ON SYSTEMS OF IMPRIMITIVITY ON LOCALLY COMPACT ABELIAN GROUPS WITH DENSE ACTIONS

BY

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1. Introduction

Let Γ be a countable dense subgroup of the group R of real numbers with usual topology. Give Γ the discrete topology and let $B=\hat{\Gamma}$ be its compact dual. For each $t \in R$, the function $\exp(it\lambda)$, $\lambda \in \Gamma$, is a character on Γ, which we denote by e_t. Then the map, $\varphi: t \to e_t$, is a continuous isomorphism of R into B and $\varphi(R)$ is dense in B. We assume that $2\pi \in \Gamma$. Let K denote the annihilator of the subgroup Γ_0 generated by 2π. The group $N = K \cap \varphi(R)$ consists of elements $\{e_n\}$, $n=0, \pm 1, \pm 2, \ldots$ and it is dense in K. In [3] Gamelin showed that every (N, K) cocycle gives rise, in a natural way, to an (R, B) cocycle, and that in any cohomology class of (R, B) cocycles there is a cocycle obtained from an (N, K) cocycle by his procedure. Gamelin considered only scalar cocycles. As a consequence of this work he was able to resolve some of the problems raised by Helson in [5 (1965)] on compact groups with ordered duals.

If a subgroup G_0 of a locally compact group G acts on G through translation, then by (G_0, G) system of imprimitivity we mean a system of imprimitivity for G_0 based on G, acting in some separable Hilbert space $\mathcal{H}$. In this paper we show that each (N, K) system of imprimitivity (V, E) gives rise to an (R, B) system of imprimitivity $(\bar{V}, \bar{E})$. If U denotes the unitary group (indexed by $\hat{K} = \Gamma/\Gamma_0$) associated with E, and F denotes the spectral measure of V (defined on Borel subsets of T, the circle group), then (U, F) is a $(\hat{K}, T)$ system of imprimitivity. We show that (U, F) gives rise in a natural way to a (Γ, R) system of imprimitivity $(\tilde{U}, \tilde{F})$, and that every (Γ, R) system of imprimitivity is equivalent to a system of imprimitivity $(\tilde{U}, \tilde{F})$. Finally if $\bar{U}$ denotes the unitary group indexed by Γ with spectral measure $\bar{E}$ and $\bar{F}$ the spectral measure of $\bar{V}$, then $(\bar{U}, \bar{F})$ and $(\tilde{U}, \tilde{F})$ are equivalent systems of imprimitivity. We thus complete the circle of ideas involved in Gamelin's work.

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Systems of imprimitivity on compact groups with ordered duals were first encountered by Helson and Lowdenslager [7] in their study of H^2 on Bohr group. Some of the subsequent papers on this are due to Helson [5], Helson and Kahane [8], Yale [13], Gamelin [3]. Muhly [11], and later Bagchi [1] generalized to vector valued case the results due to Helson [5] using (R, B) systems of imprimitivity. They used Gamelin's method to conclude that their generalization was non-trivial. In all these papers the authors work on the pair (R, B). In [3, 7, 8, 13] different methods of constructing non-trivial scalar (R, B) cocycles are given, whereas in [5, 6] deep analysis is made of the analytic structure of scalar (R, B) cocycles. The present work is at a more general level; it considers different pairs of groups, and ties up a general (R, B) system of imprimitivity with others which naturally arise from it. We are able to answer certain questions about (R, B) systems of imprimitivity by referring to the corresponding $(\hat{R}, T)$ systems of imprimitivity.

Study of systems of imprimitivity in general set up was undertaken by G. W. Mackey in various papers in connection with the theory of group representations. A connected account of this is given in Varadarajan [12]. Systems of imprimitivity associated with strictly ergodic actions are not as well studied as those associated with transitive actions. Mackey [10] has introduced the notion of virtual subgroups for study of systems associated with strictly ergodic actions. This notion is, however, not used in the present paper, though it is concerned with strictly ergodic actions.

§ 2

2.1. *Definition.* By a pair (G_0, G) we will mean that,

(i) G_0 and G are locally compact second countable abelian groups and,

(ii) there exists a one-to-one continuous homomorphism φ of G_0 into G such that $\varphi(G_0)$ is dense in G.

Given a pair (G_0, G), there arises another pair in a natural way. Consider the dual groups $\hat{G}_0$ and $\hat{G}$, and the map $\hat{\varphi}: \hat{G} \to \hat{G}_0$ defined by

$$\langle x, \hat{\varphi}(\hat{y})\rangle = \langle \varphi(x), \hat{y}\rangle, \quad x \in G_0,\ \hat{y} \in \hat{G}.$$

It can be shown that $\hat{\varphi}$ is a one-to-one continuous homomorphism of $\hat{G}$ into $\hat{G}_0$, and that $\hat{\varphi}(\hat{G})$ is dense in $\hat{G}_0$. The pair $(\hat{G}, \hat{G}_0)$ will be called the dual pair of (G_0, G). Such pairs were considered by de Leeuw and Glicksberg [2], where the map φ is not necessarily one-to-one, but simply a continuous homomorphism.

Let $\mathbf{U}(\mathcal{H})$ denote the class of unitary operators on a complex separable Hilbert space

$\mathcal{H}$, and equip $\mathbf{U}(\mathcal{H})$ with the smallest σ-algebra under which all the functions $U \to (Ux, y)$, $x, y \in \mathcal{H}$, are measurable. Let (G_0, G) be a pair. Let λ denote the Haar measure on G_0 and let μ be a σ-finite measure on G, quasi-invariant with respect to G_0. $\mathcal{C}_\mu$ will denote the measure class of μ; that is, the class of all σ-finite measures on G having the same null sets as μ. By μ_g, $g \in G_0$, we will mean the measure $D \to \mu(D + \varphi(g))$.

2.2. *Definition.* By a $(G_0, G, \mathbf{U}(\mathcal{H}))$ cocycle A relative to $\mathcal{C}_\mu$ (or relative to μ) we mean a measurable function $G_0 \times G \to \mathbf{U}(\mathcal{H})$ such that,

$$A(g_1 + g_2, x) = A(g_1, x) A(g_2, x + \varphi(g_1)) \tag{1}$$

a.e. $(\lambda \times \lambda \times \mu)$. Two cocycles are identified if they agree outside a $\lambda \times \mu$ null set. A cocycle is called a strict cocycle if (1) is satisfied everywhere. When no confusion is likely to arise, we shall refer to a $(G_0, G, \mathbf{U}(\mathcal{H}))$ cocycle relative to $\mathcal{C}_\mu$ simply as a (G_0, G) cocycle.

2.3. *Definition.* Two $(G_0, G, \mathbf{U}(\mathcal{H}))$ cocycles A_1 and A_2 relative to $\mathcal{C}_\mu$ are said to be cohomologous if there exists a measurable function $\varrho: G \to \mathbf{U}(\mathcal{H})$ such that,

$$A_1(g, x) = \varrho(x) A_2(g, x) \varrho^*(x + \varphi(g)) \quad \text{a.e. } \lambda \times \mu. \tag{2}$$

We say that A_1 is cohomologous (ϱ) to A_2. It is easy to see that 'A_1 and A_2 are cohomologous' is an equivalence relation. The equivalence classes are called cohomology classes. A cocycle A is called a coboundary if A has the form

$$A(g, x) = \varrho(x) \varrho^*(x + \varphi(g)) \quad \text{a.e. } \lambda \times \mu, \tag{3}$$

for some measurable function $\varrho: G \to \mathbf{U}(\mathcal{H})$. It is clear that every coboundary has a strict version, namely, the right hand side of (3).

2.4. LEMMA. *If G_0 is countable, then every $(G_0, G, \mathbf{U}(\mathcal{H}))$ cocycle A relative to $\mathcal{C}_\mu$ has a strict version.*

Proof. Suppose that the cocycle identity is satisfied on a subset $D \subseteq G_0 \times G_0 \times G$ of full $\lambda \times \lambda \times \mu$ measure. Since $G_0 \times G_0$ is countable, every element $(g_1, g_2) \in G_0 \times G_0$ has positive $\lambda \times \lambda$ measure, and hence D contains a rectangle $G_0 \times G_0 \times E$, where $E \subseteq G$ has full μ measure. Replacing E by $\bigcap_{g \in G_0} (E + \varphi(g))$, we may assume that $E + \varphi(g) = E$ for all $g \in G_0$. Define A' by

$$A'(g, x) = \begin{cases} A(g, x) & \text{if } x \in E \\ I & \text{if } x \notin E. \end{cases}$$

Then A' is a strict cocycle equal almost everywhere to A. Q.E.D.

§ 3

In this section we state some of the basic facts about systems of imprimitivity in a form convenient to us. For proofs of all the unproved statements, we refer to Varadarajan's book [12].

3.1. *Definition.* Let $\mathcal{H}$ be a separable Hilbert space. By a system of imprimitivity based on (G_0, G) and acting in $\mathcal{H}$ (or a (G_0, G) system of imprimitivity acting in $\mathcal{H}$) we mean a pair (U, P) where

(i) U is a representation of G_0 acting in $\mathcal{H}$, and

(ii) P is a spectral measure on Borel subsets of G, acting in the same Hilbert space $\mathcal{H}$, and such that, for each Borel set $D \subseteq G$, and for each $g \in G_0$,

$$U_g^{-1} P(D)\, U_g = P(D + \varphi(g)).$$

Two systems of imprimitivity (U, P) and (U', P') based on (G_0, G), and acting in $\mathcal{H}$ and $\mathcal{H}'$ respectively, are said to be equivalent if there exists an isometric isomorphism S of $\mathcal{H}$ onto $\mathcal{H}'$ such that,

$$SP(D)S^{-1} = P'(D),$$

and

$$SU_g S^{-1} = U'_g,$$

for each Borel set $D \subseteq G$, and each $g \in G_0$.

Let μ be a σ-finite measure on G, quasi-invariant with respect to G_0, and let A be a $(G_0, G, \mathrm{U}(\mathcal{H}))$ cocycle relative to $\mathcal{C}_\mu$. We can define a system of imprimitivity (U^A, P) based on (G_0, G) and acting in $L^2(G, \mathcal{H}, \mu)$ by setting

$$(U_g^A f)(x) = \sqrt{\frac{d\mu_g}{d\mu}}(x)\, A(g, x)\, f(x + \varphi(g)),\ x \in G, g \in G_0$$

$$P(D) f = 1_D f,$$

where 1_D stands for the characteristic function of D. (U^A, P) will be called a concrete system of imprimitivity (based on (G_0, G)) of multiplicity n, where n is the dimension of the Hilbert space $\mathcal{H}$. If A is cohomologous to A', then (U^A, P) is equivalent to $(U^{A'}, P)$. More generally, let $\mu_\infty, \mu_1, \mu_2, \ldots$ be a sequence of mutually singular Borel measures on G, each μ_i quasi-invariant under G_0 (some μ_n's may be zero measures). Let $\mathcal{H}_n$ be a Hilbert space of dimension n, $n = \infty, 1, 2, \ldots$. Let A_n be a $(G_0, G, \mathrm{U}(\mathcal{H}_n))$ cocycle relative to μ_n. Then we can define a (G_0, G) system of imprimitivity (U, P) acting on the direct sum $\Sigma\, L^2(G, \mathcal{H}_n, \mu_n)$ by requiring that the restriction of (U, P) to $L^2(G, \mathcal{H}_n, \mu_n)$ be (U^{A_n}, P_n), where P_n is the spectral measure on $L^2(G, \mathcal{H}_n, \mu_n)$ consisting of multiplication by characteristic functions. Such a system of imprimitivity will be called a concrete system of imprimitivity. If (U', P') be another (G_0, G) concrete system of imprimitivity acting in $\Sigma L^2(G, \mathcal{H}_n, \mu_n')$, with associated cocycles $A'_\infty, A'_1, A'_2, \ldots$, then (U, P) and (U', P') are equivalent if and only if for each n, $C_{\mu_n} = C_{\mu'_n}$ and A_n is cohomologous to A'_n. Finally we have the following:

3.2. THEOREM. *Every (G_0, G) system of imprimitivity acting in a separable Hilbert space $\mathcal{H}$ is equivalent to a concrete system of imprimitivity.*

For proof of this theorem we refer to Varadarajan [12] Theorem 9.11. In the sequel we shall assume that $\mathcal{H}_n$ is C^n if n is finite and l^2 if $n=\infty$.

Let (U, P) be a system of imprimitivity based on (G_0, G) and acting in $\mathcal{H}$. Apply Stone's theorem to U to yield a spectral measure Q on $\hat{G}_0$ and to P to yield a representation V of $\hat{G}$:

$$\left.\begin{aligned} U_g &= \int_{\hat{G}_0} \langle -y, g\rangle \, dQ(y), g\in G_0 \\ V_h &= \int_G \langle x, h\rangle \quad dP(x), h\in \hat{G}. \end{aligned}\right\} \tag{4}$$

Since (U, P) is a system of imprimitivity, we have

$$\begin{aligned} U_g^{-1} V_h U_g = \int_G \langle x, h\rangle \, d(U_g^{-1} P(x)\, U_g) &= \int_G \langle x, h\rangle \, dP(x+\varphi(g)) \\ &= \langle -\varphi(g), h\rangle V_h = \langle -g, \hat{\varphi}(h)\rangle V_h \end{aligned}$$

whence, $$V_h^{-1} U_g^{-1} V_h = \langle -g, \hat{\varphi}(h)\rangle U_g^{-1}$$

where $\hat{\varphi}$ is the dual map from $\hat{G}$ into $\hat{G}_0$. From (4) we have

$$\begin{aligned} V_h^{-1} U_g^{-1} V_h &= \int_{\hat{G}_0} \langle y, g\rangle \, d(V_h^{-1} Q(y)\, V_h) \\ &= \int_{\hat{G}_0} \langle y-\hat{\varphi}(h), g\rangle \, dQ(y) = \int_{\hat{G}_0} \langle y, g\rangle \, dQ(y+\hat{\varphi}(h)). \end{aligned}$$

Therefore, $V_h^{-1} Q(D)\, V_h = Q(D+\hat{\varphi}(h))$ for each Borel set $D \subseteq \hat{G}_0$, and for each $h\in\hat{G}$. Hence (V, Q) is a system of imprimitivity based on $(\hat{G}, \hat{G}_0)$ and acting in $\mathcal{H}$. We shall call (V, Q) the dual system of (U, P). We observe that a subspace of $\mathcal{H}$ reduces (U, P) if and only if it reduces (V, Q).

§ 4

4.1. *Definition.* A Bohr group B is a compact abelian group whose discrete dual Γ is a subgroup of the additive group R of real numbers, dense in the usual topology of R.

We shall consider only those B for which $\hat{B}=\Gamma$ is countable. Then both B and Γ are second countable. The inclusion map from Γ into R is a one-to-one continuous homomorphism having a dense range. This gives us the pair (Γ, R). Its dueal pair is (R, B). The continuous homomorphism of R into B will be denoted by $t\to e_t$; the elements e_t are characterised by $\langle e_t, \delta\rangle = \exp(it\delta)$, $t\in R$, $\delta\in\Gamma$.

Let B be a Bohr group with Γ countable. Assume, without loss of generality, that $2\pi\in\Gamma$. Let K be the subgroup of B defined by

$$K = \{x\in B: \langle x, 2\pi\rangle = 1\}.$$

K is a compact subgroup of B. An element e_t belongs to K if and only if t is an integer. Consider now the Borel subset $\{e_t: 0\leqslant t<1\}$ of B. It consists of exactly one element from each coset of K in B. Therefore, each $x\in B$ has a unique representation $x=y+e_t$, $y\in K$, $t\in[0, 1)$. This gives a one-to-one bimeasurable mapping $\eta: (y, t)\to x=y+e_t$ of $K\times[0, 1)$ onto B. Therefore, the Borel structure of B can be identified with that of the product space $K\times[0, 1)$.

Let $\Gamma_0=\{2\pi n: n\in N$, the integer group$\}$. Γ_0 is a closed subgroup of Γ, and K is the annihilator of Γ_0. Therefore, the dual of K is Γ/Γ_0. Since Γ is dense in R, Γ/Γ_0 is a dense subgroup of $R/\Gamma_0=T$, the circle group. This gives us the pair $(\hat{K}, T)$, where $\hat{K}=\Gamma/\Gamma_0$. Its dual is the pair (N, K), where the homomorphism is $n\to e_n$ of N into K.

Notation: B, R, N, K, $\hat{K}$, Γ, T will denote the above groups throughout the paper. We shall regard T as the interval $[0, 2\pi)$ with addition modulo 2π as the group operation. It will be convenient to regard T as a subset of R.

Consider the pairs $(\hat{K}, T)$ and (Γ, R). As Borel spaces $T\times\Gamma_0$ and R can be identified; the isomorphism being $\xi: (x, 2\pi n)\to x+2\pi n$. Let μ be a measure on T, quasi-invariant with respect to the action of $\hat{K}$, and let λ denote the Haar measure on Γ_0. Then the measure $\tilde{\mu}=(\mu\times\lambda)\circ\xi^{-1}$ on R is quasi-invariant with respect to the action of Γ on R. On the otherhand, if σ is a measure on R, quasi-invariant with respect to Γ, then σ is equivalent (in the sense of having same null sets) to a measure of the form $\tilde{\mu}=(\mu\times\lambda)\circ\xi^{-1}$, for some measure μ on T, quasi-invariant with respect to the action of $\hat{K}$. Indeed one can take μ to be the measure σ restricted to $[0, 2\pi)$. Henceforth $\tilde{\mu}$ will always denote the measure $(\mu\times\lambda)\circ\xi^{-1}$. If μ_1 and μ_2 are mutually singular measures on T, then $\tilde{\mu}_1$ and $\tilde{\mu}_2$ are also mutually singular.

Fix a finite quasi-invariant measure μ on T. Let A be a strict $(\hat{K}, T, \mathrm{U}(\mathcal{H}))$ cocycle relative to μ. (Since $\hat{K}$ and Γ are countable, by Lemma 2.4 every $(\hat{K}, T)$ cocycle and every (Γ, R) cocycle have strict versions.) Define $\tilde{A}: \Gamma\times R\to\mathrm{U}(\mathcal{H})$ by:

$$\tilde{A}(u+2\pi m,\ x+2\pi n) = A(u, x), \quad u \in \hat{K},\ x \in T,\ m, n \text{ integers.}$$

$\tilde{A}$ is clearly a strict $(\Gamma, R, \mathrm{U}(\mathcal{H}))$ cocycle (relative to $\tilde{\mu}$).

4.2. THEOREM. *Every strict $(\Gamma, R, \mathrm{U}(\mathcal{H}))$ cocycle relative to $\tilde{\mu}$ is cohomologous to a cocycle $\tilde{A}$, for some strict $(\hat{K}, T, \mathrm{U}(\mathcal{H}))$ cocycle A relative to μ. Two strict $(\hat{K}, T, \mathrm{U}(\mathcal{H}))$ cocycles A_1 and A_2 relative to μ are cohomologous if and only if the extended $(\Gamma, R, \mathrm{U}(\mathcal{H}))$ cocycles $\tilde{A}_1$ and $\tilde{A}_2$ are cohomologous.*

Proof. Let C be a strict $(\Gamma, R, \mathrm{U}(\mathcal{H}))$ cocycle relative to $\tilde{\mu}$. Define ϱ: $R \to \mathrm{U}(\mathcal{H})$ by:

$$\varrho(x+2\pi n) = C(2\pi n, x), \quad x \in T.$$

ϱ is clearly measurable. Let $\tilde{A}$ be the $(\Gamma, R, \mathrm{U}(\mathcal{H}))$ cocycle defined by:

$$\tilde{A}(g, y) = \varrho(y)C(g, y)\varrho^*(y+g), \quad y \in R,\ g \in \Gamma.$$

$\tilde{A}$ is a $(\Gamma, R, \mathrm{U}(\mathcal{H}))$ cocycle cohomologous to C. Any $y \in R$ can be written in a unique way as $y = [y] + \langle y \rangle$, where $[y] \in \Gamma_0$ and $\langle y \rangle \in T$. (This notation will be only for this proof). Now for any two integers l and k,

$$\begin{aligned}
\tilde{A}(g+2l\pi,\ y+2k\pi) &= \varrho(y+2k\pi)C(g+2l\pi,\ y+2k\pi)\varrho^*(y+2k\pi+g+2l\pi)\\
&= C([y]+2k\pi,\ \langle y\rangle)C(g+2l\pi, y+2k\pi)C^*([y+g]+2l\pi+2k\pi,\ \langle y+g\rangle)\\
&= C([y],\ \langle y\rangle)C(2k\pi,\ y)C(g+2l\pi,\ y+2k\pi)\\
&\qquad \times C^*(2l\pi+2k\pi, y+g)\, C^*([y+g],\ \langle y+g\rangle)\\
&= C([y],\ \langle y\rangle)C(g+2l\pi+2k\pi,\ y)C^*(2l\pi+2k\pi,\ y+g)\, C^*([y+g], \langle y+g\rangle)\\
&= \varrho(y)C(g,\ y)\varrho^*(y+g) = \tilde{A}(g,\ y).
\end{aligned}$$

So $\tilde{A}$ is constant on $\Gamma_0 \times \Gamma_0$-cosets. Hence $\tilde{A}$ is obtained from the $(\hat{K}, T)$ cocycle A defined by:

$$A(u, x) = \tilde{A}(u, x), \quad u \in \hat{K},\ x \in T.$$

If two $(\hat{K}, T)$ cocycles A_1 and A_2 are cohomologous (ϱ_0), then $\tilde{A}_1$ and $\tilde{A}_2$ are cohomologous (ϱ), where ϱ is defined by

$$\varrho(x+2n\pi) = \varrho_0(x), \quad x \in T.$$

If $\tilde{A}_1$ and $\tilde{A}_2$ are cohomologous (ϱ), then A_1 and A_2 are cohomologous (ϱ_0), where $\varrho_0(x) = \varrho(x)$, $x \in T$. Q.E.D.

Now let (U, F) be a concrete system of imprimitivity based on $(\hat{K}, T)$, with associated measures $\mu_\infty, \mu_1, \mu_2, \ldots$, and cocycles $A_\infty, A_1, A_2, \ldots$. Then, by $(\tilde{U}, \tilde{F})$ we shall mean the concrete system of imprimitivity based on (Γ, R), with associated measures $\tilde{\mu}_\infty, \tilde{\mu}_1, \tilde{\mu}_2, \ldots$, and cocycles $\tilde{A}_\infty, \tilde{A}_1, \tilde{A}_2, \ldots$. In view of Theorems 3.2 and 4.2, we see that any (Γ, R) system of imprimitivity acting on a separable Hilbert space is equivalent to a (Γ, R) concrete system of imprimitivity $(\tilde{U}, \tilde{F})$, for some $(\hat{K}, T)$ concrete system of imprimitivity (U, F).

Now let us consider the dual pairs (N, K) and (R, B). There is a one-to-one, bi-measurable map $\eta: K \times [0, 1)$ onto B defined by

$$\eta(y, s) = y + e_s, \quad y \in K,\ s \in [0, 1).$$

Let μ be a Borel measure on K and, with abuse of notation, let ds denote the Lebesgue measure on $[0, 1)$. By $\bar{\mu}$ we shall mean the measure $(\mu \times ds) \circ \eta^{-1}$ on B. For $t \in R$, $[t]$ will denote the integer part of t, and $\langle t \rangle$, the fractional part.

Remark. Reader familiar with the notion of a flow built under a function will note that we are expressing the action of R on B as a flow built under the constant function 1. The base space is K, and translation by e_1 in K is the base transformation.

4.3. LEMMA. *If μ is a measure on K, quasi-invariant with respect to N; then $\bar{\mu}$ on B is quasi-invariant with respect to R. Moreover,*

$$\frac{d\bar{\mu}_t}{d\bar{\mu}}(y + e_s) = \frac{d\mu_{[t+s]}}{d\mu}(y) \quad \text{a.e. } \mu \times ds.$$

(We write $\bar{\mu}_t$ for $\bar{\mu}_{e_t}$ and μ_n for μ_{e_n}).

Proof. Let A be a Borel subset of B and let $t \in R$. Then,

$$\eta^{-1}(A + e_t) = \{(y + e_{[s+t]}, \langle s+t \rangle) : (y, s) \in \eta^{-1}(A)\}.$$

Therefore,

$$(\eta^{-1}(A + e_t))_{\langle s+t \rangle} = \{y + e_{[s+t]} : (y, s) \in \eta^{-1}(A)\} = (\eta^{-1}(A))_s + e_{[s+t]}$$

where $(\eta^{-1}(A))_s$ denotes the sth section of $\eta^{-1}(A)$. Now,

$$\begin{aligned}
\bar{\mu}_t(A) = \bar{\mu}(A + e_t) &= (\mu \times ds)\,(\eta^{-1}(A + e_t)) \\
&= \int_0^1 \mu((\eta^{-1}(A + e_t))_s)\, ds = \int_0^1 \mu((\eta^{-1}(A + e_t))_{\langle s+t \rangle})\, ds \\
&= \int_0^1 \mu((\eta^{-1}(A))_s + e_{[s+t]})\, ds = \int_0^1 \mu_{[s+t]}((\eta^{-1}(A))_s)\, ds \\
&= \int_0^1 \int_{(\eta^{-1}(A))_s} \frac{d\mu_{[s+t]}}{d\mu}(y)\, d\mu(y)\, ds = \int_{\eta^{-1}(A)} \varphi(y, s)\, (d\mu \times ds)\,(y, s)
\end{aligned}$$

$$= \int_A \varphi \circ \eta^{-1}(x)\, d\bar{\mu}(x) \quad \text{where } \varphi(y, s) = \frac{d\mu_{[s+t]}}{d\mu}(y)$$

Hence,

$$\frac{d\bar{\mu}_t}{d\bar{\mu}}(x) = \varphi \circ \eta^{-1}(x).$$

Or,

$$\frac{d\bar{\mu}_t}{d\bar{\mu}}(y + e_s) = \frac{d\mu_{[s+t]}}{d\mu}(y).$$

Q.E.D.

Let A be a strict (N, K) cocycle. Since N is countable, every (N, K) cocycle has a strict version. Define $\bar{A}$: $R \times B \to \mathrm{U}(\mathcal{H})$ by

$$\bar{A}(t, y + e_s) = A([t+s], y), \quad y \in K,\ s \in [0, 1).$$

Then $\bar{A}$ is a strict cocycle. This follows from the observation that, for real numbers a, b, c,

$$[a+b+c] = [a+b] + [c + \langle a+b \rangle].$$

The method of obtaining $\bar{A}$ from an (N, K) cocycle was given by Gamelin in [3]. He considers only scalar cocycles relative to the Haar measure on K.

The next theorem is true, but we prefer to postpone its proof until the end of section 5. It was proved by Gamelin for scalar cocycles by first proving that every (R, B) cocycle has a strict version. This requires somewhat involved measure theoretic arguments which our proof will avoid. His method was used by Bagchi [1] to prove a somewhat weaker form of the theorem.

4.4. Theorem. *Every $(R, B, \mathrm{U}(\mathcal{H}))$ cocycle is cohomologous to a cocycle $\bar{A}$ for some strict $(N, K, \mathrm{U}(\mathcal{H}))$ cocycle A. If two $(N, K, \mathrm{U}(\mathcal{H}))$ cocycles A_1 and A_2 are cohomologous, then the corresponding $(R, B, \mathrm{U}(\mathcal{H}))$ cocycles $\bar{A}_1$ and $\bar{A}_2$ are cohomologous, and conversely.*

Let (V, E) be a concrete system of imprimitivity based on (N, K) with associated measures $\mu_\infty, \mu_1, \mu_2, \ldots$ and cocycles $A_\infty, A_1, A_2, \ldots$. Then by $(\bar{V}, \bar{E})$ we shall mean the concrete system of imprimitivity based on (R, B) with associated measures $\bar{\mu}_\infty, \bar{\mu}_1, \bar{\mu}_2, \ldots$ and cocycles $\bar{A}_\infty, \bar{A}_1, \bar{A}_2, \ldots$. We shall call $(\bar{V}, \bar{E})$ the Gamelin system of imprimitivity associated with the (N, K) system (V, E). In view of Theorems 3.2 and 4.4, we see that any (R, B) system of imprimitivity is equivalent to a Gamelin system of imprimitivity. It can be shown that a concrete (N, K) system of imprimitivity is irreducible if and only if the associated Gamelin system of imprimitivity is irreducible. This can be proved by using the notion of range functions. For the special case when the measure on K is Haar measure this has been done by Bagchi [1], and the same proof is valid more generally as well. This fact was also known to Muhly. See, for example, his paper [11] page 150.

§ 5

Let (V, E) be a system of imprimitivity on (N, K); $(\bar{V}, \bar{E})$ the associated Gamelin system on (R, B). Let (U, F) be the dual of (V, E), and $(\bar{U}, \bar{F})$ that of $(\bar{V}, \bar{E})$. Let $(\tilde{U}, \tilde{F})$ be the system of imprimitivity on (Γ, R) corresponding to the $(\hat{K}, T)$ system (U, F). Thus on (Γ, R) we have obtained two systems of imprimitivity $(\bar{U}, \bar{F})$ and $(\tilde{U}, \tilde{F})$, starting from the same (N, K) system (V, E). We shall show that $(\bar{U}, \bar{F})$ is equivalent to $(\tilde{U}, \tilde{F})$. We assume, without loss of generality, that F is homogeneous of multiplicity n, $1 \leqslant n \leqslant \infty$. Let $\mathcal{C}_\nu$ be the measure class of F. Then by Theorem 3.2, (U, F) is equivalent to a concrete system of imprimitivity with associated measure class $\mathcal{C}_\nu$ and a $(\hat{K}, T, \mathrm{U}(\mathcal{H}_n))$ cocycle C relative to ν. We shall show that $\bar{F}$ is also homogeneous of multiplicity n, with associated measure class $\mathcal{C}_{\tilde{\nu}}$, and that the cocycle associated with $(\bar{U}, \bar{F})$ is cohomologous to the $(\Gamma, R, \mathrm{U}(\mathcal{H}_n))$ cocycle $\tilde{C}$ extended from C. We shall also assume that (V, E) is homogeneous, but our proof with slight modifications will work for the general case. Let $\mathcal{C}_\mu$ be the measure class associated with (V, E) and A the associated $\mathrm{U}(\mathcal{H})$ cocycle.

For any finite complex valued measure ν on T, we shall denote by $\tilde{\nu}$ the measure on R, given by $\tilde{\nu} = (\nu \times \lambda) \circ \xi^{-1}$ as defined in § 4. This means that each interval $[n \cdot 2\pi, (n+1) \cdot 2\pi)$ is given the measure ν.

5.1. LEMMA. *Let ν be a finite complex valued measure on T. Then,*

$$\int_{-\infty}^{\infty} \exp(itx) \frac{(1-\exp(ix))(1-\exp(-i(x-g)))}{x(x-g)} d\tilde{\nu}(x)$$
$$= \frac{\exp(ig) - \exp(ig\langle t \rangle)}{ig} \hat{\nu}([t]) + \frac{\exp(ig\langle t \rangle) - 1}{ig} \hat{\nu}([t]+1), g \in R.$$

Proof. Let f be the function on R such that

$$\int_{-\infty}^{\infty} \exp(itx) f(x) d\tilde{\nu}(x) = \frac{1}{ig}(\exp(ig) - \exp(ig\langle t \rangle)) \hat{\nu}([t]) + \frac{1}{ig}(\exp(ig\langle t \rangle) - 1) \hat{\nu}([t]+1). \quad (6)$$

Now,
$$\int_{-\infty}^{\infty} \exp(itx) f(x) d\tilde{\nu}(x)$$
$$= \sum_{-\infty}^{\infty} \int_{2n\pi}^{2(n+1)\pi} \exp(itx) f(x) d\tilde{\nu}(x)$$
$$= \sum_{-\infty}^{\infty} \int_0^{2\pi} \exp(it(x+2n\pi)) f(x+2n\pi) d\nu(x)$$
$$= \int_0^{2\pi} \exp(i[t]x) \left(\sum_{-\infty}^{\infty} \exp(i\langle t \rangle x) \exp(2\pi int) f(x+2n\pi) \right) d\nu(x),$$

whereas the right hand side of (6) is

$$\int_0^{2\pi} \exp(i[t]x)\left\{\frac{1}{ig}(\exp(ig)-\exp(ig\langle t\rangle))+\frac{\exp(ix)}{ig}(\exp(ig\langle t\rangle)-1)\right\}d\nu(x).$$

Therefore (6) will hold if

$$\sum_{-\infty}^{\infty} \exp(2\pi int)f(x+2n\pi)$$
$$=\exp(-i\langle t\rangle x)\left\{\frac{1}{ig}(\exp(ig)-\exp(ig\langle t\rangle))+\frac{\exp(ix)}{ig}(\exp(ig\langle t\rangle)-1)\right\}.$$

Or,

$$\sum_{-\infty}^{\infty} \exp(2\pi in\langle t\rangle)f(x+2n\pi)$$
$$=\exp(-i\langle t\rangle x)\left\{\frac{1}{ig}(\exp(ig)-\exp(ig\langle t\rangle))+\frac{\exp(ix)}{ig}(\exp(ig\langle t\rangle)-1)\right\}. \tag{7}$$

Now the n^{th} Fourier coefficient of the function

$$\exp(-isx)\left\{\frac{1}{ig}(\exp(ig)-\exp(igs))+\frac{\exp(ix)}{ig}(\exp(igs)-1)\right\},\quad s\in[0,1)$$

is

$$\frac{(1-\exp(ix))(1-\exp(-i(x-g)))}{(x+2n\pi)(x+2n\pi-g)}.$$

Hence if we take

$$f(x)=\frac{1}{x(x-g)}(1-\exp(ix))(1-\exp(-i(x-g))),$$

(7) will be true and hence (6) holds. Therefore, the lemma is proved. Q.E.D.

Let $h_1, h_2 \in L^2(K, \mathcal{H}, \mu)$. Define $\bar{h}_i$ $(i=1,2)$ on B by,

$$\bar{h}_i(x+e_s)=h_i(x),\quad x\in K,\ s\in[0,1).$$

Let χ_g denote the character on B corresponding to $g\in\hat{B}=\Gamma$. Let $g=u+2k\pi$, $u\in\hat{K}$. Let

$$\nu_u^{1,2}(D)=(F(D)\chi_u h_1, h_2),\quad D \text{ a Borel subset of } T$$
$$\bar{\nu}_g^{1,2}(D')=(\bar{F}(D')\chi_g\bar{h}_1, \bar{h}_2),\quad D' \text{ a Borel subset of } R.$$

($\langle\cdot,\cdot\rangle$ will denote the inner product in $\mathcal{H}$; $(\cdot,\cdot)$ will stand for the inner product in $L^2(K, \mathcal{H}, \mu)$ or $L^2(B, \mathcal{H}, \bar{\mu})$.)

Remarks. 1. The measures $\nu_u^{1,2}$ and $\bar{\nu}_g^{1,2}$ are not the translates of the measures $\nu^{1,2}(=\nu_0^{1,2})$ and $\bar{\nu}^{1,2}(=\bar{\nu}_0^{1,2})$ respectively. 2. Except in Lemma 5.1, measures appearing without superscripts are always non-negative and subscripts to them mean their translates.

5.2. LEMMA.

$$\widehat{\nu_g^{1,2}}(t)=\frac{1}{ig}(\exp(ig)-\exp(ig\langle t\rangle))\widehat{\nu_u^{1,2}}([t])+\frac{1}{ig}(\exp(ig\langle t\rangle)-1)\widehat{\nu_u^{1,2}}([t]+1).$$

Proof.

$$\widehat{\nu_g^{1,2}}(t)=(\bar V_t\chi_g\bar h_1,\bar h_2)=\int_B\langle(\bar V_t\chi_g\bar h_1)(x),\bar h_2(x)\rangle\,d\bar\mu(x)$$

$$=\int_B\left\langle\bar A(t,x)\sqrt{\frac{d\bar\mu_t}{d\bar\mu}}(x)\,\chi_g(x+e_t)\,\bar h_1(x+e_t),\bar h_2(x)\right\rangle d\bar\mu(x).$$

By Lemma 4.3, this is

$$=\int_0^1\int_K\left\langle A([t+s],y)\sqrt{\frac{d\mu_{[t+s]}}{d\mu}}(y)\,\chi_g(y+e_{[s+t]}+e_{\langle s+t\rangle})\,h_1(y+e_{[s+t]}),h_2(y)\right\rangle d\mu(y)\,ds,$$

where $\quad x=y+e_s,\ y\in K,\ s\in[0,1)$

$$=(V_{[t]}\chi_u h_1,h_2)\int_0^{1-\langle t\rangle}\chi_g(e_{\langle s+t\rangle})\,ds+(V_{[t]+1}\chi_u h_1,h_2)\int_{1-\langle t\rangle}^1\chi_g(e_{\langle s+t\rangle})\,ds$$

$$=\frac{1}{ig}(\exp(ig)-\exp(ig\langle t\rangle))\widehat{\nu_u^{1,2}}([t])+\frac{1}{ig}(\exp(ig\langle t\rangle)-1)\widehat{\nu_u^{1,2}}([t]+1).$$

Q.E.D.

In view of Lemma 5.1, we have

COROLLARY 1. *The two measures $d\tilde\nu_g^{1,2}$ and*

$$\frac{(1-\exp(ix))(1-\exp(-i(x-g)))}{x(x-g)}\,d\tilde\nu_u^{1,2}$$

on R are the same, and

$$\frac{d\tilde\nu_g^{1,2}}{d\tilde\nu_u^{1,2}}(x)=\frac{(1-\exp(ix))(1-\exp(-i(x-g)))}{x(x-g)},\ g=u+2k\pi.$$

(Here and in the sequel $\tilde\nu_u^{i,j}$ means $\widetilde{\nu_u^{i,j}}$.)

COROLLARY 2. *When $g=0$ we get,*

$$\frac{d\tilde\nu^{1,2}}{d\tilde\nu^{1,2}}(x)=\frac{(1-\exp(ix))(1-\exp(-ix))}{x^2}=\left(\frac{\sin(x/2)}{x/2}\right)^2.$$

(By $i=1,2,\ldots,n$ we shall mean $i=1,2,\ldots,n$ if n is finite and $i=1,2,\ldots$ if n is infinite. Similarly $h_1,\ldots,h_n$ means a finite sequence if n is finite and an infinite sequence if $n=\aleph_0$.) From Lemma 5.2 we get

COROLLARY 3. *Suppose F is homogeneous of multiplicity n, $1\leqslant n\leqslant\infty$. Let $h_1,\ldots,h_n\in$ $L^2(K,\mathcal{H},\mu)$* be such that

(i) $(V_k h_i, h_j) = 0$ $i \neq j$, $1 \leqslant i, j \leqslant n$, *for all integers* k.

(ii) *the measures* $\nu(\cdot) = (F(\cdot)h_i, h_i)$ *are all same on* T, $i = 1, 2, \ldots, n$; *and*

(iii) *the closed linear span of* $\{V_k h_i : k \text{ integer}, i = 1, 2, \ldots, n\}$ *is* $L^2(K, \mathcal{H}, \mu)$.

Then: (1) $(\bar{V}_t \bar{h}_i, \bar{h}_j) = 0$ *if* $i \neq j$, *for all* t; *and*

(2) $\bar{\nu}(\cdot) = (\bar{F}(\cdot)\bar{h}_i, \bar{h}_i)$ *are same for each* $i = 1, 2, \ldots, n$.

5.3. **Lemma.** *If* $\nu(\{0\}) = 0$, *then* $\bar{F}$ *is homogeneous of multiplicity* n.

Proof. In view of Corollary 3, it is enough to show that the closed linear span of $\{\bar{V}_t \bar{h}_i : t \in R,\ i = 1, 2, \ldots, n\}$ is $L^2(B, \mathcal{H}, \bar{\mu})$. Let $f \in L^2(B, \mathcal{H}, \bar{\mu})$ be such that $(\bar{V}_t \bar{h}_i, f) = 0$ for all $t \in R$ and $i = 1, \ldots, n$. Let m be an integer, and let t_0, s_0 be real numbers such that $m \leqslant t_0 < s_0 < m+1$. Then,

$$
\begin{aligned}
0 &= ((\bar{V}_{t_0} - \bar{V}_{s_0})\bar{h}_i, f) \\
&= \int_B \langle (\bar{V}_{t_0} - \bar{V}_{s_0})\bar{h}_i(x), f(x)\rangle \, d\bar{\mu}(x) \\
&= \int_0^1 \int_K \left\{ \left\langle A([t_0+s], y) \sqrt{\frac{d\mu_{[t_0+s]}}{d\mu}}(y)\, h_i(y + e_{[t_0+s]}), f(y, s) \right\rangle \right. \\
&\qquad \left. - \left\langle A([s_0+s], y) \sqrt{\frac{d\mu_{[s_0+s]}}{d\mu}}(y)\, h_i(y + e_{[s_0+s]}), f(y, s) \right\rangle \right\} d\mu(y)\, ds \\
&= \int_\alpha^\beta ds \left(\int_K \left\{ \left\langle A(m+1, y) \sqrt{\frac{d\mu_{m+1}}{d\mu}}(y)\, h_i(y + e_{m+1}), f(y, s) \right\rangle \right. \right. \\
&\qquad \left. \left. - \left\langle A(m, y) \sqrt{\frac{d\mu_m}{d\mu}}(y)\, h_i(y + e_m), f(y, s) \right\rangle \right\} d\mu(y) \right) \\
&= \int_\alpha^\beta ((V_{m+1} h_i, f_s) - (V_m h_i, f_s))\, ds
\end{aligned}
$$

where $\alpha = m + 1 - s_0$, $\beta = m + 1 - t_0$ and f_s is the sth section of f. Varying t_0 and s_0, this is true for all α, β with $0 \leqslant \alpha < \beta < 1$. Hence the inside integral vanishes for a.e. $s \in [0, 1)$. That is, $(V_{m+1} h_i, f_s) = (V_m h_i, f_s)$ for a.e. $s \in [0, 1)$. Or, for each m, and for each $i = 1, 2, \ldots, n$,

$$(V_m h_i, f_s) = (V_0 h_i, f_s) \quad \text{for a.e. } s \in [0, 1).$$

Hence $f_s \in F(0)(L^2(K, \mathcal{H}, \mu))$ for a.e. s. Since $F(0) = 0$, $f_s = 0$ a.e. s. That is, $f(y, s) = 0$ a.e. (y, s). Q.E.D.

Let $h_1, \ldots, h_n \in L^2(K, \mathcal{H}, \mu)$ be as in Corollary 3; $\bar{h}_i$ are defined as before. Assume that $\nu(\{0\}) = 0$. Let S be the isometric isomorphism from $L^2(K, \mathcal{H}, \mu)$ onto $L^2(T, \mathcal{H}_n, \nu)$ defined by:

$$SF(D)h_i = (0, \ldots, 1_D, \ldots, 0), \quad D \text{ being a Borel subset of } T.$$

S takes h_i to $(0, \ldots, 1, 0, \ldots, 0) \in L^2(T, \mathcal{H}_n, \nu)$, where 1 occurs in the ith place.

For every Borel subset D of T,

$$\nu_u^{i,j}(D) = (F(D)\chi_u h_i, h_j) = (F(D)\chi_u h_i, F(D)h_j)$$

$$= \int_D \left\langle C(-u,x)\sqrt{\frac{d\nu_{-u}}{d\nu}}(x)\,Sh_i(x-u),\, Sh_j(x)\right\rangle d\nu(x) = \int_D c_{ij}(-u,x)\sqrt{\frac{d\nu_{-u}}{d\nu}}(x)\,d\nu(x),$$

where $C(u,x) = (c_{ij}(u,x))$, $1 \leqslant i, j \leqslant n$.

So
$$\frac{d\nu_u^{i,j}}{d\nu}(x) = c_{ij}(-u,x)\sqrt{\frac{d\nu_{-u}}{d\nu}}(x) \quad \nu \text{ a.e. } x \in T.$$

Or,
$$c_{ij}(u,x) = \frac{d\nu_{-u}^{i,j}}{d\nu}(x)\sqrt{\frac{d\nu}{d\nu_u}}(x) \quad \nu \text{ a.e. } x \in T.$$

Similarly proceeding with $\bar{h}_i$ etc. we get,

$$\bar{c}_{ij}(g,y) = \frac{d\bar{\nu}_{-g}^{i,j}}{d\bar{\nu}}(y)\sqrt{\frac{d\bar{\nu}}{d\bar{\nu}_g}}(y), \quad \text{where } \bar{C}(g,y) = (\bar{c}_{ij}(g,y))$$

is the cocycle given by the system of imprimitivity $(\bar{U}, \bar{F})$ on $(\Gamma. R)$. Hence, if $g = u + 2k\pi$, $-g = z + 2l\pi$, $0 \leqslant u, z < 2\pi$, then regarded as points in T we have $z = -u$, and

$$\bar{c}_{ij}(g,y) = \left(\frac{d\bar{\nu}_{-g}^{i,j}}{d\tilde{\nu}_z^{i,j}}\frac{d\tilde{\nu}_z^{i,j}}{d\tilde{\nu}}\frac{d\tilde{\nu}}{d\bar{\nu}}\sqrt{\frac{d\bar{\nu}}{d\tilde{\nu}}\frac{d\tilde{\nu}}{d\tilde{\nu}_u}\frac{d\tilde{\nu}_u}{d\bar{\nu}_g}}\right)(y) = \left(\frac{d\bar{\nu}_{-g}^{i,j}}{d\tilde{\nu}_z^{i,j}}\frac{d\tilde{\nu}}{d\bar{\nu}}\sqrt{\frac{d\bar{\nu}}{d\tilde{\nu}}\frac{d\tilde{\nu}_u}{d\bar{\nu}_g}}\frac{d\tilde{\nu}_z^{i,j}}{d\tilde{\nu}}\sqrt{\frac{d\tilde{\nu}}{d\tilde{\nu}_u}}\right)(y)$$

$$= \left(\frac{d\bar{\nu}_{-g}^{i,j}}{d\tilde{\nu}_z^{i,j}}\frac{d\tilde{\nu}}{d\bar{\nu}}\sqrt{\frac{d\bar{\nu}}{d\tilde{\nu}}\frac{d\tilde{\nu}_u}{d\bar{\nu}_g}}\right)(y)\,\tilde{c}_{ij}(g,y),$$

where $(\tilde{c}_{ij}(g,y)) = \tilde{C}(g,y)$ is the (Γ, R) cocycle extended from C. Using Corollaries 1 and 2 and simplifying we get,

$$\bar{c}_{ij}(g,y) = f(y)\,\tilde{c}_{ij}(g,y)\,f(y+g)^{-1}$$

where,
$$f(y) = \frac{|1-\exp(-iy)|}{1-\exp(-iy)}.$$

Therefore, $\bar{C}(g,y) = f(y)\tilde{C}(g,y)f(y+g)^{-1}$ and so $\bar{C}$ and $\tilde{C}$ are cohomologous. Therefore, we have

5.4. THEOREM. *Let (V, E) be an (N, K) system of imprimitivity such that its dual (U, F) is of uniform multiplicity and $F(0) = 0$. Let $(\tilde{V}, \tilde{E})$ be the Gamelin extension of (V, E), and $(\tilde{U}, \bar{F})$ its dual. Then $(\bar{U}, \bar{F})$ and $(\tilde{U}, \tilde{F})$ are equivalent systems of imprimitivity.*

If $\nu(\{0\}) \neq 0$, then ν can be decomposed into two quasi-invariant measures ν_1 and ν_2 such that ν_1 and ν_2 are mutually singular and ν_1 is concentrated on $\hat{K}$ and is equivalent to the Haar measure on $\hat{K}$. All $(\hat{K}, T)$ cocycles relative to ν_1 and all (Γ, R) cocycles relative to $\tilde{\nu}_1$ are coboundaries. Observe that $F(\hat{K})$ reduces the system (U, F) and hence it reduces (V, E) also. Let (U', F') be the restriction of (U, F) to $F(\hat{K})$, and (V', E') that of (V, E) to $F(\hat{K})$. Then, (U', F') and (V', E') are duals of each other. Clearly, F' is homogeneous of multiplicity n with measure class C_{ν_1} and hence cocycle of (U', F') is a coboundary. It is easy to see that E' is also homogeneous of multiplicity n, the measure class of E' is the Haar measure class on K and that the cocycle associated with (V', E') is a coboundary. Hence the multiplicity of $\bar{E}'$ is n with measure class same as the Haar measure class on B and the cocycle of $(\bar{V}', \bar{E}')$ is a coboundary. It follows that the dual system $(\bar{U}', \bar{F}')$ of $(\bar{V}', \bar{E}')$ is of uniform multiplicity n with associated measure class $C_{\tilde{\nu}}$. Hence the cocycle associated with $(\bar{U}', \bar{F}')$ is a coboundary. Thus $(\tilde{U}', \tilde{F}')$ and $(\bar{U}', \bar{F}')$ are equivalent.

Hence, combined with Theorem 5.4, we have:

5.5. THEOREM. *Let (V, E) be an (N, K) system of imprimitivity such that its dual (U, F) is of uniform multiplicity. Let $(\bar{V}, \bar{E})$ be the Gamelin extension of (V, E) and $(\bar{U}, \bar{F})$ its dual. Then $(\bar{U}, \bar{F})$ and $(\tilde{U}, \tilde{F})$ are equivalent systems of imprimitivity.*

Now we state the general theorem, proof of which is immediate.

5.6. THEOREM. *Let (V, E) be an (N, K) system of imprimitivity; (U, F) its dual. Let $(\bar{V}, \bar{E})$ be the Gamelin extension of (V, E) and $(\bar{U}, \bar{F})$ be its dual. Then $(\bar{U}, \bar{F})$ and $(\tilde{U}, \tilde{F})$ are equivalent systems of imprimitivity.*

Proof of Theorem 4.4. Let A be an $(R, B, \mathrm{U}(\mathcal{H}))$ cocycle and let (V, E) be the system of imprimitivity given by A. Let (U, F) be the (Γ, R) system of imprimitivity which is the dual of (V, E). (U, F) is equivalent to a (Γ, R) concrete system of imprimitivity $(\tilde{U}_0, \tilde{F}_0)$ for some $(\hat{K}, T)$ concrete system of imprimitivity (U_0, F_0). Let (V_0, E_0) be the dual of (U_0, F_0). Then by our theorem, $(\bar{V}_0, \bar{E}_0)$ and (V, E) are equivalent. Now, (V_0, E_0) is equivalent to a concrete system of imprimitivity given by an $(N, K, \mathrm{U}(\mathcal{H}))$ cocycle A_0. We can take A_0 to be strict since N is countable. Therefore, we see that A is cohomologous to the strict $(R, B, \mathrm{U}(\mathcal{H}))$ cocycle $\bar{A}_0$. Since A is cohomologous to a strict cocycle, A has a strict version. The other assertions in the theorem follow similarly.

§ 6

6.1. *Example.* Let Γ be the following subgroup of R, $\Gamma = \{2\pi m + n\colon m, n \text{ integers}\}$. Γ is dense in R. Γ can be identified with $N \times N$ and its dual T^2 can be written as $T^2 = [0, 1) \times [0, 2\pi)$ where elements (x_1, y_1) and (x_2, y_2) are added according to

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2 (\mathrm{mod}\ 1),\ y_1 + y_2 (\mathrm{mod}\ 2\pi)).$$

We have:

$\langle 2\pi m + n, (x, y)\rangle = \exp(2\pi i \cdot mx) \exp(iny)$ for $2\pi m + n \in \Gamma$ and $(x, y) \in T^2$. For $t \in R$, $\langle 2\pi m + n, e_t\rangle = \exp(it(2\pi m + n))$. This describes the pair (R, T^2). The annihilator of the subgroup

$\Gamma_0 = \{2\pi m: m \in N\}$ is $K = \{0\} \times [0, 2\pi)$, and the pair (N, K) is described by $n \to e_n = (0, n(\mathrm{mod}\ 2\pi))$. Thus, in this case (N, K) can be identified with (N, T). Hence the dual pair $(\hat{K}, T)$ can also be identified with (N, T).

Let q be a non-constant inner function on the circle. Let A_q be the cocycle defined by:

$$A_q(n, z) = \begin{cases} (q(z) q(z+e_1) \dots q(z+e_{n-1}), & n > 0 \\ 1, & n = 0 \\ q(z+e_{-1})^{-1} \dots q(z+e_n)^{-1}, & n < 0. \end{cases}$$

Let (V, E) be the system of imprimitivity given by A_q. Let $H_2(T)$ be the Hardy space. Then,

$$V_1 H_2(T) = \{q(\cdot) f(\cdot + e_1): f \in H_2(T)\} = q \cdot H_2(T) \subseteq H_2(T).$$

Let $(f_1, f_2, \dots)$ (this set may be finite) be a complete orthonormal system of vectors in $H_2(T) \ominus q \cdot H_2(T)$. Then the cyclic subspaces $\{V_n f_i: n \in N\}$ for $i = 1, 2, \dots$ are mutually orthogonal, and together span $L^2(T)$. Also $(V_n f_i, f_i) = \delta_{on}$ for each i. Thus, if F is the spectral measure corresponding to V, then F is homogeneous with multiplicity same as the dimension of $H_2(T) \ominus q \cdot H_2(T)$, and the measure class associated with F is the Haar measure class on T. Since (V, E) is irreducible, (U, F), the dual system of (V, E), is an irreducible system of imprimitivity based on (N, T).

(1) If q has infinitely many zeros in the disc, then $H_2(T) \ominus q \cdot H_2(T)$ is infinite dimensional. So we have an irreducible system of imprimitivity based on (N, T) and acting in $L^2(T, l^2)$.

(2) If $q(z) = \exp(ipz)$, p a positive integer, then $H_2(T) \ominus q \cdot H_2(T)$ is p-dimensional. In this case we shall calculate the cocycle C associated with (U, F).

For each k, $0 \leqslant k \leqslant p-1$, let h_k be the function $h_k(z) = \exp(ikz)$, $z \in T$. Then $h_0, h_1, \dots, h_{p-1}$ have the properties (i), (ii) and (iii) of Corollary 3. Let S be the isometric isomorphism from $L^2(T)$ onto $L^2(T, C^p)$ defined by

$$SF(D) h_k = (0, \dots, 1_D, 0, \dots, 0)$$

(1_D appears at the $(k+1)$th place), where D is a Borel subset of T; $k = 0, 1, \dots, p-1$. Then,

$$S(V_n h_k) = (0, \dots, \exp(inz), 0, \dots, 0), \quad k = 0, 1, \dots, p-1.$$

We have $(U_n h)(z) = \exp(inz)h(z)$, $h \in L^2(T)$, and

$$S U_n S^{-1} \tilde{h}(z) = C(n, z)\tilde{h}(z + e_n), \quad \tilde{h} \in L^2(T, \mathbf{C}^p).$$

Hence, taking $\tilde{h}$ to be $(1, 0, ..., 0)$, $(0, 1, ..., 0)$, ..., $(0, 0, ..., 1)$ we get

$$C(1, z) = \begin{pmatrix} 0, & 0, & \dots, & 0, & \exp(iz) \\ 1, & 0, & \dots, & 0, & 0 \\ \dots & \dots & & & \dots \\ \dots & \dots & & & \dots \\ 0, & 0, & \dots, & 1, & 0 \end{pmatrix}, \quad z \in T.$$

The cocycle C can be written in terms of $C(1, z)$. Thus, in this case we can calculate all the four cocycles.

We mention that [9] contains a construction, due to A. M. Gleason, of an (N, T) cocycle having values in 2×2 unitary matrices giving rise to irreducible systems of imprimitivity of dimension 2. The construction was communicated to A. A. Kirilov by G. W. Mackey. The paper also quotes results of O. P. Chopenko modifying this construction to exhibit irreducible systems of dimension p for any positive integer p. In Gleason's example the proof of irreducibility is direct whereas we prove irreducibility by referring to the dual system. In [11] Muhly uses Gleason's example together with Gamelin's method of obtaining an (R, B) cocycle, to exhibit an irreducible (R, B) system of imprimitivity of multiplicity 2.

§ 7

Main results of this paper are valid more generally. For example, we can take Γ to be a countable dense subgroup of R^n and $B = \hat{\Gamma}$, where Γ is given discrete topology. R^n is then densely imbedded in B, and we have dual pairs (Γ, R^n) and (R^n, B). We can again assume, without loss of generality, that the vectors $\gamma_k = (0, ..., 0, 2\pi, 0, ..., 0)$, where 2π is in the kth place, belong to Γ, and let K be the annihilator of Γ_0 = the group generated by $\{\gamma_1, ..., \gamma_n\}$. Then N^n is densely imbedded in K and dual pair of (N^n, K) is $(\hat{K}, T^n)$, $\hat{K} = \Gamma/\Gamma_0$. By modifying arguments of this paper suitably, it can be shown that systems of imprimitivity on (N^n, K), (R^n, B), (Γ, R^n) and $(\hat{K}, T^n)$ are connected with each other in a natural fashion.

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