# ANOTHER PROOF OF THE EXISTENCE OF SPECIAL DIVISORS

BY

STEVEN L. KLEIMAN and DAN LAKSOV

*M.I.T., Cambridge, Mass. USA*

In the classical literature on curves, it is repeatedly asserted that the divisors of degree $n$ and (projective) dimension at least $r$ depend on at least $(\tau+r)$ parameters, with $\tau=(r+1)(n-r)-rg$ where $g$ is the genus. However, it was only recently proved that, if $\tau$ is nonnegative, such divisors exist. Meis [17] gave an analytic proof for the case $r=1$, which involves deforming the curve over the Teichmüller space into a special curve that has an explicit $(\tau+r)$-dimensional family of such divisors. Kempf [10] and Kleiman-Laksov [12] independently gave similar algebraic proofs for the case of any $r$, valid in any characteristic, which involve constructing a global, finite cohomology complex for the Poincaré sheaf and performing a local analysis of some resulting determinantal subvarieties of the jacobian. Gunning [7], working over the complex numbers, gave a proof for the case $r=1$, which uses Macdonald's description of the homology of the symmetric product and Hensel-Landsberg's trick of choosing $n$ minimal ([8], lecture 31, § 3, p. 550); otherwise, the proof is akin to the proofs in [10] and [12].

We offer below another algebraic proof for the general case, which is conceptually simpler and more natural. Our framework is the theory of singularities of mappings. We observe that the divisors of degree $n$ and dimension at least $r$ are parametrized by the scheme $Z_r$ of first order singularities of rank at least $r$ of the canonical map from the variety of divisors of degree $n$ to the jacobian, a base point on the curve having been picked. (By definition, $Z_r$ consists of the points where the map of tangent sheaves is of rank at least $r$.) Consequently, if $Z_r$ is empty or of the right dimension, then in the Chow ring its class is given by a certain Thom polynomial in the Chern classes of the variety of divisors. The Thom polynomial is evaluated (Proposition 16), and the value is then proved nonzero if $\tau$ is nonnegative by using the projection formula and Poincaré's formulas. Hence, divisors of degree $n$ and dimension at least $r$ must exist; otherwise, the value of the Thom polynomial would be zero. In fact they abound: carried further, the proof shows that, if $\tau$ is nonnegative, every closed subvariety of the jacobian of codimension $\tau$ contains the image of at least one such divisor, and moreover the weighted number of images contained, if the number is finite and the divisors are of dimension exactly $r$, is equal to $r!...0!/(g-n+2r)!...(g-n+r)!$ times the intersection number of the subvariety with the $(g-\tau)$-th power of the theta divisor.

The map of tangent sheaves associated to the canonical map from the variety of divisors of degree $n$ to the jacobian can be represented locally by a matrix that was used by Brill and Noether [1] to find the number $(\tau+r)$; their work is discussed in Remark 6. Assuming $\tau$ is equal to zero and the divisors of degree $n$ and dimension at least $r$ form a finite number of linear series of dimension $r$, Castelnuovo [2] found the weighted number of series is

$$\frac{1!\,2!\ldots r!\,1!\,2!\ldots r'!}{1!\,2!\ldots (r+r'+1)!}\,g! \quad \text{with } r'=(g-1-n+r);$$

the same number comes out above, the subvariety now being the whole jacobian.

The Thom polynomial giving the class of the scheme $Z_r$ of first order singularities of rank at least $r$ of a map of smooth varieties was determined by Porteous [19], under a certain transversality hypothesis. This hypothesis is automatically satisfied if $Z_r$ is empty, and so Porteous' work suffices for establishing the existence of special divisors. However, the transversality hypothesis was relaxed to a more natural hypothesis, requiring simply that $Z_r$ be of the right dimension, by Kempf and Laksov [12], and their work allows us to determine the weighted number of images above, using the weighting indicated by the scheme structure of $Z_r$.

The Chern classes of the symmetric products of a curve were computed through a series of articles ([14], [13], [20], [16]). The method in Mattuck's article [16] is adapted in section 3 to the varieties of divisors. Of course, the variety of divisors of degree $n$ and the $n$-fold symmetric product have been proved isomorphic ([3, § 6], [20], [9]), but each represents a basically different point of view.

We would like to point out the influence of A. Mattuck. His article [16] inspired us to reconsider the problem of the existence of divisors from the point of view of the variety of divisors. A seminar talk he gave and many private conversations with him were very helpful.

Throughout, we work over a fixed algebraically closed ground field $k$. The term "variety" indicates a reduced and irreducible $k$-scheme of finite type. The Chow ring of a smooth quasi-projective variety refers to the ring of cycles with integer coefficients modulo rational equivalence.

## 1. Porteous' formula

Let $X$ be a smooth quasi-projective variety. For each subscheme $Z$ of $X$, let cycle $(Z)$ denote the element of the Chow ring of $X$ determined by the cycle $\Sigma n_i Z_i$, where the $Z_i$ are the irreducible components of $Z$ and $n_i$ is the length of the (Artin) local ring of $Z$ at the generic point of $Z_i$. For each locally free coherent sheaf $E$ on $X$, let

$$c(E)(t) = 1 + c_1(E)t + c_2(E)t^2 + \ldots$$

denote the Chern polynomial of $E$, whose coefficients $c_i(E)$ are the Chern classes of $E$ in the Chow ring of $X$. Finally, for each formal Laurant series with coefficients in any given ground ring,

$$c(t) = \ldots + c_{-2}t^{-2} + c_{-1}t^{-1} + c_0 + c_1 t + c_2 t^2 + \ldots,$$

and for each pair of nonnegative integers $a$, $b$, set

$$\Delta_{b,a}(c(t)) = \det \begin{bmatrix} c_b & c_{b+1} & \cdots & c_{b+a-1} \\ c_{b-1} & c_b & \cdots & c_{b+a-2} \\ \vdots & \vdots & \ddots & \vdots \\ c_{b-a+1} & c_{b-a+2} & \cdots & c_b \end{bmatrix}.$$

The formula below for cycle $(Z_r(u))$ is known as "Porteous' formula"; it is proved under the hypotheses below by Kempf and Laksov [11, Cor. 11] and under more restrictive (transversality) hypotheses by Porteous [19, Prop. 1.3, p. 298].

THEOREM 1. *Let $u\colon E \to F$ be a map between locally free coherent sheaves on $X$, say of ranks $p$ and $q$. Let $r$ be a nonnegative integer satisfying the condition, $(p-q) \leqslant r \leqslant p$. Let $Z_r(u)$ denote the scheme of zeros of the exterior power $\bigwedge^{(p-r+1)}u$.*

(i) *If $Z_r(u)$ is nonempty, then we have an estimate,*

$$\operatorname{codim}(Z_r(u), X) \leqslant r(q-p+r).$$

(ii) *If $Z_r(u)$ is empty or of pure codimension $r(q-p+r)$, then we have a formula,*

$$\text{cycle}\,(Z_r(u)) = \Delta_{(q-p+r),r}(c(F)(t)/c(E)(t)).$$

COROLLARY 2. *Let $f\colon X \to Y$ be a morphism between smooth quasi-projective varieties, say, of dimensions $p$ and $q$. Assume that each scheme-theoretic fiber of $f$ is smooth. Let $r$ be a nonnegative integer satisfying the condition, $(p-q) \leqslant r \leqslant p$, and consider the closed subset of $X$,*

$$G^r = \{x \in X \mid \dim_x (f^{-1}f(x)) \geqslant r\}.$$

(Notice that the condition on $r$ is geometrically reasonable, for by general theory in a morphism of varieties the dimension of each fiber is bounded above by the dimension of the source and below by the difference of the dimensions of the source and target.)

(i) *If $G^r$ is nonempty, then we have an estimate,*

$$\operatorname{codim}(G^r, X) \leqslant r(q-p+r).$$

(ii) *If $G^r$ is empty, then we have an equation*

$$\Delta_{(q-p+r),r}(c(f^*TY)(t)/c(TX)(t)) = 0,$$

*where $TX$ and $TY$ denote the tangent sheaves.*

*Proof.* At each closed point $x$ of $X$, we have a canonical exact sequence of tangent spaces.

$$0 \to T_x(f^{-1}f(x)) \to T_x(X) \xrightarrow{u(x)} T_{f(x)}(Y),$$

and $u(x)$ is equal to the fiber at $x$ of the canonical map of sheaves,

$$u\colon TX \to f^*TY.$$

Therefore, $x$ lies on the scheme $Z_r(u)$, clearly, if and only if the estimate,

$$\dim T_x(f^{-1}f(x)) \geqslant r,$$

holds. However, since $f^{-1}f(x)$ is smooth, the above estimate holds if and only if the estimate,

$$\dim_x (f^{-1}f(x)) \geqslant r$$

holds. Thus, $G^r$ is the set of points of the scheme $Z_r(u)$. Consequently, the corollary follows from the theorem.

## 2. Existence of divisors

Let $X$ be a projective variety. Consider the canonical morphism,

$$f\colon \mathrm{Div}_{(X/k)} \to \mathrm{Pic}_{(X/k)},$$

defined by sending a divisor to its associated invertible sheaf.

Let $L$ be a Poincaré sheaf on $X \times \mathrm{Pic}_{(X/k)}$. By [6, Thm. 7.7.6, p. 69], there exists a unique coherent sheaf $Q$ on $\mathrm{Pic}_{(X/k)}$ satisfying the equation,

$$\underline{\mathrm{Hom}}(Q, G) = p_{2*}(L \otimes p_2^* G),$$

for all quasi-coherent sheaves $G$ on $\mathrm{Pic}_{(X/k)}$. By [4, § 4, pp. 8–12], there exists a canonical isomorphism,

$$\mathbf{P}(Q) = \mathrm{Div}_{(X/k)},$$

and the structure map of $\mathbf{P}(Q)$ corresponds to $f$. Therefore, each scheme-theoretic fiber of $f$ is smooth.

Let $r$ be a nonnegative integer satisfying the condition, $(p-q) \leqslant r \leqslant p$, and consider the closed subset of $\mathrm{Div}_{(X/k)}$,

$$G^r = \{D \in \mathrm{Div}_{(X/k)} \mid \dim (f^{-1}f(D)) \geqslant r\}.$$

Corollary 2 immediately yields the following result because the tangent bundle to a group variety is trival.

PROPOSITION 3. *Let $S$ be an irreducible component of* $\mathrm{Div}_{(X/k)}$. *Assume $S$ and* $\mathrm{Pic}_{(X/k)}$ *are smooth, say, of dimensions $n$ and $g$.*

(i) *If $G^r \cap S$ is nonempty, then we have an estimate,*

$$\mathrm{codim}\,(G^r \cap S, S) \leqslant r(g-n+r).$$

(ii) *If $G^r \cap S$ is empty, then we have an equation,*

$$\Delta_{(g-n-r),\, r}(1/c(TS)(t)) = 0.$$

From now on, assume $X$ is a curve. Then, $\mathrm{Div}_{(X/k)}$ and $\mathrm{Pic}_{(X/k)}$ are smooth [3, Cor. 5.4, p. 23 and 5, Prop. 2.10 (ii), p. 16] and decompose into irreducible components $\mathrm{Div}^n_{(X/k)}$ and $\mathrm{Pic}^n_{(X/k)}$ according to the degree $n$ of the divisors and invertible sheaves they parametrize. Furthermore, $\mathrm{Div}^n_{(X/k)}$ is of dimension $n$, and $\mathrm{Pic}^n_{(X/k)}$ is of dimension $g$, the arithmetic genus of $X$. Set

$$G^r_n = G^r \cap D^n_{(X/k)}.$$

Then $G^r_n$ parametrizes the divisors of degree $n$ that vary in a linear system of projective dimension at least $r$.

Fix a (closed) base point $P_0$ in the smooth locus of $X$, and normalize the Poincaré sheaf $L$ on $X \times \mathrm{Pic}_{(X/k)}$ by requiring its fiber over $P_0$ to be trivial. Then $P_0$ is a divisor of degree 1, and sending a divisor $D$ to $(D+P_0)$ and an invertible sheaf $M$ to $M \otimes O_X(P_0)$ defines a commutative diagram,

$$\begin{array}{ccc} \mathrm{Div}^{n-1}_{(X/k)} & \rightarrow & \mathrm{Div}^n_{(X/k)} \\ \downarrow & & \downarrow \\ \mathrm{Pic}^{n-1}_{(X/k)} & \xrightarrow{\sim} & \mathrm{Pic}^n_{(X/k)} \end{array}$$

for each $n$. We obtain a morphism,

$$\phi_n\colon \mathrm{Div}^n_{(X/k)} \rightarrow \mathrm{Pic}^0_{(X/k)},$$

which sends a divisor $D$ of degree $n$ to the invertible sheaf $O_X(D-nP_0)$. We also obtain a canonical isomorphism,

$$\mathrm{Div}^n_{(X/k)} = \mathbf{P}(Q_n),$$

where $Q_n$ is the coherent sheaf on $\mathrm{Pic}^0_{(X/k)}$ satisfying the equation,

$$\underline{\mathrm{Hom}}\ (Q_n, G) = p_{2*}[p_1^* O_X(nP_0) \otimes L_0 \otimes p_2^* G],$$

for all quasi-coherent sheaves $G$ on $\mathrm{Pic}^0_{(X/k)}$, where we have set

$$L_0 = L|X \times \mathrm{Pic}^0_{(X/k)}.$$

We now introduce the last of the general notation. Set

$$z_n = c_1(O_{\mathbf{P}(Q_n)}(1));$$

$z_n$ is an element of degree 1 in the Chow ring of $\mathrm{Div}^n_{(X/k)}$. Set

$$w_{g-n} = \phi_{n*}(1);$$

$w_{g-n}$ is an element of degree $(g-n)$ in the Chow ring of $\mathrm{Pic}^0_{(X/k)}$. Finally, set

$$w(t) = 1 + w_1 t + w_2 t^2 + \dots .$$

The following proposition is the object of the remaining two sections.

PROPOSITION 4. *Under the above conditions, we have the following congruence modulo numerical equivalence:*

$$\begin{aligned}\phi_{n*}[z_n^r\, \Delta_{(g-n+r),r}(1/c(T\,\mathrm{Div}^n_{(X/k)})\,(t))] \\ \equiv [r!\dots 0!/(g-n+2r)!\dots (g-n+r)!]\, w_1^{(r+1)(g-n+r)}.\end{aligned}$$

The next result, our goal, is an immediate consequence of Propositions 3 and 4 and of Poincaré's formula (15, § 2, Formula 4] $\deg(w_1^g) = g!$.

THEOREM 5. *Assume $X$ is a smooth projective curve. Set*

$$\tau = (r+1)(n-r) - rg.$$

*Then $G_n^r$ is nonempty and of dimension at least $(\tau + r)$ if $\tau \geqslant 0$ holds.*

*Remark* 6. About a hundred years ago, Brill and Noether [1, § 9, pp. 290–293] studied the sets $G_n^r$ using a matrix, which locally represents the map of tangent sheaves,

$$u\colon\ T\,\mathrm{Div}^n_{(X/k)} \to f^* T\,\mathrm{Pic}^n_{(X/k)}.$$

Let $D$ be a divisor of degree $n$ on $X$, and also use $D$ to denote the corresponding point of $\mathrm{Div}^n_{(X/k)}$. By an argument like [18, Lecture 24, 3°, pp. 164–166], the fiber of $u$ at $D$ is easily seen to be equal to the map of cohomology groups,

$$u_D\colon H^0(O_D(D)) \to H^1(O_X),$$

arising from the short exact sequence,

$$0 \to O_X \to O_X(D) \to O_D(D) \to 0.$$

By general duality theory, the transpose of $u_D$ is equal to the natural map,

$$v\colon \quad H^0(\Omega^1_{X/k}) \to H^0(\Omega^1_{X/k} \otimes O_D).$$

Assume $D$ is a sum of $n$ distinct points $P_i$, fix a holomorphic differential $\omega$ that is nonzero at each $P_i$, and fix functions $\varphi_1, ..., \varphi_g$ such that the differentials $\varphi_1\omega, ..., \varphi_g\omega$ form a basis of $H^0(\Omega^1_{X/k})$. Then, locally at $D$, the map $v$ is represented by the $n \times g$ matrix,

$$\begin{bmatrix} \varphi_1(P_1) \ldots \varphi_g(P_1) \\ \vdots \qquad\quad \vdots \\ \varphi_1(P_n) \ldots \varphi_g(P_n) \end{bmatrix},$$

which is the matrix used by Brill and Noether.

We reasoned from the structure of $f$ that $G^r_n$ is the underlying set of $Z_r(u)$ or, in other words, that $D$ lies in $G^r_n$ if and only if all the $(n-r+1)$ minors of the above matrix vanish. Brill and Noether reasoned directly from the Riemann-Roch theorem: the vanishing of the $(n-r+1)$-minors means that the number of independent holomorphic differentials vanishing on $D$ is at least $(g-n+r)$; whence, $D$ varies in a linear system of projective dimension at least $r$.

Brill and Noether assume $X$ has general moduli. They suggest that, since the points $P_i$ are constrained to move on a curve, setting the $(n-r+1)$-minors equal to zero yields $r(g-n+r)$ independent equations; consequently, the dimension of $G^r_n$ at $D$ is precisely $(\tau+r)$. It is true that, in the affine space of all $n \times g$ matrices, the matrices whose $(n-r+1)$-minors vanish form a subvariety of codimension $r(g-n+r)$; consequently, if $G^r_n$ contains $D$, then its codimension at $D$ is at most $r(g-n+r)$, whether or not $X$ has general moduli. Thus, Brill and Noether's reasoning does yield the appropriate estimate of the dimension of $G^r_n$ for any $X$; however the exact dimension of $G^r_n$ when $X$ has general moduli remains conjectural. On the other hand, Brill and Noether do not consider the problem of proving that $G^r_n$ is nonempty if $\tau$ is nonnegative, and their reasoning does not yield a proof.

## 3. Chern Classes of $\mathrm{Div}^n_{(X/k)}$

Keep all the notation and assumptions of section 2 ($X$ is a projective curve, etc.).

LEMMA 7. *For each integer, n there is a natural embedding,*

$$j_n\colon \mathrm{Div}^{n-1}_{(X/k)} \to \mathrm{Div}^n_{(X/k)};$$

*$j_n$ is defined by a surjective map $Q_n \to Q_{n-1}$, and its image is the scheme of zeros of a section of the tautological sheaf $O_{\mathbf{P}(Q_n)}(1)$.*

*Proof.* The short exact sequence on $X$,

$$0 \to O_X((n-1)P_0) \to O_X(nP_0) \to O_{P_0} \to 0,$$

and the Poincaré sheaf $L_0$ on $X \times \mathrm{Pic}^0_{(X/k)}$ together clearly yield a short exact sequence on $X \times \mathrm{Pic}^0_{(X/k)}$,

$$\begin{aligned} 0 \to p_1^* O_X((n-1)\,P_0) \otimes L_0 \otimes p_2^* G \to p_1^* O_X(nP_0) \otimes L_0 \otimes p_2^* G \\ \to p_1^* O_{P_0} \otimes L_0 \otimes p_2^* G \to 0, \end{aligned}$$

for each quasi-coherent sheaf $G$ on $\mathrm{Pic}^0_{(X/k)}$. So, applying $p_{2*}$ and rewriting the terms, we obtain an exact sequence on $\mathrm{Pic}^0_{(X/k)}$,

$$0 \to \underline{\mathrm{Hom}}\,(Q_{n-1}, G) \to \underline{\mathrm{Hom}}\,(Q_n, G) \to \underline{\mathrm{Hom}}\,(Q', G),$$

for an appropriate coherent sheaf $Q'$ on $\mathrm{Pic}^0_{(X/k)}$. Since this sequence behaves functorially in $G$, it arises from an exact sequence,

$$Q' \to Q_n \to Q_{n-1} \to 0.$$

This last sequence defines an embedding of $\mathbf{P}(Q_{n-1})$ into $\mathbf{P}(Q_n)$ as the scheme of zeros of the composite map,

$$Q'_{\mathbf{P}(Q_n)} \to Q_{n.\mathbf{P}(Q_n)} \to O_{\mathbf{P}(Q_n)}(1).$$

However, since $L_0$ was normalized by requiring its fiber $p_1^* O_{P_0} \otimes L_0$ to be trival, $Q'$ is clearly equal to the structure sheaf on $\mathrm{Pic}^0_{(X/k)}$. Therefore, $\mathbf{P}(Q_{n-1})$ is embedded as the zeros of a section of $O_{\mathbf{P}(Q_n)}(1)$, and the proof is complete.

LEMMA 8. *For each n, we have the formulas:*

$$\phi_n j_n = \phi_{n-1} \tag{8.1}$$

$$j_n^*(z_n) = z_{n-1} \tag{8.2}$$

$$j_{n*}(1) = z_n \tag{8.3}$$

$$\phi_{n*}(z_n^i) = w_{g-n-i} \quad \text{for each } i \geqslant 0. \tag{8.4}$$

*Proof.* Formulas (8.1) and (8.2) hold because $j_n$ is defined by a surjective map $Q_n \to Q_{n-1}$ Formula (8.3) holds because the image of $j_n$ is the scheme of zeros of a section of $O_{\mathbf{P}(Q_n)}(1)$.

Combining (8.2) and (8.3) and the projection formula, we find for each $i \geqslant 0$:

$$j_{n*}(z_{n-1}^i) = j_{n*}(1\, j_n^*(z_n^i)) = j_{n*}(1)\, z_n^i = z_n^{i+1}.$$

Using this result, we easily establish the following formula:

$$j_{n*} j_{(n-1)*} \dots j_{(n-i+1)*}(1) = z_n^i \qquad \text{for each } i \geqslant 1.$$

Applying (8.1) to this formula $i$ times, we obtain (8.4).

LEMMA 9. *For $n > (2g-2)$, the sheaf $Q_n$ is locally free of rank $(n+1-g)$.*

*Proof.* The fiber of $p_1^* O_X(nP_0) \otimes L_0$ over a $k$-point of $\mathrm{Pic}^0_{(X/k)}$ is an invertible sheaf $R$ on $X$ degree $n$. So, we have:

$$H^1(X, R) = 0$$

$$\dim [H^0(X, R)] = (n+1-g).$$

By the general theory of cohomology and base change [6, § 7.8, pp. 72–76], the first equation implies $Q_n$ is locally free and the second gives its rank.

LEMMA 10. *Let $J$ be a smooth variety, $E$ a locally free sheaf on $J$ of rank $e$, say. Let $h\colon \mathbf{P}(E) \to J$ denote the structure morphism, and set*

$$\begin{aligned} z &= c_1(O_{\mathbf{P}(E)}(1)) \\ v_i &= h_*(z^{(e-1+i)}) \qquad \text{for } i = 1, 2, \dots \\ v(t) &= 1 + v_1 t + v_2 t^2 + \dots \\ E^* &= \underline{\mathrm{Hom}}\,(E, O_J). \end{aligned}$$

*Then, we have the formulas:*

(i) $c(E^*)(t) = v(t)^{-1}$;
(ii) $c(T\mathbf{P}(E)/J)(t) = (1+zt)^e[(h^*v)(t/(1+zt))]^{-1}$.

*Proof.* Recall from the general theory of Chern classes that we have an equation,

$$z^e + c_1(E^*) z^{e-1} + \dots + c_e(E^*) = 0,$$

and the formulas,

$$h_* z^{e-1} = 1$$

$$h_* z^i = 0 \qquad \text{for } i = 0, \dots, (e-2).$$

Multiply the above equation by $z^i t^{(i+1)}$ for $i = 0, 1, \dots$ and add the products. Apply $h_*$, use the projection formula and the above formulas to simplify the equation, and add 1 to both sides. The result can be put in the following form:

$$(1+c_1(E^*)t+\ldots+c_e(E^*)t^e)(1+h_*(z^e)t+h_*(z^{e+1})t^2+\ldots)=1.$$

Formula (i) follows immediately.

We have an exact sequence on $\mathbf{P}(E)$,

$$0\to O_{\mathbf{P}(E)}\to O_{\mathbf{P}(E)}(1)\otimes h^*E^*\to T\mathbf{P}(E)/J\to 0.$$

It yields the formula,

$$c(T\mathbf{P}(E)/J)(t)=c(O_{\mathbf{P}(E)}(1)\otimes h^*E^*)(t).$$

So, applying a general formula about the Chern polynomial of a tensor product with an invertible sheaf, we obtain the formula,

$$c(T\mathbf{P}(E)/J)(t)=(1+zt)^e c(h^*E^*)(t/(1+zt)).$$

Formula (ii) follows immediately from this formula and Formula (i).

PROPOSITION 11 (Mattuck). *For each $n\geqslant 0$, we have a formula,*

$$c(T\operatorname{Div}^n_{(X/k)})(t)=(1+z_nt)^{n-g-1}[(\phi_n^*w)(t/(1+z_nt))]^{-1}.$$

*Proof.* For $n>(2g-2)$, the formula results immediately from Lemma 9, Lemma 10, and Formula (8.4), and from the fact that $T\operatorname{Pic}^0_{(X/k)}$ is trivial and fits into an exact sequence.

$$0\to T\operatorname{Div}^n_{(X/k)}/\operatorname{Pic}^0_{(X/k)}\to T\operatorname{Div}^n_{(X/k)}\to T\operatorname{Pic}^0_{(X/k)}\to 0.$$

On the other hand, for any $n\geqslant 0$, there is an exact sequence,

$$0\to T\operatorname{Div}^{n-1}_{(X/k)}\to j_n^*T\operatorname{Div}^n_{(X/k)}\to j_n^*O_{\mathbf{P}(Q_n)}(1)\to 0,$$

because the image of $j_n$ is the scheme of zeros of a section of $O_{\mathbf{P}(Q_n)}(1)$. Consequently, using Formulas (8.1) and (8.2), we easily see that if the formula holds for $n$, then it holds for $n-1$.

## 4. Proof of Proposition 4

We begin with some general lemmas about the determinant $\Delta_{b,a}(c(t))$ defined in section 1.

LEMMA 12. *For each integer $m$, we have a formula,*

$$\Delta_{b,a}(t^m c(t))=\Delta_{(b-m),a}(c(t)).$$

*Proof.* The formula is obvious from the definition of $\Delta_{b,a}(c(t))$.

LEMMA 13. *Set $m=(b-a)$. Then, for each $z$, we have a formula,*

$$\Delta_{b,a}((1+zt)^m c(t/(1+zt))) = \Delta_{b,a}(c(t)).$$

*Proof*. We clearly have identities,

$$(1+zt)^m c(t/(1+zt)) = t^m \textstyle\sum_i c_{m+i}(t/(1+zt))^i, \quad c(t) = t^m \sum_i c_{m+i} t^i.$$

So, by Lemma 12, it is enough to establish the formula,

$$\Delta_{a,a}(c'(t/(1+zt))) = \Delta_{a,a}(c'(t)) \text{ with } c'(t) = \Sigma\, c_{m+i} t^i.$$

Thus, it is enough to treat the vase with $b=a$ and $m=0$.

By definition $\Delta_{a,a}(c(t))$ is the determinant of the $a\times a$-matrix $C=[c_{a+j-i}]_{i,j}$; so it is equal to the determinant of the product $ACB$ where $A$ and $B$ are the following two upper triangular matrices with 1's on the diagonals (a binomial coefficient $\binom{n}{q}$ with $q<0$ is zero by definition).

$$A=\left[\binom{j-a-1}{j-i} z^{j-i}\right]_{i,}$$

and

$$B=\left[\binom{-i}{j-i} z^{j-i}\right]_{i,j}.$$

A straightforward computation (involving setting $j=a-r$ in $A$ and $i=s+1$ in $B$) shows that the $(ij)$-th entry $p_{ij}$ in $BCA$ is given by

$$p_{ij}=\sum_{r=0}^{a-i}\sum_{s=0}^{j-1} c_{r+s+1}\binom{-1-s}{j-1-s}\binom{-1-r}{a-i-r} z^{a+j-i-r-s-1}.$$

On the other hand, $\Delta_{a,a}(c(t/(1+zt))$ is, by definition, the determinant of the $a\times a$-matrix $[d_{a+j-i}]_{i,j}$ where $d_n$ is the coefficient of $t^n$ in the expansion of $c(t/(1+zt))$. A straightforward computation (involving the binomial theorem for negative exponents) yields the formula,

$$d_n=\sum_{m=0}^{n-1} c_{n-m}\binom{m-n}{m} z^m.$$

Thus, it is enough to prove $p_{ij}$ is equal to $d_{a+j-i}$. So, clearly, it is enough to establish the identity,

$$\Sigma\binom{-1-s}{j-1-s}\binom{-1-r}{a-i-r}=\binom{m-a-j-i}{m},$$

where the summation is taken over $r$, $s$ subject to the conditions,

$$0\leqslant r\leqslant a-i,\ 0\leqslant s\leqslant j-1,\ a+j-i-r-s-1=m.$$

This identity, however, results immediately from the following well-known two:

$$\binom{-p}{q} = (-1)^q \binom{p+q-1}{q}$$

$$\sum \binom{p}{q}\binom{r}{s} = \binom{p+r}{q+s}$$

where the summation is taken over $q$, $s$ subject to the conditions,

$$0 \leqslant q \leqslant p, \ 0 \leqslant s \leqslant r, \ q+s = \text{constant}.$$

LEMMA 14. *For each $z$, we have a formula,*

$$\Delta_{b,a}((1-zt)\,c(t)) = \det \begin{bmatrix} c_b & \cdots & c_{b+a-1} & z^a \\ c_{b-1} & \cdots & c_{b+a-2} & z^{a-1} \\ \vdots & \ddots & \vdots & \vdots \\ c_{b-a} & \cdots & c_{b-1} & 1 \end{bmatrix}.$$

*Proof.* Clearly $\Delta_{b,a}((1-zt)c(t))$ is the determinant of the $a \times a$-matrix $D=[d_{b+j-i}]_{i,j}$ with $d_n = (c_n - zc_{n-1})$. On the other hand, multiplying the second row of the matrix on the right-hand side of the formula by $z$ and subtracting the product from the first row, then multiplying the third row by $z$ and subtracting the product from the second row, etc., clearly turns this matrix into one of the form $\begin{bmatrix} D & 0 \\ * & 1 \end{bmatrix}$. Thus, the formula holds.

LEMMA 15. *For each $w$, we have a formula,*

$$\Delta_{b,a}(\exp(wt)) = [(a-1)! \ldots 0!/(a+b-1)! \ldots b!]\, w^{ab}.$$

*Proof.* (Another proof is given on p. 15 of [10].) Clearly $\Delta_{b,a}(\exp(wt))$ is equal to $w^{ab}$ times the determinant of the matrix,

$$\begin{bmatrix} 1/b! & \ldots & 1/(b+a-1)! \\ \vdots & & \vdots \\ 1/(b-a+1)! & \ldots & 1/b! \end{bmatrix}.$$

Multiply the second column by $(b+1)$ and subtract the product from the first column, then multiply the third column by $(b+2)$ and subtract the product from the seond column, etc. The first $(a-1)$ entries of the first row are obviously now zero. The cofactor of the $a$-th entry is now equal to $(a-1)!\,\Delta_{b,a-1}(\exp(t))$: this can be easily seen by using the obvious identity,

$$(1/n!) - (m/(n+1)!) = (n+1-m)/(n+1)!,$$

to simplify the corresponding $(a-1)\times(a-1)$-submatrix. The formula now follows by induction on $a$.

PROPOSITION 16. *Under the conditions of section 2 ($X$ a projective curve, etc.), $\Delta_{g-n+r,r}(1/c(T\ \mathrm{Div}^n_{(X/k)}(t))$ is equal to the determinant of the $(r+1)\times(r+1)$ matrix.*

$$\begin{bmatrix} \phi_n^* w_{g-n+r} & \cdots & \phi_n^* w_{g-n+2r-1} & z_n^r \\ \vdots & & \vdots & \vdots \\ \phi_n^* w_{g-n+1} & \cdots & \phi_n^* w_{g-n+r} & z_n \\ \phi_n^* w_{g-n} & \cdots & \phi_n^* w_{g-n+r-1} & 1 \end{bmatrix}.$$

*Proof*. The formula of Proposition 11 may obviously be rewriten as follows:

$$1/c(T\mathrm{Div}^n_{(X/k)})\,(t) = (1+z_n t)^{g-n}\left[1 - z_n\left(\frac{t}{1+z_n t}\right)\right]\phi_n^* w\left(\frac{t}{1+z_n t}\right).$$

So, Lemma 13 yields the formula,

$$\Delta_{g-n+r,r}(1/c(T\mathrm{Div}^n_{(X/k)})\,(t)) = \Delta_{g-n+r,r}((1-z_n t)\,\phi_n^* w(t)).$$

Lemma 14 now yields the desired conclusion.

*Proof of Proposition* 4. By Proposition 16 and the projection formula,

$$\phi_{n*}[z_n^r\,\Delta_{g-n+r,r}(1/c(T\mathrm{Div}^n_{(X/k)})\,(s))],$$

is equal to the determinant of the $(r+1)\times(r+1)$-matrix,

$$\begin{bmatrix} w_{g-n+r} & \cdots & w_{g-n+2r-1} & \phi_{n*}\, z_n^{2r} \\ \vdots & & \vdots & \vdots \\ w_{g-n} & \cdots & w_{g-n+r-1} & \phi_{n*}\, z_n^r \end{bmatrix}$$

Using (8.4) $\phi_{n*} z_n^i = w_{g-n+i}$, we see that this determinant is equal to $\Delta_{g-n+r,r+1}(w(t))$. Now, Poincare's formulas [15, § 2, Formula 4] may be conveniently expressed in the form

$$w(t) = \exp\,(w_1 t) \qquad \text{(modulo numerical equivalence).}$$

Consequently, Lemma 15 yields the congruence asserted in Proposition 4.

## References


[1]. Brill, A. & Noether, M., Ueber die algebraischen Functionen und ihre Anvendungen in der Geometrie. *Math. Ann.* no. 7 (1874), 269–310.

[2]. Castelnuovo, G., Numero delle involuzioni razionali giancenti sopra uno curva di dato genere. *Rendiconti della R. Accademia dei Lincei*, ser. 4,5 (1889).

[3]. Grothendieck, A., Technique de descente et theorémès d'existence en géometrie algébrique, IV: Les schémas de Hilbert. *Seminaire Bourbaki, t.* 13, Mai 1961, no. 212, 20 pages.

[4]. —— Technique de descente et théorèmes d'existence en géometrie algébrique, V: Les schémas de Picard: Théorèmes d'existence. *Séminaire Bourbaki, t.* 14, Février 1962, no. 232, 19 pages.

[5]. —— Technique de descente et théorèmes d'existence en géometrie algébrique, VI: Les schémas de Picard: Propiétés générales. *Séminaire Bourbaki,* t. 14, Mai 1962, no. 236, 23 pages.

[6]. Grothendieck, A. & Dieudonné, J., *Eléménts de géométrie algébrique III: Etude cohomologique des faisceaux cohérents* (seconde partie). Publ. Math. de l'IHES, No. 17 1963, Paris, Presses universitaires de France.

[7]. Gunning, R. C., *Jacobi varieties*. Princeton Univ. Press, 1972.

[8]. Hensel, K. & Landsberg, G., *Theorie der algebraischen Functionen*, Chelsea (Reprint), New York, 1965.

[9]. Iversen, B., *Linear determinants with applications to the Picard scheme of a family of algebraic curves*. Lecture notes in mathematics, no. 174, Springer Verlag, 1970.

[10]. Kempf, G., Schubert methods with an application to algebraic curves. *Publications of Mathematisch Centrum, Amsterdam* (1971).

[11]. Kempf, G. & Laksov, D., The determinantal formula of Schubert calculus. *Acta Math.*, 132 (1973), 153–162.

[12]. Kleiman, S. & Laksov, D., On the existence of special Divisors. *Amer. J. Math.*, 94 (1972), 431–436.

[13]. Mac Donald, I. G., Symmetric products of an algebraic curve. *Topology*, 1 (1962), 319–343.

[14]. Mattuck, A., Symmetric products and jacobians. *Amer. J. Math.*, 83 (1961) 189–206.

[15]. —— On symmetric products of curves. *Proc. Amer. Math. Soc.*, 13 (1962), 82–87.

[16]. —— Secant bundles on symmetric products. *Amer. J. Math.*, 87 (1965) 779–797.

[17]. Meis, T., Die minimale Blätterzahl der Konkretisierung einer kompakten Riemannischen Fläche. *Schr. Math. Inst. Univ. Münster* (1960).

[18]. Mumford, D., *Lectures on curves on an algebraic surface*. Princeton Univ. Press, 1966.

[19]. Porteous, I. R., Simple singularities of maps. *Proceedings of Liverpool singularities-symposium 1*, Lecture notes in mathematics, no. 192, Springer Verlag 1971.

[20]. Schwarzenberger, R. L. E., Jacobians and symmetric products. *Illinois J. Math.*, 7 (1963), 257–268.


*Received April* 15, 1973