

# REPRESENTATION OF TURING REDUCIBILITY BY WORD AND CONJUGACY PROBLEMS IN FINITELY PRESENTED GROUPS

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*Introduction.* We consider the relationship between the word and conjugacy problems in finitely presented groups. In any finitely presented group  $G$  a word is equal to the identity if and only if it is conjugate to the identity. This means that if there exists an algorithm to solve the conjugacy problem for  $G$ , then there exists an algorithm to solve the word problem for  $G$ . More generally, using the language of Turing degrees of unsolvability, we can assert that if the word problem for  $G$  is of degree  $\mathbf{a}$  and the conjugacy problem for  $G$  is of degree  $\mathbf{b}$  then  $\mathbf{a}$  and  $\mathbf{b}$  are recursively enumerable degrees (r.e. degrees) and  $\mathbf{a} \leq \mathbf{b}$ . Our aim is to determine for which pairs  $\mathbf{a}$  and  $\mathbf{b}$  of r.e. degrees with  $\mathbf{a} \leq \mathbf{b}$ , there exists a finitely presented group whose word problem is of degree  $\mathbf{a}$  and whose conjugacy problem is of degree  $\mathbf{b}$ .

Now it is known that for every r.e. degree  $\mathbf{a}$  there is a finitely presented group whose word problem is of degree  $\mathbf{a}$  (Boone [4], Clapham [6] and Fridman [8, 9]). Also, for every r.e. degree  $\mathbf{b}$  there exists a finitely presented group whose conjugacy problem is of degree  $\mathbf{b}$  (Bokut' [3], the author [7] and Miller [10]). Now it happens that the groups constructed by Bokut', the author and Miller all have solvable word problem. It is therefore a priori possible (but unlikely) that if a finitely presented group has unsolvable word problem, then its conjugacy problem is of the highest possible r.e. degree. A more natural conjecture is that any two r.e. degrees  $\mathbf{a}$  and  $\mathbf{b}$  with  $\mathbf{a} \leq \mathbf{b}$  are the degrees of the word and conjugacy problems for a suitable finitely presented group. Our main result settles the issue.

**THEOREM.** *Let  $\mathbf{a}$  and  $\mathbf{b}$  be recursively enumerable Turing degrees of unsolvability such that  $\mathbf{a} \leq \mathbf{b}$ . Then there is a finitely presented group whose word problem is of degree  $\mathbf{a}$  and whose conjugacy problem is of degree  $\mathbf{b}$ .*

One point should be noted. We know from [3], [7] and [10] that for any r.e. degree  $\mathbf{b}$  there exists a finitely presented group  $G_1(\mathbf{b})$  whose word problem is solvable and whose conjugacy problem is of degree  $\mathbf{b}$ . In order to prove the theorem, it would suffice to construct,

for every r.e. degree  $\mathbf{a}$  a finitely presented group  $G_2(\mathbf{a})$  whose word and conjugacy problems were both of degree  $\mathbf{a}$ . For then the free product (or direct product) of  $G_1(\mathbf{b})$  and  $G_2(\mathbf{a})$  would have word problem of degree  $\mathbf{a}$  and conjugacy problem of degree  $\mathbf{b}$ , provided  $\mathbf{a} \leq \mathbf{b}$ . As it turns out our methods are such that this approach does not yield any essential simplification.

We begin by restating our theorem at somewhat greater length and in a form such as to justify the title of the paper.

**REPRESENTATION THEOREM.** *Let  $S_1$  and  $S_2$  be recursively enumerable sets of natural numbers. Then  $S_1$  is Turing reducible to  $S_2$  if and only if there exists a finitely presented group  $G(S_1, S_2)$  such that*

- (i) *the word problem for  $G(S_1, S_2)$  is of the same Turing degree as  $S_1$ ;*
- (ii) *the conjugacy problem for  $G(S_1, S_2)$  is of the same Turing degree as  $S_2$ .*

Our opening remarks show that if  $G(S_1, S_2)$  satisfying (i) and (ii) exists then trivially  $S_1$  is Turing reducible to  $S_2$ . It is the converse which is our main theorem.

The starting point for our argument is a theorem of Shepherdson [11] concerning Thue systems. For any Thue system  $\mathfrak{X}$  and any word  $\Phi_0$  of  $\mathfrak{X}$ , the *individual word problem* for  $\Phi_0$  in  $\mathfrak{X}$  is the problem of determining of an arbitrary word  $\Phi$  of  $\mathfrak{X}$  whether or not  $\Phi = \Phi_0$  in  $\mathfrak{X}$ . With this terminology we can state (a special case of)

**SHEPHERDSON'S THEOREM.** *Let  $S_1$  and  $S_2$  be r.e. sets of natural numbers. If  $S_1$  is Turing reducible to  $S_2$ , then there exists a Thue system  $\mathfrak{X}(S_1, S_2)$  and a word  $\Phi_0$  of  $\mathfrak{X}(S_1, S_2)$  such that*

- (i) *the individual word problem for  $\Phi_0$  in  $\mathfrak{X}(S_1, S_2)$  is of the same Turing degree as  $S_1$ ;*
- (ii) *the word problem for  $\mathfrak{X}(S_1, S_2)$  is of the same Turing degree as  $S_2$ .*

On the basis of this, our main objective is to prove

**THEOREM A.** *Let  $S_1$  and  $S_2$  be r.e. sets of natural numbers such that  $S_1$  is Turing reducible to  $S_2$ . Let  $\mathfrak{X} = \mathfrak{X}(S_1, S_2)$  and  $\Phi_0$  be as specified by Shepherdson's Theorem. Then there exists a finitely presented group  $G(\mathfrak{X}, \Phi_0)$  such that*

- (i) *the word problem for  $G(\mathfrak{X}, \Phi_0)$  is Turing equivalent to the individual word problem for  $\Phi_0$  in  $\mathfrak{X}$ ;*
- (ii) *the conjugacy problem for  $G(\mathfrak{X}, \Phi_0)$  is Turing equivalent to the word problem for  $\mathfrak{X}$ .*

The Representation Theorem is an immediate consequence of Shepherdson's Theorem and Theorem A.

**§ 0.** We shall assume throughout that the reader is familiar with [4] and [7] (and for convenience we shall refer to these as WPD and CPD respectively). In particular we shall often use, without reference, results proved in WPD and CPD. Also we shall maintain as far as possible the notation and terminology of CPD (which itself is based on that of WPD). Occasionally, however, we shall be obliged to change the meaning of a particular piece of notation—such alterations will always be specifically noted and in general the intended meaning should be clear from the context.

To begin with we make two additions; we write  $U \sim_E V$  as an abbreviation for  $U$  is conjugate to  $V$  in  $E$ . Also if  $E$  and  $E^*$  are as in § 2 of WPD (or Part I of CPD), we say that a word  $U$  of  $E^*$  is *p-infected* if  $U$  is *p-contracted* and  $U$  is not *p-free*.

To avoid continual reference to them, we assume henceforth that  $S_1$  and  $S_2$ , of degrees  $\mathbf{a}$  and  $\mathbf{b}$ ,  $\mathbf{a} \leq \mathbf{b}$ , and the corresponding  $\mathfrak{X}$  and  $\Phi_0$  defined by Shepherdson's Theorem are fixed.

In WPD, Boone gives a construction which when applied to the pair  $(\mathfrak{X}, \Phi_0)$  yields a finitely presented group  $G(\mathfrak{X}, \Phi_0)$  whose word problem is Turing equivalent to the individual word problem for  $\Phi_0$  in  $\mathfrak{X}$ . We use a variant of this construction which is as follows.

Let  $\mathfrak{X} = (s_1, s_2, \dots, s_M; P_i = Q_i, i = 1, 2, \dots, N)$  and let  $\mathfrak{X}_* = (s_1, s_2, \dots, s_M, q; P_i q = q Q_i, s_b q = q s_b, i = 1, 2, \dots, N, b = 1, 2, \dots, M)$ . As in CPD we call a word of  $\mathfrak{X}_*$  *special* if it contains exactly one  $q$ -symbol, and use  $\Delta q \Pi$  and  $\Omega q \Phi$  as variables for special words. Then as in CPD we have

LEMMA 0.1. (i) *The individual word problem for  $\Phi_0$  in  $\mathfrak{X}$  is Turing equivalent to the problem  $(? \Omega q \Phi) \Omega q \Phi =_{\mathfrak{X}_*} q \Phi_0$ .*

(ii) *The word problem for  $\mathfrak{X}$  is Turing equivalent to the problem  $(? \Delta q \Pi, \Omega q \Phi) \Delta q \Pi =_{\mathfrak{X}_*} \Omega q \Phi$ .*

*Proof.* An easy inductive argument shows that  $\Delta q \Pi =_{\mathfrak{X}_*} \Omega q \Phi$  if and only if  $\Delta \Pi =_{\mathfrak{X}} \Omega \Phi$ .

It is convenient to write the relations of  $\mathfrak{X}_*$  as  $F_i q = q K_i, i = 1, 2, \dots, P = M + N$ . Then  $G(\mathfrak{X}, \Phi_0)$  is to be the following presentation:

$$s_b, q, x, r_i, t, h, k;$$

$$x s_b = s_b x x \quad r_i s_b = s_b x r_i x$$

$$r_i \bar{F}_i q = q K_i r_i$$

$$tr_i = r_i t \quad tx = xt$$

$$hr_i = r_i h \quad hx = xh$$

$$kr_i = r_i k \quad kx = xk$$

$$k(\Phi_0^{-1}q^{-1}t^{-1}q\Phi_0h\Phi_0^{-1}q^{-1}tq\Phi_0) = (\Phi_0^{-1}q^{-1}t^{-1}q\Phi_0h\Phi_0^{-1}q^{-1}tq\Phi_0)k$$

where  $b=1, 2, \dots, M$  and  $i=1, 2, \dots, P$  and  $\bar{F}_i$  is the word obtained from  $F_i$  by replacing  $s_b$  by  $s_b^{-1}$ .

At this point we say a few words concerning the relationship between  $G(\mathfrak{X}, \Phi_0)$  above and the groups studied by Boone in WPD and by Britton in [5]. The fundamental difference is that in the final relation of  $G(\mathfrak{X}, \Phi_0)$  each side contains four  $q$ -symbols instead of two. This change does not affect the word problem in any significant way. However, the change does introduce an additional type of symmetry which makes the analysis of conjugacy in  $G(\mathfrak{X}, \Phi_0)$  above a good deal simpler than in the Boone-Britton group. The reader will find a more extended discussion of this point at the end of the paper.

For brevity we write  $G=G(\mathfrak{X}, \Phi_0)$ . We shall consider various other presentations obtained from  $G$  by deleting certain generators and relations. These are:

$G_0$  = delete  $h$  as generator and all the relations involving  $h$ ;

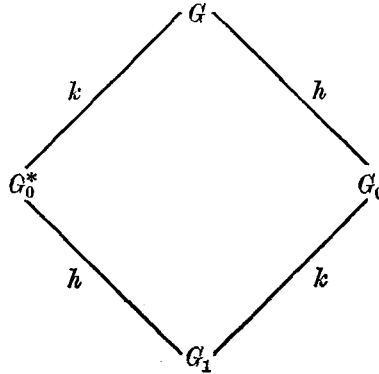
$G_0^*$  = delete  $k$  as generator and all the relations involving  $k$ ;

$G_{0h}$  = retain all the generators but delete the final relation

$$k(\Phi_0^{-1}q^{-1}t^{-1}q\Phi_0h\Phi_0^{-1}q^{-1}tq\Phi_0) = (\Phi_0^{-1}q^{-1}t^{-1}q\Phi_0h\Phi_0^{-1}q^{-1}tq\Phi_0)k;$$

$G_1$  = delete both  $k$  and  $h$  as generators and all relations involving  $k$  and  $h$ .

Some of the relationships among these presentations can be revealed by the following diagram.



The label on a descending line indicates the generator omitted in passing to the subpresentation.

It is important to compare the presentations  $G_0$ ,  $G_0^*$  and  $G_{0h}$  of this paper with the presentation  $G_0$  of CPD. The  $G_0$  of the present paper and the  $G_0$  of CPD *do* coincide while  $G_0^*$  is clearly isomorphic to  $G_0$  (relabel  $h$  as  $k$ ). On the other hand  $G_{0h}$  is not isomorphic to  $G_0$ . However  $G_{0h}$  is very similar to  $G_0$  and we shall simply assume that every theorem of CPD concerning  $G_0$  remains true when  $G_0$  is replaced by  $G_{0h}$ . In particular we assume that the conjugacy problem for  $G_{0h}$  is Turing equivalent to the problem

$$(?\Delta q\Pi, \Omega q\Phi)\Delta q\Pi =_{\mathfrak{X}_*} \Omega q\Phi$$

Where necessary this will mean that we sometimes consider presentations which differ from those in CPD by the addition of the generator  $h$ . We shall usually denote such presentations by adding a subscript  $h$  to the notation used in CPD. For example we write  $H_h$  for the presentation obtained from  $H$  by adding the generator  $h$  and the relations  $hr_i=r_ih$ ,  $hx=xh$ . Alternatively  $H_h$  can be viewed as coming from  $G_{0h}$  by the deletion of the generator  $q$  and the relations involving  $q$ . Then we assume (Reductions IX–XIII of CPD) that the conjugacy problem for  $H_h$  is solvable.

We begin with the word problem.

**THEOREM A (i).** *The word problem for  $G$  is Turing equivalent to the individual word problem for  $\Phi_0$  in  $\mathfrak{X}$ .*

*Proof.* By Lemma 0.1 it suffices to show that the word problem for  $G$  is equivalent to the problem  $(?\Omega q\Phi)\Omega q\Phi =_{\mathfrak{X}_*} q\Phi_0$ .

A proof of this can be obtained by repeating, with certain alterations, either the argument given by Boone in WPD or else the argument of L. A. Bokut' in [1].

The principal alteration is that necessitated by the introduction of the generator  $h$  and the change in the final relation whereby each side of the relation contains four  $q$ -symbols instead of two.

Although it is not an entirely routine matter, this alteration should not cause any particular difficulty to the reader familiar with the arguments of Boone and Bokut'.

The remaining changes, which are caused by the elimination of the  $y$ - and  $l_i$ -symbols, are entirely routine (compare for example Lemmas 6, 7 and 8 of [5] and Lemmas 14, 15 and 16 of CPD) and are therefore also left to the reader.

The remainder of our work is devoted to proving Theorem A (ii). One half of the equivalence is easy.

**LEMMA 0.2.** *The word problem for  $\mathfrak{X}$  is Turing reducible to the conjugacy problem for  $G$ .*

*Proof.* It suffices to show that  $(? \Delta q \Pi, \Omega q \Phi) \Delta q \Pi =_{\mathfrak{x}_*} \Omega q \Phi$  is reducible to the conjugacy problem for  $G$ . Let  $\Delta q \Pi$  and  $\Omega q \Phi$  be given and let  $U$  be  $t^{-1} \Pi^{-1} q^{-1} \bar{\Delta}^{-1} t \bar{\Delta} q \Pi$  and  $V$  be  $t^{-1} \Phi^{-1} q^{-1} \bar{\Omega}^{-1} t \bar{\Omega} q \Phi$ . We show that  $\Delta q \Pi =_{\mathfrak{x}_*} \Omega q \Phi$  if and only if  $U \sim_G V$ .

We begin by observing that  $\text{Cond}_{\text{JLB}}(G, G_0^*, k)$  holds.

If  $\Delta q \Pi =_{\mathfrak{x}_*} \Omega q \Phi$ , then by Lemma 14 of CPD (replacing  $G_0$  by  $G_0^*$ ),  $U \sim_{G_0^*} V$  and hence  $U \sim_G V$ . Conversely suppose that  $W^{-1} U W =_G V$ . If  $W$  is  $k$ -free, then the equality in fact holds in  $G_0^*$  and by Lemma 15 and 16 of CPD  $\Delta q \Pi =_{\mathfrak{x}_*} \Omega q \Phi$ . If  $W$  is not  $k$ -free, then a simple inductive argument on the number of  $k$ -symbols in  $W$  shows that there must exist a  $k$ -free word  $X$  such that  $X^{-1} U X =_{G_0^*} V$ . Again we apply Lemmas 15 and 16.

**§ 1.** In addition to  $\text{Cond}_{\text{JLB}}(G, G_0^*, k)$ , we also have  $\text{Cond}_{\text{JLB}}(G, G_0, h)$  holding, as the reader may check by rewriting the final relation. We shall use these two facts repeatedly throughout the remainder of our argument.

LEMMA 1.1. *Let  $U$  and  $V$  be words of  $G$ . Then (i)  $U \sim_G V$  if and only if  $h\{k\{U\}\} \sim_G h\{k\{V\}\}$*

(ii)  *$h\{k\{U\}\}$  is  $k$ -contracted.*

*Proof.* (i) This is an application of General Lemma 2 of CPD.

(ii) It suffices to show that if  $U$  is  $k$ -contracted, then any primitive  $h$ -contraction  $\bar{h}\{U\}$  is also  $k$ -contracted. This is proved by showing that if  $\bar{h}\{U\}$  is not  $k$ -contracted then  $U$  is not  $k$ -contracted.<sup>(1)</sup>

LEMMA 1.2. *Let  $U$  and  $V$  be words of  $G$ .*

(i) *If  $U$  and  $V$  are  $k$ -free then  $U \sim_G V$  if and only if  $U \sim_{G_0^*} V$ .*

(ii) *If  $U$  and  $V$  are  $h$ -free, then  $U \sim_G V$  if and only if  $U \sim_{G_0} V$ .*

*Proof.* These are both established by the argument of Lemma 0.2.

We call a word  $U$  of  $G$  *weakly basic* if  $U$  is  $k$ -infected and  $h$ -infected. If, in addition,  $U$  has final symbol  $k^{\pm 1}$ , we call  $U$  *basic*.

REDUCTION I. *The conjugacy problem for  $G$  is reducible to ( $?U, V$  weakly basic)  $(\exists W) W^{-1} U W =_G V$ , the conjugacy problems for  $G_0$  and  $G_0^*$  and  $(? \Omega q \Phi) \Omega q \Phi =_{\mathfrak{x}_*} q \Phi_0$ .*

*Proof.* In the process of proving Theorem A (i), the reader will have shown that the problem of computing  $k\{U\}$ , for arbitrary  $U$ , is reducible to the problem  $(? \Omega q \Phi) \Omega q \Phi =_{\mathfrak{x}_*} q \Phi_0$ .

<sup>(1)</sup> Details of a similar argument can be found in the proof of Theorem 2 of CPD. The vigilant reader will observe that in Theorem 2 of CPD, there is one possibility for an  $r$ -contraction which has not been rejected. This is the case when  $\bar{q}\{U\} \equiv U_1 (K_i r_i)^e B_0 (K_i r_i)^{-e} U_2$ , with  $U_2$  and  $U_1$   $r$ -free and the contraction occurs between  $r_i^{-e}$  and  $r_i^e$ . There is, however, no difficulty in rejecting this possibility.

It is clear that the same applies to  $h\{U\}$  (it is perhaps worth noting that here we mean  $h\{U\}$  defined with respect to  $(G, G_0, h)$ ). The reduction is now immediate from Lemma 1.2.

The main theorem of CPD asserts that the conjugacy problem for  $G_0$  is Turing equivalent to that of membership in  $S_2$ . It follows that the only problem we need consider further is  $(?U, V \text{ weakly basic})(\exists W)W^{-1}UW =_G V$ .

**§ 2.** We now begin our analysis of the problem  $(?U, V \text{ weakly basic})(\exists W)W^{-1}UW =_G V$ . In this section we try to determine necessary conditions for two weakly basic words to be conjugate. Then in § 3 we shall show how the conditions we derive enable us to reduce the above problem to the conjugacy problem in  $G_{0h}$ .

To avoid cumbersome formulae, we write  $\Sigma_0$  for  $q\Phi_0$  and  $\Gamma$  for  $\Sigma_0^{-1}t\Sigma_0$ . Also we shall not, in this section, usually specify in which group an equality holds since this will always be clear from the context.

So let  $U$  and  $V$  be given weakly basic words. If they are conjugate in  $G$ , then, by General Lemma 3 of CPD, there exist cyclic permutations  $U_0$  and  $V_0$  of  $U$  and  $V$  respectively and a word  $D$  ( $D$  is always a variable for products of  $r_i, x$  and  $\Gamma^{-1}h\Gamma$ ) such that

- (i)  $U_0$  and  $V_0$  are basic
- (ii)  $D^{-1}U_0D =_G V_0$

This suggests that we concentrate on basic words  $U$  and  $V$  and try to deduce what will result if they are conjugate by a  $D$ . We shall show that if  $U$  and  $V$  are basic and there exists  $D$  such that  $D^{-1}UD = V$ , then there exists words  $U$  and  $V$ , which are essentially cyclic permutations of certain “normalised versions” of  $U$  and  $V$ , and a word  $R$  such that  $R^{-1}UR =_G V$ . In § 3 we shall show that, after some further normalisation, we may assume that this last equality holds in  $G_{0h}$ .

A further convention we shall adopt is that any word  $D$  which appears will be assumed to be  $\delta$ -reduced, i.e. not to contain a subword  $\Gamma R\Gamma^{-1}$  such that  $R\Gamma = \Gamma R$ . A simple calculation shows that for any  $D$  there is a  $\delta$ -reduced  $D'$ , recursively computable from  $D$ , such that  $D = D'$ . Thus there is no loss of generality in this assumption. An almost equally simple calculation shows that if  $D$  is  $\delta$ -reduced then  $D$  is  $h$ -reduced. Despite its simplicity this calculation is of fundamental importance and we therefore single out its crucial feature to be stated as a separate lemma.

**LEMMA 2.1.** *Let  $t^{-\varepsilon}\Sigma_0 R \Sigma_0^{-1}t^\varepsilon$  be given,  $\varepsilon = \pm 1$ . If this is not  $t$ -reduced, then  $R$  commutes with  $\Gamma$ .*

*Proof.* If this word is not  $t$ -reduced then there exists  $R'$  such that  $\Sigma_0 R \Sigma_0^{-1} = R'$ . Then

$$\Gamma R = \Sigma_0^{-1} t R' \Sigma_0 = \Sigma_0^{-1} R' t \Sigma_0 = R \Gamma.$$

Now let the basic words  $U$  and  $V$  be given. We shall define a series of "normalisation procedures" to be applied to  $U$  and  $V$ . These procedures will yield words of a very restricted type which will be amenable to analysis. Our first procedure is to be  $\gamma$ -reduction defined as follows. If  $U \equiv U_1 k^r Z k^s U_2$  where  $r, s \neq 0$  and  $(\exists D) Z = D$  then we call  $U_1 Z k^{r+s} U_2$  a *primitive  $\gamma$ -reduction* of  $U$ . (Since  $U$  is  $k$ -reduced,  $r$  and  $s$  must have the same sign.) We adopt the usual definitions derived from a primitive reduction.<sup>(1)</sup> Clearly there is an  $S_1$ -recursive procedure to compute, for any basic word  $U$ , the  $\gamma$ -reduced word  $\gamma[U]$ . Moreover  $U = \gamma[U]$  and  $\gamma[U]$  is basic.

Let  $U \equiv X_0 k^{p_0} X_1 k^{p_1} \dots X_n k^{p_n}$  where  $p_j \neq 0$ ,  $j = 0, 1, 2, \dots, n$  and  $X_j$  is non-empty and  $k$ -free,  $j = 1, 2, \dots, n$ , be a basic word.<sup>(2)</sup> We call the  $(n+1)$ -tuple  $(p_0, p_1, \dots, p_n)$  the  *$k$ -structure* of  $U$ .

**LEMMA 2.2.** *Let  $U$  and  $V$  be basic and  $\gamma$ -reduced. If  $D^{-1}UD = V$ , then  $U$  and  $V$  have the same  $k$ -structure.*

*Proof.* Let  $U \equiv X_0 k^{p_0} X_1 \dots X_n k^{p_n}$ . Then  $(\exists D) X_j = D$  fails to hold for  $j = 1, 2, \dots, n$ . Using this fact, we establish the lemma by induction on  $l_k(U)$ . (The induction hypothesis is that if  $D_1 U D_2 = V$  then  $U$  and  $V$  have the same  $k$ -structure.)

Let  $U$  be basic and  $\gamma$ -reduced. The second normalisation procedure is designed to give us, whenever possible, a word which is *cyclically  $\gamma$ -reduced*, i.e., every cyclic permutation of the word is  $\gamma$ -reduced.

Let  $U \equiv X_0 k^{p_0} \dots X_n k^{p_n}$ . For the moment, assume  $n \geq 1$ . If  $(\exists D) X_0 = D$  then we define  $\gamma\{U\} \equiv X_0 X_1 k^{p_1} \dots X_n k^{p_n + p_0}$ . If  $X_0 \neq D$ , for any  $D$ , we let  $\gamma\{U\} \equiv U$ . In both cases  $\gamma\{U\}$  is cyclically  $\gamma$ -reduced. If  $n = 0$  and  $X_0 \neq D$ , for any  $D$ , we define  $\gamma\{U\} \equiv U$ . Here too  $\gamma\{U\}$  is cyclically  $\gamma$ -reduced.

The remaining possibility is that  $n = 0$  and  $(\exists D) X_0 = D$ . We do not define  $\gamma\{U\}$  for this case.

**LEMMA 2.3.** *Let  $U$  and  $V$  be basic,  $\gamma$ -reduced, and have the same  $k$ -structure. If  $D^{-1}UD = V$ , then  $\gamma\{V\}$  is defined if and only if  $\gamma\{U\}$  is defined and  $D^{-1}\gamma\{U\}D = \gamma\{V\}$ .*

*Conversely if  $\gamma\{U\}$  and  $\gamma\{V\}$  are defined and  $D^{-1}\gamma\{U\}D = \gamma\{V\}$ , then  $D^{-1}UD = V$ .*

*Proof.* Let  $U = X_0 k^{p_0} \dots X_n k^{p_n}$  and  $V = Y_0 k^{p_0} \dots Y_n k^{p_n}$ . Then  $(\exists D) X_0 = D$  if and only if  $(\exists D) Y_0 = D$ . The lemma follows easily from this.

<sup>(1)</sup> See CPD or WPD.

<sup>(2)</sup> Possibly  $X_0$  is empty—at any rate it must be  $k$ -free.



Lemmas 2.1–2.3 show that if  $U \equiv X_0 k^{p_0} \dots X_n k^{p_n}$  and  $V \equiv Y_0 k^{p_0} \dots Y_n k^{p_n}$  then we need consider only the following possibilities.

- Case I.  $U$  and  $V$  cyclically  $\gamma$ -reduced with  $n \geq 1$ .
- Case II.  $U$  and  $V$  cyclically  $\gamma$ -reduced with  $n = 0$ .
- Case III.  $U = D_1 k^{p_0}$  and  $V = D_2 k^{p_0}$  for some  $D_1$  and  $D_2$ .

We fix the above notation for  $U$  and  $V$ .

The most complex case is case I and we deal with this first. Our next normalisation procedure involves the notions of  $\varrho$ -reduction and  $\lambda$ -reduction which we define as follows. Let  $X$  be a  $k$ -free word. If  $X \equiv X' h^e P$  where  $P$  is  $h$ -free and  $P = R_1 \Gamma R_2$  for some  $R_1, R_2$ , then  $X' R_1 \Gamma$  is a *primitive  $\varrho$ -reduction* of  $X$ . If we observe that  $X = X' R_1 \Gamma \Gamma^{-1} h^e \Gamma R_2$  then it is clear that there is an **a**-recursive procedure to compute, for any  $X$ , the  $\varrho$ -reduced word  $\varrho[X]$  and, moreover, there is a  $D$ , also **a**-recursively computable, such that  $X = \varrho[X] D$ .

The definition of  $\lambda$ -reduction is completely dual and we can **a**-recursively compute  $\lambda[X]$  and  $D$  such that  $X = D \lambda[X]$ .

Let  $U$  be given, written as above. We firstly find  $\varrho[X_0]$  and  $D$  such that  $X_0 = \varrho[X_0] D$ . We can commute  $D$  past  $k^{p_0}$  and then in turn find  $\varrho[DX_1]$  and  $D'$  such that  $DX_1 = \varrho[DX_1] D'$ . Iterating this procedure we eventually obtain

$$U' \equiv X'_0 k^{p_0} X'_1 k^{p_1} \dots X'_n k^{p_n} D_1$$

and  $X'_j$  is  $\varrho$ -reduced,  $j = 0, 1, \dots, n$ . For our next step we  $\lambda$ -reduce  $X'_n$  and compute  $D'_1$  and  $X''_n (\equiv \lambda[X'_n])$  such that  $X'_n = D'_1 X''_n$ .

At this point we consider two subcases. If  $X''_n = R_1^* \Gamma^{-1} R^* \Gamma R_2^*$ , for some  $R_1^*, R^*$  and  $R_2^*$ , and  $R^*$  does not commute with  $\Gamma$ , then we define  $D_3 \equiv R_2^* D_1$ ,  $X_n^* \equiv \Gamma^{-1} R^* \Gamma$  and

$$U^* \equiv D_3 X'_0 k^{p_0} X'_1 k^{p_1} \dots X'_{n-1} D'_1 R_1^* k^{p_{n-1}} X_n^* k^{p_n}.$$

In this situation we shall say that  $U^*$  has been obtained *smoothly* from  $U$ .

If, on the other hand, no such  $R_1^*, R^*$  and  $R_2^*$  exist then we define  $X_n^* = X''_n$  and

$$U^* \equiv D_1 X'_0 k^{p_0} \dots X'_{n-1} D'_1 k^{p_{n-1}} X_n^* k^{p_n}.$$

We define  $V^*$  in a completely identical manner.

LEMMA 2.4. *Let  $U$  be basic and circularly  $\gamma$ -reduced. Then*

- (i)  $U^*$  is  $S_1$ -computable from  $U$ ;
- (ii)  $U^*$  is basic and circularly  $\gamma$ -reduced;
- (iii)  $(\exists D) D^{-1} U D = U^*$ .

*Proof.* (i) The only novel point here is the question of whether or not  $X'_n = R_1^* \Gamma^{-1} R^* \Gamma R_2^*$  for some  $R_1^*$ ,  $R^*$  and  $R_2^*$ . However this can be decided by an application of Theorem II of WPD. (ii), (iii). These are routine.

We fix more notation by insisting that until further notice  $U^*$  and  $V^*$  have the meaning given to them above.

**THEOREM 2.5.** *Let  $D^{-1}U^*D = V^*$  where  $D \neq R$  for every  $R$ . Then*

- (i)  $U^*$  is obtained smoothly from  $U$
- (ii)  $D = \Gamma^{-1}h^f\Gamma R$  for some  $f \neq 0$  and  $R$ .

*Proof.* (i) Let  $D \equiv R_1^{-1}\Gamma^{-1}h^e\Gamma D''$ ; using  $k$  as stable letter we obtain  $X_n^*D = D'Y_n^*$ . We claim this means that  $X_n^*$  and  $Y_n^*$  must be  $h$ -free. From the way in which  $U^*$  and  $V^*$  are defined, certainly  $X_n^*$  and  $Y_n^*$  are  $\lambda$ -reduced and  $\rho$ -reduced. This means that  $X_n^*D$ ,  $DY_n^{*-1}$ ,  $Y_n^{*-1}D'^{-1}$  and  $D'^{-1}X_n^*$  are all  $h$ -reduced. On the other hand,  $X_n^*DY_n^{*-1}D'^{-1} = 1 = D'^{-1}X_n^*DY_n^{*-1}$  and  $D$  is not  $h$ -free. This is contradictory unless both  $X_n^*$  and  $Y_n^*$  are  $h$ -free.

The equation  $X_n^*D = D'Y_n^*$  then gives  $X_n^*R_1\Gamma^{-1} = R_1'\Gamma^{-1}R_1^+$  and hence  $X_n^* = R_1'\Gamma^{-1}R_1^+\Gamma R_1^{-1}$ . Our proof is complete if we can show that  $R_1^+$  does not commute with  $\Gamma$ . However this is easily done since if  $R_1^+$  commutes with  $\Gamma$  then  $X_n^* = R_1'R_1^+R_1^{-1}$  which contradicts the fact that  $U$  is  $\gamma$ -reduced.

(ii) By (i) we know that  $X_n^* \equiv \Gamma^{-1}R^*\Gamma$  where  $R^*$  does not commute with  $\Gamma$ . Repeating the argument of (i) we obtain

$$\Gamma^{-1}R^*\Gamma R_1\Gamma^{-1} = R_1'\Gamma^{-1}R_1^+$$

By Lemma 2.1,  $R_1$  commutes with  $\Gamma$  and  $D = \Gamma^{-1}h^e\Gamma R_1D'$ .

It is obvious that  $X_n^*$  and  $\Gamma^{-1}h^e\Gamma$  commute and if we define

$$U_1 \equiv \Gamma^{-1}h^{-e}\Gamma D_3 X_0' k^{p_0} \dots X_{n-1}' D_1' R_1^* \Gamma^{-1} h^e \Gamma k^{p_{n-1}} X_n^* k^{p_n}.$$

then (a)  $\Gamma^{-1}h^{-e}\Gamma U^* \Gamma^{-1}h^e\Gamma = U_1$

and (b)  $U_1$  yields after a single primitive  $h$ -reduction a word  $U_1^*$  which is basic and circularly  $\gamma$ -reduced. (For (b) recall that  $V^*$  is either a right conjugate or a left conjugate of  $U^*$ , with respect to  $h$ , by  $D$ —see General Lemma 4 of CPD.)

The theorem now follows by iterating the above argument. (Formally we use induction to deduce that  $R_1 D'' = \Gamma^{-1}h^{f-1}\Gamma R$ , where  $(f-1)$  must in fact have the same sign as  $e$ .)

Theorem 2.5 is of great significance since it shows that if  $D$  conjugates  $U^*$  into  $V^*$  then we may assume that  $D$  has a particularly simple form.

We can make one further simplification. According to General Lemma 4 of CPD it is

enough to examine the situation in which  $V^*$  is a right conjugate of  $U^*$ , with respect to  $h$ , by a  $D$ .

We call  $U^*$  *active* if  $U^*$  has occurrences of  $h$ -symbols only within  $D_3$ .

LEMMA 2.6. *If  $U^*$  is not active and  $V^*$  is a right conjugate of  $U^*$  by  $D$ , then  $D$  is an  $R$ .*

*Proof.* Our procedure for obtaining  $U'$  from  $U$  was, roughly, to move out to the right as many  $h$ -symbols as possible. It is easy to see that if  $U^*$  is not active then the presence of  $h$ -symbols in  $D$  is inconsistent with our having performed the above procedure. More formally, a straightforward argument will show that the presence of  $h$ -symbols in  $D$  will mean that some  $X'_j$  is not  $\rho$ -reduced. This is contradictory.

It will be convenient to make a slight extension of the range of the variable  $D$ . At present a  $D$  is a formal product of  $r_i$ ,  $x$  and  $\Gamma^{-1}h\Gamma$ . There is clearly no loss of generality in extending this to include products of  $r_i$ ,  $x$  and  $\Gamma^{-1}h^m\Gamma$  for any  $m \neq 0$ .

LEMMA 2.7. *Let  $U^*$  be active and suppose  $V^*$  is a right conjugate of  $U^*$  by  $\Gamma^{-1}h^f\Gamma R$ ,  $f \geq 1$ .<sup>(1)</sup> Then either (a)  $D_3 X'_0 = R_3^* \Gamma^{-1} h^{-m} \Gamma X_0^*$ , or (b)  $D_3 X'_0 = D_3' \Gamma^{-1} h^{-k} \Gamma R_3^* \Gamma^{-1} h^{-r} \Gamma X_0^*$ ,  $r \geq 1$  and in the latter,  $R_3^*$  does not commute with  $\Gamma$ .*

*In either case  $X_0^*$  is of the form  $\Gamma^{-1} R_4 \Gamma R_4^*$  where  $R_4$  does not commute with  $\Gamma$ . Also if  $U_0 = X_0^* k^{p_0} X_1' \dots k^{p_{n-1}} X_n^* k^{p_n}$ , then  $U_0$  commutes with  $\Gamma^{-1} h \Gamma$ .*

*Proof.* Let  $D_3 \equiv D_3'' \Gamma^{-1} h^{-r} \Gamma R_3$  where  $D_3''$  is either  $h$ -free or terminates in  $h^{-1} \Gamma R_3^*$  where  $R_3^*$  does not commute with  $\Gamma$ . Using  $h$  as stable letter we deduce that  $\Gamma R_3 X_0' k^{p_0} \dots X_n^* k^{p_n} \Gamma^{-1}$  is equal to a product of  $r_i$ ,  $x$  and  $\Gamma k \Gamma^{-1}$ . Moreover it is easy to see that any such product is equal to a  $k$ -reduced word of the form  $R_4 \Gamma k^{q_0} \Gamma^{-1} R_5 \dots R_6 \Gamma k^{q_m} \Gamma^{-1} R_7$ . (Using the fact that  $X_1', \dots, X_{n-1}', X_n^*$  are  $\rho$ -reduced we can show, after a short argument, that we may even assume  $m = n$  and  $p_j = q_j$ ,  $j = 0, 1, \dots, n$ . However this is not strictly necessary.) Using  $k$  as stable letter we deduce from the above equalities, since  $\Gamma R_3 X_0' k^{p_0} \dots X_n^* k^{p_n} \Gamma^{-1}$  is certainly  $k$ -reduced, that  $\Gamma R_3 X_0' = R_4 \Gamma R_4^*$  and this gives  $X_0' = R_3^{-1} \Gamma^{-1} R_4 \Gamma R_4^*$ . Now  $R_4$  cannot commute with  $\Gamma$  because this would imply that  $U^*$  was not circularly  $\gamma$ -reduced. Thus  $D_3 X'_0 = D_3'' \Gamma^{-1} h^{-r} \Gamma X_0^*$ .

If  $D_3''$  is  $h$ -free then we obtain (a) writing  $R_3^* \equiv D_3''$  and  $m = r$ . If  $D_3''$  is not  $h$ -free then we obtain (b) and  $R_3^*$  does not commute with  $\Gamma$ .

Finally we look back at the equality between  $\Gamma R_3 X_0' k^{p_0} \dots X_n^* k^{p_n} \Gamma^{-1}$  and  $R_4 \Gamma k^{q_0} \Gamma^{-1} R_5 \dots R_6 \Gamma k^{q_m} \Gamma^{-1} R_7$ . From the definitions involved it is clear that

$$\Gamma U_0 \Gamma^{-1} = \Gamma R_3 X_0' k^{p_0} \dots X_n^* k^{p_n} \Gamma^{-1}$$

and hence  $\Gamma U_0 \Gamma^{-1}$  commutes with  $h$ . This means that  $U_0$  commutes with  $\Gamma^{-1} h \Gamma$ .

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<sup>(1)</sup> The situation when  $f \leq -1$  is exactly similar.

We write  $D_3^* \equiv R_3^* \Gamma^{-1} h^{-m} \Gamma$  or  $D_3^* \equiv D_3' \Gamma^{-1} h^{-s} \Gamma R_3^* \Gamma^{-1} h^{-k} \Gamma$  according as (a) or (b) holds. Then  $U^* = D_3^* U_0$ .

The following theorem essentially completes our analysis of case I.

**THEOREM 2.8 (a).** *Let  $U^* = R_3^* \Gamma^{-1} h^{-m} \Gamma U_0$ . Then  $V^*$  is a right conjugate of  $U^*$  by some  $\Gamma^{-1} h^f \Gamma R$ ,  $f \geq 1$ , if and only if*

- (i)  $U_0$  commutes with  $\Gamma^{-1} h \Gamma$
- (ii) there exists a cyclic permutation  $D_3^+$  of  $D_3^*$  of form  $\Gamma^{-1} h^{-s} \Gamma R_3^* \Gamma^{-1} h^{-t} \Gamma$ , where  $t + s = m$ , such that  $R^{-1} D_3^+ U_0 R = V^*$ .

- (b) Let  $U^* = D_3' \Gamma^{-1} h^{-k} \Gamma R_3^* \Gamma^{-1} h^{-r} \Gamma U_0$ .

Then  $V^*$  is a right conjugate of  $U^*$  by some  $\Gamma^{-1} h^f \Gamma R$  if and only if

- (i)  $U_0$  commutes with  $\Gamma^{-1} h \Gamma$ .
- (ii) there exists a cyclic permutation  $D_3^+$  of  $D_3^*$  of form  $\Gamma^{-1} h^{-s} \Gamma D_3' \Gamma^{-1} h^{-k} \Gamma R_3^* \Gamma^{-1} h^{-t} \Gamma$ , where  $s + t = r$ , such that  $R^{-1} D_3^+ U_0 R = V^*$ .

*Proof.* (a) If the given conditions (i) and (ii) hold then we can take  $f = s$ .

Conversely suppose  $V^*$  is a right conjugate by  $\Gamma^{-1} h^f \Gamma R$ . Then Lemma 2.7 gives us (i). If  $f \leq m$  we obtain (ii) by putting  $s = f$ .

If  $f > m$ , we obtain  $R^{-1} \Gamma^{-1} h^{-f} \Gamma R_3^* U_0 \Gamma^{-1} h^{f-m} \Gamma R$  after  $m$   $h$ -reductions. Then we must have  $\Gamma R_3^* U_0 \Gamma^{-1} = R_6 \Gamma k^{p_0} \dots k^{p_n} \Gamma^{-1} R_7$ . This gives  $\Gamma R_3^* X_0^* = R_6 \Gamma R_6^*$  and it follows that  $R_3^*$  commutes with  $\Gamma$ . From this we deduce that  $R^{-1} U^* R = V^*$ .

(b) If (i) and (ii) hold then we can just take  $f = s$ . For the converse, Lemma 2.7 gives us (i). To obtain (ii) it is enough to show that  $f \leq r$ . This follows from the fact that if  $f > r$ , then  $R_3^*$  must commute with  $\Gamma$  (by the same argument as in (a)).

We turn to case II. Since this is quite similar to case I we shall be rather sketchy as regards details.

We are given  $U \equiv X k^p$  and  $V \equiv Y k^p$  where neither  $X$  nor  $Y$  is a  $D$  (writing  $X$  for  $X_0$  etc.). To normalise we firstly obtain  $X' D_1 k^p$  where  $X'$  is  $\varrho$ -reduced and then  $D_1' X'' k^p D_1$  where  $X''$  is both  $\varrho$ -reduced and  $\lambda$ -reduced. If  $X'' = R_1^* \Gamma^{-1} R^* \Gamma R_2^* (1)$  then we define  $D_3 \equiv R_2^* D_1 D_1' R_1^*$ ,  $X^* \equiv \Gamma^{-1} R^* \Gamma$  and  $U^* \equiv D_3 X^* k^p$ . Otherwise we define  $X^* \equiv X''$ ,  $D_3 \equiv D_1 D_1'$  and  $U^* \equiv D_3 X^* k^p$ .

The definition of  $V^*$  is similar.

**LEMMA 2.9.**  *$U^*$  and  $V^*$  are  $\mathfrak{a}$ -recursively computable from  $U$  and  $V$  and are basic and circularly  $\gamma$ -reduced. Moreover  $(\exists D) D^{-1} U D = V$  if and only if  $(\exists D) D^{-1} U^* D = V^*$ .*

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(1)  $R^*$  will not commute with  $\Gamma$ .

*Proof.* This is routine.

We call  $U^*$  *active* if  $X^*$  is  $h$ -free.

LEMMA 2.10. *If  $U^*$  is not active and  $V^*$  is a right conjugate of  $U^*$  by  $D$ , then  $D$  is an  $R$ .*

*Proof.* If this were not so then  $X'$  would not be  $\rho$ -reduced.

THEOREM 2.11. *Let  $U^*$  be active and suppose  $V^*$  is a right conjugate of  $U^*$  by  $D$ . Then  $D = \Gamma^{-1}h^f\Gamma R$  for some  $f \neq 0$  and  $R$ .*

*Proof.* Using the fact that  $V^*$  is a right conjugate of  $U^*$  by  $D$  we can show that we are in the situation where  $X' = R_1^*\Gamma^{-1}R^*\Gamma R_2^*$  and thus  $X^* \equiv \Gamma^{-1}R^*\Gamma$ . We write  $D_3 \equiv D_3''\Gamma^{-1}h^{-r}\Gamma R_3$  and also  $D \equiv R_1\Gamma^{-1}h^f\Gamma D'$ . By the same argument as we have just used we deduce  $\Gamma R_3\Gamma^{-1}R^*\Gamma k^p R_1\Gamma^{-1} = R_4\Gamma k^p\Gamma^{-1}R_5$  and hence  $\Gamma R_3\Gamma^{-1}R^*\Gamma R_1\Gamma^{-1} = R_4 R_5$ .

Since  $R^*$  does not commute with  $\Gamma$  both  $R_1$  and  $R_3$  do commute with  $\Gamma$ . Now  $f_1$  and  $r$  have the same sign and since  $X^*$  commutes with  $\Gamma^{-1}h\Gamma$  the theorem follows by induction on  $l_h(D)$ .

THEOREM 2.12. *Let  $U^* \equiv D_3 X^* k^p$  and  $V^*$  be given, with  $U^*$  active. If  $V^*$  is a right conjugate of  $U^*$  by some  $\Gamma^{-1}h^f\Gamma R$ , then there exists a cyclic permutation  $D_3^+$  of  $D_3$  such that  $R^{-1}D_3^+ X^* k^p R = V^*$ .*

*Proof.* This can be shown by a repetition of the arguments of Lemma 2.7 and Theorem 2.8 (even somewhat simplified).

Finally we look at case III. Here we have  $Xk^p$  and  $Yk^p$  with  $X = D_1$  and  $Y = D_2$ . We write  $D_1$  in more detail as  $R_{10}\Gamma^{-1}h^{r_1}\Gamma R_{11} \dots \Gamma^{-1}h^{r_g}\Gamma R_{1g}$ . Since we may assume  $D_1$  is  $\delta$ -reduced,  $R_{11}, R_{12}, \dots, R_{1(g-1)}$  do not commute with  $\Gamma$ . To normalise we proceed as follows. If  $g=1$ , we define  $D_1^* \equiv R_{11} R_{10} \Gamma^{-1}h^r \Gamma$ . If, on the other hand,  $g > 1$ , then we define  $D_1^* \equiv R_{1g} R_{10} \Gamma^{-1}h^{r_1} \Gamma R_{11} \dots \Gamma^{-1}h^{r_g} \Gamma$  or  $D_1^* \equiv R_{1g} R_{10} R_{11} \Gamma^{-1}h^{r_1} \Gamma \dots \Gamma^{-1}h^{r_g+r_1} \Gamma$  according as  $R_{1g} R_{10}$  does not or does commute with  $\Gamma$ .

Let  $U^* \equiv D_1^* k^p$  and define  $V^* \equiv D_2^* k^p$  similarly.

LEMMA 2.13. *( $\exists D$ )  $D^{-1}UD = V$  if and only if ( $\exists D$ )  $D^{-1}U^*D = V^*$  and  $U^*$  and  $V^*$  are basic.*

*Proof.* Trivial.

We call  $U^*$  *porous* if  $U^* \equiv R_3^* \Gamma^{-1}h^r \Gamma k^p$  where  $R_3^*$  commutes with  $\Gamma$ .

LEMMA 2.14. *Let  $U^*$  be porous and suppose  $D^{-1}U^*D = V^*$ . Then there exists  $R$  such that  $R^{-1}U^*R = V^*$ .*

*Proof.* We shall show that  $D = \Gamma^{-1}h^t\Gamma R$  and  $R^{-1}U^*R = V^*$ .

We have two cases according as  $V^*$  is a right or left conjugate of  $U^*$  by  $D$ . If  $V^*$  is a right conjugate, then, in our usual way we write  $D \equiv R_1\Gamma^{-1}h^{t_1}\Gamma D'$  and deduce that  $R_1$  commutes with  $\Gamma$ . The lemma then follows by induction.

If on the other hand  $V^*$  is a left conjugate we obtain  $\Gamma R_1^{-1}R_3^*\Gamma^{-1} = R_4$  for some  $R_4$ . Since  $R_3^*$  commutes with  $\Gamma$  this gives  $\Gamma R_1^{-1}\Gamma^{-1}R_3^* = R_4$  whence it follows that  $R_1$  commutes with  $\Gamma$ . Again induction sees us home.

**THEOREM 2.15.** *Let  $U^* \equiv D_1^*k^p$  and  $V^* \equiv D_2^*k^p$  both be nonporous. Then there exists  $D$  such that  $D^{-1}U^*D = V^*$  if and only if there exists  $D_1^+$  and  $D_2^+$  and  $R$  such that*

- (a)  $D_1^+$  and  $D_2^+$  are cyclic permutations of  $D_1^*$  and  $D_2^*$  respectively;
- (b)  $R^{-1}D_1^+k^pR = D_2^+k^p$ .

*Proof.* If  $D_1^+$ ,  $D_2^+$  and  $R$  exist, then it is clear that there exists  $D$  such that  $D^{-1}U^*D = V$ .

Conversely suppose  $D^{-1}U^*D = V$ ; then  $D^{-1}D_1^*D = D_2^*$  and this equality holds in  $G_0^*$ . Since  $U^*$  and  $V^*$  are non-porous it follows by a straightforward argument that  $D_1^*$  and  $D_2^*$  are  $h$ -contracted. Hence by General Lemma 3 of CPD, there exist  $X_1$ ,  $Y_1$  and  $R$  such that  $R^{-1}X_1R = Y_1$  where  $X_1$  is of form

$$X_1 \equiv \Gamma R_{1t} \Gamma^{-1} h^{t+1} \Gamma R_{1(t+1)} \dots R_{1(t-1)} \Gamma^{-1} h^t$$

and  $Y_1$  of form

$$Y_1 \equiv \Gamma R_{2j} \Gamma^{-1} h^{s_j+1} \Gamma R_{2(j+1)} \dots R_{2(j-1)} \Gamma^{-1} h^{s_j}$$

We want to show that  $R$  commutes with  $\Gamma$  and we want to do so by using  $t$  as a stable letter for  $G_0^*$  over the appropriate subgroup. Firstly we verify that  $X_1$  and  $Y_1$  are  $t$ -reduced. If  $X_1$  is not  $t$ -reduced then there will be some  $R_{1k}$  which commutes with  $\Gamma$  which contradicts the fact that  $U^*$  is non-porous. The same argument applies to  $Y_1$ .

From the equality  $R^{-1}X_1R = Y_1$  we then deduce that  $\sum_0 h^{r_i} R h^{-s_j} \sum_0^{-1} = R^+$ , for some  $R^+$ . From this we promptly obtain  $r_i = s_j$ , and  $\sum_0 R \sum_0^{-1} = R^+$ . This last means that  $R$  commutes with  $\Gamma$ .

The theorem follows easily if we define

$$D_1^+ \equiv R_{1t} \Gamma^{-1} h^{t+1} \Gamma R_{1(t+1)} \dots R_{1(t-1)} \Gamma^{-1} h^t \Gamma$$

$$D_2^+ \equiv R_{2j} \Gamma^{-1} h^{s_j+1} \Gamma R_{2(j+1)} \dots R_{2(j-1)} \Gamma^{-1} h^{s_j} \Gamma.$$

This theorem completes § 2. In the introduction we asserted that we should derive certain necessary conditions for the conjugacy of  $U$  and  $V$ . These conditions are given in detail in Theorems 2.8, 2.12 and 2.15.

§ 3. As a consequence of our work, in § 2 we must examine equalities of the form  $R^{-1}UR = V$  where  $U$  and  $V$  are basic words. We shall define a last normalisation procedure which will, essentially, allow us to assume that the equality holds in  $G_{0h}$  rather than in  $G$ . (Since it is of importance in which group an equality holds, we return to our former practice of indicating this by a subscript).

If  $R^{-1}UR = {}_G V$  then in transforming  $R^{-1}UR$  into  $V$  using the relations of  $G$ , the final relation  $k\Gamma^{-1}h\Gamma = \Gamma^{-1}h\Gamma k$  may be used. Our normalisation procedure consists of anticipating all such applications.

If  $U \equiv U_1 k^\varepsilon X h^\eta U_2$ , where  $X$  is  $k$ -free and  $h$ -free and  $\varepsilon, \eta = \pm 1$ , and there exist words  $R_1$  and  $R_2$  such that  $X = R_1 \Gamma^{-1} R_2$  then a *primitive  $\tau$ -reduction* of  $U$  is  $U_1 X h^\eta X^{-1} k^\varepsilon X U_2$ . We adopt the usual definitions derived from that of a primitive reduction.

LEMMA 3.1. *Let  $U$  and  $V$  be  $k$ -reduced,  $h$ -reduced, and  $\tau$ -reduced. If  $U = {}_G V$  then  $U, V$  are  $\{h, k\}$ -parallel.*

*Proof.* If either  $U$  or  $V$  is  $k$ -free or  $h$ -free then the result is immediate from General Lemma 1 of CPD. So suppose that this is not the case; by the same General Lemma  $U$  and  $V$  must be  $k$ -parallel and can thus be written, respectively as  $U_1 k^\varepsilon U_2$  and  $V_1 k^\varepsilon V_2$  where  $U_2$  and  $V_2$  are  $k$ -free.

We proceed by induction on  $l_k(U) + l_h(U)$  and consider two cases.

*Case 1.* Suppose that  $U_2$  is  $h$ -free; now  $(\exists D) U_2 V_2^{-1} = {}_{G^*} D$  and if  $V_2$  is also  $h$ -free then in fact  $(\exists R) U_2 V_2^{-1} = {}_{G_1} R$ . The lemma then follows from the hypothesis of induction applied to  $U_1$  and  $V_1 R$ .

We therefore seek to show that  $V_2$  must be  $h$ -free. If not, then  $U_2 V_2^{-1} = {}_{G^*} D$  means  $D$  is not  $h$ -free. We may, however, assume  $D$  is  $h$ -reduced. Since  $D V_2 U_2^{-1} = {}_{G^*} 1$ , we see that  $D V_2 U_2^{-1} \equiv D_1 \Gamma^{-1} h^{-\eta} \Gamma R X h^\eta V_2' U_2^{-1}$  and  $\Gamma R X = {}_{G_1} R'$ . This contradicts the fact that  $V$  is  $\tau$ -reduced.

*Case 2.* Suppose that  $U_2$  is not  $h$ -free. Then  $V_2$  is not  $h$ -free, and we may write  $U_2 \equiv U_2' h^\eta F$  and  $V_2 \equiv V_2' h^\eta G$  where  $F$  and  $G$  are  $h$ -free. Then  $F G^{-1} = {}_{G_1} R$  and the lemma follows from the induction hypothesis applied to  $U_1 k^\varepsilon U_2' R$  and  $V_1 k^\varepsilon V_2'$ .

LEMMA 3.2. *Let  $U$  and  $V$  be  $k$ -reduced and  $h$ -reduced words which are  $\{h, k\}$ -parallel. If  $U = {}_G V$  then  $U = {}_{G_{0h}} V$ .*

*Proof.* Suppose that  $U \equiv U_1 k^\varepsilon U_2$  where  $U_2$  is  $h$ -free. Then  $V \equiv V_1 k^\varepsilon V_2$  where  $V_2$  is  $h$ -free. Hence  $U_2 V_2^{-1} = {}_{G_1} R$  and  $U_1 R = {}_G V_1$ . By the induction hypothesis (with respect to  $l_k(U) + l_h(U)$ )  $U_1 R = {}_{G_{0h}} V_1$  and hence  $U_1 k^\varepsilon R = {}_{G_{0h}} V_1 k^\varepsilon$ .

If  $U \equiv U_1 h^\varepsilon U_2$  with  $U_2$   $k$ -free, the argument is similar.

**THEOREM 3.3.** *Let  $U$  and  $V$  be  $k$ -reduced and  $h$ -reduced. Then  $R^{-1}UR =_G V$  if and only if  $R^{-1}\tau[U]R =_{G_{0h}} \tau[V]$ .*

*Proof.* Clearly  $U =_G \tau[U]$  and  $V =_G \tau[V]$ . If  $R^{-1}UR =_G V$ , then  $R^{-1}\tau[U]R =_G \tau[V]$ . But both  $R^{-1}\tau[U]R$  and  $\tau[V]$  satisfy the hypotheses of Lemma 3.1. This means that Lemma 3.2 is then applicable and so  $R^{-1}\tau[U]R =_{G_{0h}} \tau[V]$ .

The converse is trivial.

**REDUCTION II** ( $?U, V$  weakly basic) ( $\exists W$ )  $W^{-1}UW =_G V$  is reducible to the conjugacy problem for  $G_{0h}$ .

*Proof.* For each weakly basic word  $U$  we shall compile a finite list of words  $U_i$ ,  $i=1, 2, \dots, t$  such that:

- (1)  $t$  is bounded above by a recursive function of  $U$ ;
- (2) each  $U_i$  is  $S_1$ -recursively computable from  $U$ ;
- (3) each  $U_i$  is conjugate in  $G$  to  $U$ ;

(4) if  $U$  is conjugate to  $V$  then there exist  $U_i$  and  $V_j$ , which belong to the lists obtained from  $U$  and  $V$ , such that  $U_i$  is conjugate to  $V_j$  in  $G_{0h}$ . It is clear that once we have shown how to compile such a list and verified that (1)–(4) are satisfied, then the reduction is proved. For two weakly basic words  $U$  and  $V$  are conjugate in  $G$  if and only if there exist  $U_i$  and  $V_j$ , from the respective lists, which are conjugate in  $G_{0h}$ .

We compile the list in the following manner.

Let  $\bar{U}_1, \bar{U}_2, \dots, \bar{U}_r$  be the set of all cyclic permutations of  $U$  which are basic. Then we begin to normalise, following the procedure of § 2. Let  $\bar{U}_1, \bar{U}_2, \dots, \bar{U}_s$  be the set of all basic words obtainable from the above set by cyclic  $\gamma$ -reduction. We then compute  $\bar{U}_1^*, \bar{U}_2^*, \dots, \bar{U}_s^*$  as specified prior to either Lemma 2.4 (Case I), Lemma 2.9 (Case II) or Lemma 2.13 (Case III).

We make a further listing for each  $\bar{U}_k^*$ . If case I holds we list all possible words of the form  $D_3^+ U_0$ , where  $\bar{U}_k^* = D_3^* U_0$  and  $D_3^+$  is a cyclic permutation of  $D_3^*$  (see Theorem 2.8). If case II holds we list all possible words of the form  $D_3^+ X^* k^p$ , where  $\bar{U}_k^* = D_3 X^* k^p$  and  $D_3^+$  is a cyclic permutation of  $D_3$ . Finally if case III holds we list all possible words of the form  $D_1^+ k^p$ , where  $\bar{U}_k^* \equiv D_1 k^p$  and  $D_1^+$  is a cyclic permutation of  $D_1$ . Let  $\hat{U}_1, \hat{U}_2, \dots, \hat{U}_t$  be the list of all words obtainable from  $U$  in the manner we have so far described.

Our final list is obtained by putting  $U_i \equiv \tau[\hat{U}_i]$ . Properties (1), (2) and (3) are then readily verified from the various definitions involved. To verify property (4) we observe that if  $U$  is conjugate to  $V$  then there exist basic cyclic permutations  $\bar{U}_k$  and  $\bar{V}_l$  and a word  $D$  such that  $D^{-1}\bar{U}_k D =_G \bar{V}_l$ . The essential content of Lemmas 2.6, 2.10 and 2.14 and



Theorems 2.8, 2.12 and 2.15 is that if  $\bar{V}_i$  is a right conjugate of  $\bar{U}_k$  by  $D$ , then there exist  $\hat{U}_i$  and  $\hat{V}_j$  from the intermediate lists and  $R$  such that  $R^{-1}\hat{U}_iR =_G \hat{V}_j$ . By Theorem 3.3,  $R^{-1}U_iR =_{G_{0h}} V_j$ .

If on the other hand  $\bar{V}_i$  is a left conjugate of  $\bar{U}_k$  by  $D$ , then, by General Lemma 4 of CPD,  $\bar{U}_k$  is a right conjugate of  $\bar{V}_i$  by  $D^{-1}$ . The same argument as used in the previous paragraph tells us that we can find  $V_j$ ,  $U_i$  and  $R$  such that  $R^{-1}V_jR =_{G_{0h}} U_i$ . Thus in either case assertion (4) is verified and the proof of the reduction is completed.

By way of conclusion we compare our  $G$  with the presentation considered by Boone. The main distinction is in the final relation which is, in our case,

$$k(\Sigma_0^{-1}t^{-1}\Sigma_0h\Sigma_0^{-1}t\Sigma_0) = (\Sigma_0^{-1}t^{-1}\Sigma_0h\Sigma_0^{-1}t\Sigma_0)k \quad (\text{A})$$

and in Boone's presentation is

$$k(\Sigma_0^{-1}t\Sigma_0) = (\Sigma_0^{-1}t\Sigma_0)k^{(1)}. \quad (\text{B})$$

Much of the argument goes through for Boone's presentation with  $t$  in the role we have assigned to  $h$ . Thus, for example,  $t$  is a stable letter for  $G$  over an appropriate subgroup and indeed the argument of § 1 and part of § 3 can be carried through (with the obvious modifications).

The crucial difference occurs at the point in § 2 where we show that we need only look for a conjugating element  $D^{(2)}$  of a particularly simple type. We do this by showing that if  $D$  is a conjugating element of the form  $R_1\Gamma^{-1}h^*\Gamma D'$ , then  $R_1$  commutes with  $\Gamma$ . To show that  $R_1$  commutes with  $\Gamma$  we arrive at an equation

$$\Gamma^{-1}R^*\Gamma R_1\Gamma^{-1} = R_1'\Gamma^{-1}R_1^+$$

and then use  $t$  as stable letter to deduce that  $R_1$  commutes with  $\Gamma$ .

The corresponding equation for Boone's group is  $\Sigma_0^{-1}R^*\Sigma_0R_1\Sigma_0^{-1} = R_1'\Sigma_0^{-1}R_1^+$ . We can attempt to derive information by using  $q$  as stable letter. Thus, for example, we might deduce  $\Phi_0R_1\Phi_0^{-1} = B$  for some  $B$ . This does not, however, seem to imply that  $R_1$  commutes with  $\Sigma_0^{-1}t\Sigma_0$  as we should like. We might also attempt to obtain information using the  $r_i$ 's as stable letters, but there too there are unpleasant complications. It is, of course, possible that with further analysis the equivalence of the conjugacy problem for Boone's group and the word problem for  $\mathfrak{X}$  can be established. Despite a good deal of effort the author has not been able to establish this equivalence and indeed is beginning to doubt that it holds.

(<sup>1</sup>) In Boone's presentation  $\Sigma_0 \equiv q$  since  $\Phi_0$  is empty in this case. This point is trivial.

(<sup>2</sup>) For Boone's presentation,  $D$  is a variable for products of  $r_i$ ,  $x$  and  $\Gamma$ .

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CPD = [7]

WPD = [4]

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