THE ARITHMETIC COHEN–MACAULAY CHARACTER OF SCHUBERT SCHEMES

BY

DAN LAKSOV

Massachusetts Institute of Technology, Cambridge, Mass., U.S.A.

1. In this paper we prove the following theorem. (For the notation, see Section 2.)

THEOREM 1. *Let R be a Cohen–Macaulay ring. Then then homogenous coordinate ring of the Schubert scheme $\Omega(a_1 \dots a_d)$ is Cohen–Macaulay of relative dimension*

$$\left[\sum_{i=1}^{d} a_i - \tfrac{1}{2}d(d+1)+1\right].$$

The theorem was also proved by M. Hochster. For an announcement of his results, see [5].

It was proved in a weaker local case (see Theorem 12 below) by J. A. Eagon and M. Hochster in "Cohen–Macaulay rings, invariant theory and the generic perfection of determinental loci" (to appear, cf. [3]).

The proof below owes many of its ideas to Eagon and Hochster and to G. Kempf. Frequent discussions with S. Kleiman and T. Svanes have also been helpful. I am especially grateful to Kleiman for his patient help preparing this material.

The proof goes as follows. We assume by induction that the homogeneous coordinate rings of small Schubert schemes are Cohen–Macaulay (the smallest being empty). Given a Schubert scheme we intersect it properly with a given hyperplane. The intersection then breaks up into the union of smaller Schubert schemes in the way described by a classical formula of M. Pieri. (We derive this formula from a result of W. V. D. Hodge. Hodge's result is the central part of the paper and we prove it following a method of J.-I. Igusa.) A result similar to lemmas of Eagon and Hochster shows that since the smaller Schubert schemes are Cohen–Macaulay, their union is also. As the equation of the hyperplane is not a zero-divisor in the homogeneous coordinate ring of the bigger Schubert scheme, this ring is then itself Cohen–Macaulay.

2. Let R be any ring and E a free R-module with basis $e_1, \dots, e_n$. Consider the grassmannian $G_r(E)$ which parametrizes submodules of E which are direct summands of rank d where $d = n - r$. Equivalently $G_r(E)$ parametrizes projective quotient modules of E of rank r, which we call r-quotients for short. (See [8] for more precise definitions.)

An r-quotient K of E gives rise to a 1-quotient $\wedge^r K$ of $\wedge^r E$. Writing

$$p_{(i)} = e_{i_1} \wedge \dots \wedge e_{i_r},$$

we find (by Laplace expansion, see [2] Chap. III, § 6, no. 4, p. 84) that the images of the $p_{(i)}$ in $\wedge^r K$ satisfy the following quadratic relations

$$\sum_\sigma \operatorname{sign}(\sigma)\, p_{i_1 \dots i_{\lambda-1} \sigma i_\lambda \dots \sigma i_r} \otimes p_{\sigma j_1 \dots \sigma j_\lambda j_{\lambda+1} \dots j_r} = 0 \tag{*}$$

where the sum ranges over all permutations σ of $(i_\lambda \dots i_r j_1 \dots j_\lambda)$ such that $\sigma i_\lambda < \dots < \sigma i_r$ and $\sigma j_1 < \dots < \sigma j_\lambda$ (see [7], p. 310). On the other hand, it is known that a 1-quotient L of $\wedge^r E$ arises from some r-quotient K of E if the images of the $p_{(i)}$ in L satisfy the above relations (see [6], Vol. 1, Ch. VII, § 6, Theorem II, p. 312, the discussion at this point is independent of the characteristic zero assumption). Therefore $G_r(E)$ is isomorphic to the scheme of zeros of the ideal Q generated by the quadratic relations in the polynomial ring P over R in the variables $p_{(k)}$ for all (k). We thus obtain an embedding called the Plücker embedding of $G_r(E)$ in $\mathbf{P}(\wedge^r E)$, which is projective N-space over R where $N = \binom{n}{r} - 1$.

Fix $0 < a_1 < a_2 < \dots < a_d \leqslant n$. For $1 \leqslant i \leqslant d$, let A_i be the free submodule of E generated by $e_1, \dots, e_{a_i}$. The subscheme of $G_r(E)$ parametrizing the r-quotients K of E such that the canonical map $\wedge^{(a_i - i + 1)} A_i \to \wedge^{(a_i - i + 1)} K$ is zero for all i is denoted $\Omega(a_1 \dots a_d)$ and called the Schubert subscheme of $G_r(E)$ corresponding to the conditions $A_1 \subset \dots \subset A_d$ (see [8]). Intuitively $\Omega(a_1 \dots a_d)$ parametrizes those direct summands of E of rank d which intersect A_i in a module of rank at least i for $1 \leqslant i \leqslant d$.

LEMMA 2. *The homogeneous ideal $I(a_1 \dots a_d)$ of $\Omega(a_1 \dots a_d)$ (in the Plücker embedding) is generated by those $p_{(j)}$ such that $j_{(a_i - i + 1)} \leqslant a_i$ for some i and by the quadratic relations* (*).

Proof. Let K be an r-quotient of E and $\alpha\colon E \to K$ the canonical map. The map $\wedge^{(a_i - i + 1)} \alpha \colon \wedge^{(a_i - i + 1)} E \to \wedge^{(a_i - i + 1)} K$ gives rise to a commutative diagram,

$$\begin{array}{ccc}
\wedge^{(a_i-i+1)} A_i \otimes \wedge^{(r-(a_i-i+1))} K \to \wedge^{(a_i-i+1)} K \otimes \wedge^{(r-(a_i-i+1))} K & \xrightarrow{\gamma} & \wedge^r K \\
\uparrow & & \uparrow \\
\wedge^{(a_i-i+1)} A_i \otimes \wedge^{(r-(a_i-i+1))} E \to \wedge^{(a_i-i+1)} E \otimes \wedge^{(r-(a_i-i+1))} E & \xrightarrow{\delta} & \wedge^r E
\end{array}$$

where γ and δ are the canonical pairings. It is easily checked that $\wedge^{(a_i-i+1)}(\alpha \mid A_i)$ is zero if and only if the map $\wedge^{(a_i-i+1)}A_i \otimes \wedge^{(r-(a_i-i+1))}E \to \wedge^r K$ is zero. Equivalently $\wedge^{(a_i-i+1)}(\alpha \mid A_i)$ is zero if and only if $p_{(j)} = e_{j_1} \wedge \ldots \wedge e_{jr}$ is mapped to zero whenever at least $(a_i - i + 1)$ of the e_j's are in A_i for some i, that is whenever $j_{(a_i-i+1)} \leqslant a_i$ for some i.

The ring $P/I(a_1 \ldots a_d)$ is the homogeneous coordinate ring of $\Omega(a_1 \ldots a_d)$ in the Plücker embedding. We will, for short, call it the homogeneous coordinate ring of $\Omega(a_1 \ldots a_d)$.

3. A polynomial $f = \sum_{(i)(j)\ldots(l)} a_{(i)(j)\ldots(l)} p_{(i)} p_{(j)} \ldots p_{(l)}$ in P is said to be in *standard form* if, for all nonzero terms, we have $i_t \leqslant j_t \leqslant \ldots \leqslant l_t$ for $1 \leqslant t \leqslant r$. We easily check (see [6], Vol. II, Chapter XIV, § 9 Theorem I, p. 378) that any polynomial can be written in standard form modulo the quadratic relations.

Consider the matrix $M = (x_{ij})$ of the form

$$\left[\begin{array}{cccccccccc}
x_{11}\,x_{12}\ldots x_{1,a_1-1} & x_{1,a_1} & x_{1,a_1+1} & \ldots\, x_{1,a_2-1} & x_{1,a_2} & \ldots\, x_{1,a_d-1} & x_{1,a_d} & \ldots & x_{1n} \\
0 \quad x_{22} \quad \vdots & \vdots & \vdots & \cdot & \cdot & \cdot & \cdot & & \cdot \\
\ddots & & & & & & & & \\
x_{a_1-1,a_1-1} & x_{a_1-1,a_1} & x_{a_1-1,a_1+1} & \cdot & \cdot & \cdot & \cdot & & \cdot \\
 & 0 & x_{a_1,a_1+1} & \cdot & \cdot & \cdot & \cdot & & \cdot \\
 & & \ddots & & & & & & \\
 & & & x_{a_2-2,a_2-1} & x_{a_2-2,a_2} & & & & \\
0 & & & & 0 & \ddots & & & \\
 & & & & & x_{a_d-d,a_{d-1}} & x_{a_d-d,a_d} & & \\
 & & & & & & 0 & \ddots & \\
 & & & & & & & & x_{rn}
\end{array}\right]$$

where the nonzero x_{ij} are indeterminates.

Let S be the localization of the polynomial ring over R in the variables x_{ij}, for all i, j, at the $r \times r$-determinant formed from the last r columns of M. Then M defines a surjective homomorphism from E to a free S-module F of rank r. Thus M defines a point m of $G_r(E)$ (with values in S). As the $(a_i - i + 1) \times (a_i - i + 1)$ subdeterminants of the first a_i columns of M clearly vanish for all i, m lies on $\Omega(a_1 \ldots a_d)$.

PROPOSITION 3. (Hodge). *Let $f = \sum a_{(i)\,(j)\ldots(l)}\, p_{(i)}\, p_{(j)} \ldots p_{(l)}$ be a homogeneous polynomial in standard form with all $a_{(i)\,(j)\ldots(l)} \in R$. Then f vanishes at m if and only if we have $p_{(i)} \in I(a_1 \ldots a_d)$ for all (i) such that $a_{(i)\,(j)\ldots(l)} \neq 0$.*

Proof. (Igusa). The statement that f vanishes at m means that the image of f under the morphism from $P=\mathrm{Sym}_S(\wedge^r E)$ to $S=\mathrm{Sym}_S(\wedge^r F)$ induced by $\wedge^r M$ is zero, that is the following relation holds in S

$$\sum a_{(i)(j)\dots(l)}\begin{vmatrix} x_{1i_1} \dots x_{1i_r} \\ \vdots \quad\quad \vdots \\ x_{ri_1} \dots x_{ri_r}\end{vmatrix}\begin{vmatrix} x_{1j_1} \dots x_{1j_r} \\ \vdots \quad\quad \vdots \\ x_{rj_1} \dots x_{rj_r}\end{vmatrix}\dots\begin{vmatrix} x_{1l_1} \dots x_{1l_r} \\ \vdots \quad\quad \vdots \\ x_{rl_1} \dots x_{rl_r}\end{vmatrix}=0.$$

Suppose $p_{(i)}\in I(a_1\dots a_d)$. Then more than (a_i-i+1) columns in the determinant

$$\begin{vmatrix} x_{1i_1} \dots x_{1i_r} \\ \vdots \quad\quad \vdots \\ x_{ri_1} \dots x_{ri_r}\end{vmatrix}$$

are taken from the first a_i columns of M. Thus Laplace expansion along the first (a_i-i+1) columns shows that the determinant is zero. Therefore if $p_{(i)}\in I(a_1\dots a_d)$ for all (i) such that $a_{(i)(j)\dots(l)}\neq 0$, then we have that f vanishes at m.

Conversely, take an indeterminate t and let $x_{ij}=t^{\varrho_{ij}}y_{ij}$ where $\varrho_{ij}=(n-i)\ (n-i+j)$. We easily check that the lowest power of t in

$$\begin{vmatrix} t^{\varrho_{1i_1}}y_{1i_1} \dots t^{\varrho_{1i_r}}y_{1i_r} \\ \vdots \quad\quad\quad \vdots \\ t^{\varrho_{ri_1}}y_{ri_1} \dots t^{\varrho_{ri_r}}y_{ri_r}\end{vmatrix}$$

is $(\varrho_{1i_1}+\varrho_{2i_2}+\dots+\varrho_{ri_r})$ (see [6], Vol. 2, Chap. XIV, § 9, p. 381). Thus in the expression for the image of f in S the initial coefficient (that is the first nonvanishing coefficient of a power of t) is of the form

$$\sum a_{(i)(j)\dots(l)}\, x_{1i_1}\dots x_{ri_r}x_{1j_1}\dots x_{rj_r}\dots x_{1l_1}\dots x_{rl_r} \tag{**}$$

where the sum is taken over all terms where $(\varrho_{1i_1}+\dots+\varrho_{ri_r}+\varrho_{1j_1}+\dots+\varrho_{rj_r}+\dots+\varrho_{rl_r})$ is minimal and where (i) is such that $p_{(i)}\notin I(a_1\dots a_d)$ and at least one $a_{(i)(j)\dots(l)}\neq 0$. Since the image of f is zero, (**) has to be zero. However as f is expressed in standard form, we easily convince ourselves that no two of the monomials $x_{1i_1}\dots x_{ri_r}x_{1j_1}\dots x_{rl_r}$ are equal. Thus, all $a_{(i)(j)\dots(l)}$ appearing in (**) would have to be zero. Therefore we have $p_{(i)}\in I(a_1\dots a_d)$ for all (i) such that $a_{(i)(j)\dots(l)}\neq 0$.

COROLLARY 4. *We have $f\in I(a_1\dots a_d)$ if and only if $p_{(i)}\in I(a_1\dots a_d)$ for all (i) such that $a_{(i)(j)\dots(l)}$ appearing in* (3) *is nonzero.*

COROLLARY 5. *The monomials in standard form are linearly independent over R modulo Q. In particular, $P/I(a_1 \dots a_d)$ is torsionfree over R.*

Proof. With $a_i = (r+i)$, we have $I(a_1 \dots a_d) = Q$. Thus, f is in Q if and only if $a_{(i)(j)\dots(l)}$ is zero.

COROLLARY 6. *If $f \notin I(a_1 \dots a_d)$ and $p_{(m)} \notin I(a_1 \dots a_d)$, then $p_{(m)} f \notin I(a_1 \dots a_d)$.*

Proof. Suppose $p_{(m)} \cdot f \in I(a_1 \dots a_d)$. Then proceeding as in the proof of Proposition 3, we see that this relation means that

$$x_{1m_1} \dots x_{rm_r} \sum a_{(i)(j)\dots(l)} x_{1i_1} \dots x_{ri_r} x_{1j_1} \dots x_{rj_r} \dots x_{1l_1} \dots x_{rl_r} = 0$$

where the sum ranges over $(\varrho_{1i_1} + \dots + \varrho_{ri_r} + \varrho_{1j_1} + \dots + \varrho_{rj_r} + \dots + \varrho_{rl_r})$ minimal and (i) such that $p_{(i)} \notin I(a_1 \dots a_d)$. However, as no two of the monomials $x_{1i_1} \dots x_{ri_r} x_{1j_1} \dots x_{rl_r}$ are equal (since f is expressed in standard form), all $a_{(i)(j)\dots(l)}$ are zero. Therefore, we have $f \in I(a_1 \dots a_d)$, a contradiction.

Let H be the hyperplane section of $G_r(E)$ defined by the vanishing of

$$e = e_1 \wedge \dots \wedge \check{e}_{a_1} \wedge \dots \wedge \check{e}_{a_2} \wedge \dots \wedge \check{e}_{a_d} \wedge \dots \wedge e_n$$

(where $\check{}$ means the symbol has been deleted). That is, the ideal I of H in the polynomial ring P is generated by e and Q. In other words, H is the Schubert scheme of direct summands of E of rank d that (intuitively) intersect the free submodule of E generated by $e_1, \dots, \check{e}_{a_1}, \dots, \check{e}_{a_2}, \dots, \check{e}_{a_d}, \dots, e_n$ in a module of rank $\geqslant 1$.

PROPOSITION 7. (Pieri). *We have*

$$\Omega(a_1 \dots a_d) \cap H = \sum_{a_i - a_{i-1} > 1} \Omega(a_1 \dots (a_i - 1) \dots a_d);$$

that is
$$I(a_1 \dots a_d) + I = \bigcap_{a_i - a_{i-1} > 1} I(a_1 \dots (a_i - 1) \dots a_d).$$

Proof. The generators of all the ideals being known, this proposition is a "combinatorial" consequence of Proposition 3.

Indeed, the relation $e_{j_1} \wedge \dots \wedge e_{j_r} \in I(a_1 \dots a_d)$ clearly implies that $(a_i - i + 1)$ of the e_j's lie in A_i for some i. Thus, certainly $((a_i - i) - i + 1) = (a_i - i)$ of the e_j's are in the set $\{e_1, \dots, e_{(a_i-1)}\}$. Moreover, e has precisely $(a_i - i)$ of the e_j's in this module when $(a_i - a_{i-1}) > 1$. Thus the inclusion $\subset$ holds.

For the opposite inclusion let $f \in \bigcap_{a_i - a_{i-1} > 1} I(a_1 \dots (a_i - 1) \dots a_d)$ be a homogenous polynomial. Write $f=f'+f''+q$ where $f'=\sum a_{(i)(j)\ldots(l)}p_{(i)}p_{(j)}\ldots p_{(l)}$ with $p_{(i)}\notin I(a_1\ldots a_d)$ for all nonzero $a_{(i)(j)\ldots(l)}$ and $f''=\sum b_{(i)(j)\ldots(l)}p_{(i)}p_{(j)}\ldots p_{(l)}$ with $p_{(i)}\in I(a_1\ldots a_d)$ for all nonzero $b_{(i)(j)\ldots(l)}$, and where f' and f'' are in standard form and $q\in Q$. Since $(f''+q)\in I(a_1\ldots a_d)$ we have $f'\in\bigcap_{a_i-a_{i-1}>1}I(a_1\ldots(a_i-1)\ldots a_d)$. Thus by Proposition 3, $p_{(i)}\in I(a_1\ldots(a_i-1)\ldots a_d)$ for all $(a_i-a_{i-1})>1$ and for all (i) appearing in the expression for f'. However, since these $p_{(i)}$ are in $I(a_1\ldots(a_i-1)\ldots a_d)$ but not in $I(a_1\ldots a_d)$, exactly (a_j-j) of the e_i's are in the set $\{e_1,\ldots,e_{a_j-1}\}$. Therefore $p_{(i)}$ is equal to the generator e of I. Thus we have $f'\in I$ that is $f\in(I(a_1\ldots a_d)+I)$.

4. Lemma 8. *Let $I_1,\ldots,I_p$ be ideals in a local ring A and r an integer. Suppose that for any choice of (distinct) indices $k_1,\ldots,k_t$ and for any $u<t$ where $t=1,\ldots,p$, we have*

$$\operatorname{depth}(A/(I_{k_1}+\ldots+I_{k_t}))=(r-t).$$

and

$$I_{k_1}\cap\ldots\cap I_{k_u}\cap(I_{k_{u+1}}+\ldots+I_{k_t})=(I_{k_1}\cap\ldots\cap I_{k_u}\cap I_{k_{u+1}})+\ldots+(I_{k_1}\cap\ldots\cap I_{k_u}\cap I_{k_t}).$$

Then we have

$$\operatorname{depth}(A/\bigcap_{i=1}^{p}I_i)=(r-1).$$

Proof. Put $I=(I_{k_2}+\ldots+I_{k_t})$. Then we have a commutative diagram

$$\begin{array}{ccccccc}0\to I_{k_1}/(I_{k_1}\cap I) & \to & A/(I_{k_1}\cap I) & \to & A/I_{k_1}\to 0\\ \downarrow\simeq & & \downarrow & & \downarrow\\ 0\to(I_{k_1}+I)/I & \longrightarrow & A/I & \longrightarrow & A/(I_{k_1}+I)\to 0.\end{array}$$

From the bottom sequence, we conclude, looking at the long exact sequence of Ext's, that we have $\operatorname{depth}((I_{k_1}+I)/I)=(r-t+1)$ and then, from the top sequence, that we have $\operatorname{depth}(A/(I_{k_1}\cap I))=(r-t+1)$. Consequently if we put $J_2=(I_1\cap I_2),\ldots,J_p=(I_1\cap I_p)$, we have $\operatorname{depth}(A/(J_{k_2}+\ldots+J_{k_t}))=(r+1-t)$. Thus, the J_k's satisfy the first condition of the lemma, and the distributivity follows immediately from the distributivity of the I_k's. The result now follows by induction.

Corollary 9. *If $A/(I_{k_1}+\ldots+I_{k_t})$ is Cohen–Macaulay of dimension $(r-t)$, then $A/\bigcap_{i=1}^{p}I_i$ is Cohen–Macaulay of dimension $(r-1)$.*

Indeed, $\dim(A/\bigcap_{i=1}^{p}I_i)=\max.\dim(A/I_i)=(r-1)=\operatorname{depth}(A/\bigcap_{i=1}^{p}I_i)$.

Lemma 10. *Let $I_i=I(a_1\ldots(a_i-1)\ldots a_d)$ for $(a_i-a_{i-1})>1$. Then these ideals satisfy the distributivity condition of Lemma 8. Moreover*

$$\sum_{i=1}^{t}I_{k_i}=I(a_1\ldots(a_{k_1}-1)\ldots(a_{k_2}-1)\ldots(a_{k_t}-1)\ldots a_d).$$

Indeed, Lemma 10 is an easy consequence of Proposition 3 using the method of proof of Proposition 7.

LEMMA 11. *A finitely generated homogenous algebra $A = k[x_0, \ldots, x_n]$ over a field k is Cohen-Macaulay if and only if it becomes Cohen–Macaulay after localization at the irrelevant maximal ideal $(x_0, \ldots, x_n)$.*

Proof. Write A as a quotient of the polynomial ring $B = k[X_0, \ldots, X_n]$ by an ideal I. If $\dim(A) = m$, then A is (equidimensional and) Cohen–Macaulay if and only if $(\mathrm{Ext}^q_B(A, B))_N = 0$ for $q > n - m$ and all maximal ideals N of B containing I (see [1] Corollary (5.22), p. 66). However, the Ext's, being graded B-modules of finite type, are zero if and only if they become zero when localized at $(X_0, \ldots, X_n)$.

Proof of Theorem 1. Obviously $G_r(E)$ is obtained by change of ground ring from the grassmannian of r-quotients of the free Z-module with basis $e_1, \ldots, e_n$. Similarly the Schubert scheme $\Omega(a_1 \ldots a_d)$ is obtained by change of ground ring from a Schubert scheme defined over $\mathbf{Z}$. By Corollary 5 the homogeneous coordinate ring of the latter Schubert scheme is torsion free over $\mathbf{Z}$ and therefore faithfully flat over $\mathbf{Z}$. Thus the homogenous coordinate ring of $\Omega(a_1 \ldots a_d)$ is faithfully flat over R. From [4] (Chap. IV, Corollary (6.1.2), p. 135, and Proposition (6.3.1), p. 138) (or from [1], Proposition (4.2), p. 143), we conclude that we only need to prove Theorem 1 in the case when R is a field.

We will prove that the homogenous coordinate ring of $\Omega(a_1 \ldots a_d)$ is Cohen–Macaulay by induction on $\sum_{i=1}^d a_i$. So assume $P/I(b_1 \ldots b_d)$ is Cohen–Macaulay of dimension $(\sum_{i=1}^d b_i - \frac{1}{2} d(d+1) + 1)$ when $\sum_{i=1}^d b_i < \sum_{i=1}^d a_i$.

By Lemma 11, we may localize at the irrelevant maximal ideal of P and thus assume that P is local.

From Corollary 9 and Lemma 10 and the induction assumption, we conclude that P/J is Cohen–Macaulay of dimension $\delta = (\sum_{i=1}^d a_i - \frac{1}{2} d(d+1))$, where

$$J = \bigcap_{a_i - a_{i-1} > 1} I(a_1 \ldots (a_i - 1) \ldots a_d).$$

Moreover, by (7) we have $J = (I(a_1 \ldots a_d) + I)$ where I is the ideal generated by the element e and the quadratic relations Q. Thus, $P/(I(a_1 \ldots a_d) + (e))$ is Cohen–Macaulay of dimension δ. However, by Corollary 6, e is not a zero-divisor in $P/I(a_1 \ldots a_d)$. Thus $P/I(a_1 \ldots a_d)$ is Cohen–Macaulay of dimension $(\delta + 1)$.

5. THEOREM 12. *Let R be a Cohen–Macaulay ring and x_{ij} indeterminates, where $i = 1, \ldots, r$ and $j = 1, \ldots, d$. Let J be the ideal in $R[x_{ij}]$ generated by the determinants of the*

$k\times k$-submatrices of the first s_k columns of the matrix (x_{ij}), where $k=1,\dots,t$ and where the s_k are integers such that $0<s_1<\dots<s_t=d$. Then $R[x_{ij}]/J$ is a Cohen–Macaulay R-algebra of relative dimension $[\frac{1}{2}(d+t)(d-t+1)+(r+d)t-r-\sum_{i=1}^t s_i]$.

Proof. We will show that there exists an open affine subset of $\Omega(a_1\dots a_d)$, for some choice of $a_1,\dots,a_d$, whose associated ring is $R[x_{ij}]/J$. This ring is then easily seen to be Cohen–Macaulay because the homogeneous coordinate ring of $\Omega(a_1\dots a_d)$ is so, and its relative dimension is the relative dimension of $\Omega(a_1\dots a_d)$.

Let F (resp. G) be the free submodule of E generated by $e_{d+1},\dots,e_n$ (resp. $e_1,\dots,e_d$). (Recall: $n=r+d$.) We know that there is an open affine subset U of $G_r(E)$ of the form $\mathrm{Hom}_R(G,F)$ (see [8], prop. (1.2), p. 283.) (Here we identify $\mathrm{Hom}_R(G,F)$ with the subset of homorphisms of $\mathrm{Hom}_R(E,F)$ that leave e_i fixed for $i=d+1,\dots,n$).

Let $r_k=(s_k-k+1)$ for $k=1,\dots,t$ and let $a_1=1$, $a_2=2,\dots,a_{r_1}=s_1$ and $a_{r_1+1}=(s_1+2)$, $a_{r_1+2}=(s_1+3),\dots,a_{r_2}=s_2$ and $a_{r_2+1}=(s_2+2),\dots,a_{r_3}=s_3$ and $\dots,a_{r_t}=s_t$ and $a_{r_t+1}=(n-(d-r_t)+1)$, $a_{r_t+2}=(n-(d-r_t)+2),\dots,a_d=(n-(d-n_t)+(d-r_t))=n$. A point $y\in U$, with corresponding matrix $M(y)\in\mathrm{Hom}_R(G,F)$, lies in $\Omega(a_1\dots a_d)$ if and only if $\wedge^{(a_i-i+1)}(M(y)\,|\,A_i)$ is zero for $i=1,\dots,d$, or equivalently if and only if the determinants of the $(a_i-i+1)\times(a_i-i+1)$ submatrices of the first a_i columns of the matrix $M(y)$ are zero for $i=1,\dots,d$. With the above choice of a_i's, we easily see that the latter condition means that the determinants of the $k\times k$ submatrices of the first s_k columns of $M(y)$ are zero for $k=1,\dots,t$. Or, in other words, $U\cap\Omega(a_1,\dots a_d)$ looks like (Spec. of) the ring $R[x_{ij}]/J$.

As another application of the main theorem, we will compute the cohomology groups of the twisted (with respect to the Plücker imbedding) structure sheaf $O_\Omega(k)$ where $\Omega=\Omega(a_1\dots a_d)$. For this we need the following result. (For a proof see [9], prop. 2.2.4.)

Proposition 13. (Grothendieck.) *Let S_0 be a field, $S=\oplus_{n=0}^\infty S_n$ a graded algebra of finite type generated by S_1. Put $X=\mathrm{Proj}\,(S)$ and let $\alpha\colon S\to\oplus_{k=-\infty}^\infty H^0(X,O_X(k))$ be the canonical homorphism. Put $d=\mathrm{depth}\,(S_M)$ where $M=\oplus_{n=1}^\infty S_n$ and assume $d\geqslant 2$. Then α is bijective and $H^i(X,O_X(k))=0$ for all $1\leqslant i\leqslant d-2$ and all k.*

From Theorem 1 and Proposition 13 we immediately conclude that if R is a field and $\dim(\Omega)\geqslant 1$, then $H^i(\Omega,O_\Omega(k))=0$ for $0<i<\dim(\Omega)$ and all k and then $P/I(a_1\dots a_d)$ and $\oplus_{k=-\infty}^\infty H^0(\Omega,O_\Omega(k))$ are isomorphic graded algebras.

For any R, the graded pieces of $P/I(a_1\dots a_d)$ are free R-modules generated by the residue classes of the monomials of P in standard form, not contained in $I(a_1\dots a_d)$, (by Corollary 5). The rank of the kth graded piece for $k\geqslant 0$ is

$$\omega(k)=\begin{vmatrix} \binom{a_d+m-1}{m} & \binom{a_d+m-2}{m-1} & \cdots & \binom{a_d+m-d}{m-d+1} \\ \binom{a_{d-1}+m}{m+1} & \vdots & & \vdots \\ \vdots & & & \\ \binom{a_1+m+d-2}{m+d-1} & \cdots & & \binom{a_1+m-1}{m} \end{vmatrix}$$

(see [6], vol. 2, chap. XIV, § 9, Theorem III).

Assume now that R is a field. Since the higher cohomology groups of $O_\Omega(k)$ vanish for large k, the Hilbert polynomial of Ω must be equal to ω. By the above, $h^0(\Omega, O_\Omega(k))$ is equal to $\omega(k)$ for $k \geqslant 0$ and zero for $m<0$. Since $\omega(k)=[h^0(\Omega, O_\Omega(k))-h^{\dim(\Omega)}(\Omega, O_\Omega(k))]$, we must thus have $H^{\dim(\Omega)}(\Omega, O_\Omega(k))=0$ for $k \geqslant 0$.

The above results all hold for any noetherian ring R by virtue of the "property of exchange" ([4], Chap. III, 7.75) because, as we noted in the proof of Theorem 1 at the end of Section 4, Ω is flat over R. Therefore, we have established the following theorem.

THEOREM 14. *Let R be any noetherian ring, put $\Omega=\Omega(a_1 \ldots a_d)$, and assume* $\dim(\Omega) \geqslant 1$. *Then we have $H^i(\Omega, O_\Omega(k))=0$ for:* (i) $0<i<\dim(\Omega)$ *and all k:* (ii) $i=0$ *and $k<0$:* (iii) $i=\dim(\Omega)$ *and $k \geqslant 0$. For $i=0$ and $k \geqslant 0$ and for $i=\dim(\Omega)$ and $k \leqslant 0$, $H^i(\Omega, O_\Omega(k))$ is a free R-module whose rank is given by the polynomial $\omega(k)$ above.*

References

[1]. ALTMAN, A., & KLEIMAN, S. L., *Introduction to Grothendieck duality theory*. Lecture notes in mathematics no. 146, Springer Verlag, 1970.

[2]. BOURBAKI, N., *Algèbre, Chap. 3. Algèbre Multilinéaire*. Act. Sci. Ind. 1044, Hermann, 1948.

[3]. EAGON, J. A., & HOCHSTER, M., A class of perfect determinental ideals. *Bull. Amer. Math. Soc.*, 76 (1970), 1026–1029.

[4]. GROTHENDIECK, A., (with J. Dieudonné), Eleménts de géométrie algébrique. Chap. III. *Publ. Math. I.H.E.S.*, 17 (1963) and Chap. IV, *ibid.*, 24 (1965).

[5]. HOCHSTER, M., Cohen–Macaulay rings of invariants, rings generated by monomials, and polytypes. Notices Amer. Math. Soc., 18 (1971), 509.

[6]. HODGE, W. V. D. & PEDOE, D., *Methods of algebraic geometry*, vol. I and II. Cambridge Univ. Press, 1968.

[7]. IGUSA, J.-I., On the arithmetic normality of the Grassmann variety. *Proc. Nat. Acad. Sci. U.S.A.*, 40 (1954), 309–313.

[8], KLEIMAN, S. L., Geometry of Grassmannians and applications to splitting bundles and smoothing cycles. *Publ. Math. I.H.E.S.*, 36 (1969), 281–298.

[9]. KLEIMAN, S. L., & LANDOLFI, J., Geometry and deformations of special Schubert varieties. (To appear in *Compositio Math.*)

Received July 1, 1971.