THE AUTOMORPHISM GROUP OF A LOCALLY COMPACT ABELIAN GROUP

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1. Introduction

Let G be a locally compact abelian (henceforth abbreviated to "lca") group and let $\mathrm{Aut}(G)$ denote the full group of automorphisms of G with the g-topology. One asks "For which lca groups, G, is $\mathrm{Aut}(G)$ locally compact?" To answer this question in general would require a much more detailed understanding of the structure theory of lca groups than is presently available. However, in this paper we make an initial attack on the problem by answering the question for the case in which G contains a lattice non-trivially. We also give some partial results in the case that G contains a lattice trivially. We apply these results to the problem of determining those lca groups, G, for which the group, $\mathbf{B}(G)$, discussed by Weil in [16], is locally compact.

More explicitly the contents of this paper are as follows. In § 2 we review the duality and structure theory of lca groups and establish notations. In § 3 we review the definition and properties of the g-topology including a general Ascoli theorem. In § 4 the main theorem is stated. The sufficiency and necessity of the main theorem are proved in items 5 and 6 respectively. In § 7 we give partial results on the question of the local compactness of $\mathrm{Aut}(G)$ in the case that G contains a lattice trivially. In § 8 the above results are applied to the question of the local compactness of $\mathbf{B}(G)$. Finally, § 9 is a counterexample, showing that the lattice hypothesis cannot be dropped.

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2. Preliminaries

We review the Pontryagin duality theory. Let G and G' be lca groups. A function from G to G' is called a *homomorphism* if it is continuous and preserves the group operation. If this is the case, we write $f: G \to G'$. Note that not every homomorphism is an open map. By the category of lca groups we mean the category whose objects are lca groups and whose morphisms are homomorphisms.

Let $\mathbf{T}$ be the multiplicative group of complex numbers of absolute value one. Let G be any lca group. A character of G is a homomorphism from G to $\mathbf{T}$. Let G^* denote the set of all such characters. For two elements, χ and χ', of G let $\chi + \chi'$ be the character of G defined by $(\chi + \chi')(x) = \chi(x) \cdot \chi'(x)$ for all x in G. Put the compact open topology on G^*. With these definitions G^* forms a topological abelian group, called the *dual group* of G. The Pontryagin duality theorem asserts that: (i) G^* is locally compact and moreover G is discrete if and only if G^* is compact.

If $f: G \to G'$, define $f^*: G'^* \to G^*$ by $f^*(\chi) = \chi \circ f$ for each χ in G'^*. This makes $*$ into a contravariant functor on the category of lca groups. The Pontryagin duality theorem goes on to assert that: (ii) the canonical map of G into its second dual, G^{**}, is an isomorphism of topological groups.

We henceforth consider G as being identified with the dual of G^*, which we may do because of (ii) above.

If H is a closed subgroup of G, define the *annihilator of H in G^** to be the set of all characters of G which are trivial on H. It is clearly a subgroup of G^*. If there is no chance of ambiguity it will be called simply the *annihilator of H* and will be denoted, $H_\perp$.

The Pontryagin theorem says finally that: (iii) $H_\perp$ is a closed subgroup of G^*, and $H_{\perp\perp} = H$. Moreover, $H_\perp$ is isomorphic to $(G/H)^*$.

Thus "$\perp$" defines a one to one inclusion reversing correspondence between the closed subgroups of G and the closed subgroups of G^*.

A very readable proof of the duality theorem may be found in [12].

We proceed to describe the structure of lca groups. A few definitions are needed first.

Definition. Let G be an lca group. G is called a *topological torsion group* if G is equal to the union of its compact subgroups and 0 is equal to the intersection of its open subgroups. G is called a p-primary group if $\lim_h p^h x = 0$ for every x in G.

For a list of equivalent definitions of "topological torsion" and "p-primary" refer to [13]. It turns out that any p-primary group is necessarily a topological torsion group as well. In the special case that G is discrete these notions of "torsion" and "p-primary" reduce to the usual ones.

Definition. Suppose that G_i is an lca group which contains a compact open subgroup, say K_i, for each i in some index set, I. For x in $\Pi_i G_i$, let x_i denote the i-th coordinate of x. Define the *restricted product of* G_i with *respect to* K_i *over all* i *in* I to be the set of all those x in $\Pi_i G_i$ for which x_i is in K_i for almost all i. Then the restricted product with coordinatewise addition is an abelian group which contains $\Pi_i K_i$ as a subgroup. Put a topology on the restricted product by taking $\Pi_i K_i$ with the product topology to be an open subgroup. Denote this product by $\Pi_i(G_i, K_i)$. Then, clearly $\Pi_i(G_i, K_i)$ is a topological abelian group which contains $\Pi_i K_i$ as a compact open subgroup.

Definition. An lca group, G, is said to be *torsion free* if the endomorphism of G, $x \to nx$, is injective for each positive integer n. G is called *divisible* if "$x \to nx$" is surjective for each positive integer n. Let $\mathbf{R}$ and $\mathbf{Z}$ denote the additive group of reals and the additive group of integers respectively.

THEOREM. 1 *Let G be any lca group. Then*

(*i*) *$G = G_2 \oplus \mathbf{R}^n$ where n is some non-negative integer and where G_2 contains a compact open subgroup.*

Let G_1 be the union of all compact subgroups of G and let G_0 be the intersection of all open subgroups of G_1. Then

(ii) *$G_2 \supset G_1 \supset G_0$ regardless of the choice of the subgroup, G_2. Moreover G and G_0 are closed in G.*

(iii) *G_2/G_1 is torsion-free and discrete and is uniquely determined (up to isomorphism) by G independent of the choice of G_2.*

(iv) *G_1/G_0 is a topological torsion group.*

(v) *G_0 is a compact divisible group.*

(vi) *Put $G_3 = G_1/G_0$, Then for each prime, p, G_3 contains a unique maximal p-primary subgroup, say G^p. Moreover, for any compact open subgroup, K of G_3 one has*

$$G_3 = \Pi_p(G^p, G^p \cap K)$$

where the product is taken over all prime numbers, p.

Proof. The well known theorem of Pontryagin asserts that any compactly generated lca group is the direct sum of a compact group, finitely many copies of $\mathbf{Z}$, and finitely many copies of $\mathbf{R}$ [12]. (i) above is a fairly easy consequence of Pontryagin's theorem. (ii), (iii), and (iv) are easy consequences of (i). (v) follows from (iii) and a duality argument. The decomposition in (vi) was known at least as early as [2]; it also appears in [14] where a proof may be found. We remark that G^p may be found by $G^p = \{x \in G_3 \mid \lim_h p^h x = 0\}$. We regard Theorem 1 as proved.

In the light of Theorem 1 we make the following

Definition. G_2/G_1, $(G_1/G_0) = G_3$, G_0, and G^p are called respectively the *torsion free discrete, topological torsion, compact divisible, and p-primary parts* of G.

In the special case that G is discrete, $G = G_2$ and $G_0 = 0$. So that the topological torsion part of G, $G_3 = G_1$, is just the usual maximal torsion subgroup of a discrete group and $G_2/G_1 = G/G_1$ the usual torsion free part thereof.

THEOREM 2. *Let G be any lca group. Let all other notations be as in Theorem 1. Then*

(i) $G^* = (G_2)^* \oplus \mathbf{R}^n$ *where* $(G_2)^*$ *contains a compact open subgroup.*

(ii) *Consider G_1 and G_0 as subgroups of G_2. The annihilators of G_1 and G_0 in $(G_2)^*$ are $(G^*)_0$ and $(G^*)_1$ respectively.*

(iii) *The dual of the torsion free discrete part of G is the compact divisible part of G^*.*

(iv) *The dual of the topological torsion part of G is the topological torsion part of G^*.*

(v) *The dual of the p-primary part of G is the p-primary part of G^*.*

Proof. It follows easily from the Duality theorem that a subgroup, K, of G is compact if and only if $K_\perp$ is open in G^*. (i) and (ii) follow easily from this fact. The proof of (iii) and (iv) from (ii) is an easy exercise in the use of the Pontryagin duality theorem. (v) follows from (iv), theorem 1 (vi) and the fact that for any restricted product the dual of $\pi_i(G_i, K_i)$ is $\pi_i(G_i^*, K_{i\perp})$ where $K_{i\perp}$ is the annihilator of K_i in G_i^*.

Let $\mathbf{Q}$ be the field of rationals. For each rational prime, p, let $\mathbf{Q}_p$ be the additive group of the field of p-adic numbers (i.e. the completion Q with respect to its p-adic valuation). Let $\mathbf{Z}_p$ denote the additive group of p-adic integers (i.e. the closure of $\mathbf{Z}$ in $\mathbf{Q}_p$). It is easily seen that $\mathbf{Z}_p$ is a compact open subgroup of $\mathbf{Q}_p$ and that $\mathbf{Q}_p$, $\mathbf{Z}_p$, and $\mathbf{Q}_p/\mathbf{Z}_p$ are p-primary lca groups.

Definition. Let G be a p-primary lca group. G is said to have p-rank 1 if it is isomorphic to a finite cyclic group of order a power of p, to $\mathbf{Q}_p$, to $\mathbf{Z}_p$, or to $\mathbf{Q}_p/\mathbf{Z}_p$. G is said to have *p-rank n* if it is the direct sum of n groups each of which has p-rank 1. Finally G is said to have *p-rank at most n* if G has p-rank m for some $m \leqslant n$ or if $G = 0$. G is said to have finite p-rank if G has p-rank n for some positive integer n or if $G = 0$.

The dual of a p-group of p-rank n is again a p-group of p-rank n. Indeed, it is well known [15] that for any prime p, there is an ismorphism of $\mathbf{Q}_p$ onto $\mathbf{Q}_p^*$ which carries $\mathbf{Z}_p$ onto $\mathbf{Z}_{p\perp}$. Consequently, the duals of $\mathbf{Q}_p$ and $\mathbf{Z}_p$ are $\mathbf{Q}_p$ and $\mathbf{Q}_p/\mathbf{Z}_p$ respectively. Also any finite abelian group is isomorphic to its own dual. It follows that the dual of a p-rank n group again has p-rank n.

We will later have need of the following lemma.

LEMMA 1. *Suppose that A is a closed subgroup of the lca group, G, and that A^p and $(G/A)^p$ each have finite p-rank. Then G^p has finite p-rank.*

Lemma 1 is proved in a more general context in [11, Theorem 9]. Also in [11] Theorems 1 and 2 are proved in more detail and in a more general context.

We will have need of still another concept of rank.

Definition. We define the *rank of a torsion free discrete group, L,* to be the $\mathbf{Q}$-dimension of $L \otimes_{\mathbf{Z}} \mathbf{Q}$. The rank of L is also the cardinality of a maximal linearly independent subset of L. We define the *rank of a compact* divisible group, A to be the rank of A^*. (A^* is of course torsion free discrete by Theorem 2 (iii).)

3. Topology for Aut(G)

In this section we recall the definition of the g-topology and give an Ascoli criterion for local compactness in this topology. Then we describe a "nice" neighborhood system of the identity in Aut(G). It is convenient to begin the discussion in a more general context.

Definition. Let X and Y be any two topological spaces. Let $\mathscr{C}(X, Y)$ be the set of all continuous functions from X to Y. For each compact subset, C, of X and each, open subset, U, of Y let (C, U) denote the set of all continuous functions, f, from X to Y for which $f(C) \subset U$ and call (C, U) a compact open pair. The topology on $\mathscr{C}(X, Y)$ with subbase consisting of all compact open pairs is called the *compact open topology* or more simply the *C–O topology*. This topology has also been known as "the topology of uniform convergence on compacta."

Now we suppose that X is a locally compact topological space and we let $\mathrm{Aut}_0(X)$ be the group of all homeomorphisms of X to itself. If $\mathrm{Aut}_0(X)$ is given the C–O topology, the composition, $\mathrm{Aut}_0(X) \times \mathrm{Aut}_0(X) \to \mathrm{Aut}_0(X)$, is continuous, but inversion, $\sigma \to \sigma^{-1}$, is not in general continuous; cf. [1]. For this reason, one makes the following definition.

Definition. The *g-topology* on $\mathrm{Aut}_0(X)$ is the topology with subbase consisting of all sets $\mathscr{W}$ such that either $\mathscr{W}$ or $\mathscr{W}^{-1}$ is a compact open pair (where $\mathscr{W}^{-1} = \{\sigma^{-1}: \sigma \in \mathscr{W}\}$).

With the g-topology $\mathrm{Aut}_0(X)$ forms a topological group and the obvious map, $\mathrm{Aut}_0(X) \times X \to X$, is continuous. Moreover, any other topology satisfying these two properties is strictly stronger (i.e. more open sets) than the g-topology. These results were proved by Fox [5] and also Arens [1]. They justify our use of the g-topology.

We are interested in the question of the local compactness of subgroups of $\mathrm{Aut}_0(X)$.

We will first quote the well known Ascoli theorem for the compact open topology and then use it to obtain an Ascoli theorem for the g-topology.

Definition. Let X be a topological space, let Y be a uniform space, and let $\mathscr{W}$ be a set of functions from X to Y. $\mathscr{W}$ is said to be *equicontinuous* if for each x in X and each U in the uniformity of Y, there exists a neighborhood, V, of x such that $(\sigma(x), \sigma(v)) \in U$ for all v in V and all σ in $\mathscr{W}$. We recall that for x in X, the *orbit of x under $\mathscr{W}$* is the set, $\mathscr{W} \cdot x = \{\sigma(x) : \sigma \in \mathscr{W}\}$. $\mathscr{W}$ is said to *operate with bounded orbits* if for each x in X, $\mathscr{W} \cdot x$ is bounded in the uniformity of Y. In the case that Y is locally compact "bounded" becomes equivalent to "relatively compact."

THEOREM 3. (Ascoli). *Let X be a regular locally compact topological space, let Y be a locally compact uniform space, and let $\mathscr{C}(X, Y)$ be given the C–O topology. Then a subset, $\mathscr{W}$, of $\mathscr{C}(X, Y)$ is compact if and only if* (i) *$\mathscr{W}$ is equicontinuous,* (ii) *$\mathscr{W}$ operates with bounded orbits and* (iii) *$\mathscr{W}$ is closed in $\mathscr{C}(X, Y)$.*

A proof of this theorem may be found in [10].

THEOREM 4. (Ascoli). *Let X be a locally compact uniform space, and let $Aut_0(X)$ be the group of homeomorphisms of X with the g-topology. Then, a subgroup, $\mathscr{A}$, of $Aut_0(X)$ is locally compact if and only if (i) there is a neighborhood of 1 in $\mathscr{A}$ which is equicontinuous on X, (ii) there is a neighborhood of 1 in $\mathscr{A}$ which operates with bounded orbits on X, and (iii) $\mathscr{A}$ is closed in $Aut_0(X)$.*

Proof. Suppose that $\mathscr{A}$ is locally compact. Then, there is a neighborhood, say $\mathscr{W}$, of 1 in $\mathscr{A}$ which is compact in the g-topology. Then $\mathscr{W}$ is certainly compact in the coarser *C–O* topology, and hence by Theorem 3 satisfies (i) and (ii) above. $\mathscr{A}$ satisfies (iii) since a locally compact group is closed in any topological group which contains it.

Conversely, suppose that $\mathscr{A}$ satisfies (i), (ii), and (iii) above. Then we may choose a neighborhood, say $\mathscr{W}$, of 1 in $\mathscr{A}$ such that $\mathscr{W}$ is symmetric (i.e. $\mathscr{W} = \mathscr{W}^{-1}$), equicontinuous, operates with bounded orbits on X, and is closed in $\mathrm{Aut}_0(X)$. We will show that $\mathscr{W}$ is compact in the g-topology.

Our proof makes use of the uniformity, say $\mathscr{U}$, on $\mathscr{C}(X, X)$ which is known as the uniformity of uniform convergence on compact subsets. Recall that an arbitrary element of $\mathscr{U}$ may be taken to be of the form $(C \mid U)$ where C is a compact subset of X, U is an element of the uniformity on X, and $(C \mid U)$ consists of all pairs (f, g) of continuous functions on X for which $(f(x), g(x)) \in U$ for all x in C.

The map, $\sigma \to \sigma^{-1}$, of $\mathcal{W}$ to itself is uniformly continuous with respect to the uniformity $\mathcal{U}$. Indeed, let C be an arbitrary compact subset of X and let U be an arbitrary element of the uniformity on X. Then there is a compact subset, say C', of X such that $\sigma(C) \subset C'$ for all σ in $\mathcal{W}$; this is readily verified using the fact that $\mathcal{W}$ is equicontinuous and operates with bounded orbits. Similarly, there is a C'' such that $\sigma(C') \subset C''$ for all σ in $\mathcal{W}$. Also there is an element, V, of the uniformity on X such that for all $\tau \in \mathcal{W}$, $(\tau(y), \tau(z)) \in U$ whenever $(y, z) \in V$ and $y \in C$ and $z \in C''$; this follows from the uniform equicontinuity of $\mathcal{W}$ on the compact set $C \cup C''$. If (σ, τ) is in $(C' | V)$, then (σ^{-1}, τ^{-1}) is in $(C | U)$; in fact suppose $(\sigma(x), \tau(x)) \in V$ for all x in C'; then $(y, \tau\sigma^{-1}(y)) \in V$ for all y in C, because $\sigma^{-1}(C) \subset C'$; then $(\tau^{-1}(y), \sigma^{-1}(y)) \in U$ for all y in C by our choice of V; hence (σ^{-1}, τ^{-1}) is in $(C | U)$ as asserted. We have shown that $\sigma \to \sigma^{-1}$ is uniformly continuous on $\mathcal{W}$.

$\mathcal{W}$ is closed in $\mathcal{C}(X, X)$. Indeed, suppose $\sigma_\alpha \to \tau$ with each σ_α in $\mathcal{W}$ and τ in $\mathcal{C}(X, X)$. Then, $\{\sigma_\alpha^{-1}\}$ is a Cauchy net with respect to U since $\sigma \to \sigma^{-1}$ is uniformly continuous. Then σ_α^{-1} converges to some τ' in $\mathcal{C}(X, X)$ since $\mathcal{C}(X, X)$ is well known to be complete with respect to $\mathcal{U}$. Clearly, τ' is the inverse of τ. Hence τ is in $\mathrm{Aut}_0(X)$. Then τ is in $\mathcal{W}$ since $\mathcal{W}$ is assumed closed in $\mathrm{Aut}_0(X)$. This shows that $\mathcal{W}$ is closed in $\mathcal{C}(X, X)$.

Therefore, by Theorem 3, $\mathcal{W}$ is compact in the C-O topology. But the g-topology and the *C-O* topology coincide on $\mathcal{W}$, because by what we already showed, $\sigma \to \sigma^{-1}$ is continuous with respect to the *C-O* topology on $\mathcal{W}$. Therefore $\mathcal{W}$ is compact in the g-topology. Lemma 2 is proved.

We remark that Dieudonné has a result similar to Theorem 4; cf. [4, Theorem 5].

Remark. For $\mathcal{A}$ to satisfy just conditions (i) and (ii) of Theorem 4 is equivalent to $\mathcal{A}$ being locally bounded in its two-sided group uniformity. This is easily proved from Theorem 4 and the fact, proved in [1], that $\mathrm{Aut}_0(X)$ is complete in its two-sided group uniformity.

Now we specialize to the case that interests us.

Definition. Let G be an lca group. A homomorphism, σ, of G to itself is called an automorphism if there exists in inverse homomorphism, τ, of G to itself such that $\sigma\tau = \tau\sigma = 1$. Let Aut (G) denote the group of all automorphisms of G with the g-topology.

COROLLARY. (Ascoli). *A group, $\mathcal{A}$, of automorphisms of the lca group, G, is locally compact if and only if (i) there is a neighborhood of 1 in $\mathcal{A}$ which is equicontinuous, (ii) there is a neighborhood of 1 in $\mathcal{A}$ which operates with bounded orbits, and (iii) $\mathcal{A}$ is closed in Aut (G).*

Proof. This follows immediately from Theorem 4 and the fact that Aut (G) is closed in $\mathrm{Aut}_0(G)$.

Remark. The map, $\operatorname{Aut}(G) \to \operatorname{Aut}(G^*)$, defined by $\sigma \to \sigma^*$ is an anti-isomorphism of abstract groups by the Pontryagin duality theorem. Elementary considerations show that this map is bicontinuous; cf. [7 p. 429]. Hence, $\operatorname{Aut}(G)$ is anti-isomorphic as a topological group to $\operatorname{Aut}(G^*)$. This fact will be used continually without further reference.

We proceed now to exhibit a very convenient neighborhood system of 1 in $\operatorname{Aut}(G)$.

An lca group is called *elementary* if it is isomorphic to

$$\mathbf{R}^n \oplus \mathbf{Z}^r \oplus (\mathbf{R}/\mathbf{Z})^s \oplus F$$

where n, r, and s are finite and F is a finite abelian group.

Let G be any lca group. (H, K) is called an *elementary pair* in G to mean that H is an open subgroup of G, K is a compact subgroup of H, and H/K is an elementary group.

In any lca group, G, there exist elementary pairs; for instance, if $G = G_2 \oplus \mathbf{R}^n$ where G_2 contains a compact open subgroup, say K, as in Theorem 1, then $(K + \mathbf{R}^n, K)$ is an elementary pair in G.

If (H, K) is an elementary pair in G, then let $\operatorname{Aut}(G: H, K)$ be the set of all automorphisms, σ, of G for which $\sigma(H) = H$ and $\sigma(K) = K$. This is clearly a subgroup of $\operatorname{Aut}(G)$. Moreover, we have a homomorphism,

$$\pi_{H,K}: \operatorname{Aut}(G: H, K) \to \operatorname{Aut}(H/K)$$

where for each σ in $\operatorname{Aut}(G: H, K)$, $\pi_{H,K}(\sigma)$ is the automorphism, σ, first restricted to H and then projected to an automorphism of H/K.

LEMMA 2. *Let (H, K) be an elementary pair in the lca group G. Then*

(i) $\operatorname{Aut}(G: H, K)$ is an open subgroup of $\operatorname{Aut}(G)$.

(ii) $\pi_{H,K}$ is a homomorphism of topological groups. In particular, if $\mathcal{V}$ is an open subset of $\operatorname{Aut}(H/K)$, then its inverse image under $\pi_{H,K}$ is open in $\operatorname{Aut}(G)$.

(iii) Moreover, one obtains in this way a neighborhood system of 1 in $\operatorname{Aut}(G)$. Specifically, given any neighborhood, $\mathcal{W}$, of the identity in $\operatorname{Aut}(G)$, there exists an elementary pair, (H, K), in G and neighborhood, $\mathcal{V}$, of the identity in $\operatorname{Aut}(H/K)$ such that the inverse image of $\mathcal{V}$ under $\pi_{H,K}$ is contained in $\mathcal{W}$.

Proof. (i) The set, $\mathcal{S}$, of all σ in $\operatorname{Aut}(G)$ for which $\sigma(H) = H$ is open in $\operatorname{Aut}(G)$. Indeed, from the fact that (H, K) is an elementary pair it is readily seen that H is generated by some compact set, say C. Then, $\mathcal{S}$ is just the set of all σ in $\operatorname{Aut}(G)$ for which $\sigma(C)$ and $\sigma^{-1}(C)$ are both contained in L, which is open in $\operatorname{Aut}(G)$ by the definition of the g-topology.

From the fact that (H, K) is an elementary pair in G it can be shown that $(K_\perp, H_\perp)$

is an elementary pair in G^*. Moreover, $\sigma(K)=K$ if and only if $\sigma^*(K_\perp)=K_\perp$. Hence, the same argument applied to $(K_\perp, H_\perp)$ along with the bicontinuity of * shows that the set, $\mathcal{S}'$, of all σ in Aut (G) for which $\sigma(K)=K$ is open in Aut (G). Therefore, Aut $(G: H, K)=\mathcal{S}\cap\mathcal{S}'$ is open in Aut(G). This proves (i).

For each compact subset, C, of G and each neighborhood, U, of 0 in G, let $(C; U)$ be the set of all σ in Aut (G) for which $(\sigma(x)-x)\in U$ and $(\sigma^{-1}(x)-x)\in U$ for all x in C; and call $(C; U)$ a modified compact open pair. Elementary considerations show that the modified compact open pairs form a neighborhood system of 1 in Aut(G).

(ii) is proved easily using modified compact open pairs. We omit the details.

To prove (iii), let $\mathcal{W}=(C; U)$ be an arbitrary modified compact open pair in Aut (G). By making $(C; U)$ smaller, if necessary, we may assume that U has compact closure in G. If H is any compactly generated lca group and V is any neighborhood of 0 in H, then there is a compact subgroup, K, of H such that $K\subset V$ and H/K is elementary; this follows for any compact H by a duality argument and then for any compactly generated H by theorem 1 (i). So, let H be the subgroup of G generated by C and U, let V be a neighborhood of 0 in G such that $V+V\subset U$, and let K be a compact subgroup of H such that $K\subset V$ and H/K is elementary. Then (H, K) is an elementary pair in G. Let C' and V' be the projections of C and V respectively to H/K. Then $\mathcal{V}=(C'; V')$ is a modified compact open pair in Aut (H/K). Moreover, $\mathcal{W}=(C; U)$ contains the inverse image of $\mathcal{V}=(C'; V')$ by $\pi_{H,K}$; in fact, suppose that $\pi_{H,K}(\sigma)$ is in $(C'; V')$; then $\sigma(x)-x\in V+K$ and $\sigma^{-1}(x)-x\in V+K$ for all x in C; but $V+K\subset V+V+U$; hence, σ is in $(C; U)$. This proves (iii).

4. Statement of the main theorem

Definition. Let G be an lca group. A subgroup, L, of G is called a *lattice in G* if L is discrete and G/L is compact. If there exists such a subgroup in G, then G is said to contain a lattice.

Not every lca group contains a lattice.

A group which is the direct sum of a discrete group and a compact group is said to *contain a lattice trivially*. A group which contains a lattice but which cannot be so decomposed is said to *contain a lattice non-trivially.*

Important examples of groups which contain a lattice non-trivially are the real vector group, R^n, which contains Z^n as a lattice, and the ring of adeles, A_k, of the number field k which contains k as a lattice.

If G contains a lattice, say L, then G^* also contains a lattice, namely $L_\perp$; this follows easily from the Pontryagin duality theorem.

For groups which contain a lattice non-trivially the main theorem answers the question of when Aut (G) is locally compact.

THEOREM 5. (Main theorem.) *Let G be a group which contains a lattice non-trivially. Then Aut (G) is locally compact if and only if the following three conditions are satisfied.*

(i) *G^p has finite p-rank for every prime number p.*

(ii) *G_2/G_1 has finite rank.*

(iii) *G_0 has finite rank.*

We remark that not every group satisfying (i), (ii), and (iii) contains a lattice. A characterization of the groups with lattice may be found in [11].

5. Proof of sufficiency of the main theorem

Notation. If G is a group and $\mathscr{W}$ is a subset of Aut (G), let $\mathscr{W}^*$ denote the corresponding subset $\{\sigma^*: \sigma \in \mathscr{W}\}$ of Aut (G^*).

LEMMA 3. *Let G be a group and let $\mathscr{W}$ be a subset of Aut (G) which contains 1. Then there is a neighborhood of 1 in $\mathscr{W}$ which operates with bounded orbits on G if and only if there is a neighborhood of 1 in $\mathscr{W}^*$ which is equicontinuous on G^*.*

Proof. Suppose that $\mathscr{W}'$ is a neighborhood of 1 in $\mathscr{W}$ which operates with bounded orbits on G. Let U be an open neighborhood of 0 in G whose closure, U^c, is compact. Let U_1 be an open set which contains U^c and whose closure, U_1^c, is compact. By making $\mathscr{W}'$ smaller, if necessary, we may and do assume that $\sigma(U^c) \subset U_1$ for all σ in $\mathscr{W}'$.

For any compact subset, C, of G, the set $\mathscr{W}C = \{\sigma(x): \sigma \in \mathscr{W}' \; x \in C\}$ is contained in some compact subset, C', of G. Indeed, by the compactness of C, there is a finite set, $\{x_1, \ldots, x_n\}$, of elements of G such that $\bigcup_i (x_i + U)$ covers C. Then $\mathscr{W}'C$ is contained in $U_i(\mathscr{W}'x_i + U_1^c)$. So, take C' to be the closure of this latter set. C' is compact since $\mathscr{W}'$ operates with bounded orbits.

Now, we show that $\mathscr{W}'^*$ is equicontinuous at 0 in G^*. Let V be an arbitrary neighborhood of 0 in G^*. Recalling that G^* is the set of characters of G with the compact open topology, we see that we may take $V = (C, N)$ in the notation of § 3 where C is a compact subset of G and N is a neighborhood of 1 in T. For this C, let C' be as in the above paragraph. Then $V' = (C', N)$ is a neighborhood of 0 in G^*, and it is easily checked that $\sigma^*(V') \subset V$ for all $\sigma \in \mathscr{W}'$. This shows that $\mathscr{W}'^*$ is equicontinuous at 0 in G^* as asserted. From this and the fact that the elements of $\mathscr{W}'^*$ are homomorphisms it follows that $\mathscr{W}'^*$ is equicontinuous at every point of G^*. But $\mathscr{W}'^*$ is a neighborhood of 1 in $\mathscr{W}^{*\prime}$. The first half of the lemma is proved.

For the second half of the proof it is convenient to interchange G and G^*. Thus, we assume that $\mathcal{W}'$ is a neighborhood of 1 in $\mathcal{W}$ which is equicontinuous on G and we show that there is a neighborhood, namely $\mathcal{W}'^*$, of 1 in $\mathcal{W}^*$ which operates with bounded orbits on G^*. Indeed, if χ is any character of G, then its orbit under $\mathcal{W}'^*$ is just $\mathcal{W}'^*\chi = \{\chi \circ \sigma: \sigma \in \mathcal{W}'\}$. But this is a set of equicontinuous characters of G by the equicontinuity of $\mathcal{W}'$ and the continuity of χ. Hence, by the Ascoli theorem, theorem 3, $\mathcal{W}'^*\chi$ has compact closure in G^* (recall that T is compact). This shows that $\mathcal{W}'^*$ operates with bounded orbits on G^*. The proof of the lemma is complete.

PROPOSITION 1. *Let G be group such that*

(i) *G^p has finite p-rank for each prime p.*

(ii) *G_2/G_1 has finite rank.*

(iii) *G_0 has finite rank.*

Then Aut (G) is locally compact.

We remark that Proposition 1 establishes the sufficiency of the conditions given by the main theorem. Notice that no assumptions about lattices are needed to prove Proposition 1. They are needed only for the proof of necessity.

Proof of Proposition 1. If G satisifies the above hypotheses, then so does G^*, because of Theorem 2. Hence, by the Ascoli corollary and Lemma 3 we need only show that there is a neighborhood of 1 in Aut (G) which operates with bounded orbits on G.

Put $G = G_2 \oplus \mathbf{R}^n$ as in Theorem 1. Let $S = \{s_1, ..., s_k\}$ be a finite subset of G_2 which is maximal linearly independent modulo G_1; the existence of S is guaranteed by hypothesis (ii). Let K be a compact open subgroup of G_2. Let U be the open unit ball in R^n, $U = \{x \in \mathbf{R}^n: |x| < 1\}$. For each positive real, r, let rU be the open ball of radius r in R^n. Let V be the closure of $(\frac{1}{2})U$. Let $C = S \cup K \cup V$. Then, with notation as in the proof of Lemma 2, $\mathcal{W} = (C; K+U)$ is a neighborhood of 1 in Aut (G). We will show that $\mathcal{W}$ operates with bounded orbits on G.

$\mathcal{W}$ operates with bounded orbits on $\mathbf{R}^n$, since any element of $\mathbf{R}^n$ can be written as kv with $k \in \mathbf{Z}$ and $v \in V$, and $\mathcal{W}(kv)$ is contained in $kv + (K + kU)$ which is relatively compact.

So it is sufficient to show that $\mathcal{W}$ operates with bounded orbits on G_2. Let x be an arbitrary element of G_2. Then for some integer m, $mx = x' + \sum_{i=1}^{k} a_i s_i$ where x' is in G_1, and the a_i are integers by our choice of S. Moreover, by multiplying the above equation through by the order x' modulo K, if necessary, we may assume that x' is in K. (G_1/K is a discrete torsion group.) Then, by our choice of $\mathcal{W}$, $m(\sigma(x) - x) \in (K + rU)$ for all σ in $\mathcal{W}$, where $r = \Sigma_{i=1}^{n} |a_i|$. But the set $\{y \in G_2\colon my \in K\} = C_1$ is compact because each p-primary part of G_2 has finite p-rank, and the set $\{z \in R^n\colon mz \in rU\} = C_2$ is clearly compact. Thus, $\mathcal{W}x$ is contained in the compact set, $x + C_1 + C_2$. We have shown that $\mathcal{W}$ operates with bounded orbits on G. The proposition is proved.

6. Proof of necessity

Proving necessity is somewhat more difficult than proving sufficiency. One must show that G satisfies conditions (i), (ii), and (iii) of the main theorem by arguing that otherwise there would be "too many" automorphisms of G for $\mathrm{Aut}(G)$ to be locally compact. Thus, one must construct automorphisms of G, and this is where the difficulty lies.

LEMMA 4. *Let G be a group which contains a lattice. Then G contains a lattice non-trivially if and only if G contains a closed subgroup isomorphic to* $\mathbf{R}$.

Proof. G could not split into the direct sum of a discrete group and compact group if it contained $\mathbf{R}$, since $\mathbf{R}$ has no compact open subgroup. This proves the "if" part.

Conversely, suppose that G does not contain R (or equivalently by Theorem 1 that G contains a compact open subgroup). Suppose also that G contains a lattice. We wish to show that G contains a lattice trivially.

If G/G_0 contains a lattice trivially, then so does G. Indeed, suppose $G/G_0 = K \oplus L$ with K compact and L discrete. Let K' and H be the inverse images of K and L respectively by the canonical map, $G \to G/G_0$. Then, $H = G_0 \oplus L'$ for some subgroup L' by the divisibility of G_0 and the decomposition is topological since G_0 is open in H. Then $G = K' \oplus L'$ with K' compact and L' discrete. We have shown that if G/G_0 contains a lattice trivially then so does G.

Moreover, if G contains a lattice, say L, then the image of L by the map, $G \to G/G_0$, is a lattice in G/G_0. Therefore, we may assume that $G_0 = 0$. By a duality argument, one sees that we may also assume that the torsion free discrete part of G is 0. Since G has no subgroup isomorphic to $\mathbf{R}$, this amounts to saying that G is a topological torsion group.

Thus we assume that G is a topological torsion group with lattice, say L, and compact open subgroup, say K. We wish to show that G is the direct sum of a compact group and a discrete group. Indeed, $K + L$ has finite index in G since G/K is discrete and G/L is compact. But G is the union of its compact subgroups. So, we may, by making K larger if necessary, assume that $K + L = G$. Moreover, $K \cap L$, being both compact and discrete, is finite. An exercise in duality and in the theory of discrete groups shows that a finite subgroup of a compact topological torsion group is contained in a finite direct summand thereof. Thus $K = F \oplus K'$ where F is finite and contains $K \cap L$. Then, clearly, $G = K' \oplus (F + L)$ where K' is compact and $F + L$ is discrete. Lemma 4 is proved.

LEMMA 5. *Suppose that Aut* (G) *is locally compact. Suppose further that* G *contains a divisible subgroup,* D, *whose closure in* G *is not compact. Then, the torsion free discrete part of* G *has finite rank.*

Proof. By the Ascoli corollary, there exists a neighborhood, say $\mathcal{W}$, of 1 in Aut (G) which operates with bounded orbits on G. Suppose that G_2/G_1 has infinite rank. We will derive a contradiction by constructing automorphisms of G which are in $\mathcal{W}$ and which fail to operate with bounded orbits.

There exists a compact open pair, (H, K), in G such that $\mathcal{W} \supset \pi_{H,k}^{-1}\ (\mathcal{V})$ for some neighborhood, $\mathcal{V}$, of 1 in Aut (H/K); this follows from Lemma 2.

By choosing a slightly smaller unbounded divisible subgroup, if necessary, we may assume that the image of D under the canonical map, $G \to G/(G_1 \oplus \mathbf{R}^n)$ has finite rank. Then $G' = G/(G_1 + \mathbf{R}^n + D + H)$ has infinite torsion free rank, since $G/(G_1 \oplus \mathbf{R}^n) = G_2/G_1$ has infinite torsion free rank, since H is compactly generated, and by our choice of D. Therefore, there is an element, z, of G' which has infinite order. Let Z be the infinite cyclic subgroup of G' generated by z. Then for each t in D, there is a homomorphism, $\psi_t: Z \to D$, such that $\psi_t(z) = t$. Now we use the crucial fact that D is divisible. It is well known [6] that if A is a discrete abelian group with subgroup B, then a homomorphism of B into a divisible group can always be extended to a homomorphism of A into that group. In particular, ψ_t, can be extended to a homomorphism, $\psi_t': G' \to D$, such that $\psi_t'(z) = t$. Moreover ψ_t' is continuous, since its domain is discrete. Let μ be the endomorphism of G defined by $\mu_t = \phi_1' \circ \psi_t' \circ \phi_0$, where ϕ_0 is the canonical projection, $\phi_0: G \to G/(G_1 + \mathbf{R}^n + D + H) = G'$, and ϕ_1 is the canonical injection, $\phi_1: D \to G$. Then $1 + \mu_t$ is an automorphism of G; in fact, it has an inverse, namely $1 - \mu_t$, since $\mu_t \circ \mu_t = 0$. Moreover, $1 + \mu_t$ is in $\mathcal{W}$, since it restricts to the identity on H. Now, let x be an element of G which projects to z modulo $(G_1 + \mathbf{R}^n + D + H)$. Then the orbit of x under $\mathcal{W}$ contains $\{(1 + \mu_t)(x) = x + t \mid t \in D\}$, which does not have compact closure in G, since D does not have compact closure in G. This contradicts the fact that $\mathcal{W}$ operates with bounded orbits.

PROPOSITION 2. *If* G *contains a lattice nontrivially and Aut* (G) *is locally compact, then the torsion free discrete and compact divisible parts of* G *each have finite rank.*

Proof. The torsion free discrete part of G has finite rank by Lemmas 4 and 5 and by the fact that R is divisible and not compact. The same argument applied to G^* shows that the torsion free discrete part of G^* has finite rank. Then the compact divisible part of G has finite rank by Theorem 2 (iii). This proves Proposition 2.

We have established the necessity of conditions (ii) and (iii) of the main theorem. We proceed now to establish the necessity of (i).

Notation. If G is a group and p is a prime number, let $G_p = \{x \in G: px = 0\}$.

LEMMA 6. *Suppose that G is an lca group and that $Aut(G)$ is locally compact. Let p be a prime. Then either G_p is compact or else $pG+H=G$ for some compactly generated open subgroup, H, of G.*

Proof. By the Ascoli corollary, there is a neighborhood, $\mathcal{W}$, of 1 in Aut(G) which operates within bounded orbits on G. Then, there exists an elementary pair, (H, K), in G such that $\mathcal{W} \supset \pi_{H,K}^{-1}(\mathcal{V})$ for some neighborhood, $\mathcal{V}$, of 1 in Aut(H/K); this follows by Lemma 2. Then H is a compactly generated open subgroup of G. We will assume that $pG+H \neq G$ and prove from this that G_p is compact.

$G/(pG+H)$ is a discrete vector space over the field of p-elements. If $pG+H \neq G$, then there exists a homomorphism of $G/(pG+H)$ onto the cyclic group, $\mathbf{Z}/p\mathbf{Z}$, of p-elements (just factor out by a subspace of codimension 1). Composing this homomorphism with the canonical projection from G, one obtains a surjection, $\phi_0: G \to \mathbf{Z}/p\mathbf{Z}$, whose kernel, say N, contains H. Let z be a generator of $\mathbf{Z}/p\mathbf{Z}$. Then for each t in $N \cap G_p$, there exists a unique homomorphism, $\psi_t: (\mathbf{Z}/p\mathbf{Z}) \to G$, such that $\psi_t(z) = t$. Moreover, ψ_t is continuous since its domain is discrete. Put $\mu_t = \psi_t \circ \phi_0$. Then for each t in $N \cap G_p$, $(1+\mu_t)$ is an automorphism of G; in fact, $(1+\mu_t)$ has an inverse, namely $1-\mu_t$, since $\mu_t \circ \mu_t = 0$. Moreover, $(1+\mu_t)$ is in $\mathcal{W}$ since it restricts to the identity on H. Now, let x be an element of G which projects to z modulo N. Then $\{(1+\mu_t)(x) = x+t \mid t \in N \cap G_p\}$ has compact closure, since it is contained in the orbit of x under $\mathcal{W}$. It follows that $N \cap G_p$ is compact, since it is closed and its translate by x has compact closure. Then G_p is compact, since N has finite index in G. This proves Lemma 6.

LEMMA 7. *Suppose that $Aut(G)$ is locally compact. Suppose further that D is a discrete subgroup of G which is a divisible p-group. Then D has finite p-rank.*

Proof. It is well known that a discrete divisible p-group is isomorphic to $(\mathbf{Q}_p/\mathbf{Z}_p)^\alpha$ for some cardinal α [6]. We must show that α is finite.

By the Ascoli theorem, there is a neighborhood, $\mathscr{W}$, of 1 in Aut (G) which operates with bounded orbits on G. Then there is an elementary pair, (H, K), in G such that $\mathscr{W}$ contains $(\pi_{H,K})^{-1}(\mathscr{V})$ for some neighborhood, $\mathscr{V}$, of 1 in Aut (H/K).

From the fact that H is compactly generated and D is a discrete divisible p-group, it can be shown that $D = D' \oplus D''$ where D' has trivial intersection with H and where D'' has finite p-rank. Let M be a subgroup of G, maximal with respect to the properties of containing H and having trivial intersection with D'. Then $G = D' \oplus M$; this can be shown from the divisibility of D' and the maximality of M. Moreover, the direct sum decomposition is topological, since M, by virtue of containing H, is an open subgroup of G. $D' = (\mathbf{Q}_p/\mathbf{Z}_p)^{\beta}$ for some cardinal β. The automorphisms of G which restrict to the identity on M and which permute the various direct summands of D' are clearly in $\mathscr{W}$. If β were infinite, then these automorphisms would fail to operate with bounded orbits. Hence, β is finite. This shows that D' has finite p-rank. Hence, $D = D' \oplus D''$ has finite p-rank (or equivalently, α is finite). The lemma is proved.

PROPOSITION 3. *Suppose that G is a group which contains a lattice and that Aut (G) is locally compact. Then G^p has finite p-rank for each prime number, p.*

Proof. Let p be an arbitrary prime. Let L be a lattice in G. By Lemma 1 it is sufficient to show that the p-primary parts of L and G/L each have finite p-rank. However, i we can show that the p-primary part of L has finite p-rank, then the same argument applied to G^* will show that the p-primary part of $L_{\perp} = (G/L)^*$ has finite p-rank, and then by Theorem 2 (v), $(G/L)^p$ would have finite p-rank and we would be done. Thus it is sufficient to show that the p-primary part of L has finite p-rank.

If G_p is compact, then L_p, being both compact and discrete, is finite. From this, the discreteness of L, and elementary group theory it can be shown that L^p has finite p-rank.

Therefore we may assume that G_p is not compact. Then by Lemma 6, $pG + H = G$ for some compactly generated open subgroup, H, of G. Then $L^p/p(L^p)$ is finite. Indeed, $L + H$ has finite index in G, since G/L is compact and G/H is discrete. Hence $pL + H$ has finite index in $pG + H = G$. Hence, G/pL is compactly generated. But a discrete torsion subgroup of a compactly generated lca group must be finite; this follows from Pontryagin's decomposition of compactly generated groups, mentioned in the proof of Theorem 1. Hence, L/pL is finite. Since the p-primary part, L^p, of L is a pure subgroup thereof, it follows that $L^p/p(L^p)$ is also finite which is what we wanted to show.

Now, any discrete p-group, and L^p in particular, decomposes into the direct sum of a divisible p-group and a reduced p-group (a group is called "reduced" if it contains no non-trivial divisible subgroups) cf. [6]. An easy exercise in group theory shows that if L' is a reduced discrete p-group for which L'/pL' is finite, then L' is finite. Therefore, in particular the reduced component of L^p is finite. But the divisible component of L^p has finite p-rank by Lemma 7. Hence, L^p has finite p-rank which is what we wanted to prove. Proposition 3 is proved. The necessity of the main theorem is now established.

7. The case of trivial lattice

Suppose that G contains a lattice trivially, say $G = K \oplus L$ with K compact and L discrete. Then Aut(G) is locally compact if and only if Aut(K) and Aut(L) are both locally compact. Indeed, the automorphisms, σ, of G for which $\sigma(K) = K$ form an open subgroup of Aut(G), call it $\mathscr{A}$. Any σ in $\mathscr{A}$ may be written as a two by two matrix with entries, α, β, 0, and δ where $\alpha \in$ Aut (K), $\delta \in$ Aut (L), $\beta: L \to K$, and 0 is the trivial map from K to L. The β's operate equicontinuously and with bounded orbits since L is discrete and K is compact. Therefore, $\mathscr{A}$, and hence also Aut (G) is locally compact if and only if Aut (L) and Aut (K) are locally compact which is what we wanted to prove.

Since Aut (K) is anti-isomorphic to Aut (K^*) where K^* is discrete, this reduces the problem of the local compactness of Aut(G) in the case that G contains a lattice trivially to the case that G is discrete. For this case we have the following partial results.

THEOREM 6. *Let G be a discrete group. If Aut(G) is locally compact, then either*

(a) *G/G_1 has finite rank and for each prime, p, G^p has finite rank, or else*

(b) *G/G_1 has infinite rank and for each prime, p, G^p is finite.*

Moreover, case (a) *is conclusive; if G satisfies* (a) *then Aut(G) is locally compact. However, case* (b) *is inconclusive; for some groups satisfying* (b) *Aut(G) is locally compact and for some it is not.*

Proof. If Aut(G) is locally compact and G is discrete but does not satisfy (a), then G must satisfy (b); this follows from Proposition 3, Lemma 5, and the fact that a discrete p-group of finite p-rank is the direct sum of a finite group and finitely many copies of the infinite divisible group, $\mathbf{Q}_p/\mathbf{Z}_p$. Conversely, if G satisfies (a), then Aut(G) is locally compact by Proposition 1. It remains to show that case (b) is inconclusive. It is easy to find examples for which G satisfies (b) and Aut(G) is not locally compact; for instance let G be the direct sum of infinitely many copies of $\mathbf{Z}$. Finding examples for which Aut(G) is locally compact is more difficult. However, Fuchs has obtained the following startling result. Define a *rigid system* to be a collection, $\{G_i\}$, of torsion free discrete abelian groups such that any homomorphism of G_i into G_j is trivial in the case that $i \neq j$ and is multiplication by a rational in the case that $i=j$. Then, Fuchs [7] has shown that to every infinite cardinal, α, there exists a rigid system consisting of 2^α torsion free groups each of power α. Clearly, if α is uncountable, then for any of the G_i, G_i satisfies (b) and $\mathrm{Aut}(G_i)$ is locally compact, in fact discrete. This proves Theorem 6.

We remark that attempting to determine which groups, G, satisfying (b), have $\mathrm{Aut}(G)$ locally compact, seems hopelessly difficult. For examples of how pathological can be the structure of torsion free discrete groups even in the case of finite rank, we refer the reader to [3].

8. Application to B(G)

Let G be any locally compact abelian group. Let U be the group of unitary operators on $L^2(G)$. Let $\mathbf{A}(G)$ be the subgroup of U generated by the regular representation of G and the Fourier transform of the regular representation of G^*. It turns out that $\mathbf{A}(G)$ is a two step nilpotent locally compact group. Let $\mathbf{B}(G)$ be the normalizer of $\mathbf{A}(G)$ in U. It is of some interest to know when $\mathbf{B}(G)$ is locally compact. For instance, Igusa has shown [9] that if C is a compact subset of $\mathbf{B}(G)$ and $\mathcal{S}(G)$ is the space of Schwartz-Bruhat functions on G, then the map, $C\times\mathcal{S}(G)\to\mathcal{S}(G)$ is continuous.

By theorems of Segal and Igusa (cf. [9], [16]) it turns out that $\mathbf{B}(G)/\mathbf{A}(G)$ is isomorphic as a topological group to $\mathrm{Sp}(G)$ which is a certain subgroup of $\mathrm{Aut}(G\times G^*)$. This makes it possible for us to apply our earlier results to the problem of the local compactness of $\mathbf{B}(G)$.

We still have to tell the reader what $\mathrm{Sp}(G)$ is. We follow the notation of [16]. Write the elements, w, of $G\times G^*$ as pairs, $w=(x, x^*)$, and write the automorphisms, σ, of $G\times G^*$ as two by two matrices

$$\sigma=\begin{vmatrix}\alpha & \beta\\ \gamma & \delta\end{vmatrix} \tag{\#}$$

which operate by matrix multiplication on the right, so that $\sigma(w)=w\cdot\sigma=(x\alpha+x^*\gamma,\ x\beta+x^*\delta)$. If σ is an automorphism of $G\times G^*$, then σ^* is an automorphism of $G^*\times G$. Let η be the isomorphism of $G\times G^*$ onto $G^*\times G$, defined by $(x, x^*)\to(-x^*, x)$. For each automorphism, σ, of $G^*\times G$, define $\sigma^I=\eta^*\eta^{-1}$. Then σ^I is also an automorphism of $G\times G^*$, and it is easily checked that if σ is given by #, then

$$\sigma^I=\begin{vmatrix}\delta^* & -\beta^*\\ -\gamma^* & \alpha^*\end{vmatrix}$$

Define $\mathrm{Sp}(G)$ to be the subgroup of $\mathrm{Aut}(G\times G^*)$ consisting all σ in $\mathrm{Aut}(G\times G^*)$ for which $\sigma\sigma^I=1$. Then, $\mathrm{Sp}(G)$ is clearly a closed subgroup of $\mathrm{Aut}(G\times G^*)$.

THEOREM 7. *Let G be a group which contains a lattice non-trivially. Then $\mathbf{B}(G)$ is locally compact if and only if the following three conditions are satisfied.*

(i) *G^p has finite p-rank for each rational prime, p.*

(ii) *G_2/G_1 has finite rank.*

(iii) *G_0 has finite rank.*

Proof. $\mathbf{B}(G)$ is locally compact if and only if $\mathrm{Sp}(G)$ is locally compact, because of the above mentioned isomorphism. (For a proof that a locally compact extension of a locally compact group is locally compact, we refer to [8, p. 39].)

If $\mathrm{Sp}(G)$ is locally compact, then so is $\mathrm{Aut}(G)$, since the map

$$\alpha \rightarrow \begin{vmatrix} \alpha & 0 \\ 0 & 1 \end{vmatrix}$$

is an isomorphism of $\mathrm{Aut}(G)$ to a closed subgroup of $\mathrm{Sp}(G)$. Therefore, by Theorem 5, G must satisfy the above conditions whenever $\mathrm{Sp}(G)$ is locally compact.

Conversely, suppose that G satisfies the above conditions. Then, clearly, so does $G \times G^*$. Hence, $\mathrm{Aut}(G \times G^*)$ is locally compact by Theorem 5. It follows that $\mathrm{Sp}(G)$, being a closed subgroup of $\mathrm{Aut}(G \times G^*)$, is also locally compact. This proves Theorem 7.

The following proposition reduces the problem of the local compactness of $\mathbf{B}(G)$ in the case that G contains a lattice trivially to the problem of the local compactness of $\mathrm{Aut}(G)$ in the case that G is discrete. For this latter problem, partial results were given by Theorem 6.

PROPOSITION 4. *Suppose that G contains a lattice trivially, say $G = K \oplus L$ with K compact and L discrete. Then, $\mathbf{B}(G)$ is locally compact if and only if $Aut(L \oplus K^*)$ is locally compact.*

Proof. Because of the above mentioned isomorphism of $\mathbf{B}(G)/\mathbf{A}(G)$ to $\mathrm{Sp}(G)$ we may replace $\mathbf{B}(G)$ by $\mathrm{Sp}(G)$ in the statement of the proposition.

Suppose that $\mathrm{Aut}(L \oplus K^*)$ is locally compact. Then $\mathrm{Aut}(L^* \oplus K)$ is also locally compact. Then, by the remarks at the beginning of § 7, $\mathrm{Aut}(L^* \oplus K \oplus L \oplus K^*)$ is also locally compact. But this last group is just $\mathrm{Aut}(G \oplus G^*)$ of which $\mathrm{Sp}(G)$ is a closed subgroup. Hence $\mathrm{Sp}(G)$ is locally compact.

Conversely, if $\mathrm{Sp}(G)$ is locally compact then so is $\mathrm{Aut}(L \oplus K^*)$. Indeed, write the automorphisms of $L \oplus K^*$ as two by two matrices which operate on $L \oplus K^*$, written as ordered pairs, by matrix multiplication on the right. Also write the elements of $G \times G^*$ as four tuples in $L \oplus K \oplus L^* \oplus K^*$, and write the automorphisms of $G \times G^*$ as four by four matrices which operate on $(G \times G^*) = (L \oplus K \oplus L^* \oplus K^*)$ by matrix multiplication on the right. If σ is an automorphism of $L \oplus K^*$ with

$$\sigma = \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} \quad \text{and} \quad \sigma^{-1} = \begin{vmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{vmatrix}$$

then it is easily checked that

$$\sigma'' = \begin{vmatrix} \alpha & 0 & 0 & \beta \\ 0 & \delta'^* & \beta'^* & 0 \\ 0 & \gamma'^* & \alpha'^* & 0 \\ \gamma & 0 & 0 & \delta \end{vmatrix}$$

is an element of Sp (G). Then, the correspondence $\sigma \to \sigma''$, maps Aut $(L \oplus K^*)$ isomorphically onto a closed subgroup of Sp(G). Therefore, if Sp(G) is locally compact, so is Aut$(L \oplus K^*)$. This proves Proposition 4.

9. A counterexample

We show that the hypothesis that G contain a lattice cannot be dropped from Proposition 3. Let p be a fixed rational prime. For each positive integer, h, let M_h be the cyclic group of order p^{2h} and let N_h be its unique cyclic subgroup of order p^h. Let $G = \prod_{h=1}^{\infty}(M_h, N_h)$.

Then G is a p-primary group which does not have finite p-rank. We will show that Aut(G) is nevertheless locally compact. Indeed, let $K = \prod_h N_h$. Then K is a compact open subgroup of G. Then by Lemma 2, $\mathscr{A} = \{\sigma \in \text{Aut}(G) \,|\, \sigma(K) = K\}$ is an open subgroup of Aut(G). $\mathscr{A}$ operates with bounded orbits on G; in fact, $\mathscr{A}$ leaves invariant each of the compact sets, $G_h + K$, where $G_h = \{x \in G \,|\, p^h x = 0\}$; but $\bigcup_{h=1}^{\infty}(G_h + K)$ is all of G, since $G_{2h} + K = \sum_{m=1}^{h} M_n + K$; hence $\mathscr{A}$ operates with bounded orbits on G as asserted. $\mathscr{A}$ also operates equicontinuously on G; in fact, $\mathscr{A}$ leaves invariant each of the sets, $(p^h G) \cap K$, $h = 1, 2, 3, \ldots$, and these sets form a neighborhood system of 0 in G, since $(p^{2h} G \cap K) \subset \prod_{n=h}^{\infty} N_n$; it follows that $\mathscr{A}$ operates equicontinuously on G as asserted. Therefore, by the corollary of Theorem 4, Aut(G) is locally compact which is what we wanted to show. This shows that the hypothesis that G contain a lattice cannot be dropped from Proposition 3.

Remark. Let us carry the above example a bit further. Suppose that $G = \prod_h(M_h, N_h)$ where each M_h is a p-primary group of p-rank 1 with compact open subgroup, N_h, and that h ranges over some index set. Then, it can be shown without much difficulty that Aut(G) is locally compact if and only if there are only finitely many N_h of any given order (be it finite or infinite), and there only finitely many M_h/N_h of any given order. Vilenkin [14] has given conditions under which a p-primary group will decompose into the restricted product of groups of p-rank 1.

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