POLYTOPE PAIRS AND THEIR RELATIONSHIP TO LINEAR PROGRAMMING

BY

VICTOR KLEE

University of Washington, Seattle, Washington 98195, USA

Introduction

As the terms are used here, a *polyhedron* is the intersection of a finite number of closed halfspaces in a finite-dimensional real vector space, a *pointed polyhedron* is one whose vertex set is nonempty, and a *polytope* is a bounded polyhedron; equivalently, a polytope is the convex hull of a finite set of points. Prefixes indicate dimension, and the $(d-1)$-faces of a d-polyhedron are its *facets*. A polyhedron of *class* (d, n) is one that is pointed, d-dimensional, and has precisely n facets; necessarily, $n \geqslant d$, with $n > d$ in the case of polytopes. A pointed d-polyhedron is *simple* provided that each of its vertices is incident to precisely d edges or, equivalently, to precisely d facets. A polytope is *simplicial* provided that each of its facets is a simplex. For properties of polyhedra and polytopes that are used here without explicit reference, are Grünbaum [10]. In particular, basic properties of the duality or polarity of polytopes are used freely [10, pp. 46–49].

Two landmarks in the theory of polytopes were the proofs that as P ranges over all simple polytopes of class (d, n), the minimum and maximum of $v(P)$ (number of vertices of P) are equal respectively to

$$(n-d)(d-1)+2$$

and to

$$\gamma(d, n) = \binom{n - \left[\frac{d+1}{2}\right]}{n-d} + \binom{n - \left[\frac{d+2}{2}\right]}{n-d}.$$

These results, due respectively to Barnette [1] and McMullen [22], are here extended to certain pairs consisting of a polytope and one of its facets.

For $3 \leqslant d \leqslant u < n$, a pair (P, F) is called a *polytope pair of class* (d, n, u) provided that P is a simple polytope of class (d, n) and F is a facet intersecting precisely u other facets of P; F is then a simple polytope of class $(d-1, u)$. The set of all such pairs is denoted by

$\mathbf{P}(d, n, u)$. Part of the interest in polytope pairs arises from the fact that if $(P, F) \in \mathbf{P}(d, n, u)$ and T is a projective transformation carrying F into the hyperplane at infinity, then $P \sim F$ is carried by T onto an unbounded polyhedron Q of class $(d, n-1)$ having precisely u unbounded facets; conversely, each such Q is projectively equivalent to $P \sim F$ for some $(P, F) \in \mathbf{P}(d, n, u)$.

Polytope pairs of class $(d, n, n-1)$ are called *Kirkman pairs of class* (d, n), and the fact that (P, F) is a Kirkman pair is also expressed by saying that P is a *Kirkman polytope with base* F, or *based on* F. Kirkman 3-polytopes were studied in detail by Kirkman [12], [13], Rademacher [26], and others, and they are closely related to a number of combinatorial or algebraic problems that seem at first to have no geometric content (Brown [3], Ordman [25]). As we shall see, Kirkman d-polytopes are related to several aspects of linear programming.

The main results of the present paper are stated below. The assertions about minima and maxima are proved in sections 1 and 2 respectively, and section 3 discusses some connections between Kirkman pairs and neighborly polytopes. The final section 4 is concerned with the relationships of polytope pairs to linear programming, including the d-step conjecture and a recent algorithm of Mattheiss [21] for finding all vertices of a polytope defined by a system of linear inequalities. (For background material on the relationship of polytopes to linear programming, see Dantzig [5] and Klee [16].)

THEOREM 1. *Suppose* $3 \leqslant d \leqslant u < n$. *As* (P, F) *ranges over all polytope pairs of class* (d, n, u), *the minima and maxima of certain functions are as follows:*

function	*minimum*	*maximum*
$v(F)$	$(u-d)(d-2)+d$	$\gamma(d-1, u)$
$v(P)$	$(n-d)(d-1)+2$	(*see Theorem* 3)
$v(P \sim F)$	$(n-u-2)(d-1)+u$	$\gamma(d, n-1)+d-u-1$
$\dfrac{v(P \sim F)}{v(F)}$	(*see Theorem* 2)	$\dfrac{\gamma(d, n-1)+d-u-1}{(u-d)(d-2)+d}$

THEOREM 2. *For* $3 \leqslant d \leqslant t \leqslant u < n$, *let*

$$\beta(d, t, u, n) = \frac{\binom{t-1-[d/2]}{[(d-1)/2]} + u - t + (n-u-1)(d-1)}{\gamma(d-1, t) + (u-t)(d-2)}.$$

Then
$$\min_{(P, F) \in \mathbf{P}(d, n, u)} \frac{v(P \sim F)}{v(F)} \leqslant \min_{d \leqslant t \leqslant u} \beta(d, t, u, n),$$

with equality if $d \leqslant 4$ *or* $u = d$ *or* $u = n-1$. *(In these cases both minima are equal to* $\beta(d, d, u, n)$.) *For all* d,

$$\lim_{d,\,u \text{ fixed},\, n\to\infty} \frac{1}{n} \min_{(P,F)\in \mathbf{P}(d,n,u)} \frac{v(P \sim F)}{v(F)} = \frac{d-1}{\gamma(d-1, u)}$$

and

$$\lim_{d,\,n-u \text{ fixed},\, n\to\infty} \min_{(P,F)\in \mathbf{P}(d,n,u)} \frac{v(P \sim F)}{v(F)} = \frac{1}{d-2}.$$

THEOREM 3. *For* $3 \leqslant d \leqslant u < n$,

$$\gamma(d, n-1) + (u-d)(d-3) + d - 1 \leqslant \max_{(P,F)\in \mathbf{P}(d,n,u)} v(P) \leqslant \gamma(d, n),$$

with equality on the left if $d \leqslant 5$ *or* $u = d$ *and equality on the right if and only if* $d = 3$ *or* $u = n-1$. *Also*

$$\max_{(P,F)\in \mathbf{P}(d,d+3,d+1)} v(P) = \begin{cases} 2k^2 + 2k + 1 & \text{when } d = 2k \\ 2k^2 & \text{when } d = 2k-1. \end{cases}$$

COROLLARY 1. *Suppose* $3 \leqslant d \leqslant u < n$. *As* P *ranges over all simple polyhedra of class* (d, n) *having precisely* u *unbounded facets, the minimum and maximum of* $v(P)$ *are respectively*

$$(n-u-2)(d-1)+u \quad \text{and} \quad \gamma(d, n-1)+d-u-1.$$

COROLLARY 2. *Suppose* $2 \leqslant d < n$. *As* (P, F) *ranges over all Kirkman pairs of class* $(d+1, n+1)$, *the minima and maxima of certain functions are as follows:*

function	*minimum*	*maximum*
$v(F)$	$(n-d)(d-1)+2$	$\gamma(d, n)$
$v(P)$	$(n-d)d+2$	$\gamma(d+1, n+1)$
$v(P \sim F)$	$n-d$	$\gamma(d+1, n)+d-n$
$\dfrac{v(P \sim F)}{v(F)}$	$\dfrac{n-d}{(n-d)(d-1)+2}$	$\dfrac{\gamma(d+1, n)+d-n}{(n-d)(d-1)+2}$

Though our main results are all stated in terms of simple polytopes or polyhedra, most of the proofs involve dual formulations in terms of simplicial polytopes. Since many of the results can be extended, at least in part, to simplicial complexes more general than the boundary complexes of simplicial polytopes (see Barnette [2] and Klee [15] for an indication of methods), there would have been advantages in emphasizing the simplicial rather than the simple approach. Nevertheless, I have chosen to emphasize the simple approach because of its greater intuitive appeal and its more obvious relevance to linear programming.

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1. Minima

With $3 \leqslant d \leqslant t \leqslant u < n$, suppose that (P_t, F_t) is a polytope pair of class $(d, t+1, t)$. For $t < i \leqslant u$, form (P_i, F_i) from (P_{i-1}, F_{i-1}) by truncating P_{i-1} at a vertex of F_{i-1}, whence

$$v(F_i) = v(F_{i-1}) + d - 2 \quad \text{and} \quad v(P_i \sim F_i) = v(P_{i-1} \sim F_{i-1}) + 1.$$

For $u < j < n$, form (P_j, F_j) from (P_{j-1}, F_{j-1}) by truncating P_{j-1} at a vertex of $P_{j-1} \sim F_{j-1}$, whence

$$v(F_j) = f(F_{j-1}) \quad \text{and} \quad v(P_j \sim F_j) = v(P_{j-1} \sim F_{j-1}) + d - 1.$$

It follows that F_{n-1} and P_{n-1} are simple polytopes of classes $(d-1, u)$ and (d, n) respectively, with

$$v(F_{n-1}) = v(F_t) + (u-t)(d-2) \quad \text{and} \quad v(P_{n-1}) = v(P_t) + u - t + (n-u-1)(d-1).$$

If the above construction is started from a pair of class $(d, d+1, d)$—that is, if P_t is a d-simplex and F_t one of its facets—then

$$v(F_{n-1}) = (u-d)(d-2) + d \quad \text{and} \quad v(P_{n-1} \sim F_{n-1}) = (n-u-2)(d-1) + u.$$

Hence the first three minima of Theorem 1 do not exceed the values stated there. It is immediate from Barnette's theorem [1] that the minima of $v(F)$ and $v(P)$ are equal to the stated values.

In discussing the minima of $v(P \sim F)$ and $v(P \sim F)/v(F)$, we consider first the case in which $u = n-1$. Then $P \sim F$ is projectively equivalent to an unbounded simple polyhedron of class $(d, n-1)$ and it follows from a remark of Klee [17, p. 230] that

$$v(P \sim F) \geqslant (n-1) - d + 1 = n - d,$$

the desired conclusion in this instance. To handle $v(P \sim F)/v(F)$, let G denote the graph formed by the vertices and edges of P that are disjoint from F, and note that G is connected. Let $r = v(G) = v(P \sim F)$ and let $k_1, \ldots, k_r$ denote the G-valences of the various vertices of G, whence $\Sigma_{i=1}^r k_i \geqslant 2r - 2$ by a general property of connected graphs. Since each vertex of P is d-valent in P, and each vertex of F is joined to G by a unique edge of P, it follows that

$$v(F) = \sum_{i=1}^{r} (d - 1 - k_i) \leqslant r(d-1) + 2$$

and $$\frac{v(P \sim F)}{v(F)} \geqslant \frac{r}{r(d-1)+2} \geqslant \frac{n-d}{(n-d)(d-1)+2}.$$

We assume henceforth that $u < n-1$.

It is easily verified that a vertex p of $P \sim F$ has precisely $d-1$ neighbors in F if and only if p is a vertex of a $(d-1)$-simplex S that is a facet of P intersecting F. Any such vertex p is called *special*. A special vertex and the associated facet F can be "removed" by constructing a pair (P_1, F_1) whose combinatorial structure is obtained from that of (P, F) by collapsing S into a single vertex of F_1 and making the appropriate adjustments in the other faces of P that intersect S. The simple polytopes P_1 and F_1 will be of classes $(d, n-1)$ and $(d-1, u-1)$ respectively, F_1 being a facet of P_1 such that

$$v(F_1) = v(F) - (d-2) \quad \text{and} \quad v(P_1 \sim F_1) = v(P \sim F) - 1.$$

To effect the removal of S, let H_0 be the hyperplane determined by S and let $H_1, \ldots, H_d$ be the hyperplanes determined by the other d facets of P that intersect S. By slightly perturbing these facets if necessary, we may assume that $\bigcap_1^d H_j$ is nonempty, whence it consists of a single point q. If q is on the opposite side of H_0 from P itself, let $P_1 = \text{con}\,(P \cup \{q\})$ and $F_1 = \text{con}\,(F \cup \{q\})$. If q is on the same side of H_0 as P, then (since P is not a simplex) P is disjoint from the hyperplane through q parallel to H_0 and the situation is easily reduced by a projective transformation to the one just considered. (The removal process can be described even more easily in terms of a simplicial polytope polar to P.)

If $P_1 \sim F_1$ has a special vertex, another facet of P_1 is removed, and after $k(\geqslant 0)$ steps of this sort we arrive at a pair (P_k, F_k) consisting of simple polytopes of classes $(d, n-k)$ and $(d-1, u-k)$ respectively, F_k being a facet of P_k such that

$$v(F) = v(F_k) + k(d-2), \tag{1}$$

$$v(P \sim F) = v(P_k \sim F_k) + k, \tag{2}$$

and $P_k \sim F_k$ does not have a special vertex. If some vertex of $P_k \sim F_k$ has d neighbors in F_k, then P_k is a simplex, whence $u-k=d$ and $n-k=d+1$. But then $u=n-1$, a case that has already been settled. Thus we assume henceforth that no vertex of $P_k \sim F_k$ has more than $d-2$ neighbors in F_k.

Let l $\langle$resp. $m\rangle$ denote the number of vertices of $P_k \sim F_k$ that have more than one $\langle$resp precisely one$\rangle$ neighbor in F_k. Then

$$v(F_k) \leqslant m + l(d-2). \tag{3}$$

By Barnette's theorem [1],

$$v(F_k) \geqslant (u-k-d+1)(d-2)+2 = (u-k-d)(d-2)+d, \tag{4}$$

and hence with $s=u-d-k-l$ it follows from (3) that

$$l+m \geqslant v(F_k)-l(d-2)+l \geqslant u-k+s(d-3). \tag{5}$$

Let $\mathbf{C}$ denote the complex formed by the $n-u$ facets of P_k that miss F_k, along with all faces of those facets, and let i denote the number of vertices of $\mathbf{C}$ that are *interior* in the sense that all their neighbors belong to $\mathbf{C}$. It is easily verified (for example, by looking at the polar of P_k) that $\mathbf{C}$ is a *strong* $(d-1)$-*cell complex* in the sense of Sallee [27, p. 470], whence it follows from the reasoning of Barnette [1, p. 123] that

$$i \geqslant (n-u-2)(d-1) \tag{6}$$

and $\qquad$ $\mathbf{C}$ has at least d exterior vertices. $\qquad$ (7)

Now if $u-d-k-l \leqslant 0$ it follows from (2), (6), and (7) that

$$v(P \sim F) = v(P_k \sim F_k)+k \geqslant v(\mathbf{C})+k+l \geqslant i+d+k+l \geqslant (n-u-2)(d-1)+u,$$

while if $u-d-k-l=s \geqslant 0$ the desired conclusion about $v(P \sim F)$ follows by combining (2), (5), and (6) to show that

$$v(P \sim F) = v(P_k \sim F_k)+k \geqslant i+k+l+m \geqslant (n-u-2)(d-1)+u+s(d-3).$$

The minimum of $v(P \sim F)/v(F)$ must still be considered for the case in which $u<n-1$. Note, however, that the results already established are enough to justify the statements about minima in Corollaries 1 and 2.

With $3 \leqslant d \leqslant t \leqslant u < n$, let C_t be a cyclic $(d-1)$-polytope whose t vertices lie on the moment curve M_{d-1} in R^{d-1}. It follows from the reasoning of Gale [8, p. 227] that the simplicial polytope C_t has a total of $\gamma(d-1, t)$ facets and the number of facets missing the first vertex (in the natural order on M_{d-1}) of C_t is equal to

$$\binom{t-1-[d/2]}{[(d-1)/2]}.$$

(See Proposition 1 in section 3.)

Let F_t be a polytope dual to C_t, so that F_t is a simple polytope of class $(d-1, t)$, and let G_t be the facet of F_t that corresponds under the duality to the first vertex of C_t. Let P_t be a *wedge over* F_t *with foot* G_t, in the sense of Klee and Walkup [19, p. 57–58]. whence (P_t, F_t) is a polytope pair of class $(d, t+1, t)$ with

$$v(F_t) = \gamma(d-1, t) \text{ and } v(P_t \sim F_t) = v(F_t \sim G_t) = \binom{t-1-[d/2]}{[(d-1)/2]}.$$

The inequality of Theorem 2 then follows from the construction in the first paragraph of this section. For the first limit assertion, note that if $(P, F) \in \mathbf{P}(d, n, u)$ then

$$\frac{(n-u-2)(d-1)+u}{\gamma(d-1,u)} \leqslant \frac{v(P\sim F)}{v(F)} \leqslant \beta(d,u,u,n) = \frac{\binom{u-1-[d/2]}{[(d-1)/2]} + (n-u-1)(d-1)}{\gamma(d-1,u)}.$$

For the second limit assertion, note that if $(P, F) \in \mathbf{P}(d, n, n-c)$ and P is not a simplex, then

$$\frac{1}{d-2} \leqslant \frac{v(P\sim F)}{v(F)} \leqslant \beta(d,d,n-c,n) = \frac{(c-2)(d-1)+n-c}{(n-c-d)(d-2)+d}.$$

In the discussion of minima, there remain only the cases of equality in Theorem 2. That

$$\min_{(P,F)\in\mathbf{P}(d,n,u)} \frac{v(P\sim F)}{v(F)} = \frac{(n-u-2)(d-1)+u}{(u-d)(d-2)+d} = \beta(d,d,u,n) \tag{8}$$

has already been established when $u=n-1$. To see that (8) holds also when $d=3$, $d=4$, or $u=d$, note that in these cases

$$v(F) = (u-d)(d-2)+d \quad \text{and} \quad v(P\sim F) \leqslant (n-u-2)(d-1)+u.$$

2. Maxima

When $3 \leqslant d \leqslant u < n$, each simple polytope F of class $(d-1, u)$ appears in some pair $(P, F) \in \mathbf{P}(d, n, u)$ (form a wedge over F and then truncate at vertices not in F). It therefore follows from McMullen's theorem [22] that $\gamma(d-1, u)$ is the maximum of $v(F)$ in Theorem 1. As the first step toward discussing the other maxima in Theorems 1 and 3, we are going to construct a simplicial d-polytope P' in R_d having a vertex z_1 such that

the number of vertices of P' is n; (9)

the number of edges of P' incident to z_1 is u; (10)

the number of facets of P' incident to z_1 is $(u-d)(d-2)+d$; (11)

the number of facets of P' not incident to z_1 is $\gamma(d, n-1)+d-u-1$. (12)

When P' and z_1 are available, we can let P be a polytope dual to P' and F the facet of P that corresponds under the duality to z_1. By (9) and (10), (P, F) is a polytope pair of class (d, n, u), with

$$v(F) = (u-d)(d-2)+d$$

by (11) and

$$v(P\sim F) = \gamma(d, n-1)+d-u-1$$

by (12). But then

$$v(P) = v(F) + v(P \sim F) = \gamma(d, n-1) + (u-d)(d-3) + d - 1.$$

That will establish the lower bounds stated in Theorems 1 and 3 for the maxima of $v(P)$, $v(P \sim F)$, and $v(P \sim F)/v(F)$.

The polytope P' is constructed by an elaboration of a procedure used by Grünbaum [10, p. 125] and McMullen (in an unpublished manuscript) for purposes related to our present one. It is convenient first to establish the following lemma, using the terms *beneath* and *beyond* in the sense of [10, p. 78].

LEMMA. *Suppose that C is a d-polytope in R^d, G is a proper face of C; $S_0, \ldots, S_m$ are the facets of C that contain G; and $H_0, \ldots, H_m$ are the hyperplanes determined by those respective facets. Then there exist a relatively interior point s_0 of S_0 and a point h_m of $H_m \sim S_m$ such that the closed segment $[s_0, h_m]$ is beneath all facets of C other than $S_0, \ldots, S_m$ and the hyperplanes $H_1, \ldots, H_{m-1}$ are crossed one at a time (not necessarily in that order) by the open segment $]s_0, h_m[$.*

Proof. Let s_m and g be relatively interior points of S_m and G respectively. For each $\lambda > 0$ the point

$$y_\lambda = (1+\lambda)g - \lambda s_m$$

belongs to $H_m \sim S_m$ and is beyond $S_0, \ldots, S_{m-1}$, and by letting $h_m = y_\lambda$ for a sufficiently small λ we assure that h_m is beneath all facets of C not containing G. For each relatively interior point q of S_0, the segment $[q, h_m]$ is beneath all facets not containing G and the segment $]q, h_m[$ crosses $H_1, \ldots, H_{m-1}$. Let s_0 be a q such that $]q, h_m[$ misses $\bigcup_{0<k<l<m}(H_k \cap H_l)$.

Having proved the lemma, we are now ready to construct the polytope P'. If $n-1=d$ then $u=d$ and P' is a d-simplex. Suppose, then, that $n-1>d$ and let C denote a cyclic d-polytope whose $n-1$ vertices $z_2, \ldots, z_n$ lie on the moment curve in R^d—say $z_i = (\tau_i, \tau_i^2, \ldots, \tau_i^d)$ with $\tau_2 < \tau_3 < \ldots < \tau_n$. The polytope P' will be the convex hull of C and one additional point z_1. In order to avoid the computation that would otherwise be required, we shall choose z_1 with the aid of the Lemma and establish its properties with the aid of a theorem of Bruggesser and Mani [4].

$$\text{Let} \quad S_0 = \text{con}\,\{z_2, \ldots, z_d, z_n\} \quad \text{and} \quad S_j = \text{con}\,\{z_2, \ldots, z_{d-1}, z_{d-1+j}, z_{d+j}\} \tag{13}$$

for $1 \leqslant j \leqslant n-d$. With $m = n-d$ and $G = \text{con}\,\{z_2, \ldots, z_{d-1}\}$, it follows from Gale's evenness condition [8], [10, p. 62] that the hypotheses of the lemma are satisfied. Let s_0 and h_m be as in the lemma, let $H_{c(1)}, \ldots, H_{c(m-1)}$ be the order in which the hyperplanes $H_1, \ldots, H_{m-1}$ are crossed by $]s_0, h_m[$ starting from s_0, and let the points $w_0, \ldots, w_{m-1}$ be such that w_i is on $]s_0, h_m[$ between H_i and H_{i+1}. For $0 \leqslant k < m$, let W_k denote the set of all boundary points of C that are visible from w_k, whence

$$W_k = \bigcup_{l=0}^{k} S_{c(l)}. \tag{14}$$

By a result of Bruggesser and Mani [4, p. 202], each of the sets W_k is a topological $(d-1)$-ball, and $S_{c(k)} \cap W_{k-1}$ is a topological $(d-2)$-ball for $0<k<m$. The latter condition implies that W_{k-1} contains at least one $(d-2)$-face of $S_{c(k)}$ and hence (since $S_{c(k)}$ is a simplex) omits at most one vertex of $S_{c(k)}$. But we see from (13) that W_{m-1} includes $m-1$ vertices of C that are not in S_0, and it follows that for $0<k<m$, W_{k-1} omits precisely one vertex of S_k. In view of (13) and (14), this implies that $r(i)=i$ for $0<i<m$.

Now with $z_1 = w_{u-d}$ and

$$P' = \operatorname{con}(\{z_1\} \cup C) = \operatorname{con}\{z_1, z_2, \ldots, z_n\},$$

the assertions (9)–(12) are consequences of Gale's evenness condition and Grünbaum's main result [10, p. 78] on the inductive construction of polytopes. Plainly z_1 is a vertex of P', and z_i is a vertex for $2 \leqslant i \leqslant n$ because z_1 is beneath a facet of C that includes z_i; that settles (9). For (10), note that W_{u-d} has precisely u vertices and each of them is also in a facet of C that z_1 is beneath. For (11), note that if r_i is the number of $(d-2)$-faces A of C such that A is incident to S_i and also to some facet of C that z_1 is beneath, then $r_0=d$ if $u=d$ and otherwise $r_0=d-1$, $r_1=\ldots=r_{u-d-1}=d-2$, and $r_{u-d}=d-1$. For (12) note that C has $\gamma(d, n-1)$ facets and z_1 is beyond $u-d+1$ of them.

To complete the proof of Theorem 1 it suffices to show that

$$\max_{(P,F)\in \mathbf{P}(d,n,u)} v(P \sim F) \leqslant \gamma(d, n-1) + d - u - 1,$$

for plainly
$$\max \frac{v(P\sim F)}{v(F)} \leqslant \frac{\max v(P\sim F)}{\min v(F)}.$$

Suppose, then, that (P, F) is a polytope pair of class (d, n, u) with $P \subset R^d$. For each facet G of P, let H_G denote the hyperplane determined by G and J_G the closed halfspace bounded by H_G and containing P. By a slight perturbation of these halfspaces, we may assume that each d of the hyperplanes H_G have a common point and there are d facets $G(1), \ldots, G(d)$ of P other than F such that the set $H_F \cap (\bigcap_1^d J_{G(i)})$ is a $(d-1)$-simplex. Let the point $q \in R^d \sim H_F$ be such that $\bigcap_1^d H_{G(i)} = \{q\}$, and suppose at first that $q \notin J_F$. Then let

$$K = \bigcap\nolimits_{G \neq F} J_G, \; K^+ = P \sim F, \quad \text{and} \quad K^- = K \sim J_F,$$

so that $v(K) = v(K^-) + v(K^+)$. Since K is a simple polytope of class $(d, n-1)$, it follows from McMullen's theorem [22] that $v(K) \leqslant \gamma(d, n-1)$, and since K^- is projectively equivalent to an unbounded simple polyhedron of class (d, u) it follows from a result used earlier [17, p. 230] that $v(K^-) \geqslant u-d+1$. But then

$$v(P \sim F) = v(K^+) = v(K) - v(K^-) \leqslant \gamma(d, n-1) + d - u - 1.$$

Now suppose, on the other hand, that $q \in J_F$, and let ϕ be an affine functional on R^d such that $J_F = \phi^{-1}[0, \infty[$. Then either (i) $\max \phi P < \phi(q)$ or (ii) q is a vertex of P and is the only point of P at which ϕ attains a maximum. In each case there exists $\delta \in]0, \phi(q)[$ such that the halfspace $D = \phi^{-1}]-\infty, \delta[$ contains all vertices of P other than q. Now define the projective transformation T by setting

$$T(x) = \frac{1}{\delta - \phi(x)} x \quad \text{for all } x \in D.$$

Let $P^* = T(P \cap D)$, for each facet G of P let $G^* = T(G \cap D)$, and define the hyperplane H_{G^*} and halfspaces J_{G^*} in the natural way. Since the point $\bigcap_1^d H_{G(i)^*}$ does not belong to J_{F^*}, the reasoning of the preceding paragraph applies directly in case (i). In case (ii), $v(P \sim F)$ is equal to $v(P^* \sim F^*) + 1$ rather than to $v(P^* \sim F^*)$, but since K^* is unbounded it follows from McMullen's theorem is conjunction with a remark of Klee [15, p. 718] that $v(K) < \gamma(d, n-1)$. Once again, $v(P \sim F) \leqslant \gamma(d, n-1) + d - u - 1$.

Having completed the proof of Theorem 1, we turn now to Theorem 3. Its left-hand inequality follows from the first part of this section and its right-hand inequality from McMullen's theorem. Note also that if a simple polytope P of class (d, n) has $\gamma(d, n)$ vertices then its dual P' is a simplicial d-polytope P' with n vertices and $\gamma(d, n)$ facets, whence P' is neighborly (McMullen [22]); but then $d = 3$ or P is a Kirkman polytope with every one of its facets as a base. (See the first paragraph of section 3.) That completes the discussion of equality on the right in Theorem 3.

If (P, F) is a polytope pair of class (d, n, d), the facet F may be removed (as in the fourth paragraph of Section 1) to produce a polytope P_1 of class $(d, n-1)$ with $v(P) = v(P_1) + d - 1$; hence $v(P) = \gamma(d, n-1) + d - 1$ in this case. Since all simple 3-polytopes with n facets have $2n - 4$ vertices, only the cases $d = 4$ and $d = 5$ of Theorem 3 remain in the discussion of equality on the left in Theorem 3.

If (P, F) is a polytope pair of class (d, n, u) and P' is dual to P, then P' is a simplicial d-polytope with n vertices and the vertex that corresponds to F is incident to only u edges of P'. With $f_i(\cdot)$ denoting number of i-faces, it follows that

$$f_1(P') \leqslant \binom{n-1}{2} + u.$$

But then we can use certain solutions of the Dehn-Sommerville equations [15, p. 527] [10, pp. 161 and 425] to see that

$$f_0(P)=f_3(P')=f_1(P')-f_0(P')\leqslant \binom{n-1}{2}+u-n \quad \text{when } d=4$$

and

$$f_0(P)=f_4(P')=2f_1(P')-6f_0(P')+12\leqslant 2\binom{n-1}{2}+2u-6n+12 \quad \text{when } d=5.$$

In each case the right-hand side is equal to

$$\gamma(d,n-1)+(u-d)(d-3)+d-1.$$

The final assertion of Theorem 3 is equivalent under duality to the following:

For simplicial d-polytopes having $d+3$ vertices and $\binom{d+2}{2}+d+1$ edges, the maximum number of facets is

$$2k^2+2k+1 \quad \text{when} \quad d=2k, \quad \text{and} \quad 2k^2 \quad \text{when} \quad d=2k-1. \tag{15}$$

Rather than constructing the maximizing polytopes explicitly, we rely on the technique of Gale diagrams developed by Micha Perles and described in [19, pp. 85–90, 108–114]. If X is the vertex set of a simplicial d-polytope Q with $d+3$ vertices, there is a mapping $\hat{}$ of X into the unit circumference $S^1=\{(\xi,\eta): \xi^2+\eta^2=1\}$ such that

for some odd m with $3\leqslant m\leqslant d+3$, $\hat{X}$ consists of m equally spaced points of S^1 [10, p. 111]; (16)

a nonempty set $Y\subset X$ is a *coface* of X (that is, $X\sim Y$ is the vertex set of a face of Q) if and only if con $\hat{Y}$ includes the origin [10, p. 88]. (17)

(Conditions (16) and (17) become more complicated when Q is not simplicial, but we are concerned only with the simplicial case.) Defining the *multiplicity* of a point p of $\hat{X}$ as the cardinality of $\{x\in X: \hat{x}=p\}$, it is clear that

the sum of the multiplicities is $d+3$. (18)

The *Gale diagram* of X consists of the set $\hat{X}$ with each point of $\hat{X}$ labeled by its multiplicity. Conversely, for each labeled subset $\hat{X}$ of S^1 satisfying (16) and (18) there exists a simplicial d-polytope Q with $d+3$ vertices such that $\hat{X}$ is a Gale diagram of Q. Note that:

a triple $Y\subset X$ is a cofacet of X if and only if Y is the vertex set of a triangle whose interior includes the origin; (19)

a set $Y\subset X$ is a face of X if and only $Y\sim X$ is a union of cofacets. (20)

Now with $r>1<s$ and $r+s=d\geqslant 4$, let Q be a simplicial d-polytope whose Gale diagram consists of the successive vertices $p_1, \ldots, p_5$ of a regular pentagon, their respective

multiplicities being 1, r, 1, 1, and s. The total number of cofacets (and hence of facets) of Q is $2rs+r+s+1$, for by (19) the number of cofacets mapping onto $\{p_1, p_2, p_4\}$ (resp. $\{p_1, p_3, p_4\}$, $\{p_1, p_3, p_5\}$, $\{p_2, p_3, p_5\}$, $\{p_2, p_4, p_5\}$) is r (resp. 1, s, rs, rs). With $r=[d/2]$ and $s=d-r$, that is the number of facets mentioned in (15). Further, $\binom{d+2}{2}+d+1$ is the total number of edges of Q, for by (20) the only pair of Q's vertices not determining an edge is the pair mapping onto $\{p_3, p_4\}$.

To conclude the proof of Theorem 3 we show that if Q is a simplicial d-polytope with $d+3$ vertices and there is a pair of vertices of Q that does not determine an edge, then the number of facets of Q does not exceed the numbers mentioned in (15). With the aid of the reasoning of Gale [9, pp. 14–16], that is seen to be a consequence of the following:

If a complete graph with $d+3$ vertices is oriented in such a way that every cyclic triangle includes at least one of two vertices p_1 and p_2, then the total number of cyclic triangles is at most $2k^2+2k+1$ when $d=2k$ and at most $2k^2$ when $d=2k-1$. (21)

To prove (21), let Z denote the set of all vertices other than p_1 and p_2, and note that each admissible orientation provides a linear ordering of Z. That is, the members of Z can be arranged in a sequence $z_0, \ldots, z_d$ such that the arc $(z_j, z_{j'})$ belongs to the oriented graph G if and only if $j<j'$. Assuming without loss of generality that $(p_1, p_2)\in G$ and that the sequence $z_0, \ldots, z_d$ is given, the orientation may then be specified by means of a 2-*by*-$(d+1)$ binary matrix (a_{ij}), where a_{ij} is 0 or 1 according as $(p_i, z_j)\in G$ or $(z_j, p_i)\in G$. Note that:

the number of cyclic triangles involving p_1 but not p_2 is equal to the number of pairs (j, j') such that $j<j'$, $a_{1j}=0$, and $a_{1j'}=1$; (22_1)

the number of cyclic triangles involving p_2 but not p_1 is equal to the number of pairs (j, j') such that $j<j'$, $a_{2j}=0$, and $a_{2j'}=1$; (22_2)

the number of cyclic triangles involving both p_1 and p_2 is equal to the number of indices j such that $a_{1j}=1$ and $a_{2j}=0$. (23)

If a 1 precedes a 0 in the ith row of the matrix, interchanging the two entries increases the number (22_i) and does not decrease the number (23) by more than 1. Hence the number of cyclic triangles is maximized by a matrix whose ith row consists, for some r_i, of r_i 0's followed by $d+1-r_i$ 1's. If $r_1>r_2$, interchanging the two rows increases the number (23). Thus we may assume also that $r_1\leqslant r_2$. The number of cyclic triangles is then

$$r_1(d+1-r_1)+r_2(d+1-r_2)+(r_2-r_1) = g_1(r_1)+g_2(r_2),$$

where $$g_1(r_1) = dr_1-r_1^2 \quad\text{and}\quad g_2(r_2) = (d+2)r_2-r_2^2.$$

When $d=2k$ the maxima of g_1 and g_2 are attained respectively at $r_1=k$ and $r_2=k+1$, yielding $2k^2+2k+1$ as the maximum number of cyclic triangles. When $d=2k-1$ the maxima of the g_i's are attained (subject to the integrality constraint) for $r_1 \in \{k-1, k\}$ and $r_2 \in \{k, k+1\}$, and hence the maximum number of cyclic triangles is $2k^2$.

3. Neighborly polytopes, Kirkman pairs, and K-specificity

A d-polytope is said to be *r-neighborly* provided that each set of r vertices is the vertex set of a face, and *neighborly* provided that it is $[d/2]$-neighborly [10, pp. 122–129]. For even d the neighborly d-polytopes are simplicial, and for all d the cyclic polytopes are both neighborly and simplicial. Neighborly polytopes are of interest in the study of Kirk man polytopes because a polytope P is a Kirkman polytope based on each of its facets if and only if P's dual is 2-neighborly and simplicial. When $d \geqslant 6$ the dual of a neighborly simplicial d-polytope may be regarded as a sort of "super Kirkman polytope", for not only is the dual a Kirkman polytope based on each of its facets but the same is true of each of its $(d-j)$-faces for $0 \leqslant j \leqslant [d/2]-2$.

For $2 \leqslant d < n$ and $1 \leqslant m \leqslant n$ let $C(d, n)$ denote a cyclic d-polytope with n vertices and $\gamma_m(d, n)$ the number of facets of $C(d, n)$ that miss the mth vertex of $C(d, n)$ in the natural ordering on the moment curve M_d. If P is a polytope dual to $C(d, n)$ and F is the facet of P corresponding to the mth vertex of $C(d, n)$, then (P, F) is a Kirkman pair of class (d, n) with $v(P)=\gamma(d, n)$ and $v(F)=\gamma(d, n)-\gamma_m(d, n)$. The fact that

$$\gamma_1(d, n) = \binom{n-[(d+3)/2]}{[d/2]},$$

used in the proof of Theorem 2, is established below along with some related results. Proposition 1 may be regarded as a first step toward solving the problem (24) below. A complete solution of (24) would lead to a greater understanding of neighborly polytopes, which could be of important because of the key role that they have played (and probably will continue to play) in the theory of polytopes.

If P is a simple polytope of class (d, n) that has the maximum possible number of vertices for its class (that is, if P's dual is neighborly and simplicial), what can be said (in terms of d and n) about the sequence $(v(F_1), \ldots, v(F_n))$ listing the numbers of vertices of the various facets of P? In particular, what are the possibilities for card $\{v(F_i): 1 \leqslant i \leqslant n\}$ and (for given j) for card $\{i: v(F_i)=j\}$? (24)

In view of Gale's evenness condition [8], [10, p. 62] characterizing the facets of $C(d, n)$, results on the numbers $\gamma_m(d, n)$ may be stated in purely combinatorial terms.

PROPOSITION 1. *Suppose that d, m, and n are positive integers with $d<n$ and $m\leqslant n$. Let $\mathbf{F}(d, n)$ denote the set of all subsets X of $\{1, \ldots, n\}$ such that X is of cardinality d and between any two members of $\{1, \ldots, n\}\sim X$ there is an even number of members of X. Let $\gamma_m(d, n)$ denote the cardinality of the set $\mathbf{F}_m(d, n)=\{X\in\mathbf{F}(d, n): m\notin X\}$. For all d, $\gamma_m(d, n)=\gamma_{n+1-m}(d, n)$ and $\gamma_m(d, n)$ is constant for $[(d+1)/2]<m<n+1-[(d+1)/2]$. If d is even (say $d=2k$) then*

$$\gamma_m(d, n)=\binom{n-1-k}{k} \quad \textit{for all } m.$$

If d is odd (say $d=2k-1$), the numbers $\gamma_m(d, n)$ are determined by either of the recursions

$$\gamma_m(d, n)=\gamma_{m-1}(d, n-1)+\gamma_{m-2}(d-2, n-2) \quad (3\leqslant m\leqslant n) \tag{25}$$

$$\gamma_m(d, n)=\gamma_m(d, n-1)+\gamma_m(d-2, n-2) \quad (1\leqslant m\leqslant n-2) \tag{26}$$

in conjunction with the boundary conditions

$$\gamma_1(d, n)=\binom{n-1-k}{k-1},\ \gamma_2(d, n)=\binom{n-1-k}{k-1}+\binom{n-2-k}{k-1}, \tag{27}$$

$$\gamma_m(d, d)=0 \textit{ for all } d \textit{ and } m,\ \gamma_m(1, n)=2 \textit{ for } 1<m<n. \tag{28}$$

Proof. For positive integers s and p, let $\pi(s, p)$ denote the number of ordered partitions of s into p nonnegative parts, whence $\pi(s, p)$ is also the number of ordered partitions of $2s$ into p nonnegative even parts. As is well known,

$$\pi(s, p)=\binom{p+s-1}{s}. \tag{29}$$

When $d=2k$ it follows from (29) and the reasoning of Gale [8, p. 227] that

$$\gamma_1(d, n)=\pi(d/2, n-d)=\binom{n-1-k}{k}.$$

Further, in the case of even d the condition for membership in $\mathbf{F}(d, n)$ is equivalent to the corresponding condition relative to the cyclic rather than the linear ordering of $\{1, \ldots, n\}$, and consequently $\gamma_m(d, n)=\gamma_1(d, n)$ for $1\leqslant m\leqslant n$. It is interesting to note that, for the "regular cyclic polytopes" of Gale [8, pp. 230–231], it is actually the group of isometries and not merely the group of combinatorial symmetries that is transitive on the vertex set.

Now suppose that d is odd—say $d=2k-1$. Then the members of $\mathbf{F}_1(d, n)$ are precisely the members of $\mathbf{F}(d, n)$ that include n, whence

$$\gamma_1(d, n)=\pi((d-1)/2, (n-2)-(d-1)+1)=\binom{n-1-k}{k-1}.$$

The number of members of $\mathbf{F}_2(d, n)$ that omit ⟨resp. include⟩ 1 is

$$\pi((d-1)/2,\ (n-3)-(d-1)+1)\ \langle\text{resp. } \pi((d-1)/2,\ (n-2)-(d-1)+1\rangle,$$

whence with the aid of (29) it follows that

$$\gamma_2(d, n) = \pi(k-1, n-2k) + \pi(k-1, n+1-2k) = \binom{n-2-k}{k-1} + \binom{n-1-k}{k-1}.$$

That takes care of the boundary conditions (27). The conditions (28) are obvious, and the first of them could of course be replaced by $\gamma_m(d, d+1) = 1$.

Now suppose that $3 \leqslant m$, let $\mathbf{F}'_m(d, n)$ ⟨resp. $\mathbf{F}''_m(d, n)$⟩ denote the set of all members of $\mathbf{F}_m(d, n)$ that omit ⟨resp. include⟩ $m-1$, and note that the members of $\mathbf{F}''_m(d, n)$ include $m-2$ as well as $m-1$. For each $X \in \mathbf{F}'_m(d, n)$ let

$$\xi X = \{x: x \in X \text{ and } 1 \leqslant x \leqslant m-2\} \cup \{x-1: x \in X \text{ and } m+1 \leqslant x \leqslant n\}$$

and for each $X \in \mathbf{F}''_m(d, n)$ let

$$\eta X = \{x: x \in X \text{ and } 1 \leqslant x \leqslant m-3\} \cup \{x-2: x \in X \text{ and } m+1 \leqslant x \leqslant n\}.$$

Then ξ ⟨resp. η⟩ is a one-to-one mapping of $\mathbf{F}'_m(d, n)$ onto $\mathbf{F}_{m-1}(d, n-1)$ ⟨resp. $\mathbf{F}''_m(d, n)$ onto $\mathbf{F}_{m-2}(d-2, n-2)$⟩, thus establishing (25). The recursion (26) is a consequence of (25), for if $m \leqslant n-2$ then $n+1-m \geqslant 3$ and

$$\gamma_m(d, n) = \gamma_{n+1-m}(d, n) = \gamma_{n-m}(d, n-1) + \gamma_{n-1-m}(d-2, n-2) = \gamma_m(d, n-1) + \gamma_m(d-2, n-2).$$

It remains only to show that

$$\gamma_m(2k-1, n) = \gamma_{k+1}(2k-1, n) \quad \text{whenever} \quad k+1 \leqslant m \leqslant n-k+1. \tag{30}$$

Suppose there exists a triple (k, m, n) for which (30) fails. Among all such triples, let (k_0, m_0, n_0) be one for which k_0 is minimum and such that n_0 is minimum for the given k_0. It follows from (26) that

$$\gamma_{m_0}(2k_0-1, n_0) = \gamma_{m_0}(2k_0-1, n_0-1) + \gamma_{m_0}(2k-3, n_0-2) \tag{31'}$$

and
$$\gamma_{k_0+1}(2k_0-1, n_0) = \gamma_{k_0+1}(2k_0-1, n_0-1) + \gamma_{k_0+1}(2k_0-3, n_0-2). \tag{31''}$$

By the choice of k_0 and n_0, the first terms of the right sides of (31′) and (31″) are equal if $k_0+1 \leqslant m_0 \leqslant (n_0-1)-k_0+1$ and the second terms are equal if $(k_0-1)+1 \leqslant m_0 \leqslant (n_0-2)-(k_0-1)+1$. Since (30) is assumed to fail for (k_0, m_0, n_0), it follows that $m_0 = n_0 - k_0 + 1$. But then (30) is obvious and the proof of Proposition 1 is complete.

With the aid of (25) it can be verified that for $1 \leqslant m \leqslant n-2$ the values of $\gamma_m(2k-1, n)$ are as shown in the table below. However, I do not know of any general formula for $\gamma_m(2k-1, n)$ that is simple enough to be useful.

m	Value of $\gamma_m(2k-1,n)$ for $1 \leqslant m \leqslant n-2$
3	$\binom{n-1-k}{k-1}+\binom{n-3-k}{k-1}$
4	$\binom{n-1-k}{k-1}+\binom{n-3-k}{k-2}+\binom{n-4-k}{k-1}$
5	$\binom{n-1-k}{k-1}+2\binom{n-4-k}{k-2}+\binom{n-5-k}{k-1}$
6	$\binom{n-1-k}{k-1}+\binom{n-3-k}{k-2}+\binom{n-5-k}{k-2}+\binom{n-6-k}{k-1}.$

Let us say that a simple polytope is *K-specific* provided that all Kirkman polytopes based on it have the same number of vertices. Thus, for example, all simplices are K-specific. Plainly the property of K-specificity is a projective invariant, but as we shall see it is not a combinatorial invariant. The second part of the following result generalizes the well-known fact that if (P, F) is a Kirkman pair and F is 2-dimensional, then $v(P \sim F) = v(F) - 2$.

PROPOSITION 2. *If a polytope is projectively equivalent to a d-cube it is K-specific. If F is a simple polytope of class $(2k, n)$ such that each k facets of F have nonempty intersection then F is K-specific with*

$$v(F) = \frac{n}{n-k}\binom{n-k}{k} \quad \text{and} \quad v(P \sim F) = \frac{n-2k}{n}\, v(F) = \binom{n-k-1}{n-k}$$

for every Kirkman polytope P based on F.

Proof. For the second assertion, note that each k facets of P have nonempty intersection, and since P is simple each such intersection is of dimension $k+1$. But then the dual polytopes P' and F' are both neighborly and simplicial, whence it follows [10, pp. 124, 163] that the numbers of facets of P' and F' (and hence the numbers of vertices of P and F) are respectively $2\binom{n-k}{k}$ and $\frac{n}{n-k}\binom{n-k}{k}$.

For the first assertion of Proposition 2 it suffices to show that the d-cube

$$F = \{(x_1, \ldots, x_d) \colon 0 \leqslant x_1 \leqslant 1\}$$

is K-specific, and it is not hard to verify that each Kirkman polytope based on F is affinely equivalent to a set P defined by a system of linear inequalities of the form

$$x_i + t_i x_{d+1} \leqslant 1 \quad (1 \leqslant i \leqslant d) \tag{32}$$

in the nonnegative variables $x_1, \ldots, x_{d+1}$, where $t_1 \leqslant t_2 \leqslant \ldots \leqslant t_d$ and $t_d > 0$. Now let us consider the problem of maximizing on P the linear function x_{d+1}. The introduction of a slack variable for each of the constraints in (32) leads to the following linear programming tableau, in which the last column contains the "constants" and the last row represents the objective function. (The tableau is shown explicitly for $d=4$.)

$$\begin{array}{cccccccccc}
1 & & & & 1 & & & & t_1 & 1 \\
& 1 & & & & 1 & & & t_2 & 1 \\
& & 1 & & & & 1 & & t_3 & 1 \\
& & & 1 & & & & 1 & t_4 & 1 \\
& & & & & & & & 1 & 0
\end{array} \tag{33}$$

There are 2^d possible choices for the initial feasible basis, corresponding to the 2^d vertices of F. After a pivot on t_d (the $(2d+1)$th column and the dth row) the tableau takes the form

$$\begin{array}{cccccccccc}
1 & & & & 1 & & & & & 1-t_1/t_4 \\
& 1 & & & & 1 & & & & 1-t_2/t_4 \\
& & 1 & & & & 1 & & & 1-t_3/t_4 \\
& & & 1/t_4 & & & & 1/t_4 & 1 & 1/t_4 \\
& & & -1/t_4 & & & & -1/t_4 & & -1/t_4,
\end{array} \tag{34}$$

which is optimal for the given objective function. There are 2^{d-1} ways of choosing a subset B of $\{1, \ldots, d-1\} \cup \{d, \ldots, 2d-1\}$ so that $B \cup \{2d+1\}$ is the set of indices corresponding to an optimal feasible basis. That $t_{d-1} < t_d$ follows from the assumption that P is simple, and hence these 2^{d-1} ways correspond to 2^{d-1} vertices of $P \sim F$. Since the tableaux (33) and (34) correspond respectively to the minimum and maximum values of the objective function, all vertices have been accounted for and we conclude that $v(P \sim F) = 2^{d-1}$.

PROPOSITION 3. *For each K-specific simple polytope F there is an integer j having the following properties:*

(i) *each facet of F has precisely j vertices;*

(ii) *whenever G_1 and G_2 are disjoint facets of F and H is a hyperplane that contains the intersection of the hyperplanes determined by G_1 and G_2 (or is parallel to them when they are parallel), intersects the interior of F, and contains no vertex of F, then H intersects precisely j edges of F;*

(iii) *whenever G_1 and G_2 are intersecting facets of F and H is a hyperplane that contains*

$G_1 \cap G_2$, intersects the interior of F, and contains no vertex of $F \sim (G_1 \cap G_2)$, then j is equal to the sum of the number of vertices of $G_1 \cap G_2$ and the number of edges of $P \sim (G_1 \cap G_2)$ intersected by H.

Proof. To establish (i), recall that if G is a facet of F and W is a wedge over F with foot G (in the sense of [19, pp. 57–58]) then W is a Kirkman polytope based on F and $v(W \sim F) = v(F \sim G)$.

For (ii) and (iii) it is convenient to define

$$R_d = \{x = (x_1, \ldots, x_d, x_{d+1}) \in R^{d+1}: x_{d+1} = 0\},$$

and by subjecting the d-polytope F to a suitable projective transformation we may assume that

$$F \subset \{x \in R_d: x_1 \geq 0, x_2 \geq 0\} \quad \text{and} \quad G_i = F \cap \{x \in R_d: x_i = 0\} \quad (i = 1, 2).$$

The hyperplane H in R_d then has the form

$$H = \{x \in R_d: (x_1, x_2) \in R(z_1, z_2)\}$$

for a suitable point $z \in R_d$ with $z_1 > 0$, $z_2 > 0$, and $z_i = 0$ for $3 \leq i \leq d+1$. With $u = (0, \ldots, 0, 1) \in R^{d+1}$, let C denote the cylinder $F + [0, \infty[u$ and let J_i denote the closed halfspace in R^{d+1} that contains F and whose bounding hyperplane H_i contains both G_i and the line $R(z+u)$. Since

$$\{x \in J_i: x_{d+1} \geq 0\} \subset \{x \in R^d: x_i \geq 0\},$$

it is not hard to verify that the set

$$P = C \cap J_1 \cap J_2$$

is a Kirkman polytope based on F. The fact that P is simple is dependent, of course, on the assumption about H's not including certain vertices of F. There is a vertex of $P \sim F$ on each edge of P that is parallel to the line R_u, and the number of such edges is equal to

$$v(F) - v(G_1) - v(G_2) + v(G_1 \cap G_2).$$

The additional vertices of $P \sim F$ are the intersections of the $(d-1)$-flat $H_1 \cap H_2$ with the 2-dimensional faces of the cylinder $C^+ = \{c \in C: c_{d+1} > 0\}$, and these project in a one-to-one manner onto the intersections of H with the edges of $P \sim (G_1 \cap G_2)$. That completes the proof.

PROPOSITION 4. *A simple 3-polytope is K-specific if and only if it is a simplex or is projectively equivalent to a cube.*

Proof. It is a well-known consequence of Euler's theorem (see, for example, the equation (*) in [10, p. 254]) that if F is a simple polytope of class $(3, n)$ and all facets of F have the same number r of vertices, then (n, r) is $(4, 3)$ or $(6, 4)$ or $(12, 5)$. Plainly F is a simplex in the first instance, and it can be verified that F is combinatorially equivalent to a cube in the second instance and to a regular dodecahedron in the third.

In the case of the dodecahedron let G_1 and G_2 be two facets of F that are opposite to each other with respect to the combinatorial structure of the dodecahedron. Since they are disjoint we may assume (with the aid of a suitable projective transformation) that they lie in parallel planes H_1 and H_2. Let H_1' be the first translate of H_1 (in moving toward H_2) that contains a vertex of $F \sim G_1$. If H is a translate of H_1 that is beyond H_2' by a sufficiently small positive amount, H misses all vertices of F and intersects at least six edges of F. It follows from (11) of Proposition 3 that F is not K-specific.

Now suppose, finally, that F is K-specific and is combinatorially equivalent to a 3-cube. From (iii) of Proposition 3 it follows that if K_1 and K_2 are opposite facets of F, E_1 is an edge of K_1, and E_2 is the edge of K_2 that is opposite to E_1, then E_1 and E_2 are coplanar. With the aid of a suitable projective transformation we may assume that a particular pair G_1 and G_2 of opposite facets are parallel and that G_1 is a square. The coplanarity condition then implies G_2 is a rectangle with sides parallel to those of G_1, and the desired conclusion can be derived from further applications of the coplanarity condition.

It follows from the results of Grünbaum and Sreedharan [11, p. 448] that there are precisely two combinatorial types of Kirkman 4-polytopes P having a base F that is combinatorially equivalent to a 3-cube. For one type (a wedge over a facet of F), $v(P \sim F) = 4$, while $v(P \sim F) = 5$ for the other. See [11, p. 442] for the two Schlegel diagrams.

4. Polytope pairs and linear programming

The results of this paper are related to several questions from linear programming. At present, for example, the best mathematical upper bound on the number of iterations required by the simplex algorithm is provided by McMullen's upper bound [22] on the number of vertices of a simple polytope and hence, for nondegenerate linear programs whose feasible region is bounded, on the number of feasible bases. While that can hardly be a sharp bound on the number of simplex iterations, the examples of Klee and Minty [18] show that it is good in a certain asymptotic sense. In any case, the feasible region of a linear program is often unbounded and hence there is interest in the numbers of vertices of unbounded simple polyhedra. Sharp lower and upper bounds are provided by Corollary 1. Only the latter are of direct interest in connection with linear programs *per se*, but both lower and upper bounds are of interest in connection with other problems of mathematical programming—for example, minimizing a concave function subject to linear constraints, or finding all vertices of a polyhedron.

Kirkman polytopes are of interest in connection with the famous d-step conjecture and Hirsch conjecture of linear programming [5, pp. 160, 168], [6], [19]. In particular, if the bounded d-step conjecture is valid for all $d<e$ and yet there is a simple e-dimensional Dantzig figure (P, x, y) (in the sense of [19, pp. 57, 59]) for which the conjecture fails, then P is a Kirkman polytope with several bases. Indeed, for each edge $[x, \bar{x}]$ $\langle$resp. $[y, \bar{y}]\rangle$ of P, P is based on the facet that is incident to x but not $\bar{x}$ $\langle$resp. $\bar{y}$ but not $y\rangle$. For more on this matter, see [19] and especially [14, pp. 608–610].

A principal motivation for the present paper is a recent algorithm of Mattheiss [21] which, under suitable nondegeneracy assumptions or suitable perturbations of the constraints, finds all vertices of a polytope F defined by a system of n linear inequalities,

$$\begin{array}{l} a_{11}x_1+\ldots+a_{1d}x_d\leqslant b_1 \\ \vdots \qquad\qquad\quad \vdots \quad\; \vdots \\ a_{n1}x_1+\ldots+a_{nd}x_d\leqslant b_d \end{array} \tag{35}$$

in the d real variables $x_1, \ldots, x_d$. Like several other algorithms for that purpose (e.g. Manas and Nedoma [20]), Mattheiss's procedure applies the tableaux and pivot operations of the simplex algorithm to a system of linear equalities (in nonnegative variables) obtained from (35) by the addition of slack variables. However, rather than working directly with F he finds all vertices of the larger polytope P defined by the system

$$\begin{array}{r} a_{11}x_1+\ldots+a_{1d}x_d+t_1y\leqslant b_1 \\ \vdots \qquad\qquad\quad \vdots \qquad \vdots \quad\; \vdots \\ a_{n1}x_1+\ldots+a_{nd}x_d+t_ny\leqslant b_n \\ y\geqslant 0 \end{array} \tag{36}$$

(or of a suitably perturbed version of this), where the constants t_i are given by

$$t_i=\left(\sum_{j=1}^{d}a_{ij}^2\right)^{\frac{1}{2}} \quad (1\leqslant i\leqslant n), \tag{37}$$

The vertices of F are identified as the vertices of P for which $y=0$. The computation starts with a vertex of $P\sim F$ and proceeds by means of pivot operations to construct a spanning tree in the graph of P such that each vertex is of valence 1 in the tree. Hence, as Mattheiss emphasizes, it is never necessary to produce the tableau (but only the list of basic variables) associated with a given vertex of F. That is regarded as important and useful because of his observation that the inequality

$$v(P \sim F) < v(F) \tag{38}$$

is sometimes valid and his conjecture that it always holds. There seems to be little point in his algorithm when (38) fails.

While the condition (37) plays a certain role in the case of redundant constraints, the fact that the t_i's are given by (37) is not used in any essential way in Mattheiss's procedure for finding vertices. The same algorithm applies for any choice of t_i's subject to

the t_i's are all nonnegative and at least one of them is positive, (39)

though of course the specific pivots and the actual number of vertices of any $P \sim F$ will vary from one choice of t_i's to another. We speak of the *restricted* or *unrestricted* form of Mattheiss's algorithm according as the t_i's are given by (37) or required only to satisfy (39).

Let us say that a Kirkman pair (P, F) is *equiangular with angle* θ provided that the dihedral angles made by F with the other facets of P are all equal to θ. Necessarily, $\theta \in]0, \pi/2[$. A pair (P, F) that is equiangular with any angle whatever can be carried onto a pair that is equiangular with specified angle θ by means of an affine transformation that is the identity on F.

The following result relates Kirkman pairs to the pairs involved in Mattheiss's algorithm.

PROPOSITION 5. *If the simple d-polytope F and the simple $(d+1)$-polytope P are defined by means of the systems* (35) *and* (36) *respectively, with the t_i's satisfying* (39), *and if no inequality in* (35) *is redundant, then (P, F) is a Kirkman pair of class $(d+1, n+1)$. When the t_i's are given by* (37), *(P, F) is equiangular with angle $\pi/4$. Conversely, for each Kirkman pair (P, F) of class $(d+1, n+1)$ there is a nonsingular projective transformation T that carries F and P onto a pair of sets defined by systems of the forms* (35) *and* (36) *respectively, with $b_i > 0 < t_i$ for all i. If (P, F) is equiangular with angle $\pi/4$, T may be taken to be an isometry and the t_i's to satisfy* (37).

Proof. All except perhaps the third assertion are obvious, so we confine our attention to it. With (P, F) denoting a Kirkman pair of class $(d+1, n+1)$, we may assume without loss of generality that P lies in the halfspace $\{(x, y): x \in R^d, y \geqslant 0\}$ of R^{d+1}, F in the hyperplane $H = \{(x, y): y = 0\}$, and the origin is relatively interior to F. By hypothesis, the intersection of F with any other facet of P is a $(d-1)$-face of P and a facet of F, whence plainly (identifying R^d with H in the usual way) F may be defined by a system of the form (35) with all $b_i > 0$ and there are real constants t_i such that P is defined by (36). However, some of these t_i may be negative.

For a sufficiently small $\varepsilon > 0$, the point $(0, -\varepsilon)$ is beneath all facets of P except for

F, and is of course beyond F. Hence all of P except the relative boundary of F is interior to the convex cone C formed by the open rays that issue from $(0, -\varepsilon)$ and pass through the various points of F. Now for all $(x, y) \in R^{d+1}$ with $y > -\varepsilon$, let

$$T(x, y) = \frac{\varepsilon}{y+\varepsilon}(x, y).$$

The projective transformation T is the identity on R^d, is permissible for C, and carries the rays that make up C onto a system of rays parallel to the ray from $(0, -\varepsilon)$ through $(0, 0)$. Hence all of the polytope TP except for $F = TF$ is interior to the cylinder $\{(x, y)\colon x \in P, y \geqslant 0\}$, whence the pair (TP, F) plainly has the desired form.

In view of Proposition 5, questions concerning the efficiency of Mattheiss's algorithm lead naturally to questions concerning the relationship of a simple polytope F to the various Kirkman polytopes P based on F. There is concern, in particular, with the minima and maxima discussed in Corollary 2 and with the corresponding minima and maxima as (P, F) ranges over all equiangular Kirkman pairs of class $(d+1, n+1)$. It is easily seen that Corollary 2's results on minima are not changed by the restriction to equiangular pairs, but perhaps some of the maxima are reduced in the restricted case.

The quotient $v(P \sim F)/v(F)$ is of special interest, for

$$\frac{d+1}{d}\,\frac{v(P\sim F)}{v(F)}$$

seems to be a reasonable estimate for the ratio of the number of arithmetic operations required in finding F's vertices by applying pivot operations directly to (35) to the number required in applying Mattheiss's restricted pivots to (36). By Corollary 2 the minimum of $v(P \sim F)/v(F)$ is always between $1/(d+1)$ and $1/(d-1)$, while the maximum (over *all* Kirkman pairs of class (d, n)) is equal to $1/(d+1)$ when $n = d+1$ and $1/2$ when $n = d+2$, and for large d is approximately $d/12$ when $n = d+3$ and $d^2/96$ when $n = d+4$. To estimate the maximum for other values of n, let us fix a multiplier $\mu > 2$ and suppose that n is equal to μ times the least integer not less than $d/2$; that is, $n = \mu k$ where $d = 2k$ or $d = 2k-1$. Then $\gamma(d+1, n)$ is equal to the product of

$$\frac{(\mu k - k - 1)!}{k!\,(\mu k - 2k)!} \tag{40}$$

by $2(\mu-2)k$ or by μk according as d is even or odd. Replacing the factorials in (40) by Stirling's approximation, we conclude that for large d the maximum of $v(P \sim F)/v(F)$ is approximately

$$\frac{e}{(2\pi)^{1/2}}\left(\frac{\mu k - k - 1}{\mu k - 2k}\right)^{\mu k - k - 1/2}(\mu-2)^{k-1}k^{-5/2} \tag{41}$$

when d is even and approximately $\mu/(2\mu-4)$ times (41) when d is odd.

The above numbers suggest that the unrestricted form of Mattheiss's algorithm is not useful because in most cases the possible loss in computational efficiency (as compared to the result of working directly with (35)) greatly exceeds the possible gain. In particular, (38) can fail badly in the unrestricted case. I expect the same to be true when the t_i's are given by (37), but in fact have no counterexample to (38) in that case. When there are no redundant constraints in (35), the validity of (38) may be assured by letting one t_i be positive and the rest zero, for then P is a wedge over a facet of F. However, such slight reductions in the number of tableaux do not seem worth the effort and, as Mattheiss emphasizes, one would hope to have (38) hold "strongly".

Branko Grünbaum has remarked that when d is 3 every combinatorial type of Kirkman d-polytope can be realized by ones that are equiangular. That probably is not true for $d \geq 4$. However, not combinatorial types but merely numbers of vertices are involved in the main open question related to the extreme behavior of the restricted form of Mattheiss's algorithm: *What are the analogues for equiangular Kirkman pairs of Corollary 2's results on maxima?*

The above discussion is all concerned with the extreme behavior of Mattheiss's algorithm. However, the minimum and maximum of $v(P \sim F)/v(F)$ are not as important for computational considerations as is the expected value of $v(P \sim F)/v(F)$, in some sense appropriately related to both theory and computational practice. I have no information on the expected value, but it seems conceivable that the second part of Proposition 2 is relevant. Though all 2- and 3-polytopes are both neighborly and dual neighborly, it is natural to regard the neighborly and dual neighborly d-polytopes as being rather "unusual" for $d > 3$, since most of the familiar higher-dimensional polytopes except the simplices are not of either sort. However, there are indications that, in various senses, both sorts of polytopes may become very "common" as $d \to \infty$ (Gale [7, p. 262], Grünbaum [10, p. 129], Motzkin [23, p. 249], Motzkin and O'Neil [24, p. 254]).

Does there exist a positive function ϕ on R^d such that $v(P \sim F) < v(F)$ whenever (P, F) is a Kirkman pair of class $(d+1, n+1)$ with F defined by (35) *and P by* (36) *with $t_i = \phi(a_{i1}, \ldots, a_{id})$ $(1 \leq i \leq n)$?* Mattheiss conjectures that $\phi(a_1, \ldots, a_d) = (\Sigma_1^d a_j)^{1/2}$ is such a function, and I have no counterexample even though I am inclined to doubt the conjecture. If there is an easily computed ϕ which insures that $v(P \sim F) < v(F)$ "strongly", or that in some other respect the combinatorial structure of $P \sim F$ is significantly simpler than that of F, that could be useful not only for finding all vertices of F but also for solving linear programs having F as feasible region.

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