unique, being given by (11.20) and (11.21), and their analogues.

We now apply Theorem C′ to the set $S_{3,2}^{(1)} + S_{3,3}^{(1)}$ and those obtained from it by cyclic permutation of indices. The relevant fixed point is obviously the solution of

$$T(F) - (10, 22) = (1, 1) - F,$$

which is $\left(\frac{18}{39}, \frac{19}{39}\right)$; and so the only possible exceptional points of $S_{3,2}^{(1)} + S_{3,3}^{(1)}$ lie on the line

$$\frac{5-\sqrt{61}}{6}\left(x - \frac{18}{39}\right) = y - \frac{19}{39}. \tag{11.35}$$

Similarly, we apply Theorem C′ (with T^{-1} for T) to the set $S_{3,2}^{(1)} + S_{4,2}^{(1)}$ and those obtained from it by cyclic permutation of indices. The relevant fixed point is the solution of

$$T^{-1}(F) - (-8, 3) = (0, 1) + U(F)$$

which is $\left(\frac{141}{305}, \frac{148}{305}\right)$; and so the only possible exceptional points of $S_{3,2}^{(1)} + S_{4,2}^{(1)}$ lie on the line

$$\frac{5+\sqrt{61}}{6}\left(x - \frac{141}{305}\right) = y - \frac{148}{305}. \tag{11.36}$$

In particular we may deduce from this that the only possible exceptional point of $S_{3,2}^{(1)}$ is given by the intersection of (11.35) and (11.36), that is

$$P_1 = \left(\frac{18}{39} + \frac{3+\sqrt{61}}{7\,930}, \frac{19}{39} - \frac{23-\sqrt{61}}{23\,790}\right).$$

The transforms of P_1 under positive and negative powers of T, their transforms under U, U^2 and U^3, and the points congruent to them form a set at every point of which $M(f; P)$ has the same value. We call this set C.

Now consider the possible exceptional points not in the set C. Then none of their transforms under (positive or negative) powers of T can lie in any $S^{(i)}_{i+2,i+1}$, so that we must always take the first alternative in (11.32), (11.33) and the analogous results. We can now deduce from the extension of Theorem D analogous to Theorem C′ that the only other possible exceptional points in any $S^{(i)}_{i-1,i+1}$ are the points (11.5) and the only other possible exceptional points in any $S^{(i)}_{i+2,i+2}$ are the points (11.6).

(v) It remains only to establish the value of $M(f; P)$ at the various possible exceptional points we have now obtained. It is convenient to return to the original form (11.2), the sets (11.5) and (11.6) now becoming (11.3) and (11.4) respectively.

(a) As appeared in the above work, the set (11.3) is permuted modulo 1 by the fundamental automorph. Since

$$f_{61}\left(\frac{67}{305}+2,\ -\frac{134}{305}\right)=\frac{1\,611}{1\,525},$$

it is sufficient to prove that the inequality

$$|f_{61}(x, y)| < \frac{1\,611}{1\,525} \tag{11.37}$$

is impossible for points (x, y) congruent to any point of the set (11.3). But Theorem B shows that if (11.37) has such a solution, it has one with

$$y^2 < \frac{1\,611}{1\,525}\cdot\frac{u^2}{t} < 1.$$

Since $f_{61}(x, y) = f_{61}(-x, -y)$, we need only consider the cases

$$x = u + \frac{66}{305},\quad y = -\frac{132}{305};$$

$$x = u + \frac{66}{305},\quad y = \frac{173}{305};$$

$$x = u + \frac{67}{305},\quad y = -\frac{134}{305};$$

$$x = u + \frac{67}{305},\quad y = \frac{171}{305};$$

where u is integral. However, none of these values of (x, y) satisfy (11.37), since they give respectively

$$\left| f_{61}\left(u + \frac{66}{305}, -\frac{132}{305}\right) \right| = \left| u^2 - \frac{4\,356}{1\,525} \right| \geq \frac{1\,744}{1\,525};$$

$$\left| f_{61}\left(u + \frac{66}{305}, \frac{173}{305}\right) \right| = \left| \left(u + \frac{1}{2}\right)^2 - \frac{29\,929}{6\,100} \right| \geq \frac{2\,049}{1\,525};$$

$$\left| f_{61}\left(u + \frac{67}{305}, -\frac{134}{305}\right) \right| = \left| u^2 - \frac{4\,489}{1\,525} \right| \geq \frac{1\,611}{1\,525};$$

$$\left| f_{61}\left(u + \frac{67}{305}, \frac{171}{305}\right) \right| = \left| \left(u + \frac{1}{2}\right)^2 - \frac{29\,241}{6\,100} \right| \geq \frac{2\,221}{1\,525}.$$

(b) A precisely similar argument holds for the set (11.4). Since this set is permuted by the fundamental automorph and

$$f_{61}\left(\frac{8}{39} + 2, -\frac{17}{39}\right) = \frac{41}{39},$$

it is sufficient to prove that the inequality

$$|f_{61}(x, y)| < \frac{41}{39} \tag{11.38}$$

is impossible for points (x, y) congruent to any point of the set (11.4). As above, we need only consider the cases

$$x = u + \frac{8}{39}, \quad y = -\frac{17}{39};$$

$$x = u + \frac{8}{39}, \quad y = \frac{22}{39};$$

$$x = u + \frac{9}{39}, \quad y = -\frac{17}{39};$$

$$x = u + \frac{9}{39}, \quad y = \frac{22}{39};$$

where u is integral. However, none of these values of (x, y) satisfy (11.38), since they give respectively

$$\left| f_{61}\left(u + \frac{8}{39}, -\frac{17}{39}\right) \right| = \left| \left(u - \frac{1}{78}\right)^2 - \frac{17\,629}{6\,084} \right| \geq \frac{41}{39};$$

$$\left| f_{61}\left(u + \frac{8}{39}, \frac{22}{39}\right) \right| = \left| \left(u + \frac{19}{39}\right)^2 - \frac{7\,381}{1\,521} \right| \geq \frac{52}{39};$$

$$\left| f_{61}\left(u + \frac{9}{39}, -\frac{17}{39}\right) \right| = \left| \left(u + \frac{1}{78}\right)^2 - \frac{17\,629}{6\,084} \right| \geq \frac{41}{39};$$

$$\left| f_{61}\left(u + \frac{9}{39}, \frac{22}{39}\right) \right| = \left| \left(u + \frac{20}{39}\right)^2 - \frac{7\,381}{1\,521} \right| \geq \frac{52}{39}.$$

(c) It remains only to consider the set C. It is clear that it contains no rational points, and the argument which gave (11.35) shows further that the points (11.6) are limit points of C. Thus from Theorem F we have

$$M(f; C) \leq \frac{41}{39}. \tag{11.39}$$

By an obvious extension of Theorem B it is possible to show that we have in fact equality in (11.39) and that the minimum is unattained; but the proof, while introducing no new ideas, involves such a mass of arithmetic that we do not give it.

A slight extension of the ideas of this proof would show that there are an infinity of incongruent rational points having

$$M(f_{61}; P) > \frac{411}{390} - \frac{\sqrt{61}}{1\,586} > 1;$$

but we have no satisfactory way of specifying these points, or the values of the minimum at them. Indeed, we are even unable to show that this constant is best possible, though this is probably so.

12. Of the forms listed in (10.1) and (10.2), we have also obtained the values of $M(f_m)$ when

$$m = 19, 22, 43, 58, 59, 70;\; 33, 89, 97; \tag{12.1}$$

and the values of $M_2(f_m)$ for

$$m = 3, 6, 10, 15, 26, 30, 35, 42, 43, 82;\; 17, 33, 37, 65, 101. \tag{12.2}$$

We omit the proof of these results, since no essentially new principle is involved.

The table below gives all results which are now known for the first and second minima of forms $f_m(x, y)$ with $m \leq 101$. The second column of the table gives the theorem in this paper in which a result is proved. The last column consists mainly of acknowledgements to previous work, references being to the bibliography at the end of the paper. All results for which no acknowledgement is given are here stated for the first time. The entries under the values (12.1) and (12.2) of m contain a complete set of incongruent points P for which $M(f_m; P) = M(f_m)$ or $M(f_m; P) = M_2(f_m)$; these we denote by C_1, C_2 respectively.

The value of $M(f_m)$ is not yet known when

$$m = 46, 67, 71, 86, 94;\; 57, 73.$$

In these cases we have given the best upper bound known for $M(f_m)$.

$f_m(x, y) = x^2 - my^2, \quad m \equiv 2$ or $3 \pmod 4$.

m	Theorem	$M(f_m)$	$M_2(f_m)$	Acknowledgements and Remarks
2		$\frac{1}{2}$	$\frac{1}{4}$	Varnivides [2] gives all $M_i(f)$ $(i = 1, 2, 3, \ldots)$.
3		$\frac{1}{2}$	$\frac{1}{3}$	Heinhold [1] gives M; $C_2 = \pm\left(0, \frac{1}{3}\right)$.
6		$\frac{3}{4}$	$\frac{1}{2}$	Heinhold [1] gives M; $C_2 = \left(0, \frac{1}{2}\right)$.
7	5	$\frac{9}{14}$	$\frac{1}{2}$	Bambah [1] gives M_2; Varnavides [3], Inkeri [3, 5] give M.
10		$\frac{3}{2}$	$\frac{39}{40}$	Heinhold [1] gives M; $C_2 = \pm\left(0, \frac{9}{20}\right), \pm\left(\frac{1}{2}, \frac{7}{20}\right)$.
11	4	$\frac{19}{22}$		Varnavides [2] states result; Bambah [1]; Inkeri [5].
14		$\frac{5}{4}$		Heinhold [1].
15		$\frac{3}{2}$	$\frac{7}{5}$	Heinhold [1] for M; $C_2 = \pm\left(\frac{1}{2}, \frac{2}{5}\right)$.
19		$\frac{31}{38}$		$C_1 = \pm\left(\frac{1}{2}, \frac{9}{38}\right)$.
22		$\frac{27}{22}$		$C_1 = \pm\left(0, \frac{7}{22}\right)$.
23	2	$\frac{77}{46}$		
26		$\frac{5}{2}$	$\frac{207}{104}$	Heinhold [1] gives M; $C_2 = \pm\left(0, \frac{25}{52}\right), \pm\left(\frac{1}{2}, \frac{21}{52}\right)$.
30		$\frac{3}{2}$	$\frac{29}{20}$	Heinhold [1] gives M; $C_2 = \pm\left(\frac{1}{2}, \frac{2}{5}\right)$.
31	8	$\frac{45}{31}$		
34		$\frac{9}{4}$		Heinhold [1].
35		$\frac{5}{2}$	$\frac{17}{7}$	Heinhold [1] for M; $C_2 = \pm\left(\frac{1}{2}, \frac{3}{7}\right)$.
38		$\frac{11}{4}$		Heinhold [1].
39		$\frac{5}{2}$		Heinhold [1].
42		$\frac{7}{4}$	$\frac{41}{24}$	Heinhold [1] gives M; $C_2 = \pm\left(0, \frac{7}{12}\right)$.
43		$\frac{11\,829}{6\,962}$	$\frac{5\,902}{3\,483}$	$C_1 = \pm\left(\frac{1}{118}, \frac{1}{2}\right)$; $C_2 = \pm\left(0, \frac{193}{387}\right)$.

m	Theorem	$M\,(f_m)$	$M_2\,(f_m)$	Acknowledgements and Remarks
46				$M < \frac{7}{4}$, Varnivides [1], Inkeri [3].
47	2	$\frac{253}{94}$		Inkeri [3], states result only.
51	4	$\frac{287}{102}$		
55		$\frac{9}{4}$		Heinhold [1].
58		$\frac{3}{2}$		$C_1 = \left(0, \frac{1}{2}\right)$.
59		$\frac{125}{59}$		$C_1 = \pm\left(0, \frac{19}{59}\right)$.
62		$\frac{13}{4}$		Heinhold [1].
66		$\frac{15}{4}$		Heinhold [1].
67				$M < \frac{9}{4}$, Inkeri [3].
70		$\frac{891}{500}$		$C_1 = \pm\left(\frac{1}{2}, \frac{6}{25}\right)$.
71				$M < 2.40$, Inkeri [3].
74		$\frac{5}{2}$		Heinhold [1].
78		$\frac{7}{2}$		Heinhold [1].
79	2	$\frac{585}{158}$		
82		$\frac{9}{2}$	$\frac{1\,311}{328}$	Heinhold [1] gives M; $C_2 = \pm\left(0, \frac{81}{164}\right), \pm\left(\frac{1}{2}, \frac{73}{164}\right)$.
83	4	$\frac{631}{166}$		
86				$M < 2.24$, Inkeri [3].
87	6	$\frac{169}{58}$		
91		$\frac{5}{2}$		Heinhold [1].
94				$M < \frac{5}{2}$.
95		$\frac{7}{2}$		Heinhold [1].

$$f_m(x, y) = x^2 + xy - \frac{1}{4}(m-1)y^2, \quad m \equiv 1 \pmod 4.$$

m	Theorem	$M(f_m)$	$M_2(f_m)$	Acknowledgements and Remarks
5		$\frac{1}{4}$	$\frac{1}{5}$	Davenport [1] gives all $M_i(f)$ $(i = 1, 2, 3, \ldots)$.
13	7	$\frac{1}{3}$	$\frac{4}{13}$	Inkeri [4].
17		$\frac{1}{2}$	$\frac{8}{17}$	Heinhold [1] for M; $C_2 = \pm\left(\frac{5}{17}, \frac{7}{17}\right), \pm\left(\frac{3}{17}, \frac{6}{17}\right)$.
21	3	$\frac{5}{7}$		
29	7	$\frac{4}{5}$	$\frac{23}{29}$	
33		$\frac{29}{44}$	$\frac{6}{11}$	$C_1 = \pm\left(\frac{7}{22}, \frac{4}{11}\right)$; $C_2 = \pm\left(\frac{3}{11}, \frac{5}{11}\right)$.
37		$\frac{3}{4}$	$\frac{27}{37}$	Heinhold [1] for M; $C_2 = \pm\left(\frac{8}{37}, -\frac{16}{37}\right), \pm\left(\frac{11}{37}, -\frac{15}{37}\right)$.
41	9	$\frac{23}{32}$		
53	7	$\frac{9}{7}$	$\frac{68}{53}$	
57				$M < 0.89$, Inkeri [3].
61	11	$\frac{1\,611}{1\,525}$	$\frac{41}{39}$	Rédei [1] proves $M \geq \frac{41}{39}$.
65		1	$\frac{64}{65}$	Heinhold [1] gives M; $C_2 = \pm\left(\frac{18}{65}, \frac{29}{65}\right), \pm\left(\frac{14}{65}, \frac{28}{65}\right)$.
69	5	$\frac{25}{23}$		Inkeri [3].
73				$M < 1$; Inkeri [3], Rédei [1].
77	1	$\frac{19}{11}$		
85	7	$\frac{16}{9}$	$\frac{151}{85}$	
89		$\frac{1\,004\,287}{1\,000\,004}$		$C_1 = \pm\left(\frac{1\,497}{9\,434}, -\frac{1\,497}{4\,717}\right), \pm\left(\frac{1\,503}{9\,434}, -\frac{1\,503}{4\,717}\right)$.
93	10	$\frac{44}{31}$		
97		$\frac{33\,679\,354}{31\,404\,817}$		$C_1 = \pm\left(\frac{14\,845}{55\,193}, -\frac{29\,690}{55\,193}\right), \pm\left(\frac{15\,529}{55\,193}, -\frac{31\,058}{55\,193}\right)$.
101		$\frac{5}{4}$	$\frac{125}{101}$	Heinhold [1] gives M; $C_2 = \pm\left(\frac{23}{101}, -\frac{46}{101}\right), \pm\left(\frac{28}{101}, -\frac{45}{101}\right)$.

All minima given in the table are isolated. This follows from Theorem G, except in the cases $m = 61$ (Theorem 11), $m = (2n + 1)^2 + 4$ (Theorem 7), where a slight modification in the proof will clearly isolate $M_2(f_m)$.

It will be noted that the results stated in the table for $m = 97$ show that the field $k(\sqrt{97})$ is not Euclidean. In view of the interest of this fact, we now give a direct proof of the existence of a rational point P for which (2.8) is false. We could of course use the points quoted in the table, but these require the consideration of a larger number of cases.

Theorem 12. *If*

$$f(x, y) = f_{97}(x, y) = x^2 + xy - 24y^2,$$

then[1]

$$M\left(f; \frac{374}{1\,401}, \frac{2\,587}{5\,604}\right) = \frac{3\,001}{2\,802}. \tag{12.3}$$

The field $k(\sqrt{97})$ therefore does not possess a Euclidean Algorithm.

Proof: Write

$$P = \left(\frac{374}{1\,401}, \frac{2\,587}{5\,604}\right), \quad K = \frac{3\,001}{2\,802}.$$

Since

$$f\left(2 + \frac{374}{1\,401}, \frac{2\,587}{5\,604}\right) = K,$$

we have

$$M(f; P) \leq K.$$

Hence if (12.3) is false there exists a solution of

$$|f(x, y)| < K, \quad (x, y) \equiv P. \tag{12.4}$$

We now apply Theorem B. The fundamental automorph of $f(x, y)$ is

$$T = \begin{pmatrix} 5\,035 & 27\,312 \\ 1\,138 & 6\,173 \end{pmatrix},$$

corresponding to the solution $t = 11\,208$, $u = 1\,138$ of $t^2 - 97u^2 = -4$. It is easily verified that

$$T(P) = (13\,952, 3\,153) + P \equiv P.$$

Theorem B now shows that if (12.4) has a solution, it has one with

$$y^2 < \frac{3\,001}{2\,802} \cdot \frac{u^2}{t} = \frac{971\,606\,761}{(2\,802)^2},$$

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[1] The constant $\frac{3\,001}{2\,802}$ is probably $M_2(f_{97})$; but the verification of this conjecture is of no real interest.

and so with

$$|y| < \frac{31\,171}{2\,802} < 11 + \frac{2\,587}{5\,604}. \tag{12.5}$$

To prove the theorem, it is therefore sufficient to show that the inequality $|f(x, y)| < K$ has no solution with

$$x = u + \frac{374}{1\,401}, \quad y = v + \frac{2\,587}{5\,604},$$

where u, v are integers and, by (12.5),

$$-11 \leq v \leq 10. \tag{12.6}$$

We write

$$g(u, v) = f\left(u + \frac{374}{1\,401},\ v + \frac{2\,587}{5\,604}\right) =$$

$$= u^2 + u\left(v + 1 - \frac{25}{5\,604}\right) - 24\,v^2 - \frac{30\,670}{1\,401}\,v - \frac{6\,893}{1\,401}.$$

For each integer v satisfying (12.6), it is a simple matter to determine the value of u for which $|g(u, v)|$ is a minimum. The results are tabulated below, and show that in all cases we have

$$|g(u, v)| \geq \frac{3\,001}{2\,802} = K.$$

$$g(u, 0) = u^2 + \left(1 - \frac{25}{5\,604}\right)u - 5 + \frac{112}{1\,401},$$

$$|g(2, 0)| = \frac{3\,001}{2\,802} = K;$$

$$g(u, 1) = u^2 + \left(2 - \frac{25}{5\,604}\right)u - 50 - \frac{1\,137}{1\,401},$$

$$|g(-8, 1)| = 2 - \frac{50}{1\,401} + \frac{1\,137}{1\,401} > 2;$$

$$g(u, 2) = u^2 + \left(3 - \frac{25}{5\,604}\right)u - 144 - \frac{985}{1\,401},$$

$$|g(11, 2)| = 10 - \frac{275}{5\,604} - \frac{985}{1\,401} > 9;$$

$$g(u, 3) = u^2 + \left(4 - \frac{25}{5\,604}\right)u - 286 - \frac{833}{1\,401},$$

$$|g(-19, 3)| = 1 + \frac{2\,857}{5\,604} > \frac{3}{2};$$

$$g(u, 4) = u^2 + \left(5 - \frac{25}{5\,604}\right) u - 476 - \frac{681}{1\,401},$$

$$|g(-24, 4)| = 20 - \frac{150}{1\,401} + \frac{681}{1\,401} > 20;$$

$$g(u, 5) = u^2 + \left(6 - \frac{25}{5\,604}\right) u - 714 - \frac{529}{1\,401},$$

$$|g(24, 5)| = 6 - \frac{150}{1\,401} - \frac{529}{1\,401} > 5;$$

$$g(u, 6) = u^2 + \left(7 - \frac{25}{5\,604}\right) u - 1\,000 - \frac{377}{1\,401},$$

$$|g(-35, 6)| = 20 - \frac{875}{5\,604} + \frac{377}{1\,401} > 20;$$

$$g(u, 7) = u^2 + \left(8 - \frac{25}{5\,604}\right) u - 1\,334 - \frac{225}{1\,401},$$

$$|g(33, 7)| = 19 - \frac{825}{5\,604} - \frac{225}{1\,401} > 18;$$

$$g(u, 8) = u^2 + \left(9 - \frac{25}{5\,604}\right) u - 1\,716 - \frac{73}{1\,401},$$

$$|g(-46, 8)| = 14 - \frac{1\,150}{5\,604} + \frac{73}{1\,401} > 13;$$

$$g(u, 9) = u^2 + \left(10 - \frac{25}{5\,604}\right) u - 2\,146 + \frac{79}{1\,401},$$

$$|g(42, 9)| = 38 - \frac{1\,050}{5\,604} + \frac{79}{1\,401} > 37;$$

$$g(u, 10) = u^2 + \left(11 - \frac{25}{5\,604}\right) u - 2\,623 - \frac{1\,170}{1\,401},$$

$$|g(-57, 10)| = 1 - \frac{1\,425}{5\,604} + \frac{1\,170}{1\,401} > \frac{3}{2};$$

$$g(u, -1) = u^2 - \frac{25}{5\,604} u - 7 - \frac{40}{1\,401},$$

$$|g(3, -1)| = 2 - \frac{75}{5\,604} - \frac{40}{1\,401} > \frac{3}{2};$$

$$g(u, -2) = u^2 - \left(1 + \frac{25}{5\,604}\right) u - 57 - \frac{192}{1\,401},$$

$$|g(-7, -2)| = 1 - \frac{175}{5\,604} + \frac{192}{1\,401} = \frac{6\,197}{5\,604} > K;$$

$$g(u, -3) = u^2 - \left(2 + \frac{25}{5\,604}\right) u - 156 + \frac{1\,057}{1\,401},$$

$$|g(14, -3)| = 12 - \frac{350}{5\,604} + \frac{1\,057}{1\,401} > 12;$$

$$g(u, -4) = u^2 - \left(3 + \frac{25}{5\,604}\right) u - 302 + \frac{905}{1\,401},$$

$$|g(19, -4)| = 2 - \frac{475}{5\,604} + \frac{905}{1\,401} > 2;$$

$$g(u, -5) = u^2 - \left(4 + \frac{25}{5\,604}\right) u - 496 + \frac{753}{1\,401},$$

$$|g(-20, -5)| = 16 - \frac{500}{5\,604} - \frac{753}{1\,401} > 15;$$

$$g(u, -6) = u^2 - \left(5 + \frac{25}{5\,604}\right) u - 738 + \frac{601}{1\,401},$$

$$|g(30, -6)| = 12 - \frac{750}{5\,604} + \frac{601}{1\,401} > 12;$$

$$g(u, -7) = u^2 - \left(6 + \frac{25}{5\,604}\right) u - 1\,028 + \frac{449}{1\,401},$$

$$|g(-29, -7)| = 13 - \frac{725}{5\,604} - \frac{449}{1\,401} > 12;$$

$$g(u, -8) = u^2 - \left(7 + \frac{25}{5\,604}\right) u - 1\,366 + \frac{297}{1\,401},$$

$$|g(41, -8)| = 28 - \frac{1\,025}{5\,604} + \frac{297}{1\,401} > 28;$$

$$g(u, -9) = u^2 - \left(8 + \frac{25}{5\,604}\right) u - 1\,752 + \frac{145}{1\,401},$$

$$|g(-38, -9)| = 4 - \frac{950}{5\,604} - \frac{145}{1\,401} > 3;$$

$$g(u, -10) = u^2 - \left(9 + \frac{25}{5\,604}\right) u - 2\,186 - \frac{7}{1\,401},$$

$$|g(-42, -10)| = 44 - \frac{1\,050}{5\,604} + \frac{7}{1\,401} > 43;$$

$$g(u, -11) = u^2 - \left(10 + \frac{25}{5\,604}\right) u - 2\,668 - \frac{159}{1\,401},$$

$$|g(57, -11)| = 11 - \frac{1\,425}{5\,604} - \frac{159}{1\,401} > 10.$$

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