# On the classification of $G$-spheres I: equivariant transversality

by

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This paper is the first in a series of three. Stated in geometric terms the papers examine locally linear group actions on spheres for odd order groups $G$. In essentially equivalent homotopy theoretic terms the papers study the homotopy types of the spaces $PL_G(V)$ and $\mathrm{Top}_G(V)$ of equivariant $PL$-homeomorphisms and homeomorphisms of a linear representation $V$. In fact, it is the homogeneous spaces $F_G(V)/PL_G(V)$ and $\mathrm{Top}_G(V)/PL_G(V)$ we study where $F_G(V)$ is the space of proper equivariant homotopy equivalences. Our results generalize theorems of Haefliger, Kirby-Siebenmann, Sullivan and Wall, and others.

The link between classification of manifolds and classification of homotopy types is transversality and this is the subject of the present first paper. The second paper generalizes Wall's classification of fake lens spaces to the classification of $G$-spheres which are equivariantly homotopy equivalent to a given linear action. This involves the determination of the equivariant homotopy type of the $G$-space $F/PL$ and a complete calculation of the $PL$ equivariant surgery sequence for a linear $G$-sphere. As a result we show that the homotopy groups of $PL_G(V\oplus U)/PL_G(V)$ vanish in dimensions less than $\dim V^G$. In the third paper we study $G$ triangulation theory. In particular we study the homotopy groups of $\mathrm{Top}_G(V\oplus U)/\mathrm{Top}_G(V)$ in a range. The homotopy groups of the Stiefel spaces $PL_G(V\oplus U)/PL_G(V)$ and $\mathrm{Top}_G(V\oplus U)/\mathrm{Top}_G(V)$ are in turn needed for equivariant transversality, so the three papers are locked together in an inductive fashion. The reader is referred to § 4 below and to the individual Parts II and III for more information.

We now give a brief discussion of the content of the present paper. Let $G$ denote an arbitrary finite group. We consider the $G$-transversality question in the *locally linear* category of $PL$ or topological $G$-manifolds. The general theory is the same in the two cases so to avoid unnecessary repetition we write *cat* for either case. For $X$ a $G$-space we let $A(X)$ be the simplicial set of cat automorphisms of $\Delta_n \times X$ commuting with projection. It is a $G$-simplicial set and we will denote its fixed point set by $A_G(X)$.

Our general problem is the following: We are given $G$-manifolds $M$, $X$ and $Y$, with $M \subset Y$ and a $G$-map $f: X \to Y$. We wish to homotop $f$ through $G$-maps so that it becomes "transverse" to $M$. We will give a precise meaning to this, analyze the obstructions to making a map transverse, and give criteria for them to vanish.

Our approach is through the theory of $G$-submersions. For example, if $M=\{y\}$ is a point then $f$ is transverse to $y$ provided there exists a neighborhood $U$ of $y$ in $Y$ such that $f|f^{-1}(U): f^{-1}(U) \to U$ is a $G$-submersion. Therefore, at least in the case where $M$ is a point, the transversality question can be translated to a submersion question, which in turn can be treated by homotopy theory. By globalizing the idea we can treat the more general situation where $M$ is a $G$-submanifold of $Y$ with a nice normal tube.

Our main results are a 'stable' $G$-transversality theorem in the $PL$ category for odd order groups: if $X$ is a $PL$ $G$-manifold and $f: X \to E_+$ is a $G$-map into the Thom space of a $PL$ $G$-bundle $E$ then $f$ is equivariantly homotopic to a map which is transversal to the zero-section of $E$, provided $X$ and $E$ satisfy certain gap-conditions. In the locally linear topological category we set up an obstruction theory for stable transversality. This is particularly effective when $G$ is abelian.

Combined with results from [16] we can then prove that oriented locally linear topological bundles are oriented w.r.t. equivariant $KO$-theory localized away from 2.

In turn, this implies that the usual cannibalistic classes $\varrho^k(\xi)$ of $G$-vector bundles are topological invariants, and for representations $V$ we show via Franz' independence lemma that the classes $\varrho^k(V)$ in fact determines $V$ when $G$ has odd order. Hence we arrive at the result which motivated the whole study: topological similar representations of odd order groups are linearly similar. This result was independently proved by Hsiang and Pardon in [12] by rather different methods.

Equivariant transversality was introduced by A. Wassermann [28] in the smooth category. An obstruction theory to smooth equivariant transversality was developed by T. Petrie, [25]. The obstruction theory presented in this paper in the $PL$ and Top categories is similar in spirit to the theory from [25], but the details are quite different. Of importance is that our obstruction groups are often zero even when the obstruction groups to smooth transversality are non-zero. Thus it can happen that a map from a smooth manifold into a vector bundle which cannot be deformed to a smoothly transversal map can be deformed to a $PL$-transversal map.

Our topological transversality results (Theorems 4.10 and 4.12 below) are probably not optimal. First, the restriction to abelian groups might not be necessary. Second, we only present a hierarchy of 'stepwise' obstructions, where one would like to have the transversality obstructions a priori defined. However, several years have passed since we published the outline in [18], and we find it unreasonable to delay the publication of detailed proofs any further, even if we think that we might be able to treat more general cases in the future.

Finally, it is in order to point out that the restriction to consider only groups of odd order is in fact necessary. For $G=\mathbf{Z}/2$ stable transversality fails even in the $PL$ category. This is discussed in [19].

The paper is divided into the following sections:



## §1. Notations, basic notions, the $G$-submersion theorem

To carry out the above program rigorously requires a rather formidable amount of technical machinery and notation. We will attempt to steer a middle course, providing enough detail to convince the skeptical without belaboring technical points to the extent that this paper becomes unreadable. For the benefit of specialists we will note some of the finer technical points in comments without dwelling on them. The experts can provide their own favorite treatment of these details while the more general reader can ignore them without great loss.

*Definition* 1.1. Let $X$ be a cat manifold and $x\in X$. A coordinate patch around $x$ is a pair $(\lambda, U)$ where $U$ is an invariant open set in a $G_x$-representation and $\lambda: U\to X$ a cat $G_x$-isomorphism onto a neighborhood of $x$.

We will assume when convenient that $0\in U$ and $\lambda(0)=x$. We will also, as customary, sometimes pretend $U\subset X$ and $\lambda=\text{Identity}$.

*Definition* 1.2. Let $H_1\subset H_2\subset G$. A linear submersion is a triple $(\lambda, U, V)$ where $U$ is an $H_1$-invariant open subset of an $\mathbf{R}H_1$-module $A$, $V$ is an $H_2$-invariant open subset of an

$\mathbf{R}H_2$-module $B$, and $\lambda: U \to V$ is the restriction of an $H_1$-linear epimorphism $\lambda: A \to B$. Again when convenient we will assume $U$, $V$ contain 0.

*Definition* 1.3. Let $X$ and $Y$ be cat $G$-manifolds. A cat $G$-map $f: X \to Y$ is called a cat $G$-submersion if it is locally equal to a linear submersion.

A technical comment is in order. We will also need the above definition for cat manifolds with boundary. To handle boundary points we can either generalize the notion of linear submersion to half spaces or extend the boundary to an open collar. Using the strong uniqueness of open collars one can easily show the two approaches are equivalent.

Our notion of $G$-submersion is stronger than the notion of submersion (in the non-equivariant sense) which happens to be a $G$-map. One could also develop a general submersion theory in this weaker sense. However we cannot calculate the obstruction groups there.

*Definition* 1.4. Let $X$, $Y$ be cat $G$-manifolds. We write $\mathrm{SUB}_G(X, Y)$ for the simplicial set ($\Delta$-set) whose $k$-simplices are $G$-submersions $f: \Delta_k \times X \to \Delta_k \times Y$ over the identity on $\Delta_k$, i.e. $f(t, x) = (t, f_1(t, x))$.

Degeneracy operators can be defined, but are unnecessary (cf. [23]). However it is important (and easy to see) that $\mathrm{SUB}_G(X, Y)$ is a Kan complex.

The reduction of submersion theory to homotopy theory involves comparing $\mathrm{SUB}_G(X, Y)$ to a more tractable complex, via the use of bundle theory. Following [14], a $G$-$\mathbf{R}^n$ bundle is a locally linear $G$-cat bundle with fiber $\mathbf{R}^n$. If $X$ is a cat manifold, $TX$ will denote the tangent bundle. This is most directly defined as a $G$-micro bundle, and not as a $G$-$\mathbf{R}^n$ bundle. However, the equivariant Kister theorem (cf. [14, § 2]) allows us to replace the micro bundle by an actual $G$-$\mathbf{R}^n$ bundle, unique up to a canonical isomorphism. We shall therefore ignore the micro bundle viewpoint and consider $TX$ as a $G$-$\mathbf{R}^n$ bundle.

*Definition* 1.5. Let $E_1$ be a $G$-$\mathbf{R}^n$ bundle over $X_1$ and $E_2$ a $G$-$\mathbf{R}^m$ bundle over $X_2$. A fiberwise cat $G$-map $\lambda: E_1 \to E_2$ over $\bar{\lambda}$ is called a bundle $G$-epimorphism if the following local condition is satisfied for each $x \in X_1$:

There exist $G_x$-neighborhoods $U_1$ and $U_2$ of $x$ and $\bar{\lambda}(x)$ with $\bar{\lambda}(U_1) \subseteq U_2$ and $G_x$-bundle isomorphisms

$$\lambda_i\colon E_i|U_i \xrightarrow[\cong]{} U_i \times V_i.$$

Here $V_1$, $V_2$ are representations of $G_x$ such that $\lambda$ factors as the composite

$$E_1|U_1 \xrightarrow[\cong]{\lambda_1} U_1 \times V_1 \xrightarrow{\bar{\lambda}\times\varrho} U_2 \times V_2 \xleftarrow[\cong]{\lambda_2} E_2|U_2$$

for a linear $G_x$-epimorphism $\varrho\colon V_1 \to V_2$.

The above definition is somewhat complicated but these complications are necessary to compare bundle epimorphisms with submersions. One must take account of the fact that a $G$-$\mathbf{R}^n$ locally linear bundle structure is radically non-homogeneous. The condition on a neighbourhood of a point will depend on the isotropy of that point.

*Definition* 1.6. Let $\mathrm{EPI}_G(E_1, E_2)$ be the simplicial set whose $k$-simplicies are bundle $G$-epimorphisms $\alpha\colon \Delta_k \times E_1 \to \Delta_k \times E_2$ which preserve the $\Delta_k$-coordinate.

Here $\Delta_k \times E_1$ denotes the bundle over $\Delta_k \times X$ induced from $E_1$ via the projection $\Delta_k \times X \to X$ and similarly for $\Delta_k \times E_2$. The definitions have been set up so that there exists a natural map

$$d\colon \mathrm{SUB}_G(X, Y) \to \mathrm{EPI}_G(TX, TY).$$

Call a $G$-manifold *non-closed* if for each maximal isotropy subgroup $H$ the fixed set $X^H$ has no non-empty closed manifold components.

We can now state the Equivariant submersion theorem.

THEOREM 1.7. *If $X$ is a non-closed $G$-manifold then $d$ is a homotopy equivalence.* □

In [13] this is proven in the topological category. The same argument works in the (locally linear) *PL* category once we have a $G$-isotopy extension theorem. This in turn can be proved in the equivariant setting following the $G$-trivial case from [10]; we stress that the isotopy extension theorem does not hold without the local linearity, see e.g. [22, § 5] for counter-examples.

## § 2. Obstruction theory for $G$-submersions

The $G$-submersion theorem allows us to substitute $\mathrm{EPI}_G(TX, TY)$ for $\mathrm{SUB}_G(X, Y)$. We wish to reduce the study of the former to a question of cross sections of bundles, allowing us to utilize $G$-obstruction theory for these problems. The reduction follows the well-known procedure in the non-equivariant case, with one interesting extra complication which we will discuss. But first a technical comment is in order. The spaces (say, $X$ and $Y$) we are dealing with are locally linear $G$-manifolds. In the $PL$ case these spaces have two useful properties which we shall use repeatedly:

(2.1) (I). "Nice" subspaces of $X$ and $Y$, in particular $X^H$ and $Y^H$ where $H \subset G$, have $G$-simplicial structures.

This is needed since the auxillary bundles we introduce below are simplicial bundles (or fibrations), and further, the $G$-obstruction theory is defined directly when the spaces involved are $G$-complexes.

(2.1) (II). "Nice" subspaces of $X$ and $Y$, in particular $X^H$ and $Y^H$ are $G$-neighborhood deformation retracts.

This allows us to extend $G$-cross sections to $G$-neighborhoods in a more or less canonical way.

We do not have direct analogues for (I) and (II) in the locally linear topological category, however up to $G$-homotopy they are true in the following sense: $X$, $Y$ have $G$-$\mathbf{R}^n$ bundle neighborhoods $\tilde{X}$, $\tilde{Y}$ in $G$-representations [14, p. 277], which because they are smooth have canonical $G$-simplicial structures. If $\tilde{E}(X)$, $\tilde{E}(Y)$ are the pullbacks over $\tilde{X}$, $\tilde{Y}$ of $TX$, $TY$, then up to homotopy

$$\mathrm{EPI}_G(E(\tilde{X}), E(\tilde{Y})) \text{ is the same as } \mathrm{EPI}_G(TX, TY).$$

One might consider then replacing $X$ and $Y$ by $\tilde{X}$ and $\tilde{Y}$ once and for all so as to obtain properties (I) and (II). However it is not really convenient to do so. For example, we want to analyze obstructions lying in $H^*_{N(H)}(X^H, W; \mathbf{B})$, where $W$ is a $N(H)$-invariant subspace of $X^H$, $\mathbf{B}$ a local coefficient system and $H^*_G$ denotes Bredon cohomology [5]. We will use the obvious fact that $H^k_{N(H)}(X^H, W; \mathbf{B})=0$ for $k > \dim X^H$. We thus wish to keep hold of the dimensions of $X$ and $Y$. Hence a completely detailed treatement involves moving back and forth between $\tilde{X}^H, \tilde{W}$ and $X^H, W$ where $\tilde{W}=\pi^{-1}(W)$, $\pi: \tilde{X}^H \to X^H$.

Doing this would not involve any real conceptual difficulties, but it would involve dragging along an extra bit of notation in a procedure that is already rather complicated. Therefore we shortcircuit this whole complication by assuming:

*$X$ and $Y$ have $G$-simplicial structures.*

Observe that this does *not* collapse the topological category into the $PL$ category since the question of topological $G$-submersions is different from that of $PL$ $G$-submersions even when the domain and range are piece-wise linear.

The reason we can assume that $X$ and $Y$ have $G$-simplicial structures is that we are interested in questions of transversality and the universal vanishing of the obstructions to transversality. While it turns out that the obstructions are in general global obstructions, the conditions which we derive for their vanishing are dimensional conditions and hence purely local, and in our categories the spaces involved are by definition locally triangulable. Even from the point of view of the general theory of $G$-submersions the difficulties encountered by dispensing with this triangulability assumption are purely questions of complication of detail. They can always be overcome by utilizing the device of passage to the normal tube, mentioned above. Alternatively, one could use nice coverings and the Čech version of Bredon cohomology, pulling the bundles back from bundles over the nerve of the covering.

We are attempting to build a $G$-bundle $\mathrm{Epi}(TX, TY)$ over $X$ so that cross-sections of it correspond to $G$-epimorphisms from $TX$ to $TY$. We recall how one proceeds in the case $G=\{1\}$, following Haefliger and Poenaru [11].

We begin with $\mathbf{R}^{n_i}$ bundles $E_i \to X_i$ over simplicial complexes $X_1$ and $X_2$. Actually, it is more convenient to work with $\Delta$-sets; thus we order the vertices of $X_i$ and identify $X_i$ with the $\Delta$-set it generates (each $j$-simplex of $X_i$ corresponds to a unique $\sigma: \Delta_j \to X_i$).

We define a $\Delta$-set $\mathrm{Epi}(E_1, E_2)$. The $j$-simplices are triples $(\sigma_1, \sigma_2, \varrho)$ where $\sigma_1$ and $\sigma_2$ are $j$-simplices of $X_1$ and $X_2$ and $\varrho$ is a map

$$\begin{tikzcd} \sigma_1^*(E_1) \arrow[rr, "\varrho"] \arrow[dr] & & \sigma_2^*(E_2) \arrow[dl] \\ & \Delta_j & \end{tikzcd}$$

which can be written as a composite of the form

$$\varrho: \sigma_1^*(E_1) \overset{\psi_1}{\cong} \Delta_j \times \mathbf{R}^{n_1} \xrightarrow{1\times p} \Delta_j \times \mathbf{R}^{n_2} \overset{\psi_2}{\cong} \sigma_2^*(E_2).$$

Here $\sigma_i: \Delta_j \to X_i$, $\psi_i$ is a bundle equivalence and $p: \mathbf{R}^{n_1} \to \mathbf{R}^{n_2}$ is a linear surjection.

Projection onto $X_1$ defines a bundle (in fact a twisted Cartesian product bundle, [15, chapter IV])

$$\omega: \mathrm{Epi}(E_1, E_2) \to X_1.$$

It is direct from the definitions that the $\Delta$-set of sections of $\omega$ is equal to the $\Delta$-set of bundle epimorphisms $\mathrm{EPI}(E_1, E_2)$ (where $E_1$ and $E_2$ are viewed as $\Delta$-set bundles).

If $E_1$ and $E_2$ are locally linear $G$-$\mathbf{R}^{n_i}$ bundles then $\mathrm{Epi}(E_1, E_2)$ has a natural action of $G$, but $G$-sections of $\omega$ do *not* correspond to $G$-epimorphisms, i.e. to elements of $\mathrm{EPI}_G(E_1, E_2)$ as defined in Definition 1.6. Indeed, $G$-sections of $\omega$ correspond to epimorphisms which are $G$-maps; this is a weaker notion than $G$-epimorphisms.

To rectify this we proceed as follows: over $(X_1 \times X_2)^H$ we define $\mathrm{Epi}_H(E_1, E_2)$ exactly as we defined $\mathrm{Epi}(E_1, E_2)$ above but with the one further requirement that $\psi_1$ and $\psi_2$ be $H$-invariant with respect to some linear $H$-structures on $\mathbf{R}^{n_1}$ and $\mathbf{R}^{n_2}$. We get an $N(H)$-bundle

$$\omega_H\colon \mathrm{Epi}_H(E_1, E_2) \to X_1^H,$$

and the following is immediate:

PROPOSITION 2.2. *The G-epimorphisms* $\mathrm{EPI}_G(E_1, E_2)$ *correspond uniquely to systems of* $N(H)$ *cross-sections* $\psi_H\colon X_1^H \to \mathrm{Epi}_H(E_1, E_2)$ *of* $\omega_H$ *which satisfy*:

(i) *If* $H_1$ *is conjugate to* $H_2$ *then* $\psi_{H_1}$ *corresponds to* $\psi_{H_2}$ *under conjugation and*

(ii) *If* $H \subset K$ *then* $\psi_H | X_1^K = \mathrm{Res} \circ \psi_K$ *where* $\mathrm{Res}\colon \mathrm{Epi}_K(E_1, E_2) \to \mathrm{Epi}_H(E_1, E_2)$ *is the natural inclusion.* □

It is convenient to formalize the situation by regarding $\{\mathrm{Epi}_H(E_1, E_2) \to X_1\}$ as a functor from the category $\mathcal{O}_G$ of orbits to the category of bundles and fiberwise maps. We do this in the following:

*Definition* 2.3. A bundle functor $\pi$ is a functor from $\mathcal{O}_G$ to the category of bundles and fiberwise maps. Precisely, for each $H \subset G$, $\pi(H)$ is a map $E(H) \to B(H)$, where $E(H)$ and $B(H)$ are $\Delta$-sets and $\pi(H)$ a bundle map which is also a natural transformation of the $\Delta$-set valued functors $E(H)$, $B(H)$.

A cross-section of $\pi$ will then be a natural transformation $j(H)\colon B(H) \to E(H)$ such that $\pi(H) \circ j(H) = \mathrm{Id}$. The natural example of a bundle functor is given by a $G$-bundle $E \xrightarrow{\pi} B$ where $\pi(H)$ is the fixed point bundle, $E^H \to B^H$, but as we have seen not all bundle functors are of this form. In particular $\mathrm{Epi}(E_1, E_2)$ which to $G/H$ assigns $\mathrm{Epi}_H(E_1, E_2)$ is not of this type.

Given a bundle functor $\pi$, the family of homotopy groups $\pi_n(E(H)_x)$, $x \in B(H)$ form a local coefficient system in the sense of [5]. The obstruction theory from [5] or [14] goes through word for word in this slightly more general case. In fact the use of Bredon cohomology is in some sense more transparent and natural in the above context. One might still ask

*Question.* Is each $G$-bundle functor fiber homotopy equivalent to a $G$-bundle?

It follows that we have an obstruction theory for constructing elements of $\mathrm{EPI}_G(E_1, E_2)$. In particular for the tangent bundles $E_1=TX$ and $E_2=TY$, the obstructions lie in Bredon cohomology of $X$ with local coefficients in $\pi_* \mathrm{EPI}_H(T_xX, T_y Y)$. The next step is to analyze the coefficient system for these obstructions, that is, the homotopy groups $\pi_k \mathrm{EPI}_H(T_xX, T_y Y)$ for $x \in X^H$, $y \in Y^H$.

A 0-simplex of $\mathrm{EPI}_H(T_xX, T_y Y)$ is a cat $H$-map $\lambda\colon T_xX \to T_y Y$ which factors as a composition

$$T_xX \overset{\psi_1}{\cong} W \xrightarrow{\varrho} V \overset{\psi_2}{\cong} T_yY,$$

with $\psi_1$ and $\psi_2$ cat $H$-isomorphisms, $V$ and $W$ $\mathbf{R}H$-modules and $\varrho$ a linear $H$-equivariant surjection. Thus we consider all pairs of $\mathbf{R}H$-modules $V$, $W$ and linear $H$-surjections $\varrho\colon W \to V$ such that $W$ is cat $H$-isomorphic to $T_xX$ and $V$ cat $H$-isomorphic to $T_yY$. We set $(W, V, \varrho) \sim (\bar{W}, \bar{V}, \bar{\varrho})$ if there exist cat $H$-isomorphisms $\alpha_1$ and $\alpha_2$ such that the following diagram commutes,

$$\begin{array}{ccc} W & \xrightarrow{\varrho} & V \\ {\scriptstyle\alpha_1}\downarrow & & \downarrow{\scriptstyle\alpha_2} \\ \bar{W} & \xrightarrow{\bar{\varrho}} & \bar{V} \end{array}$$

Each $k$-simplex of $\mathrm{EPI}_H(T_xX, T_yY)$ determines a unique equivalence class $[(W, V, \varrho)]$ and $\mathrm{EPI}_H(T_xX, T_yY)$ breaks up into the disjoint union of sub-complexes indexed by these classes:

$$\mathrm{EPI}_H(T_xX, T_yY) = \coprod_{\alpha} \mathrm{EPI}_H^{\alpha}(T_xX, T_yY),$$

where $\alpha$ runs over the equivalence classes $[(W, V, \varrho)]$.

Remember that $A_H(V)$ is the $\Delta$-group of cat automorphisms of $V$. The product

$A_H(T_xX)\times A_H(T_yY)$ acts on $\mathrm{EPI}_H(T_xX, T_yY)$, transitively on each $\mathrm{EPI}_H^\alpha(T_xX, T_yY)$. If this set is non-empty we have an $H$-linear splitting $W=V\oplus V_1$ and we can absorb the action of $A_H(T_yY)$ into the action of $A_H(T_xX)=A_H(W)$. That is we can identify $\mathrm{EPI}_H^\alpha(T_xX, T_yY)=A_H(W)/A_0$ where

$$A_0=\{\gamma\in A_H(W)\,|\,\gamma(v,v_1)=(v,\gamma_1(v,v_1))\}.$$

In other words, $A_0$ is the group of cat $H$-bundle automorphisms over the identity,

$$\begin{tikzcd} W=V\oplus V_1 \arrow[rr, "\gamma"] \arrow[dr] & & V\oplus V_1 \arrow[dl] \\ & V & \end{tikzcd}$$

Since $V$ is $H$-contractible, $A_0$ has the group of cat $H$-automorphisms of $V$ as a deformation retract. In fact a retraction is given by $\gamma_t(v,v_1)=(v,\gamma_1(tv,v_1))$. It follows that we have a homotopy equivalence

$$\mathrm{EPI}_H^\alpha(T_xX, T_yY)\simeq A_H(W)/A_H(V_1), \quad \text{where} \quad W=V\oplus V_1 \tag{2.4}$$

Notice, since we are dealing with $\Delta$-groups that

$$\pi_k(A_H(W)/A_H(V_1))\cong\pi_k(A_H(W), A_H(V_1)).$$

Suppose now we know that any two linear representations of $H$ which are cat $H$-isomorphic are linearly isomorphic. Then it follows that there is only one equivalence class $[(W, V, \varrho)]$, so $\mathrm{EPI}_H(T_xX, T_yY)=\mathrm{EPI}_H^\alpha(T_xX, T_yY)$. Since for the group $G$ we will be considering we will have this as an inductive assumption for $G$ and all subgroups we can and will drop the subscript $\alpha$.

## § 3. Transversality obstructions

The link between submersion and transversality is the following basic definition.

*Definition* 3.1. Let $X$ and $Y$ be cat manifolds $y\in Y^G$, $f\colon X\to Y$ a cat $G$-map. Then $f$ is transversal to $y$ ($f\pitchfork y$) if there exists a $G$-neighborhood $U$ of $f^{-1}(y)$ such that $f\colon U\to Y$ is a $G$-submersion.

We wish to analyze the obstruction to deforming $f$ so that $f\pitchfork y$. More generally suppose $A\subset X$ is a closed $G$-invariant subspace. We say $f\pitchfork y$ on $A$ if $f$ is transversal to $y$

when restricted to a $G$-neighborhood of $A$. It is usually more convenient to approximate $f$ with respect to some metric on $Y$ than to actually construct a deformation. By standard methods for locally linear manifolds, close enough such approximations are always $G$-homotopic to $f$ via a $G$-homotopy which doesn't move points too far from $f$. For example one could embed $Y$ in a linear representation and use standard linear approximations techniques there. Notice that the closeness of the approximation necessary to construct the deformation depends only on $Y$. It has nothing to do with $f$ or $X$. First we specify the setting.

*Notation* 3.2. We assume given a cat $G$-map $f: X \to Y$, a closed $G$-subspace $A$ of $X$ and a subgroup $H$ of $G$ such that $A \cap X^H$ contains $X_0^H = \{x \in X^H | G_x > H\}$. We further assume $f \pitchfork y$ on $A$ where $y \in Y^G$, and that $f^H: X^H \to Y^H$ is transverse to $y$ considered as an $N(H)$-map. We ask, when we can find a cat $G$-map $\bar{f}: X \to Y$ such that $\bar{f}|W \cup GX^H = f|W \cup GX^H$ where $W$ is a $G$-neighborhood of $A$ and $\bar{f} \pitchfork y$ on $A \cup GX^H$. If we can do this we say we can solve the $(f, A, H)$-problem.

Notice that to solve this problem we may replace $X$ by any $G$-neighborhood of $A \cup GX^H$ since the $G$-homotopy extension in our category implies that any extension of $f|A \cup GX^H$ on a neighborhood of $A \cup GX^H$ extends (on a smaller neighborhood) to a cat $G$-map on all of $X$. Further by cutting down our neighborhood of $A \cup GX^H$ and then extending once more we can make $\bar{f}$ as close to $f$ as we like. Secondly, it suffices to solve the problem for $G = N(H)$, since any $N(H)$-invariant map on a neighborhood of $A \cup GX^H$ which is a $G$-map on a neighborhood of $A$ extends uniquely to a $G$-map on a neighborhood of $A \cup GX^H$ which is transversal to $y$ as a $G$-map if the original map was transversal to $y$ as an $N(H)$-map. Thus without loss of generality we can (and often will) assume that $H$ is normal in $G$.

THEOREM 3.3. *With the notation from Notation* 3.2, *let* $Z^H = (f^H)^{-1}(y)$ *and suppose that* $V$ *is a* $G$*-linear neighborhood of* $y$ *in* $Y$. *The obstructions to solve the* $(f, A, H)$*-problem lie in the Bredon cohomology groups* $H^k_{N(H)}(Z^H, Z^H \cap A; \pi_k(\alpha))$ *where*

$$\alpha: A_H(T_x X)/A_H(V_1) \to A(T_x X^H)/A(V_1^H)$$

*restricts to* $H$ *fixed set and* $T_x X = V \oplus V_1$.

*Proof.* We assume $H \lhd G$ and choose a $G$-neighborhood $W$ of $A$ such that $f|W$ is transversal to $y$. Let $V$ be a small $G$-neighborhood of $y$ such that for $U = f^{-1}(V)$,

(i) $f|U\cap W$ is a $G$-submersion, and

(ii) $f^H|U^H$ is a $G$-submersion. Let $Z^H=(f^H)^{-1}(y)$.

Suppose we can find a $G$-neighborhood $U_0$ of $Z^H$ in $U$ and a $G$-submersion $\bar{\bar{f}}\colon U_0\to Y$ such that $\bar{\bar{f}}=f$ on a neighborbood of $U_0\cap A$ in $U_0$ and $\bar{\bar{f}}^H=f^H$ on a neighborhood of $Z^H$ in $X^H$. Then we can solve the $(f, A, H)$-problem. Indeed, we can use $f$ to extend $\bar{\bar{f}}$ to $\bar{f}$ on $W_0\cup U_1\cup X^H$ where $W_0$ is a $G$-neighborhood of $A$ in $X$ and $\bar{f}|W_0=f|W_0$, $U_1\subset U_0$ is a $G$-neighborhood of $Z_H$, $\bar{f}|U_1=\bar{\bar{f}}|U_1$ and $\bar{f}|X^H=f|X^H$. Now $\bar{f}$ can be extended to a $G$-neighborhood $N$ of $X^H$ (using the $G$ Tietze extension theorem or a $PL$ version of it). If $N$ is small enough $\bar{f}$ will be close to $f$ on $N$ and since $y\notin f(N-U_1)$ we can, by making $N$ even smaller, assume $y\notin\bar{f}(N-U_1)$. Then $\bar{f}$ is transversal to $y$ on $W_0\cup U_1\cup N$ and we have solved the $(f, A, H)$-problem.

Fixing $U_0$ as above we wish to apply the obstruction theory of the last section to find $\bar{\bar{f}}$. We can replace $A$ by a small $G$-neighborhood which we can assume to be a $G$-subcomplex of $X$. Then we have the obstruction to finding a submersion $\bar{\bar{f}}\colon U_0\to V\subset Y$ agreeing with $f$ on $U_0\cap A$ lies in the Bredon cohomology groups $H^*_G(U_0, U_0\cap A; \pi_*)$ with $\pi_*=\pi_*\operatorname{Epi}(TU, TV)$. We also wish $\bar{\bar{f}}^H=f^H$ on $U_0^H$. In terms of cross-sections of bundles this can be formulated as follows: The $G$-bundle functor $\operatorname{Epi}(TU_0, TV)$ over $U_0$ restricts to a $G$-bundle functor over $U_0^H$. Since $\mathcal{O}_{G/H}\subset\mathcal{O}_G$ this restriction is a $G/H$-bundle functor and in this case cross-sections correspond bijectively when considered as a $G$ or $G/H$-bundle functor since $(U_0^H)^K=U_0^{(H\cdot K)}$. On the other hand considered as a $G/H$-bundle functor there is a natural transformation

$$\alpha_1\colon \operatorname{Epi}(TU_0, TV)|U_0^H\to\operatorname{Epi}(TU_0^H, TV^H)|U_0^H.$$

The submersion $f^H$ yields a cross-section of this second bundle functor, which we denote by $\bar{\lambda}$. Then using elementary covering homotopy properties for $G$-submersions we can find $\bar{\bar{f}}\colon U_0\to V$ with the desired properties if we can find a cross-section $\lambda$ of $\operatorname{Epi}(TU_0, TV)$ such that

(a) $\lambda|A=\lambda_f$, the cross-section determined by $f$ in a neighborhood of $A$,

(b) $\lambda$ corresponds over $U_0^H$ to $\bar{\lambda}$, i.e. $\alpha_1\circ\lambda=\bar{\lambda}$.

Note that $\bar{\lambda}$ and $\lambda_f$ agree on $A^H$.

We further simplify the problem as follows: $U_0$ is by choice a $G$-simplicial complex. Since we are only interested in arbitrarily small neighborhoods of $Z^H$ we can further assume that $U_0$ is a $G$-regular neighborhood of $U_0^H$, and since $A$ is a subcomplex we can assume $U_0 \cap A$ is a $G$-regular neighborhood of $U_0^H \cap A^H = (U_0 \cap A)^H$. In particular cross-sections of $\mathrm{Epi}(TU_0, TV)$ over $U_0$ fixed on $A$ correspond, up to $G$-homotopy, uniquely to cross-sections of $\mathrm{Epi}(TU_0, TV)|U_0^H$, fixed on $A^H$.

Over $U_0^H$ we have the map $\alpha_1$ and cross-section $\bar{\lambda}$,

$$\mathrm{Epi}(TU_0, TV) \xrightarrow{\alpha_1} \mathrm{Epi}(TU_0^H, TV^H)$$

$$\bar{\lambda}\colon U_0^H \to \mathrm{Epi}(TU_0^H, TV^H).$$

We are looking for a cross-section $\lambda$ with $\alpha_1(\lambda) = \bar{\lambda}$ (and equal to $\lambda_f$ on $A$). Define a bundle $E$ over $U_0^H$ by the pullback diagram:

$$\begin{tikzcd} E \arrow[r] \arrow[d, "\alpha_2"] & \mathrm{Epi}(TU_0, TV) \arrow[d, "\alpha_1"] \\ U_0^H \arrow[r] & \mathrm{Epi}(TU_0^H, TV^H). \end{tikzcd}$$

The cross-section we seek is exactly a cross-section of $\alpha_2$, fixed on $A$. The obstructions lie in $H_G^*(U_0, U_0 \cap A; \pi_*(\alpha_2)) \cong H_G^*(U_0^H, U_0^H \cap A; \pi_*(\alpha_1))$.

Since $V$ is a small $G$-neighborhood of $y \in Y^G$ we may as well take it as a linear $G$-space so that for $v \in V$, $T_v V = V$. Then for $x \in U_0^H$ the fiber of $\alpha_1(H)$ is the fiber of $\mathrm{Epi}_H(T_x X, V) \to \mathrm{Epi}(T_x X^H, V^H)$. By (2.4) this fiber is equal to the fiber of

$$\alpha\colon A_H(T_x X)/A_H(V_1) \to A(T_x X^H)/A(V_1^H).$$

We need not consider the fiber of $\alpha_1(K)$, when $K > H$ because we have assumed that in this case $U_0^K \subset A \cap U_0^H$ and the cross-section $\lambda$ is already defined on $A$. In other words using excision we need only consider obstructions in

$$H_G^*(U_0^H - \mathrm{int}\, A, U_0^H \cap \partial A; \pi_*(\alpha_1))$$

where $U_0^H - A$ has only one orbit type $G/H$. In particular we can replace the Bredon cohomology with the ordinary local coefficient cohomology groups

$$H^*(U_0^H - \mathrm{int}\, A/G, U_0^H \cap \partial A/G; \pi_*(\alpha_1))$$

$$\alpha_1\colon \mathrm{Epi}_H(T_x X, V) \to \mathrm{Epi}(T_x X^H, V^H).$$

We have analyzed the obstruction for finding a submersion $\bar{\bar{f}}$ of $U_0$, a fixed $G$-neighborhood of $Z^H$. We are however only concerned with finding some $G$-neighborhood of $Z^H$ for which $\bar{\bar{f}}$ exists, and since obstruction theory is natural with respect to restrictions, the obstructions to our problem can be considered to lie in

$$\operatorname{colim} H_G(U_i^H, U_i^H \cap A; \pi_*).$$

Here $U_i$ varies over all $G$ neighborhoods of $Z^H$ in $X$. Since the intersection of $U_i^H$ for such $U_i$ is precisely equal to $Z^H$ we have

$$\operatorname{colim} H_G^*(U_i^H, U_i^H \cap A; \pi_*) = H_G^*(Z^H, Z^H \cap A; \pi_*(\alpha))$$

This completes the proof. $\square$

The cohomology groups in Theorem 3.3 vanish for $k > \dim Z^H$. On the other hand we will show in the $PL$ category that under suitable 'gap-conditions',

$$\pi_k(\alpha) = 0 \quad \text{for} \quad k \leq \dim Z^H,$$

thus forcing the obstructions to vanish. In the topological category there will however be obstructions to transversality in general.

Now suppose we begin with an arbitrary $G$-map $f: X \to Y$, and $y \in Y^G$. Then by starting with $A = \varnothing$ and working up the subgroups $H \subset G$ we can reduce the problem of deforming $f$ to a map transversal to $y$ to a finite sequence of $(f, A, H)$-problems. Thus when all above obstructions vanish we can stepwise deform $f$ to $\bar{f}$ which is transversal to $y$.

We wish to consider transversality in a more general form. Ideally, one would like to work most generally with $f$: $X \to Y$ and $M \subset Y$ a submanifold. While one can define transversality in this context there are technical difficulties in proving anything about it. So we consider instead the problem of $G$-transversality assuming the existence of a fixed $G$-$\mathbf{R}^n$ normal tube $\gamma(M) \subset Y$. Then the generalization goes through rather smoothly. Since such a tube exists and is unique stably this is no real restriction for stable transversality questions.

A $G$-$\mathbf{R}^n$ bundle $E \xrightarrow{\pi} M$ has a special family of coordinate patches corresponding to $E|U$ where $U \subset M$ is a coordinate patch for $M$. We call such coordinate patches, bundle coordinate patches.

Let $x \in X$. If $W$ is a neighborhood of 0 in an $\mathbf{R}G_x$-module, and $x \in W \subseteq X$ is a $G_x$-embedding we call $W$ a slab through $x$. A *coordinate patch $E$* around $x$ with *slab $W$* is a trivial $G_x$-$\mathbf{R}^n$ bundle $\pi: E \to W$ embedded as a $G_x$-neighborhood $x \in E \subseteq X$ such that the 0-section of $\pi$ becomes the given embedding of the slab $W$.

*Definition* 3.4. A $G$-map $f: X \to Y$ is $G$-transverse to $M$ with respect to the normal tube $\gamma(M)$, if for each $x \in f^{-1}(M)$ there exists a coordinate patch $E$ around $x$ with slab $W$, and a bundle coordinate patch $\gamma(M)|U$ around $f(x)$ such that $f$ agrees near $W$ with a bundle map $\hat{f}: E \to \gamma(M)|U$ over the map $f: W \to U$.

When the conditions in Definition 3.4 are satisfied we write $f \pitchfork M$ w.r.t. $\gamma(M)$.

THEOREM 3.5. *Let $f: X \to Y$ be a G-map with $f \pitchfork M$* w.r.t. $\gamma(M)$. *Let $N = f^{-1}(M)$. Then $N$ has a normal tube $\gamma(N) \subset X$, with $f = \hat{f}$ near $N$, where $\hat{f}$: $\gamma(N) \to \gamma(M)$ is a bundle map covering $f: N \to M$. If $j$: $\gamma(N) \subset X$ and $j'$: $\gamma'(N) \subset X$ are two such normal tubes, there exists an isotopy $j_t$ with $j_0 = j$ and $(j')^{-1} j_1$: $\gamma(N) \to \gamma'(N)$ a G-bundle isomorphism.*

*Proof.* Clearly such a normal bundle exists in a neighborhood of each $x \in N$. Thus the problem of constructing $\gamma(N)$ consists of piecing them together. It follows that the existence and uniqueness up to isotopy of $\gamma(N)$ will follow from the local existence of the isotopy $j_t$. We can then assume that $\gamma = \gamma(N)|W$ and $\gamma' = \gamma'(N)|W$ are product bundles over the slab $W$, say $k$: $W \times \mathbf{R}^s \cong \gamma$ and $k'$: $W \times \mathbf{R}^s \cong \gamma'$. Let $\hat{f}: \gamma \to U \times \mathbf{R}^s$, $\hat{f}': \gamma' \to U \times \mathbf{R}^s$ be the bundle maps agreeing with $f$ near $W$. By choosing trivializations $k$ and $k'$ correctly we can assume $\hat{f} \circ k = \hat{f}' \circ k' = f|W \times \mathrm{Id}$ near $W \times 0$. Without loss of generality we may assume $\gamma \subset \gamma'$. Construct $j_t$ as follows: for $z \in \gamma_x$ and $t > 0$ set $j_t(z) = k'(1/t)k'^{-1}ktk^{-1}(z)$. Observe that $\pi_2 \hat{f} j_t(z) = \pi_2 \hat{f}(z)$, where $\pi_2$: $U \times \mathbf{R}^s \to \mathbf{R}^s$. It follows that as $t \to 0$, $j_t(z) \to z'$, where $z'$ is the unique point in $\gamma'_x$ with $\hat{f}'(z') = \hat{f}(z)$. □

The existence of a normal tube $\gamma(N)$ which satisfies the conclusion of Theorem 3.5 implies $f \pitchfork M$ w.r.t. $\gamma(M)$. Hence it is a necessary and sufficient condition. Any such $\gamma(N)$ will be denoted by $f^* \gamma(M)$. As a consequence we have

COROLLARY 3.6. *Suppose we have G-maps $T \xrightarrow{g} X \xrightarrow{f} Y$. If $f \pitchfork M$ w.r.t. $\gamma(M)$ and $g \pitchfork f^{-1}(M)$ w.r.t. $f^*\gamma(M)$ then $f \circ g \pitchfork M$ w.r.t. $\gamma(M)$ and $(f \circ g)^*\gamma(M) = g^*(f^*(\gamma(M)))$.* □

This result on the composition of transverse maps doesn't seem to carry over to any weaker definition of transversality, and can be seen as a technical reason for why we need the normal tube $\gamma(M)$.

Let $f: X \to Y$ be a $G$-map and $M \subset Y$ with normal tube $\gamma(M)$. Let $U = f^{-1}(\gamma(M))$. Then $f \pitchfork M$ w.r.t. $\gamma(M)$ if and only if $(f|U) \pitchfork M$ w.r.t. $\gamma(M)$. If $f \simeq_G f'$ with $f' \pitchfork M$ w.r.t. $\gamma(M)$ then

$f|U \simeq_G f'|U$ and $(f'|U) \pitchfork M$ w.r.t. $\gamma(M)$. It is not so obvious that if $f|U$ is $G$-homotopic to a map transversal to $M$ w.r.t. $\gamma(M)$ then the same is true for $f$ since an arbitrary homotopy may not be extendable without changing $f^{-1}(M)$. However, a transversal approximation to $f$ is so extendable, and with a little care we can then reduce the problem of deforming $f$ to a map transverse to $M$ w.r.t. $\gamma(M)$ to the situation where $f(X) \subset \gamma(M)$. Suppose further $\gamma(M) = M \times V$, where $V$ is a $G$-module. Consider the composite $X \xrightarrow{f} M \times V \xrightarrow{\pi} V$. By the above, $f \pitchfork M$ w.r.t. $M \times V$ implies that $(\pi \circ f) \pitchfork 0$ w.r.t. $V$. The homotopy converse is also true.

LEMMA 3.7. *If* $(\pi \circ f) \pitchfork 0$ *w.r.t.* $V$ *with* $N = (\pi \circ f)^{-1}(0)$ *then* $f \simeq_G g$, *and* $g \pitchfork M$ *w.r.t* $M \times V$ *with* $N = g^{-1}(M \times 0)$ *and* $g^*(M \times V) = (\pi \circ f)^*(V)$.

*Proof.* $(\pi \circ f)^*(V)$ is trivial and we can choose the trivialization so that we have the following commutative diagram,

$$\begin{tikzcd} X \arrow[r, "f"] & M \times V \arrow[r, "\pi"] & V \\ \cup & & \\ N \times V \arrow[uur, "\psi"'] & & \end{tikzcd}$$

where $\psi(x, v) = (\psi_1(x, v), v)$. Let

$$\begin{aligned} \varrho(v, t) &= t, \quad |v| \leqslant 1 \\ \varrho(v, t) &= |v| - 1 + (2 - |v|)\, t, \quad 1 \leqslant |v| \leqslant 2 \\ \varrho(v, t) &= 1, \quad |v| \geqslant 2 \end{aligned}$$

Let $\chi_t : N \times V \to M \times V$ be defined by $\chi_t(x, v) = (\psi_1(x, \varrho(v, t)\, v), v)$. Then $\chi_1 = \psi$, $\chi_t = f$ for $|v| \geqslant 2$, and $\chi_0$ is a bundle map over $f|N$ in a neighborhood of the 0-section. We let $g$ be the extension of $\chi_0$ by $f$. $\square$

We can now globalize Theorem 3.3. Suppose we have a cat $G$-map $f: X \to Y$ and a $G$-submanifold $M$ of $Y$ with normal tube $\gamma(M)$. We are given a closed $G$-subspace $A \subset X$. We assume

$$f \pitchfork M \quad \text{w.r.t.} \quad \gamma(M) \quad \text{on} \quad A \tag{3.8i}$$

$$X_0^H = \{x \in X^H | G_x > H\} \subset A, \tag{3.8ii}$$

$$f^H : X^H \to Y^H \quad \text{is} \quad N(H)\text{-transversal} \quad \text{to} \quad M^H \quad \text{w.r.t.} \quad \gamma(M)^H \tag{3.8iii}$$

The global $(f, A, H)$-problem is to find $\bar{f}\colon X\to Y$ with $\bar{f}|W\cup GX^H=f|W\cup GX^H$ for a $G$-neighborhood $W$ of $A$ and with $\bar{f}\pitchfork M$ w.r.t. $\gamma(M)$ on $A\cup GX^H$. As in the discussion following Theorem 3.3, if we can solve this problem stepwise for $H\subset G$, we can stepwise approximate $f$ by a map $X\to Y$ which is transversal to $M$ w.r.t. $\gamma(M)$.

By Corollary 3.6 and Lemma 3.7 we can reduce the global problem to a sequence of local problems by filtering $X$ by submanifolds $X_0\subset X_1\subset \dots \subset X_k=X$, with $X_i-X_{i-1}$ "small" and letting $A_i=A\cap X_i\cup X_{i-1}$ and $f_i=f|X_i$ and solving the sequence of local problems $(f_i, A_i, H)$. Again without loss of generality we assume $H\lhd G$. Let $Z^H=(f^H)^{-1}(M^H)$. By Theorem 3.3. we can solve these local problems step by step if we can find a cross-section of the bundle functor $\mathrm{Epi}\,(TX|Z^H, (f^H)^*(\gamma(M)|M^H))$ which restricts to the one induced by $f$ over a $G$-neighborhood of $A$ and to the cross-section of $\mathrm{Epi}\,(TX^H|Z^H, (f^H)^*(\gamma(M)^H))$ induced by $f^H$.

Suppose for simplicity that $X^H$ and $M^H$ are connected. For $x\in X^H$ we can then write $T_xX=T$ and for $y\in f(M)^H$, $\gamma_y(M)=V$. Suppose $T=V\oplus V_1$. Then we have the global version of Theorem 3.3; it follows by essentially the same argument as used for Theorem 3.3 and the discussion above.

THEOREM 3.9. *Under the hypothesis of* (3.8) *and with* $Z^H=(f^H)^{-1}(M^H)$, *the obstructions to solving the global* $(f, A, H)$-*problem lie in* $H^k_{N(H)}(Z^H, Z^H\cap A; \pi_k(\alpha))$, *where* $\alpha\colon A_H(T)/A_H(V_1)\to A(T^H)/A(V_1^H)$ *is restriction to the* $H$ *fixed set.* □

If we do not assume $X^H$ and $M^H$ connected we must allow the $T$, $V$ to vary with different components of $Z^H$. This is only notationally more complicated, it does not introduce any new conceptual problems.

In the above we have formulated transversality entirely within the locally linear manifold categories. However, since we found it necessary to specify the normal bundle $\gamma(M)$ we are really dealing with transversality for maps into a locally linear bundle. More specifically, suppose $\xi$ is a cat $G$-$\mathbf{R}^n$ bundle over the $G$-complex $Y$ and $f\colon X\to T(\xi)$ is a $G$-map of a cat $G$-manifold into the Thom space of $\xi$.

Assume $f$ is $G$-transverse on the closed $G$-subspace $A\subset X$ and that $f$ satisfies the obvious analogues of (3.8). For $x\in Z^H$ and $y=f(x)$ suppose

$$T_xX = V_{x,y}\oplus\xi_y$$

and let

$$\alpha\colon A_H(T_x)/A_H(V_{x,y})\to A(T_x^H)/A(V_{x,y}^H).$$

Then we have the local coefficients $\pi_*(\alpha)$ and

PROPOSITION 3.10. *The obstruction to solving the global* $(f, A, H)$*-problem lies in* $H^k_{N(H)}(Z^H, Z^H \cap A; \pi_k(\alpha))$. □

*Remark* 3.11. We have studied transversality for mappings into (normal) bundles. Stably, normal bundles always exist, but unstably it is more natural to consider transversality for maps into block bundles. It appears likely (if not obvious) that there is a block version of the above. In particular, Proposition 3.10 should be valid when we replace $A_H$ by the space $\tilde{A}_H$ of block automorphisms.

## §4. The transversality theorems

We can now state precisely the results we aim to prove in this work. First we recall the standard stability conditions, also sometimes called the 'strong gap-conditions'.

*Definition* 4.1. An $\mathbf{R}G$-module $V$ is called *stable* if for all subgroups $K \subset H$ of $G$ for which $V^H \neq (0)$ and $V^K \neq V^H$, we have $10 < 2 \dim V^H < \dim V^K$.

In the *PL* category the basic homotopy theoretic result is the following theorem which will be proved in Part II, [19].

THEOREM 4.2. *Let $G$ be a group of odd order and $V \subset T$ stable representations of $G$. If $V$ and $T$ have the same isotropy subgroups then $PL_G(T)/PL_G(V)$ is* $\dim V^G - 1$ *connected.* □

Theorem 3.9 and the discussion following it allows us to translate the above into a transversality theorem. We first set up some terminology.

*Definition* 4.3. Let $\xi$ be a cat $G$-$\mathbf{R}^n$ bundle over a $G$-space $M$, and let $X$ be a cat $G$-manifold. The pair $(X, \xi)$ will be called *stable* if for each subgroup $H \subset G$ and points $x \in X^H$, $y \in M^H$,

$$T_x X \cong V_{x,y} \oplus \xi_y,$$

where $V_{x,y}$ and $T_x X$ are stable $\mathbf{R}H$-modules with the same set of isotropy subgroups.

Suppose now given a $G$-map $f: X \to T(\xi)$ of a *PL* $G$-manifold $X$. From Theorem 4.2 and Proposition 3.10 we have

THEOREM 4.4. *Suppose $G$ has odd order and $\xi$ is a PL $G$-bundle.*

(i) *If* $(X, \xi)$ *is stable then any* $f: X \to T(\xi)$ *is G-homotopic to a G-map which is transversal to the zero section* $Y$.

(ii) *If f is already transversal to Y on a G-neighborhood of a closed G-subspace A then the homotopy can be taken constant on a possible smaller G-neighborhood of A.*

□

This theorem can be extended to a transversality result in the *PL* manifold category: A $G$-map $f: X \to Y$ between $G$-manifolds can be made transversal to a submanifold $M \subseteq Y$ w.r.t. to a specified normal bundle $\gamma(M)$, cf. section 3.

In the topological category Theorem 4.2 and therefore Theorem 4.4 are not true. This has been known for many years in the relatively free case (cf. [1], [14]): the algebraic $K_{-1}(\mathbf{Z}G)$ enters in. There is then a stable obstruction to transversality, one which cannot be eliminated by making dimension and gap assumptions. We will now formulate our necessarily weaker and more delicate results in this category.

Recall first the 'lower' $K$-groups $\tilde{K}_i(\mathbf{Z}G)$ for $i \leq 1$: if $i=1$, $\tilde{K}_1(\mathbf{Z}G)$ is the Whitehead group, if $i=0$, $\tilde{K}_0(\mathbf{Z}G)$ is the reduced class group and for $i \leq -1$, $\tilde{K}_{-i}(\mathbf{Z}G) = K_{-i}(\mathbf{Z}G)$ is the usual lower $K$-group, defined to be the invariant part of the Whitehead group $\mathrm{Wh}(\mathbf{Z}[G \times \mathbf{Z}^{i+1}])$. More precisely, let $n$ be a positive number. Consider the inclusion $n: \mathbf{Z}^{i+1} \to \mathbf{Z}^{i+1}$ which multiplies each coordinate by $n$. Let $n^*$ denote the induced endomorphism of $\mathrm{Wh}(\mathbf{Z}[G \times \mathbf{Z}^{i+1}])$ in the contravariant structure. This induces an action of $\mathbf{N}$ and $\tilde{K}_{-i}(G)$ is the subgroup which is left invariant, i.e.

$$\tilde{K}_{-i}(\mathbf{Z}G) = \mathrm{Wh}(\mathbf{Z}[G \times \mathbf{Z}^{i+1}])^{\mathbf{N}}, \quad i \geq -1.$$

Let $\mathcal{F}$ be any family of subgroups of $G$, closed under conjugacy and intersection. Write $\mathcal{K}_{-i}(G; \mathcal{F})$ for the corresponding equivariant lower $K$-groups (cf. [24]):

$$\mathcal{K}_{-i}(G; \mathcal{F}) = \sum_{\Gamma \in (\mathcal{F})}^{\oplus} \tilde{K}_{-i}(\mathbf{Z}[N\Gamma/\Gamma]).$$

Here $(\mathcal{F})$ denotes the set of conjugacy classes of subgroups in $\mathcal{F}$. We shall be particularly interested in the case where $\mathcal{F} = \mathrm{Iso}(V)$ is the family of isotropy groups of an $\mathbf{R}G$-module $V$. This set is closed under conjugacy and intersection, see e.g. [21].

In the case of a relatively free representation $V$,

$$\pi_i(\mathrm{Top}_G(V \oplus \mathbf{R}), \mathrm{Top}_G(V)) = \tilde{K}_{i-k}(\mathbf{Z}G) \quad \text{for} \quad i \leq k+1$$

where $k = \dim V^G$, cf. [1] and [14]. For more general representations one would maybe expect the similar result with $\tilde{K}_{i-k}(\mathbf{Z}G)$ replaced by $\mathcal{K}_{i-k}(G; \mathrm{Iso}(V))$. But this is not the case. The result is more complicated.

For $\Gamma \in \mathcal{F}$ let $\mathcal{F} \cap \Gamma$ be the family of subgroups of $\Gamma$ which belong to $\mathcal{F}$. For each such $\Gamma$ there is an (induction) homomorphism

$$\mathrm{Ind}\colon \mathcal{K}_{-i}(\Gamma; \mathcal{F}\cap\Gamma) \to \mathcal{K}_{-i}(G; \mathcal{F})$$

which maps $\tilde{K}_{-i}(\mathbf{Z}[N_\Gamma(H)/H])$ to $\tilde{K}_{-i}(\mathbf{Z}[N_G(H)/H])$ by the usual covariant structure. We define

$$\bar{\mathcal{K}}_{-i}(G; \mathcal{F}) = \mathrm{Cok}\left\{ \sum_{\Gamma \in (\mathcal{F})} \mathrm{Ind} \right\}. \tag{4.5}$$

The proof of the next result is given in Part III, cf. [20].

THEOREM 4.6. *Let $G$ be of odd order, and $V \subset T$ a pair of stable representations with* $\dim V^G = \dim T^G = k$. *Then*

(a) $\pi_i(\mathrm{Top}_G(V\oplus\mathbf{R}), \mathrm{Top}_G(V)) = \bar{\mathcal{K}}_{i-k}(G; \mathrm{Iso}(V))$ *for* $i \leq k-1$.

*The group vanishes for* $i \leq k-2$.

(b) $\pi_i(\mathrm{Top}_G(T), \mathrm{Top}_G(V)) = 0$ *for* $i \leq k-2$. □

Unfortunately, the non-vanishing of $\pi_{k-1}$ in Theorem 4.6(a) implies that there are global obstructions to stable topological transversality, cf. proof of Theorem 3.9. However, in special cases (where the fixed sets have simply connected components, see below) not all elements of $\pi_{k-1}(\mathrm{Top}_G(V\oplus\mathbf{R}), \mathrm{Top}_G(V))$ can appear as obstructions. In order to examine the obstructions carefully we need the following further information from Part III.

The triples

$$\mathrm{Top}_G(V\oplus\mathbf{R}^{j+2}) \supset \mathrm{Top}_G(V\oplus\mathbf{R}^{j+1}) \supset \mathrm{Top}_G(V\oplus\mathbf{R}^{j})$$

give rise to an exact couple converging to $\mathrm{Top}_G(V\oplus\mathbf{R}^\infty)$. The first non-trivial differential is given by the triple boundary map

$$\partial_{k+j}\colon \pi_{k+j}(\mathrm{Top}_G(V\oplus\mathbf{R}^{j+2}), \mathrm{Top}_G(V\oplus\mathbf{R}^{j+1})) \to \pi_{k+j-1}(\mathrm{Top}_G(V\oplus\mathbf{R}^{j+1}), \mathrm{Top}_G(V\oplus\mathbf{R}^{j})).$$

Both the domain and the range are identified with the group $\bar{\mathcal{K}}_{-1}(G; \mathrm{Iso}(V))$ by Theorem 4.6(a). On the other hand, $\mathcal{K}_{-1}(G)$ and $\bar{\mathcal{K}}_{-1}(G)$ comes equipped with the standard involution from algebraic $K$-theory and hence with a Tate differential

$$d_r\colon \bar{\mathscr{K}}_{-1}(G)\to\bar{\mathscr{K}}_{-1}(G),\quad d_r(x)=x-(-1)^r x^*.$$

From [20, § 5] we have

PROPOSITION 4.7. *Under the identification in Theorem* 4.6(a)

$$\partial_{k+j}(x)=x-(-1)^{k+j}x^*=d_{k+j}(x)$$

*where* $k=\dim V^G$ *and* $(x)^*$ *is the usual (algebraic) involution on* $\bar{\mathscr{K}}_{-1}(G)$. □

The Tate cohomology of $\bar{\mathscr{K}}_{-1}(G)$ is denoted $\hat{H}^*(\bar{\mathscr{K}}_{-1}(G))$, more precisely

$$\hat{H}^r(\bar{\mathscr{K}}_{-1}(G))=\operatorname{Ker} d_r/\operatorname{Im} d_{r-1}.$$

For odd order groups $G$, $\mathscr{K}_{-1}(G;\mathscr{F})$ is always torsion free. Moreover, in [20] we prove

THEOREM 4.8. *For G abelian,* $\bar{\mathscr{K}}_{-1}(G;\mathscr{F})$ *is torsion free.* □

Strangely enough, this result is no longer true for groups which are not abelian, and consequently $\pi_{k-2}(\mathrm{Top}_G(V\oplus\mathbf{R}),\mathrm{Top}_G(V))$ can have torsion. We shall not go into any further details here of this fact.

We now apply the above theorems to questions of transversality in the topological category. In order to make the notation more manageable we drop the family $\mathscr{F}=\mathrm{Iso}(V)$ from the notation and just write $\mathscr{K}_{-1}(G)$ and $\bar{\mathscr{K}}_{-1}(G)$.

With these preliminaries we are ready to present our obstruction theory for solving the global $(f,A,H)$-problem of Section 3. We assume given a $G$-$\mathbf{R}^n$ bundle $\xi$ over a $G$-manifold $M$ and a $G$-map

$$f\colon X\to T(\xi)$$

where $X$ is compact topological $G$-manifold. We make the following assumption for the rest of this section:

$$(X,\xi)\ \textit{is stable in the sense of}\ (4.3).$$

Let $H\subset G$. We assume that $f^H\colon X^H\to T(\xi)^H$ *is* $N(H)$*-transversal to* $M^H$ *and that the restriction of f to A is G-transversal.* Here $A$ is a closed $G$-neighborhood of the singular subset $X_0^H=\{x\in X|G_x>H\}$ of $X^H$.

For $x\in X^H$ and $y\in M^H$ we write $T=T_xX$ and $V=V_{x,y}$ where $T_xX=\xi_y\oplus V_{x,y}$. With this simplified notation we neglect that the representations may change from component to component of $X^H$ and $M^H$. The reader can easily supply the details of the more general situation where the representations do indeed vary.

Set $Z^H=(f^H)^{-1}(M^H)$. By Proposition 3.10, the obstructions to solve the $(f, A, H)$-problem lie in $H^*_{N(H)}(Z^H, Z^H\cap A; \pi_*(\alpha))$ where $\alpha=\mathrm{Fix}^H$,

$$\alpha\colon \mathrm{Top}_H(T)/\mathrm{Top}_H(V)\to \mathrm{Top}(T^H)/\mathrm{Top}(V^H).$$

Let $k=\dim Z^H=\dim V^H$. Since $\mathrm{Top}(T^H)/\mathrm{Top}(V^H)$ is $(k-1)$-connected,

$$\pi_r(\alpha)=\pi_{r-1}(\mathrm{Top}_H(T)/\mathrm{Top}_H(V))$$

when $r\le k$ and by Theorem 4.6 this group vanishes when $r\le k-1$. Thus there is a single obstruction

$$\tau(f, A, H)\in H^k_{N(H)}(Z^H, Z_H\cap A; \pi_{k-1}(\mathrm{Top}_H(T), \mathrm{Top}_H(V)))$$

for solving the $(f, A, H)$-problem.

We now make some further assumptions. The first one is

(I) Suppose $V\subset V\oplus\mathbf{R}^2\subset T$ as $H$-representations.

Here $\mathbf{R}^2$ has trivial $H$-action. Then Theorem 4.6 and the homotopy exact sequence of (I) yield the following diagram

$$\begin{tikzcd}
0 & & & \\
\pi_k(\mathrm{Top}_H(T), \mathrm{Top}_H(V\oplus\mathbf{R})) \arrow[u] \arrow[r] & \pi_{k-1}(\mathrm{Top}_H(V\oplus\mathbf{R}), \mathrm{Top}_H(V)) \arrow[r] & \pi_{k-1}(\mathrm{Top}_H(T), \mathrm{Top}_H(V)) \arrow[r] & 0 \\
\pi_k(\mathrm{Top}_H(V\oplus\mathbf{R}^2), \mathrm{Top}_H(V\oplus\mathbf{R})). \arrow[u] \arrow[ur, "\partial_k"'] & & &
\end{tikzcd}$$

Hence

$$\pi_{k-1}(\mathrm{Top}_H(T), \mathrm{Top}_H(V))\cong\bar{\mathscr{K}}_{-1}(H)/d_k(\bar{\mathscr{K}}_{-1}(H)).$$

There is an inclusion of $\hat{H}^{k-1}=\hat{H}^{k-1}(\bar{\mathscr{K}}_{-1}(H), d)$ in $\bar{\mathscr{K}}_{-1}(H)/d_k(\bar{\mathscr{K}}_{-1}(H))$. This gives an injection

$$i\colon H^k_G(Z^H, Z^H\cap A; \hat{H}^{k-1})\to H^k_G(Z^H, Z^H\cap A; \pi_{k-1}).$$

We wish to examine when our obstruction lies in the image of $i$, which is a finite 2-group. We will assume for simplicity of discussion that $\partial X=\varnothing$. The case of non-empty boundary is only notationally, not conceptually, more complicated.

We begin with a simple lemma which allows us to consider the singular set $Z_0^H$ as a submanifold of $Z^H$.

LEMMA 4.9. *For each $N(H)$-neighborhood $U$ of $Z_0^H$ there exists a closed $N(H)$-neighborhood $B\subset U\cap Z^H$ of $Z_0^H$ such that $B$ is a $N(H)$-manifold with $H_0(B,\partial B)=0$ and $H_1(B,\partial B)=0$. Moreover, when each component of $Z^H$ is simply connected then the same can be assumed for $Z^H-B$.*

*Proof.* Write $G=N(H)$. It acts freely on $Z^H-Z_0^H$. For each compact set $C\subset(Z^H-Z_0^H)/G$ there exists a compact manifold $\bar{W}$ which contains $C$ in the interior and such that the inclusion

$$j\colon \bar{W}\to(Z^H-Z_0^H)/G$$

is $[k/2]$-connected, $k=\dim Z^H$. $\bar{W}$ is called a compact core containing $C$. Its existence is proven in [26] (se also [29]).

Let $W\subset Z^H-Z_0^H$ be the inverse image of $\bar{W}$ under the covering map $(Z^H-Z_0^H)\to(Z^H-Z_0^H)/G$. Then the complement $B=Z^H-\operatorname{int}(W)$ satisfies the requirements of the lemma. Indeed it is clear that $\partial B$ and $B$ have the same number of components and

$$H_1(B,\partial B)\cong H_1(Z^H,W)\cong H_1(Z^H,Z^H-Z_0^H).$$

The latter group vanishes because

$$\operatorname{codim}(Z^H,Z_0^H)=\operatorname{codim}(V^H,V_0^H)$$

by transverality, and because $V=V_{x,y}$ is assumed stable: in particular

$$\operatorname{codim}(V^H,V_0^H)>2.$$

Finally, if $Z^H$ is simply connected so is $Z^H-Z_0^H$ and therefore $W$ by our requirements to a compact core. □

We now add the assumptions

(II) Each component of $X^H$ and of $M^H$ is simply connected.

(III) $V=V_{-1}\oplus\mathbf{R}$ as $H$-representations with $V_{-1}$ stable.

Notice from (II) and standard surgery arguments that one may deform

$$f^H\colon X^H\to T(\xi)^H(\operatorname{rel} X^H\cap A)$$

so that

(II′) $Z^H$ has simply connected components.

It is only this consequence of (II) which will be used below. For abelian groups $G$ (of odd order) and for certain nice choices of $A$ we can now show that the obstruction to the $(f, A, H)$-problem lies in a finite 2-group.

THEOREM 4.10. *Let $G$ be abelian. Let $A$ be a closed neighborhood of $Z_0^H$ such that $B=A\cap Z^H$ satisfies the properties in Lemma* 4.9. *Under the assumptions* (I), (II′) *and* (III) *the obstruction to solving the global $(f, A, H)$-problem lie in*

$$H_G^k(Z^H, Z^H\cap A; \hat{H}^{k-1}(\tilde{\mathscr{K}}_{-1}(H))) \quad \textit{where} \quad k=\dim Z^H.$$

*Proof.* Since $f$ is already transverse on $A$ we have a partial section $\lambda$ of the bundle functor $\operatorname{Epi}(T(X)|Z^H, f^*(\xi|M^H))$ defined over $A$. To solve the $(f, A, H)$-problem is equivalent to extend $\lambda$ over all of $Z^H$. We divide into two cases.

Case (1): $H=G$.

Let $\varepsilon^1$ be the trivial 1-dimensional bundle. There is a surjection of bundle functors

$$p\colon \operatorname{Epi}(T(X)|Z^H, f^*(\xi|M^H)\oplus\varepsilon^1)\to \operatorname{Epi}(T(X)|Z^H, f^*(\xi|M^H))$$

whose "fiber" is $\operatorname{Top}_H(V)/\operatorname{Top}_H(V_{-1})$. We first want to lift $\lambda|Z^H\cap A$ to a section $\tilde{\lambda}$ with $p\circ\tilde{\lambda}=\lambda|Z^H\cap A$. The obstructions lie in $H^r(A\cap Z^H; \pi_{r-1}(\operatorname{Top}_H(V), \operatorname{Top}_H(V_{-1})))$. This group vanishes for $r\neq k,\ k-1$ by Theorem 4.6 and for $r=k,\ k-1$ by Poincaré duality because $H_i(Z^H\cap A, \partial)=0$ for $i=0,1$ by our assumptions.

Let $W$ be the closure of $Z^H-A$ so that $Z^H=W\cup(Z^H\cap A)$. Then $H^{k-1}(W, \partial)=H_1(W)=0$ by assumption, so there is just a single obstruction to extend $\tilde{\lambda}|\partial W$ over $W$. It lies in

$$\pi_{k-1}(\operatorname{Top}_H(T), \operatorname{Top}_H(V_{-1}))\cong H^k(W, \partial W; \pi_{k-1}(\operatorname{Top}_H(T), \operatorname{Top}_H(V_{-1}))),$$

and it maps by $p$ into the single obstruction to extend $\lambda|\partial W$ to $W$. Hence the latter obstruction, which a priori belongs to $\pi_{k-1}(\operatorname{Top}_H(T), \operatorname{Top}_H(V))$, is in fact in the image of

$$p_*\colon \pi_{k-1}(\operatorname{Top}_H(T), \operatorname{Top}_H(V_{-1}))\to \pi_{k-1}(\operatorname{Top}_H(T), \operatorname{Top}_H(V)).$$

But the image of $p_*$ is equal to the kernel of

$$\pi_{k-1}(\operatorname{Top}_H(T), \operatorname{Top}_H(V))\xrightarrow{\partial_{k-1}} \pi_{k-2}(\operatorname{Top}_H(V), \operatorname{Top}_H(V_{-1})).$$

This proves Theorem 4.10 in the case $H=G$.

Case (2): $H\subset G$, $H\neq G$.

The obstructions are also natural with respect to restriction of groups. Considered as $G$-spaces the obstructions lie in the Bredon cohomology,

$$H^k_G=H^k_G(W,\partial W;\pi_{k-1}(\mathrm{Top}_H(T),\mathrm{Top}_H(V))).$$

This group is isomorphic with $\pi_{k-1}=\pi_{k-1}(\mathrm{Top}_H(T),\mathrm{Top}_H(V))$. Considered as $H$-spaces the obstructions lie in $H^k_H\cong\pi_{k-1}$. With this identification, and observing that in this case of one orbit type we can identify the Bredon cohomology with the ordinary cohomology of the quotient, we can identify the restriction of groups map $H^k_G\to H^k_H$ with multiplication by the index $|G{:}H|$. Thus, if $a\in\pi_{k-1}$ is the obstruction for the $G$-problem, then $|G{:}H|a$ is the obstruction for the $H$-problem. We then have the homomorphism $d_{k-1}\colon\pi_{k-1}\to\tilde{\mathcal{K}}_{-1}(H)$ and by case (1) $d_{k-1}(|G{:}H|a)=0$. Hence $|G{:}H|d_{k-1}(a)=0$, and Theorem 4.8 implies that $d_{k-1}(a)=0$. □

*Addendum* 4.11. The conclusions of Theorem 4.10 are also true when $X$ has a non-empty boundary provided $f$ is already transversal to $M$ near $\partial X$. Indeed, at each stage, $A$ should be taken to include $\partial X$.

Theorem 4.10 translates into a transversality result in the topological $G$-category for abelian groups of odd order as follows.

Let $(X,\xi)$ satisfy Definition 4.3. Let $H_1, H_2, \ldots$ be an admissible ordering of the occuring isotropy groups, i.e. $H_j\subset H_i$ only if $j>i$.

THEOREM 4.12. *Let $G$ be abelian and $|G|$ odd, and let $f\colon X\to T(\xi)$ be a $G$-map which satisfies Definition* 4.3 *and* (I), (II), (III). *Then there exists a hierarchy of obstructions,*

$$\tau_i(f)\in\hat{H}^*(\tilde{\mathcal{K}}_{-1}(H_i)),$$

*with $\tau_j(f)$ defined provided $\tau_i(f)=0$ for $i<j$. In this case $\tau_j(f)$ is a $G$-bordism invariant of the $G$-map $f$ relative to smaller fixed sets, and is additive with respect to bordism addition. If $\tau_i(f)$ vanishes for all $i$ then $f$ is $G$-homotopic to a map which is transversal to the* 0*-section of $\xi$. The reasonable relative version is also valid.* □

We can note that the (stepwise) bordism invariance of the transversality obstructions follows because the obstructions are concentrated on the top cycle of

$(Z^H, Z^H \cap A)$. This uses the vanishing of the groups in Theorem 4.6(a), and is in the end a consequence of the algebraic fact that $\tilde{K}_{-i}(\mathbf{Z}G)=0$ for $i\geq 2$, [8].

In Theorem 4.10 we were forced to add the unpleasant assumption that $G$ be abelian. On the expense of stabilizing by replacing $X$ with $X\times Q$ for a suitable (non-equivariant) manifold we can however obtain a version of Theorem 4.10 for all odd order groups. This will be of importance for our applications in Section 5.

Let $Z^H=(f^H)^{-1}(M^H)$ as above and consider the bundle functor

$$\mathrm{Epi}\,(T(X)|Z^H, f^*(\xi|M^H)).$$

In the $(f, A, H)$-problem we ask for a section relative to $Z^H\cap A$. We now examine instead the $(f\times Q, A\times Q, H)$-problem where $Q$ is a closed smooth manifold with trivial $G$-action and $f\times Q$ is the composite of $f$ with the projection $X\times Q\to X$.

PROPOSITION 4.13. *Let $G$ be any odd order group. Suppose $(X\times Q, \xi)$ satisfies Definition* 4.3 *and make the assumptions* (I) *and* (III) *above. Let $A$ be as in Theorem* 4.10. *The obstruction to solve the global $(f, A, H)$-problem lies in $H^k_{N(H)}$ $(Z^H, Z^H\cap A; \hat{H}^{k-1}(\tilde{\mathcal{K}}_{-1}(H))$, provided $Q$ has trivial Euler characteristic* ($k=\dim Z^H$).

*Proof*. Consider the bundle functors

$$\mathscr{E} = \mathrm{Epi}\,(T(X)|\, Z^H\times T(Q), f^*(\xi|M^H))$$
$$\mathscr{E}_1 = \mathrm{Epi}\,(T(X)|\, Z^H\times T(Q), f^*(\xi|M^H)\oplus\varepsilon^1)$$

are $Z^H$. The projection from $f^*(\xi|M^H)\oplus\varepsilon^1$ to $f^*(\xi|M^H)$ defines a map of bundle functors $p\colon \mathscr{E}_1\to\mathscr{E}$.

From the proof of Theorem 4.10 we know that there exists a section $\gamma_{k-1}$ defined over the $(k-1)$-skeleton $(Z^H)_{k-1}$ of $\mathrm{Epi}\,(T(X)|Z^H, f^*(\xi|M^H))$, extending the given section on $Z^H\cap A$. Since $Q$ has vanishing Euler characteristic its tangent bundle splits off a trivial bundle. Using a projection $TQ\to\varepsilon^1$, the section $\gamma_{k-1}$ implies a section of $\mathscr{E}_1$ defined over $(Z^H)_{k-1}\times Q$. Thus there is only a single obstruction to having a section of $\mathscr{E}_1$ (rel $(Z^H\cap A)\times Q$); it lies in

$$\begin{aligned}&\pi_{k+l-1}(\mathrm{Top}_H(T\oplus\mathbf{R}^l), \mathrm{Top}_H(V\oplus\mathbf{R}^{l-1}))\\ &\qquad\cong H^{k+l}_{N(H)}(Z^H\times Q, (Z^H\cap A)\times Q; \pi_{k+l-1}(\mathrm{Top}_H(T\oplus\mathbf{R}^l), \mathrm{Top}_H(V\oplus\mathbf{R}^{l-1}))).\end{aligned} \tag{*}$$

Here $l=\dim Q$ and $T$, $V$ are as in the proof of Theorem 4.10. The single obstruction to a section of $\mathscr{E}_1$ maps to the single obstruction to a section of $\mathscr{E}$. Thus the latter obstruction, which is the obstruction to solving the global $(f\times Q, A\times Q, H)$-problem, lies in the image of

$$\pi_{k+1-1}(\mathrm{Top}_H(T\oplus\mathbf{R}^l), \mathrm{Top}_H(V\oplus\mathbf{R}^{l-1}))\to\pi_{k+l-1}(\mathrm{Top}_H(T\oplus\mathbf{R}^l), \mathrm{Top}_H(V\oplus\mathbf{R}^l))$$

under the identification (*). This gives the obstruction group $\hat{H}^{k+l-1}(\tilde{\mathscr{K}}_{-1}(H))$, cf. Theorem 4.10. □

There is also a 'stable' analogue of Theorem 4.12. At each level in the hierarchy one needs to multiply by $Q$ in order to make the obstruction to lie in the stated Tate cohomology group. Thus we must assume that $(X\times Q\times \ldots \times Q, \xi)$ is stable where the number of factors $Q$ is equal to the number of different orbit types of $X$.

THEOREM 4.14. *Let $G$ be an odd order group and $f\colon X\to T(\xi)$ a $G$-map. Suppose $X$ has $N$ orbit types. Let $Q$ be a closed manifold with trivial $G$-action and zero Euler characteristic, and suppose that $(X\times Q^N, \xi)$ satisfies Definition* 4.3, (I), (III). *Then there exists a hierarchy of obstructions*

$$\tau_i(f\times Q^N)\in\hat{H}^*(\tilde{\mathscr{K}}_{-1}(H_i))$$

*for deforming $f\times Q^N\colon X\times Q^N\to T(\xi)$ to a $G$-transverse map. The obstructions are $G$-bordism invariants and additive in the sense of Theorem* 4.12. □

## § 5. Applications: $K_G$-orientations

A celebrated result of D. Sullivan asserts that a topological $\mathbf{R}^n$-bundle is oriented with respect to $K$-theory localized away from 2. It is the purpose of this section to prove a similar theorem for $G$-$\mathbf{R}^n$ bundles, when $G$ has odd order. There are two ingredients to the proof. One is the relation between the geometric bordism and $K$-theory in the equivariant smooth category, and the other is equivariant stable transversality.

We recall briefly the results on equivariant smooth $G$-bordism, referring the reader to [16] for details. The set of $G$-bordism classes of $G$-maps $f\colon X\to T$, from a smooth $G$-oriented $G$-manifold $X$ into the $G$-space $T$, is denoted $\Omega_*^G(T)$. It is an abelian group under disjoint union. Homotopy theoretically it can be calculated as

$$\Omega_n^G(T)=\varinjlim_V\,[S^{V+n}, T^+\wedge MSO_{|V|}(V\oplus\mathbf{R}^\infty)]^G,$$

where $V$ runs over all representation spaces for $G$, $S^{V+n}$ is the one point compactification of $V\oplus\mathbf{R}^n$, and $MSO_k(V\oplus\mathbf{R}^\infty)$ denotes the Thom space of the classifying bundle over the Grassmannian of $G$-oriented $k$-planes in $V\oplus\mathbf{R}^\infty$.

If $KO_G(T)$ denotes the usual equivariant (orthogonal) $K$-groups of $T$ we let

$$E^i_G(T)=\widetilde{KO}_G(S^i(T^+))\otimes\mathbf{Z}[\tfrac{1}{2}].$$

These groups form a periodic equivariant cohomology theory of period 4, so below we take $i\in\mathbf{Z}/4$.

It is well-known that oriented $G$-vector bundles are oriented with respect to the cohomology theory $E^*_G(-)$ in sense that there is a "linear" Thom class

$$\Delta(\eta)\in E^k_G(T\eta).$$

There are several good choices of Thom classes; we use

$$\Delta(\eta)=u(\eta)\cdot\lambda_1(\eta\otimes\mathbf{C})^{-1},$$

where $u(\eta)$ is the universal symbol class of the $D^+$-operator, [3]. Using the homotopy theoretic description of $\Omega^G_*(T)$, $\Delta$ gives a transformation of functors

$$\Delta\colon\ \Omega^G_*(T)\to E^G_*(T)$$

where $E^G_*(T)$ is the $G$-homology theory associated with the $G$-spectrum which defines $E^*_G(T)$. With our choice of $\Delta(\eta)$, $\Delta$ reduces to the $G$-signature when $T$=pt.

From [16], we have that $\Delta$ induces an isomorphism

$$\mathrm{Hom}_{E^G_*}(E^G_*(T),E^G_*)\cong\mathrm{Hom}_{\Omega^G_*}(\Omega^G_*(T),E^G_*).$$

On the right hand side the action of $\Omega^G_*$ on $E^G_*$ is via the $G$-signature. The universal coefficient theorem for equivariant $K$-theory gives, cf. [16]:

THEOREM 5.1. *There is an exact sequence*

$$0\to\mathrm{Ext}^1_{E^G_*}(E^G_*(T),E^G_*)\to E^*_G(T)\xrightarrow{\mu}\mathrm{Hom}_{\Omega^G_*}(\Omega^G_*(T),E^G_*)\to 0.\qquad\Box$$

For $x\in E^k_G(T)$, $\mu(x)$ can be described as follows. Suppose

$$\gamma\colon\ S^{V+n}\to MSO_{|V|}(V\oplus\mathbf{R}^\infty)\wedge T^+$$

represents an element of $\Omega_n^G(T)$. Let $\Delta \in E_G^{|V|}(MSO_{|V|}(V\oplus\mathbf{R}^\infty))$ be the universal Thom class. Then

$$\mu(x)(\gamma) = \gamma^*(\Delta \cdot x) \cap \Delta(V\oplus\mathbf{R}^n) \in E_G^{k-n}.$$

Here $\Delta(V\oplus\mathbf{R}^n) \in E_G^{|V|+n}(S^{V+n})$.

Let $\xi \to Y$ be an oriented topological $G$-$\mathbf{R}^n$ bundle over $Y$. Suppose that $\dim \xi^G > 2$. We shall use Theorem 4.14 to construct a homomorphism

$$\delta\colon \Omega_*^G(T(\xi)) \to \underset{\bullet}{R}(G) \otimes \mathbf{Z}[\tfrac{1}{2}].$$

If $f\colon X \to T(\xi)$ is a $G$-map which is transversal to the base space $Y$ then $f^{-1}(Y)$ is a $G$-manifold and we can take

$$\delta(X,f) = \operatorname{sign}_G(f^{-1}(Y)), \tag{5.2}$$

zero when $f^{-1}(Y)$ is odd-dimensional. But the problem is that not every element of $\Omega_*^G(T(\xi))$ is represented by a transversal map. Indeed, Theorem 4.14 indicates obstructions to transversality, even stably.

We must stabilize $f\colon X \to T(\xi)$ to meet the requirements of Theorem 4.14. On the other hand we must pick $Q$ with some care since we want to use the $G$-signature to get an invariant. Indeed, we want $Q$ with $\chi(Q)=0$ and $\operatorname{sign}(Q) \neq 0$.

Since the Euler characteristic and the signature are congruent (mod 2) we cannot get $\operatorname{sign}(Q)=1$ and $\chi(Q)=0$. However, there exists a smooth closed 8-dimensional manifold $Q$ with

$$\chi(Q) = 0, \quad \operatorname{sign}(Q) = 2. \tag{5.3}$$

(cf. [27]). Second, recall from [16, § 3] that for any pair $(M, \xi)$ of a $G$-manifold $M$ and a $G$-bundle $\xi$ (over $Y$) there exists a smooth closed $G$-manifold $P$ with

$$\operatorname{sign}_G(P) = 1 \quad \text{and} \quad (M\times P, \xi\times P) \quad \text{stable.} \tag{5.4}$$

Indeed, $P$ can be taken to be the complex projective space of (many) copies of the regular representation. Let $N$ be the number of subgroups of $G$ and choose $P$ so that $(X\times P\times Q^N, \xi)$ is stable. Write $R = P\times Q^N$ and

$$f_R\colon X\times R \to X \to T(\xi). \tag{5.5}$$

We want to make $f_R$ $G$-transverse to $Y$. The conditions (I) and (III) follow from our assumptions on $\xi$, namely from

$$\dim \xi^G \geq 2. \tag{5.6}$$

We can apply Theorem 4.14 step by step. First make $f_R^G \pitchfork Y^G$ and set

$$Z^G = (f_R^G)^{-1}(Y^G) \subset X \times R.$$

The first transversality obstruction is then in

$$\tau(f_R) \in \hat{H}^*(\mathcal{K}_{-1}(G)).$$

If it vanishes we can make $f_R$ $G$-transverse near $Z^G$.

The obstruction is of order 2, so if we replace the original $f: X \to T(\xi)$ by $2 \cdot f: 2 \cdot X \to T(\xi)$ where $2 \cdot X$ is the connected sum along a fixed point(¹) of $X$ with itself then

$$((2 \cdot f_R)^G)^{-1}(Y^G) = 2 \cdot Z^G = Z^G \# Z^G$$

and $\tau_G(2 \cdot f_R) = 2\tau_G(f_R) = 0$. We can continue up the stratification (following an admissible ordering of the orbit types). Thus for sufficiently large $k$,

$$(2^k \cdot f)_R: 2^k \cdot X \times R \to T(\xi)$$

can be approximated by a mapping $g_R$ which is $G$-transversal to $Y$. Therefore (5.2) can be extended to all bordism elements by defining

$$\delta(X, f) = 1/2^{k+N} \operatorname{sign}_G(g_R^{-1}(Y)) \in R(G) \otimes \mathbf{Z}[\tfrac{1}{2}]. \tag{5.7}$$

In fact, $\delta(X,f) \in E_r^G$ where $r = \dim g_R^{-1}(Y) = \dim X - \dim \xi$.

More importantly, $\delta(X,f)$ is a cobordism invariant as one can see by making $F_P: W \times R \to T(\xi)$ $G$-transversal (rel. $\partial W \times R$) whenever $(W, F)$ is a bordism of $(X,f)$. Note also that $\delta(X,f)$ is independent of the choice of $R$, since we may choose $g_{R_1}^{-1}(Y) \times R_2$ $G$-cobordant to $g_{R_2}^{-1}(Y) \times R_1$.

PROPOSITION 5.8. *There is a homomorphism*

$$\delta: \Omega_{n+k}^G(T\xi) \to E_k^G.$$

---

(¹) Every cobordism class $\{X, f\}$ has a representative with $X^G \neq \emptyset$. Indeed one can always replace $X$ with the connected sum along an orbit with a linear sphere (with a fixed point) to create a fixed point of $X$. This process does not change the cobordism class.

*which satisfies*

(i) *for* $\{M\} \in \Omega_*^G$, $\delta(\{X,f\} \times \{M\}) = \delta\{X,f\} \cdot \mathrm{sign}_G(M)$;

(ii) *for* $G$-$\mathbf{R}^{n_i}$ *bundles* $\xi_1$ *and* $\xi_2$

$$\begin{tikzcd}
\Omega^G_{n_1+k_1}(T\xi_1) \otimes \Omega^G_{n_2+k_2}(T\xi_2) \arrow[r] \arrow[d, "\delta_1 \otimes \delta_2"] & \Omega^G_{n_1+n_2+k_1+k_2}(T\xi_1 \wedge T\xi_2) \arrow[d, "\delta"] \\
E^G_{k_1} \otimes E^G_{k_2} \arrow[r, "\otimes"] & E^G_{k_1+k_2}
\end{tikzcd}$$

*is commutative.*

*Proof.* If $X \xrightarrow{f} T\xi$ is $G$-transversal map then $f_M: X \times M \to X \to T\xi$ is $G$-transversal and $f^{-1}(Y) \times M = f_M^{-1}(Y)$; since $\mathrm{sign}_G(\ )$ is multiplicative (i) follows. Similarly, if $f_i: X_i \to T\xi_i$ are $G$-transversal maps then

$$f_1 \times f_2 \colon X_1 \times X_2 \to T\xi_1 \wedge T\xi_2 = T(\xi_1 \times \xi_2)$$

is $G$-transversal to $Y_1 \times Y_2$ and the counter-image of $Y_1 \times Y_2$ is the product of the counter images of $Y_1$ and $Y_2$. □

We can now combine Theorem 5.1 and Proposition 5.8 to construct an $E_G^*$-Thom isomorphism for Top and $PL$ $G$-$\mathbf{R}^n$ bundles; the procedure is the same as in the $G$-trivial case (cf. [17]).

THEOREM 5.9. *Let $\xi$ be an oriented $G$-$\mathbf{R}^n$ bundle over $Y$ and let $|G|$ be odd. Suppose further that $\xi$ satisfies* (5.6). *Then there is an isomorphism*

$$\Phi_\xi \colon E_G^k(Y) \to \tilde{E}_G^{k+n}(T\xi),$$

*such that*

(i) $\hat{f}^* \Phi_{\xi_2}(y_2) = \Phi_{\xi_1}(f^*(y_2))$ *when* $(\hat{f}, f): (\xi_1, Y_1) \to (\xi_2, Y_2)$ *is a bundle map, and* $y_2 \in E_G^k(Y_2)$.

(ii) $\Phi_{\xi_1 \times \xi_2}(1) = \Phi_{\xi_1}(1) \otimes \Phi_{\xi_2}(1)$, *at least rationally.*

*Proof.* We construct $\Phi$ for the universal oriented $G$-$\mathbf{R}^n$ bundle, and get it in general from (i). Then $\Phi_\xi$ will automatically be natural with respect to bundle maps.

From Proposition 5.8 we have the homomorphism

$$\delta\colon \Omega^G_{*+n}(T\zeta_n) \to E^G_*$$

where $\zeta_n$ is the universal bundle over $BS\mathrm{Top}_n(G)$, and by Theorem 5.1 we obtain an element

$$\Delta_{\mathrm{top}}(\zeta_n) \in E^n_G(T\zeta_n; \infty).$$

We must show that $\Delta_{\mathrm{top}}(\zeta_n)$ is a Thom class. As usual it suffices to check that $\Delta_{\mathrm{top}}(\zeta_n)$ restricts to a Thom class on each fiber; we show it restricts to the linear Thom class on fibers.

Consider a $G$-map $f\colon (X, \partial X) \to (S^W, \infty)$ which is $G$-transverse to $0 \in W \subset S^W$ and let $M = f^{-1}(0) \subset X$ be the corresponding submanifold with trivial normal bundle $M \times W$. The map $f$ represents a bordism class $\{X, f\} \in \Omega^G_{n+|W|}(S^W, \infty)$. With the notation of Theorem 5.1 we shall see below that

$$\mu(\Delta(W))\,\{X, f\} = \mathrm{sign}_G(M).$$

Since $\mu(\Delta_{\mathrm{top}}(W))\{X, f\} = \mathrm{sign}_G(M)$ by definition it will follow that $\Delta(W) = \Delta_{\mathrm{top}}(W)$ at least rationally, and hence integrally, as $E^*_G(S^W)$ is torsion free.

Embed $X$ in a representation, say $(X, \partial X) \subseteq (DV, SV)$. Let $g$ be the composition

$$g\colon DV/SV \xrightarrow{c_X} T(\nu_X)/T(\nu_X|\partial X) \xrightarrow{\hat{f}} T(\nu_X) \wedge S^W,$$

where $c$ is the collapse map and $\hat{f}$ is induced from $\mathrm{id} \times f\colon (X, \partial X) \to X \times (S^W, *)$.

The normal bundle $\nu_X$ is classified by $\gamma_k(V)$ over the Grassman manifold of $k$-planes in $V$, so composing with $g$ we have

$$S^V \xrightarrow{g} T(\nu_X) \wedge S^W \to MSO_k(V) \wedge S^W.$$

This composition represents $\{X, f\}$ in the homotopy theoretic interpretation of $\Omega^G_n(S^W)$. It follows that

$$\mu(\Delta(w))(\{X, f\}) = g^*(\Delta(\nu_X) \otimes \Delta(W)) \cap \Delta(V).$$

On the other hand, the normal bundle of $M \subset X \subset V$ is $\nu_X|M \oplus (M \times W)$. We denote the collapse map for this embedding by $c_M$ and get a commutative diagram

$$\begin{tikzcd} S^V \arrow[r, "c_M"] \arrow[dr, "g"'] & T(\nu_X|M) \wedge S^W = T(\nu_M) \arrow[d, "T(i)\wedge \mathrm{id}"] \\ & T(\nu_X) \wedge S^W \end{tikzcd}$$

Hence $g^*(\Delta(\nu_X))\otimes\Delta(W))=c_M^*(\Delta(\nu_M))$ and by the Atiyah-Singer $G$-signature theorem (in the formulation of [16, (1.12)],

$$c_M^*(\Delta(\nu_M)) \cap \Delta(V) = \mathrm{sign}_G(M).$$

This completes the proof that $\Delta_{\mathrm{top}}(\xi)$ is a Thom class, and gives the isomorphism $\Phi_\xi$, which satisfies condition (i) of the theorem.

Condition (ii) follows from Proposition 5.8 (ii) and the fact that the kernel of $\mu$ in Theorem 5.1 is a torsion group. □

*Question.* Is $\mathrm{Ext}^1_{E^G}(E_*^G(B\mathrm{Top}(G)), E_*^G)=0$?

The answer is yes in the $G$-trivial case. The proof is based upon Sullivan's decomposition of the $p$-local $B\mathrm{Top}_{(p)}$ as $BSO_{(p)}\times B\mathrm{Cok}\,J_p$, the Hodgkin-Snaith theorem that $\mathrm{Cok}\,J_p$ has trivial $K$-theory and the description of $K$-theory of $BSO$ in terms of the completed representation ring. We refer the reader to [17] for more details. It does not appear unreasonable that the line of arguments for the $G$-trivial case can be carried through for general odd order groups. The necessary analysis of the equivariant $(F/PL)_{(p)}$ is done in [19] and $(F/\mathrm{Top})_{(p)}\simeq(F/PL)_{(p)}$ for $p$ odd by results from [20]. One then needs an equivariant $\mathrm{Cok}\,J_{(p)}$-subspace of $F_{(p)}$ and an equivariant Hodgkin-Snaith calculation.

We saw in the proof of Theorem 5.9 that the Thom class $\Phi_\xi(1)=\Delta_{\mathrm{top}}(\xi)$ restricts to the "linear" Thom class $\Delta(\xi_x)$ on each fiber. In fact, the same argument shows that $\Delta_{\mathrm{top}}(\xi)\otimes\mathbf{Q}=\Delta(\xi)\otimes\mathbf{Q}$ for a $G$-vector bundle $\xi$ over a $G$-simply connected space. In particular, $\Delta(\xi)\otimes\mathbf{Q}$ is a topological invariant. But moreover, since $E_G^*(MSO(G))\cong E_G^*(BSO(G))$ is torsion free we in fact get

COROLLARY 5.10. *For oriented G vector bundles with $|G|$ odd the K-theory Thom class $\Delta(\xi)\otimes\mathbf{Z}[\frac{1}{2}]$ is a topological invariant.* □

## §6. Topological similarity of representations

In this section we shall exploit Proposition 5.8 for $G$-vector bundles and in particular for representations of $G$, where $G$ has odd order. As pointed out in the last section the Thom class in $E_G^*$-theory of a $G$ vector bundle is a topological invariant. Therefore, any characteristic class which is derived from an operation on $\Delta(\xi)$ is also a topological invariant. We shall use this for the Euler class $e(\xi)$ and for the "cannibalistic" class $\varrho^k(\xi)$ defined respectively as

$$\begin{aligned} e(\xi) &= s^*(\Delta(\xi)), \\ \psi^k(\Delta(\xi)) &= k^n\varrho^k(\xi)\,\Delta(\xi), \quad 2n = \dim \xi. \end{aligned} \tag{6.1}$$

Here $s: X \to T(\xi)$ is the zero section and $\psi^k$ is the Adams operation. Both classes are exponential,

$$e(\xi_1 \oplus \xi_2) = e(\xi_1)\,e(\xi_2), \quad \varrho^k(\xi_1 \oplus \xi_2) = \varrho^k(\xi_1)\,\varrho^k(\xi_2)$$

and for a complex $G$-bundle they are given by

LEMMA 6.2. *Let $L$ be a complex $G$-line bundle over $X$. Then*
(i) $e(L)=(1-L)/(1+L) \in E_G^2(X)$,
(ii) $k\varrho^k(L) = \begin{cases} (1-L^k)(1+L)/(1-L)(1+L^k) \in E_G^2(X) & \text{if } L \neq 1 \\ k \quad \text{if } L = 1 \end{cases}$.

*Proof.* The argument follows from the relation $\Delta_L = \lambda_L \cdot \gamma_1(L)^{-1}$ where $\gamma_L \in K_G(TL)$ is the usual Thom class, and from the well-known relations:

$$s^*(\lambda_L) = 1 - L$$
$$\psi^k(\lambda_L) = (1 + L + \ldots + L^{k-1})\,\lambda_L$$

(see [2, §2.6]). □

We can take the base space $X$ to be a single point. Then $K_G(X) = R(G)$ is the complex representation ring, and we obtain strong topological invariants of representations. Since representations are detected by restrictions to cyclic subgroups we may assume $G$ is cyclic to start with.

Let $G$ be a cyclic group of odd order $m$. Choose a monomorphism $\chi: G \to \mathbf{C}^\times$, so we can list representations of $G$ as sums

$$V=\sum n_j\chi^j,\quad j\in \mathbf{Z}/m$$

where $n_j$ are integers and $\chi^j(g)=\chi(g)^j$. From Lemma 6.2 we get

$$\begin{aligned} e(V)(g) &= \prod_j (1-\chi(g)^j/1+\chi(g)^j)^{n_j} \\ \varrho^k(V)(g) &= k^{|V^g|-|V|}\prod_j (\varepsilon(\chi(g)^{jk})/\varepsilon(\chi(g)^j))^{2n_j-n_{j/2}}. \end{aligned} \tag{6.3}$$

Here $j$ varies over $\mathbf{Z}/m$, $\varepsilon(\zeta)=1-\zeta$ if $\zeta\neq 1$ and $\varepsilon(1)=1$, and $|V|$ denotes the complex dimension of $V$. Note that $e(V)(g)=0$ when $V^g\neq 0$ and that $\varrho^k(V)(g)=\varrho^k(V-V^g)(g)$.

Next, recall the Franz' independence lemma in the formulation of [6]. Let $\zeta_m$ denote a primitive $m$th root of 1.

THEOREM 6.4 (Bass, Franz). *Suppose given integers $a_j$ for $j\in\mathbf{Z}/m$ such that $a_j=a_{-j}$. If*

$$\prod_{j=1}^{m-1}\varepsilon(\zeta_m^{dj})^{a_j}=1$$

*for all divisors $d$ of $m$, then $a_j=0$ for all $j=1,\dots,m-1$.* □

COROLLARY 6.5. *Let $G$ be a cyclic group of odd order $m$. Two $\mathbf{R}G$-modules $V$ and $W$ with $V^G=W^G$ and $\varrho^k(V)=\varrho^k(W)$ for $(k,m)=1$ are isomorphic.*

*Proof.* We can assume $V^G=W^G=0$, and since $m$ is odd we can view $V$ and $W$ as $\mathbf{C}G$-modules; and must show that $V\oplus\bar V=W\oplus\bar W$. Write

$$V\oplus\bar V=\sum n_j\chi^j$$

$$W\oplus\bar W=\sum m_j\chi^j$$

with $j\in\mathbf{Z}/m$. Then $n_j=n_{-j}$ and $m_j=m_{-j}$ and $m_0=n_0=0$. Let

$$a_j=2n_j-n_{j/2}-2m_j+m_{j/2},$$

so by assumptions and (6.3)

$$k^{|V^g|-|W^g|-|V|+|W|}\prod[\varepsilon(\chi(g)^{jk})/\varepsilon(\chi(g)^j)]^{a_j}=1.$$

Since each term $\varepsilon(\chi(g)^{jk})/\varepsilon(\xi(g)^j)$ is a unit of $\mathbf{Z}[\chi(g)]$ we conclude that $|V^g|-|V|=|W^g|-|W|$ for all $g\in G$. Hence $V$ and $W$ have the same fixed set dimensions, and

$$\prod_j \varepsilon(\chi(g)^{jk})^{a_j}=\prod_j \varepsilon(\chi(g)^j)^{a_j} \tag{*}$$

for each $k$ with $(k, m)=1$.

Let $N_n\colon \mathbf{Q}(\zeta_m)\to\mathbf{Q}$ be the norm. It is well-known that $1-\xi_m^j$ is a unit unless $j|m$ and $m/j$ is a $p$-power. In fact, $N_d(1-\zeta_d)=1$ if $d$ is composit and $N_d(1-\zeta_d)=p$ if $d=p^i$. Hence we have

$$N_m(\varepsilon(\chi(g)^j))=\begin{cases}1 & \text{if } |\langle g^j\rangle| \text{ is composite}\\ p^{|G:\langle g^j\rangle|} & \text{if } |\langle g^j\rangle| \text{ is a power of } p.\end{cases}$$

Since $V$ and $W$ have the same fixed set dimensions $\Sigma\{a_j|\ |\langle g^j\rangle|=d\}=0$, and we get

$$\prod_{k\in(\mathbf{Z}/m)^\times}\ \prod_{j\in\mathbf{Z}/m}\varepsilon(\chi(g)^{jk})^{a_j}=1.$$

Then (*) implies $\Pi\,\varepsilon(\chi(g)^j)^{a_j}=1$, and from (6.4) we conclude that $a_j=0$ for all $j\in\mathbf{Z}/m-\{0\}$. Repeated use of the equation

$$n_j-m_j=2(n_{2j}-m_{2j})$$

shows that $n_j=m_j$ for all $j$. □

We can combine this result with the fact that $\varrho^k$ is a topological invariant to get the following main theorem.

THEOREM 6.6. *For groups of odd order, topologically conjugate representations are linearly conjugate.* □

This result, which in fact motivated our whole study, has been proved independently by W.C. Hsiang and W. Pardon in [12] using rather different methods. For groups of order $4m$, $m>1$. Theorem 6.6 is known to be false, [7].

Our results should also make it possible to get information about the kernel of the maps

$$KO_G(X)\to KPL_G(X)\to K\mathrm{Top}_G(X)$$

in a number of interesting situations, but we have not made any specific calculations on this problem.

## § 7. The topological $G$-signature theorem

The Thom isomorphism in $E_G^*$-theory from § 5 can be used to define a Gysin homomorphism for a $G$-map between oriented topological $G$-manifolds

$$f\colon M^m \to N^n.$$

For a sufficiently large representation $V$ the composition

$$f_V\colon M^m \to N^n \to N^n \times V$$

can be approximated by a $G$-embedding which has a $G$-normal bundle $\nu(f_V)$, cf. [14, p. 247]. We get a collapse map

$$(N^n \times V)^+ \xrightarrow{c} T\nu(f_V).$$

The composition

$$E_G^*(M) \underset{\cong}{\xrightarrow{\Phi_\nu}} E_G^*(T\nu(f_V)) \xrightarrow{c^*} E_G^*((N^n \times V)^+) \underset{\cong}{\xleftarrow{\Phi_V}} E_G^*(N)$$

is independant of the choice of $V$, at least after we tensor by $\mathbf{Q}$ (cf. Theorem 5.9), and is called the Gysin homomorphism. We denote it as usual by

$$f_!\colon E_G^i(M) \to E_G^{i+n-m}(N).$$

In particular for $N=\mathrm{pt}$ we have

$$f_!\colon E_G^0(M) \to E_G^{-m} \subset R(G) \otimes \mathbf{Q},$$

and there is the following topological version of the $G$-signature theorem:

THEOREM 7.2. *Let $G$ be a group of odd order, and let $M$ be an oriented topological $G$-manifold. The Gysin homomorphism*

$$f_!\colon E_G^0(M) \to E_G^{-m} \subset R(G) \otimes \mathbf{Q}$$

*maps* 1 *into* $\operatorname{sign}_G(M)$.

*Proof.* We have a $G$-transversal diagram

$$\begin{array}{ccc} S^V = S(V\oplus \mathbf{R}) & \xrightarrow{c} & T(\nu) \to MO_k(V) \\ \cup & & \cup \\ M & \xrightarrow{\text{id}} & M \end{array}$$

By the construction of $\Delta_{\text{top}}(\nu)$ (cf. the proof of Theorem 5.9):

$$c^*(\Delta_{\text{top}}(\nu)) = \text{sign}_G(M)\cdot\Delta(V) \qquad \square$$

For an oriented $G$-manifold $M$ the topological signature $\text{sign}_G(M)$ can be considered in the usual fashion as a complex (class) function of $G$, and

$$\text{sign}_G(M)(g)=\text{sign}_{\langle g\rangle}(M)(g).$$

Let $g$ be a generator of the cyclic odd order group $G$. Assuming that $M^g\subset M$ has a (locally linear) topological normal bundle $\nu(M^g, M)$ we have its $K$-theoretic Euler class

$$e(\nu(M^g, M)) \in K_G(M^g) \cong R(G)\otimes K(M^g).$$

Let $R(G)_g$ be the ring $R(G)$ localized at all characters which vanish at $g$ (explicitly, $R(G)_g \cong \mathbf{Q}(\zeta_{|G|})$), and let $K_G(M^g)_g$ be the corresponding localized module.

From [4], $e(\nu(M^g, m)$ is invertible in $K_G(M^g)_g$ and the Gysin homomorphism

$$i_!\colon K_G(M^g)_g \to K_G(M)_g$$

takes $e(\nu(M^g\colon M))^{-1}$ to 1 (since $i^*i_!$ is multiplication by $e(\nu(M^g, M)$, and $i^*$ is a local isomorphism). If $f\colon M\to \text{pt}$ is the constant map then $f_! = (f^G)_! \circ i_!$. Summarizing, we obtain the formula

$$\text{sign}_G(M)(g) = (f^g)_!(e(\nu(M^g\colon M))^{-1}) \tag{7.3}$$

valid for all odd order groups $G$ and all oriented topological $G$-manifolds for which (locally linear) normal bundles $\nu(M^g, M)$ exist.

The question of existence of normal tubes will be taken up in Part II, III (cf. [19], [20]) in the $PL$ and topological cases, respectively. We can notice that it suffices to have block bundle neighborhoods; which is automatic in the $PL$ category. Indeed, stably block bundles contain locally linear bundles. Thus (7.3) gives a $G$-signature theorem for every $PL$ $G$-manifold.

In closing, let us note for $\nu(M^g, M)$ trivial, say equal to $M^g \times W$, that $W$ is a fixed point free $\langle g \rangle$-module and (7.3) becomes

$$\mathrm{sign}_G(M)(g) = \mathrm{sign}(M) \cdot e(W)(g)^{-1}. \tag{7.4}$$

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