THE LIMIT SET OF A FUCHSIAN GROUP

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§ 1. Introduction

In this paper we shall consider the limit set of a finitely generated Fuchsian group of the second kind. In particular, we shall attempt to calculate the Hausdorff dimension of the limit set. If the group, G, has no parabolic elements this is actually achieved in section 4, whereas, if G has parabolic elements we can obtain partial results which are discussed in section 5. The proof of these results involves the construction of a measure supported on the limit set, from which, at least in principle, we can obtain a lower bound for the Hausdorff dimension. This bound is the same as the upper bound found by Beardon [4].

The actual measure which we construct proves to be intimately related to the theory of the Laplace operator on $G\backslash\mathbf{H}$, and consequently we obtain new insights into both of these. This allows us, for example, to give a new proof of a theorem of Beardon [4] (see the Corollary to Theorem 7.1).

Section 2 recalls some well-known, but difficult to locate, results on the description of the geometry of Fuchsian groups of the second kind. These are used in making various estimates.

By way of notation, we shall take all Fuchsian groups to be finitely generated, although several of the results will be valid without this restriction (in particular those of section 3). These groups will be assumed to act on the unit disc Δ unless stated otherwise. For a domain D we shall write Con (D) for the group of conformal homeomorphisms of D onto D. We can, as usual, represent an element $g \in$ Con (Δ) as a matrix

$$\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \quad (\alpha, \beta \in \mathbf{C}, |\alpha|^2 - |\beta|^2 = 1),$$

and if this is so we shall write

$$\mu(g) = 2(|\alpha|^2 + |\beta|^2).$$

Likewise, if $g \in \mathrm{Con}(\mathbf{H})$, where $\mathbf{H}$ is the upper half-plane, g can be represented as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (a, b, c, d \in \mathbf{R}, ad - bc = 1).$$

We shall write here

$$\mu(g) = a^2 + b^2 + c^2 + d^2.$$

The unit circle, $\partial \Delta$, will be denoted by S.

Some of the results of this paper were contained in a Ph.D. thesis submitted to the University of Cambridge. I would like to express my thanks to my research supervisor, Dr A. F. Beardon; some extent of my debt to him will be clear from this paper.

§ 2. Geometry of the fundamental domain and the limit set

Let G be a Fuchsian group acting on Δ. Then there is an (open) fundamental domain D in Δ so that

(a) ∂D is a finite collection $\{u_j\}$ of geodesic arcs,

(b) $\{u_j\}$ splits uniquely into pairs u_k, u'_k in such a way that there is $\gamma_k \in G$ so that $u_k = \gamma_k(u'_k)$, the γ_k are distinct and generate G,

(c) $0 \in D$,

(d) if u_j, u_k meet at $p \in S$ then p is a parabolic vertex; $u_j = u'_k$ and γ_k generates G_p. No other u_l meets $G\{p\}$.

A justification of this can be found in [8]. Let L_G be the limit set of G.

L_G is a closed subset of S. Thus $\Omega = L_G^c (= S \backslash L_G)$ is open and so is a countable union of disjoint intervals, say $\Omega = \bigcup_j \Omega_j$. We shall say that Ω_j and Ω_k are *equivalent* if there is $g \in G$ so that $g\Omega_j = \Omega_k$. It is clear, as L_G is invariant under G, that either $g\Omega_j = \Omega_k$ or $g\Omega_j \cap \Omega_k = \emptyset$. Now we have

THEOREM 2.1. *There are only a finite number of equivalence classes of Ω_j. There is a hyperbolic subgroup (but no larger subgroup), G_j say, preserving Ω_j.*

This is proved in [8] but it is not stated formally.

Let η_j, η'_j be the end-points of Ω_j and let λ_j be that arc of a circle lying in Δ, joining η_j, η'_j, making an internal angle $\alpha > 0$ with Ω_j. The collection of all λ_j is a figure invariant under G as G preserves $\{\Omega_j\}$, angles and orientation. Let $\Lambda_j = \Lambda_j(\alpha)$ be the *open* region between λ_j and Ω_j; it is lens-shaped and we call it an *α-lens*. As G is non-elementary the Ω_j have distinct end-points (consider the action of G_j on $S \backslash \Omega_j$) and the λ_j cannot inter-

sect if $\alpha \leqslant \pi/2$. In particular, if $\alpha \leqslant \pi/2$ the $\Lambda_j(\alpha)$ are disjoint and are permuted by G. Let $K_G(\alpha) = \Delta \diagdown \bigcup_j \Lambda_j(\alpha)$. If $\alpha \leqslant \pi/2$, $K_G(\alpha)$ is hyperbolically convex.

The next point to note is that $D \cap K_G(\alpha)$ has finite (hyperbolic) area and if G has no parabolic elements $D \cap K_G(\alpha)$ is relatively compact. This follows as the only infinite parts of D are those adjacent to free sides (i.e. $\{\Omega_j \cap \bar{D}\}$) and cusps. Any free side is in some $\bar{\Lambda}_j$. All this is a direct consequence of the description of D given above.

Suppose now that G has parabolic elements and let $p_1, ..., p_r$ be the parabolic vertices lying on ∂D. We call an open disc contained in Δ and tangent to S at p a *horocycle* at p. Construct horocycles C_j' at p_j. Now refer this to $\mathbf{H}$ with $p_1 = \infty$. We know that the diameter of a horocycle $g(C_j')$ $(g \in G, g(p_j) = \infty)$ is bounded ([9]) and so we can find $C_1 \subseteq C_1'$, a horocycle at ∞ so that

(a) C_1 meets no image of $\{C_j'\}$ under G other than C_1' (and hence no image of $\{C_1, C_2', ..., C_r'\}$, other than C_1, under G),

(b) if $\alpha \leqslant (\pi/2)$ C_1 meets no $\Lambda_k(\alpha)$.

(b) follows as the length of Ω_k is clearly bounded by the translation length of G_∞; hence the height of Λ_k is bounded.

We can repeat this argument to each j. Thus

PROPOSITION 2.1. *There is a horocycle C_p at each parabolic vertex p and a $\pi/2$-lens Λ_j on each Ω_j so that*

(i) *$\{C_p, \Lambda_j\}$ are disjoint,*

(ii) *$C_{g(p)} = g(C_p)$,*

(iii) *$D \diagdown (\bigcup_p C_p \cup \bigcup_j \Lambda_j)$ is relatively compact in Δ,*

(iv) *D meets only a finite number of C_p and Λ_j.*

If p is any parabolic vertex then $p = g(p_j)$ for some j; then define $C_p = g(C_{p_j})$. Then the construction above is sufficient to imply the proposition.

Now let p be a parabolic vertex. If we conjugate G to act on $\mathbf{H}$ with p at ∞ and G_p generated by $z \mapsto z + 1$ then C_p becomes $\{x: \mathrm{Im}(z) > d\}$ $(d > 0)$. (G_p is the subgroup of G fixing p and we have used the same notation for G and its conjugate). A fundamental domain for the action of G on this is

$$\{z: \mathrm{Im}(z) > d, \ |\mathrm{Re}(z)| < 1/2\}.$$

Such a region, or its image in Δ, we call *standard*.

Furthermore, if we conjugate Ω_j to be $[0, \infty]$, (G acting on $\mathbf{H}$) $\Lambda_j(\alpha)$ becomes $\{z: 0 < \arg(z) < \alpha\}$ and G_j is generated by an element of the form

$$z \mapsto \exp(\varkappa_j)z. \quad (\varkappa_j > 0).$$

Then the set

$$\{z: 1 < |z| < \exp(\varkappa_j), 0 < \arg(z) < \alpha\}$$

is a fundamental domain for the action of G_j on $\Lambda_j(\alpha)$. This, or its image in Δ, will be called *standard*.

Now let A be a finite set so that the $\Omega_j(j \in A)$ are inequivalent under G and A is maximal subject to this restriction. Let P be a maximal set of inequivalent parabolic vertices; this is also finite. It now follows from Proposition 2.1 that the following assertion holds:

PROPOSITION 2.2. *Let the notations as be above. Then there is a (possibly disconnected) fundamental domain D for G of the form*

$$D = K_\alpha \cup \bigcup_{j \in A} D_j(\alpha) \cup \bigcup_{p \in P} D_p^*$$

where

(i) K_α *is relatively compact in* Δ,

(ii) $D_j(\alpha)$ *is a standard fundamental domain for G_j on $\Lambda_j(\alpha)$, and,*

(iii) D_p^* *is a standard fundamental domain for G_p on C_p.*

§ 3. The construction of of a measure

In this section G will denote an arbitrary Fuchsian group acting on the unit disc. If $z, w \in \Delta$ we set,

$$h(z, w) = \frac{|1 - \bar{z}w|^2}{(1 - |z|^2)(1 - |w|^2)}.$$

From this we form the Dirichlet series for $z, w \in \Delta$,

$$f(z, w; s) = \sum_{g \in G} h(z, g(w))^{-s}. \tag{3.1}$$

Let us recall some known facts concerning this series. It converges if $\mathrm{Re}\,(s) > 1$ and diverges for $\mathrm{Re}\,(s) \leq 0$. It has an 'exponent of convergence', $\delta(G)$; it diverges if $\mathrm{Re}\,(s) < \delta(G)$ and converges if $\mathrm{Re}\,(s) > \delta(G)$. Trivially $\delta(G) \in [0, 1]$. Beardon ([3], [4]) has shown that $\delta(G) \in]0, 1[$ if G is of the second kind and it is a classical fact that $\delta(G) = 1$ if G is of the

first kind. We shall in our work require Beardon's estimate $\delta(G) > 0$, which is the easiest part. Beardon has also shown in [3] that if G has parabolic elements $\delta(G) > \frac{1}{2}$.

The construction which we are about to perform relies on the following elementary lemma.

LEMMA 3.1. *Let $\Sigma_{n=1}^{\infty} a_n^{-s}$ be a Dirichlet series with exponent of convergence $e > 0$. There is a positive increasing function k on $[0, \infty]$ so that*

$$\sum_{n=1}^{\infty} \frac{k(a_n)}{a_n^s}$$

has exponent of convergence e and diverges at $s = e$. Also if $\varepsilon > 0$ is given there is y_0 so that for $y > y_0$, $x > 1$

$$k(xy) \leqslant x^{\varepsilon} k(y).$$

Proof. By reordering the series we may suppose that (a_n) is an increasing sequence and as $e > 0$, $a_n \to \infty$. Let (ε_n) be a decreasing sequence such that $\varepsilon_n > 0$, $\varepsilon_n \to 0$ as $n \to \infty$. We are going to define a sequence (X_n), with $X_n \to \infty$, and k on the interval $[X_n, X_{n+1}]$ inductively. Take $X_1 = 1$ and $k(x) = 1$ on $[0, 1]$.

Suppose k is defined on $[0, X_n]$. Then choose X_{n+1} so that

$$\frac{k(X_n)}{X_n^{\varepsilon_n}} \sum_{p:\, X_n < a_p \leqslant X_{n+1}} a_p^{-(e-\varepsilon_n)} \geqslant 1. \tag{3.2}$$

This can always be done as $\Sigma_p a_p^{-(e-\varepsilon_n)}$ diverges. Now define, for $x \in [X_n, X_{n+1}]$,

$$k(x) = k(X_n) \left(\frac{x}{X_n}\right)^{\varepsilon_n}. \tag{3.3}$$

With this choice $\Sigma k(a_p) \cdot a_p^{-e}$ diverges as

$$\Sigma k(a_p) a_p^{-e} = \sum_{n=1}^{\infty} \sum_{p:\, a_p \in [X_n, X_{n+1}]} k(X_n) \cdot (a_p/X_n)^{\varepsilon_n} \cdot a_p^{-e} \geqslant \sum_{n-1}^{\infty} 1$$

by (3.2). By (3.3) k is positive and increasing.

Suppose we are given $\varepsilon > 0$. Choose n so that $\varepsilon > \varepsilon_n$. Then if $x \geqslant X_n \log k(x)$ is, by (3.3), a piecewise linear continuous function of $\log x$ and the slope of each component is $\leqslant \varepsilon_n$. Hence if $y > X_n$, $x > 1$

$$\log k(xy) - \log k(y) \leqslant \varepsilon \log x$$

which is precisely what we want.

Finally, we have to show that if $s > e$ then $\Sigma k (a_p) \cdot a_p^{-s}$ converges. Choose $\varepsilon > 0$ so that $e + \varepsilon < s$. Then, as $p \to \infty$ $k(a_p) = O(a_p^{\varepsilon})$ by what we have just proved. The convergence of $\Sigma k (a_p) \cdot a_p^{-s}$ follows at once. This proves the lemma.

We apply the lemma to the series (3.1). Hence there is a function k with the properties described in the lemma so that, for fixed $\alpha, \beta \in \Delta$,

$$M(s) = \sum_{g \in G} k(h(\alpha, g(\beta))) \cdot h(\alpha, g(\beta))^{-s}$$

has exponent of convergence $\delta (= \delta(G))$ and which diverges at $s = \delta$. As the terms are positive, as $s \to \delta$,

$$M(s) \to \infty.$$

Let δ_x be the Dirac measure at x. For $s > \delta$ we define the following measure on $\bar{\Delta}$, for fixed $\alpha, \beta \in \Delta$,

$$\mu_s = M(s)^{-1} \sum_{g \in G} k(h(\alpha, g(\beta))) h(\alpha, g(\beta))^{-s} \delta_{g(\beta)}. \tag{3.4}$$

μ_s is a family of probability measures supported on $\bar{\Delta}$. We can apply Helly's theorem; that is, if (s_j') is a sequence $s_j' \to \delta$ we can extract a subsequence (s_j) so that (μ_{s_j}) converges weakly on $\bar{\Delta}$. Let μ be the limit on some such sequence.

As there are only a finite number of images of β in any closed set in Δ, and as $M(s_j) \to \infty$, μ is concentrated on the unit circle S. In fact the same reasoning shows that it is concentrated on the limit set L_G. As $\bar{\Delta}$ is compact μ is a probability measure.

μ is the measure we have been seeking. In order to investigate its properties we introduce an auxillary function.

Let us define, for $z \in \Delta$, $\zeta \in \bar{\Delta}$, the Poisson 'kernel',

$$P(z, \zeta) = \frac{1 - |z|^2}{|1 - \bar{z}\zeta|^2}.$$

Then we define

$$F(z) = \int P(z, \zeta)^\delta d\mu(\zeta). \tag{3.5}$$

As $\mu_{s_j} \to \mu$ weakly we have

$$F(z) = \lim_{j \to \infty} \int P(z, \zeta)^\delta d\mu_{s_j}(\zeta).$$

But, for z fixed there is a constant c so that, if $\zeta \in \bar{\Delta}$, $|\delta - s_j| < 10$,

$$|P(z, \zeta)^\delta - P(z, \zeta)^{s_j}| \leqslant c(z) \cdot |s_j - \delta|.$$

As the μ_{s_j} are probability measures we have

$$F(z) = \lim_{j \to \infty} \int P(z, \zeta)^{s_j} d\mu_{s_j}(\zeta).$$

If we take $\alpha = 0$ and use (3.4) this becomes

$$F(z) = \lim_{j \to \infty} \frac{1}{M(s_j)} \sum_{g \in G} \frac{k(h(0, g(\beta)))}{h(z, g(\beta))^{s_j}}. \tag{3.6}$$

Let $\varepsilon > 0$ and $\gamma \in G$. Then as for any $a, b, c \in \Delta$,

$$h(a, c) \leqslant 2h(a, b)h(b, c)$$

(see [7]) and as k is increasing

$$\begin{aligned}
F(\gamma(z)) &= \lim_{j \to \infty} \frac{1}{M(s_j)} \sum_{g \in G} \frac{k(h(0, g(\beta))}{h(\gamma(z), g(\beta))^{s_j}} \\
&= \lim_{j \to \infty} \frac{1}{M(s_j)} \sum_{g \in G} \frac{k(h(0, \gamma g(\beta)))}{h(z, g(\beta))^{s_j}} \\
&\leqslant \lim_{j \to \infty} \frac{1}{M(s_j)} \sum_{g \in G} \frac{k(2h(0, \gamma(0)) \cdot h(\gamma(0) \gamma g(\beta)))}{h(z, g(\beta))^{s_j}}.
\end{aligned}$$

Now there is y_0 so that if $y > y_0$, $x \geqslant 1$

$$k(xy) \leqslant x^\varepsilon k(y).$$

For all but a finite number of $g \in G$

$$h(\gamma(0), \gamma g(\beta)) = h(0, g(\beta)) \geqslant y_0.$$

As $M(s_j) \to \infty$ as $j \to \infty$ this finite collection makes no contribution to the limit. Hence

$$F(\gamma(z)) \leqslant 2h(0, \gamma(0)))^\varepsilon \lim_{j \to \infty} \frac{1}{M(s_j)} \sum_{g \in G} \frac{k(h(0, g(\beta)))}{h(z, g(\beta))^{s_j}} = (2h(0, \gamma(0)))^\varepsilon F(z).$$

This is true for any $\varepsilon > 0$ and hence

$$F(\gamma(x)) \leqslant F(z).$$

Apply this with $\gamma(z)$ replacing z and γ^{-1} replacing γ. Thus

$$F(\gamma(z)) \geqslant F(z).$$

Hence

$$F(\gamma(z)) = F(z)$$

for any $\gamma \in G$. This proves

THEOREM 3.1. *If μ is a probability measure constructed as above then*

(i) *μ is supported on L_G,*

(ii) *$F(z) = \int P(z, \zeta)^\delta d\mu(\zeta)$ is automorphic under G.*

The following facts are worth remarking. Firstly this holds for any Fuchsian group, even infinitely generated ones. Next, F is infinitely differentiable and satisfies

$$DF = -\delta(1-\delta)F$$

where D is the Laplace-Beltrami operator. This follows as $P(z, \zeta)^\delta$ satisfies the same equation; compare [7]. This means that these problems are susceptible to the methods of Selberg which we shall exploit in later sections.

§ 4. The Hausdorff dimension of the limit set

In this section we use the measure of the last section to calculate the Hausdorff dimension of the limit set of a Fuchsian group of the second kind without parabolic elements. There are two reasons for this last restriction. Firstly, the consideration of parabolic elements makes the estimates much more difficult although they still can be made. Secondly the results are not quite as complete in the excluded case.

In order to explain the genesis of this problem we recall some theorems of Beardon ([2], [3], [4], [5]). In this series of papers he proves that, for a Fuchsian group of the second kind,

$$0 < d(L_G) \leqslant \delta(G) < 1 \tag{4.1}$$

and the sharper result if G has parabolic elements

$$\tfrac{1}{2} < d(L_G) \leqslant \delta(G) < 1. \tag{4.2}$$

Here we have denoted the Hausdorff dimension of a set E by $d(E)$. Clearly, if G is of the first kind

$$d(L_G) = \delta(G) = 1.$$

One would hope that in general $d(L_G) = \delta(G)$.

Such an assertion has been claimed, for Schottky groups, by Akaza, [1]. The methods of both Beardon and Akaza depend on a close, direct analysis of the action of G on S. In this chapter we will use the measure μ, constructed in the last chapter, with the methods of harmonic analysis on thin sets to solve this problem.

THEOREM 4.1. *If G is of the second kind and has no parabolic elements then*

$$d(L_G) = \delta(G).$$

The restriction that G be of the second kind is clearly inessential. If we admit parabolic elements we can only prove that $d(L_G) = \delta(G)$ subject to $\delta(G) \geqslant 2/3$, which is unsatisfactory. The reason for this is that μ is not the equilibrium measure and if $\frac{1}{2} < \delta(G) < \frac{2}{3}$ the difference is too great to give the expected dimension. In general we cannot even do as well as Beardon's estimate (4.2). We shall return to this point in the next section.

For the proof, which will occupy us to the end of the section, we require the new auxillary function

$$F_s(z) = \int P(z, \zeta)^s d\mu(\zeta) \quad (z \in \Delta). \tag{4.3}$$

Note that $F(z) = F_\delta(z)$.

PROPOSITION 4.1. (i) *If $z \in \Delta$, $g \in G$ then*

$$F_s(g(z)) \leqslant \mu(g)^{|s-\delta|} F_s(z). \tag{4.4}$$

(ii) *There is a fundamental domain D for G so that $d(L_G, D) > 0$ (d is the Euclidean distance). Fix such a D. There is $c_1 = c_1(G, D) > 0$ so that if $z \in D$, $s \in [0, 1]$*

$$0 < F_s(z) \leqslant c_1(1 - |z|^2)^s. \tag{4.5}$$

Proof. We prove (ii) first. The existence of D follows from Proposition 2.2. If $\zeta \in L_G$, $z \in D$ then $|\zeta - z| \geqslant d(L_G, D)$ and hence there is $c_1 > 1$ so that

$$P(z, \zeta) \leqslant c_1(1 - |z|^2).$$

On substituting into (4.3) we obtain (4.5).

Now we prove (i). By Theorem 2.1 if $g \in G$

$$F(g(z)) = F(z).$$

Thus as

$$F(g(z)) = \int P(g(z), \zeta)^\delta d\mu(\zeta)$$

$$= \int P(g(z), g(\zeta))^\delta d\mu(g(\zeta))$$

$$= \int P(z, \zeta)^\delta |g'(\zeta)|^{-\delta} d\mu \circ g(\zeta)$$

it follows that

$$\int P(z, \zeta)^\delta d\mu(\zeta) = \int P(z, \zeta)^\delta |g'(\zeta)|^{-\delta} d\mu \circ g(\zeta).$$

If we write $z = re^{i\theta}$ then $\int P(z, \zeta)^\delta d\mu(\zeta)$ can be expanded as a Fourier series

$$2\pi \sum_{n=-\infty}^{+\infty} \frac{\Gamma(\delta + |n|)}{\Gamma(\delta)|n|!} r^{|n|}(1 - r^2)^\delta F(\delta, \delta + |n|; |n| + 1; r^2)\, \hat\mu(n)\, e^{in\theta}$$

(cf. [7, § 2]). $\hat\mu(n)$ is the nth Fourier coefficient of μ.

As $\delta > 0$ (by (4.1)) $\int P(z, \zeta)^\delta d\mu(\zeta)$ determines $\hat\mu$ for all n, and hence μ also. So, if, $g \in G$,

$$\mu \circ g = |g'|^\delta \mu.$$

Observe that here μ satisfies the sort of functional equation one would get for a δ-dimensional Hausdorff measure; this is the first hint that we are on the right path.

$$F_s(g(z)) = \int P(g(z), \zeta)^s d\mu(\zeta)$$

$$= \int P(g(z), g(\zeta))^s d\mu \circ g(\zeta)$$

$$= \int P(z, \zeta)^s |g'(\zeta)|^{\delta - s} d\mu(\zeta).$$

(4.4) will be a result of this and the following lemma.

LEMMA 4.1. *For any* $g \in \mathrm{Con}\,(\Delta),\ \zeta \in S$

$$\mu(g)^{-1} \leqslant |g'(\zeta)| \leqslant \mu(g).$$

Proof.

Let
$$g = \begin{pmatrix} \alpha & \beta \\ \bar\beta & \bar\alpha \end{pmatrix}.$$

Then, one the one hand

$$|g'(\zeta)|^{-1} = |\bar\beta\zeta + \bar\alpha|^2 \leqslant 2(|\alpha|^2 + |\beta|^2) = \mu(g).$$

On the other hand, as $|\alpha|^2 - |\beta|^2 = 1$,

$$|g'(\zeta)| = |\bar\beta\zeta + \bar\alpha|^{-2} \leqslant (|\alpha| - |\beta|)^{-2} = (|\alpha| + |\beta|)^2 \leqslant 2(|\alpha|^2 + |\beta|^2) = \mu(g).$$

This proves the lemma and with it the proposition.

The remainder of the proof rests on an identity. In order to express this we need some notation and preliminaries. Let

$$d\sigma(z) = \frac{|dz \wedge \overline{dz}|}{(1 - |z|^2)^2}$$

be the hyperbolic area. Let ξ, η, ζ be three distinct points of S. Then the integral

$$\int P(z, \xi)^s P(z, \eta)^t P(z, \zeta)^u d\sigma(z)$$

converges if

$$s+t+u > 1, \quad s+t > u, \quad t+u > s, \quad u+s > t. \tag{4.6}$$

To see this we consider the integral on $\mathbf{H}$ with $\xi = 0$, $\eta = 1$, $\zeta = \infty$ and an elementary manipulation shows that the integral is, up to a constant multiple,

$$\int_0^\infty \int_{-\infty}^{+\infty} y^{s+t+u-2}(x^2 + y^2)^{-s}((x-1)^2 + y^2)^{-t} dx\, dy.$$

In $A = \{(x, y) \mid (x-1)^2 + y^2 \leqslant \frac{1}{4}\}$ the integral is bounded by

$$c_1(s) y^{s+t+u-2} \cdot ((x-1)^2 + y^2)^{-t}.$$

Thus the integral over A, is bounded by (on changing to polar co-ordinates centred at $(1, 0)$)

$$c_1(s) \int_0^{1/2} \int_0^\pi (\sin \theta)^{s+t+u-2} r^{s+t+u-1} dr\, d\theta.$$

The conditions for this to converge are

$$s+t+u > 1, \quad s+t > u.$$

Consider $B = \{(x, y): (x-1)^2 + y^2 > \frac{1}{4}\}$. In this region, if $t \geqslant 0$, the integrand is bounded by

$$c_2(t) y^{s+t+u-2}(x^2 + y^2)^{-s} \min (1, (x^2 + y^2)^{-t}).$$

Converting to polars shows that this converges if

$$s+t+u > 1, \quad t+u > s, \quad u+s > t.$$

The case $t < 0$ is incompatible with (4.6).

Thus we obtain (4.6) as the condition for convergence.

PROPOSITION 4.2. *If (4.6) holds there is a function $D(s, t, u)$ so that, if ξ, η, $\zeta \in S$ are unequal,*

$$\int_{\Delta} P(z, \xi)^s P(z, \eta)^t P(z, \zeta)^u d\sigma(z) = \frac{D(s, t, u)}{|\xi - \eta|^{s+t-u}|\eta - \zeta|^{t+u-s}|\zeta - \xi|^{u+s-t}}.$$

This is the identity which we require.

Proof: Let

$$R(z, \xi, \eta, \zeta) = |\xi - \eta|^{s+t-u}|\eta - \zeta|^{t+u-s}|\zeta - \xi|^{u+s-t} P(z, \xi)^s P(z, \eta)^t P(z, \zeta)^u.$$

If $g \in \mathrm{Con}\,(\Delta)$ then one checks that

$$R(g(z), g(\xi), g(\eta), g(\zeta)) = R(z, \xi, \eta, \zeta).$$

Integrating this over Δ and using the invariance of σ shows that the function

$$S(\xi, \eta, \zeta) = \int R(z, \xi, \eta, \zeta) d\sigma(z),$$

satisfies

$$S(g(\xi), g(\eta), g(\zeta)) = S(\xi, \eta, \zeta).$$

But if ξ, η, $\zeta \in S$ are unequal there is $g \in \mathrm{Con}\,(\Delta)$ so that $\{g(\xi), g(\eta), g(\zeta)\} = \{1, -1, i\}$. This shows that

$$S(\xi, \eta, \zeta) = S(1, -1, i) \quad \text{or} \quad S(-1, 1, i).$$

Conjugating shows these are equal and the conclusion follows on setting

$$D(s, t, u) = S(1, -1, i).$$

Now fix $\zeta \in (\bar{D} \cap S)^{\circ}$. Observe in passing that there is a constant $c_4 > 0$ so that if $z \in D$, $g \in G \setminus \{I\}$ then

$$|z - g(\zeta)| \geqslant c_4. \tag{4.7}$$

For we can take c_4 to be the smallest Euclidean distance to any side (in Δ) of D.

Suppose s, t, u satisfy (4.6). Then $s, t, u > 0$ and

$$\int_\Delta P(z, \xi)^s P(z, \eta)^t P(z, \zeta)^u \, d\sigma(z) \geqslant 2^{-2u} D(s, t, u) |\xi - \zeta|^{-(s+t-u)},$$

as $|\xi - \zeta|$, $|\zeta - \eta| \leqslant 2$. Integrating this with respect to $d(\mu \otimes \mu)(\xi, \eta)$ and using Tonelli's theorem (the integrand being positive) to interchange the order of integration, we obtain

$$\int_\Delta F_s(z) \cdot F_t(z) \, P(z, \zeta)^u \, d\sigma(z) \geqslant 2^{-2u} D(s, t, u) \int \frac{d\mu(\xi) \, d\mu(\eta)}{|\xi - \eta|^{s+t-u}} = 2^{-2u} D(s, t, u) \, I_{s+t-u}(\mu) \quad (4.8)$$

where $I_k(\mu)$ is the k-energy of μ (cf. [6, § III]).

Now suppose $s \geqslant \delta$, $t \geqslant \delta$. Then

$$\int_\Delta F_s(z) \, F_t(z) \, P(z, \zeta)^u \, d\sigma(z = \sum_{g \in G} \int_D F_s(g(z)) \cdot F_t(g(z)) \, P(g(z), \zeta)^u \, d\sigma(z).$$

If $g \neq I$, by (4.7)

$$P(g(z), \zeta) \leqslant c_4^{-2} \cdot (1 - |g(z)|^2) = c_4^{-2} \cdot |g'(z)| (1 - |z|^2). \tag{4.9}$$

We require a lemma.

LEMMA 4.2. *There is $c_5 > 0$ so that, if $z \in D$,*

$$|g'(z)| \leqslant c_5 \cdot \mu(g)^{-1}.$$

Proof. $\{g^{-1}(\infty) | g \in G\}$ lies in $\{|z| > 1\}$ and accumulates on L_G. Thus, as $d(L_G, D) > 0$, there is $c_6 > 0$ so that, if $z \in D$, $g \in G$

$$|g^{-1}(\infty) - z| \geqslant c_6.$$

But trivially $|g^{-1}(\infty) - z| \geqslant |g^{-1}(\infty)| - 1$.

Combining these, there is $c_7 > 0$ so that

$$\left| \frac{z}{g^{-1}(\infty)} - 1 \right| \geqslant c_7.$$

It is easy to check that

$$|g'(z)| = \frac{4}{2 + \mu(g)} \left| \frac{z}{g^{-1}(\infty)} - 1 \right|^{-2},$$

and the lemma follows immediately.

Applying the lemma to (4.9) we obtain that, if $z \in D$, $g \in G \setminus \{I\}$

$$P(g(z), \zeta) \leqslant c_8 \, \mu(g)^{-1}(1 - |z|^2).$$

Recall that by Proposition 4.1 (i), if $s \geqslant \delta, t \geqslant \delta$,

$$F_s(g(z)) \cdot F_t(g(z)) \leqslant \mu(g)^{s+t-2\delta} F_s(z) F_t(z).$$

Combining everything we obtain

$$\int_\Delta F_s(z) F_t(z) P(z, \zeta)^u d\sigma(z) \leqslant \int_D F_s(z) F_t(z) P(z, \zeta)^u d\sigma(z)$$

$$+ c_8^u \sum_{g \in G} \mu(g)^{s+t-u-2\delta} \int_D F_s(z) F_t(z) (1 - |z|^2)^u d\sigma(z).$$

Let ξ, η be two arbitrary distinct points of L_G. Then on D $F_s(z)$ (resp. $F_t(z)$) is majorized by $P(z, \xi)^s$ (resp. $P(z, \eta)^t$)) (Proposition 4.1 (ii)). Thus, by Proposition 4.2 both integrals are finite if (4.6) holds. The sum $\Sigma_{g \in G} \mu(g)^{s+t-u-2\delta}$ converges if $s+t-u < \delta$. From (4.8) we have shown that if

$$s+t+u > 1, \quad s+t > u, \quad s+u > t, \quad t+u > s, \quad s \geqslant \delta, \quad t \geqslant \delta, \quad s+t-u < \delta$$

then

$$I_{s+t-u}(\mu) < \infty.$$

We will now deduce from this that if k is such that $0 < k < \delta$ then

$$I_k(\mu) < \infty. \tag{4.10}$$

Suppose first that $\delta \geqslant \frac{1}{3}$. Let $s = \delta$, $t = \delta$, $u = 2\delta - k(>\delta)$. Then $s+t+u = 4\delta - k > 3\delta \geqslant 1$, $s+t-u = k > 0$. The other inequalities are trivial and (4.10) follows.

Now suppose that $\delta \leqslant \frac{1}{3}$. Set $s = t = (1+\delta)/4$, $u = (1+\delta-2k)/2$. Then $s+t+u = (2+2\delta-2k)/2 > 1$. $s+t-u = k$ so that $0 < s+t-u < k$. Also $0 < (1+\delta-2\delta)/2 < u$. Finally $s \geqslant \delta$ as $\delta \leqslant \frac{1}{3}$. This verifies the inequalities and (4.10) follows again.

We have shown that if $k < \delta$ then $I_k(\mu) < \infty$. As μ is supported on L_G this shows that $d(L_G) \geqslant \delta(G)$, (see [6: § III, Theorem 5; § IV, Theorem 4.1]). By (4.1) $d(L_G) \leqslant \delta(G)$. Hence $d(L_G) = \delta(G)$. This completes the proof of the theorem.

§ 5. Estimates if G has parabolic elements

Now we shall indicate the modifications necessary if G has parabolic elements but is still finitely generated. We shall make use of the fundamental domain described in Proposition 2.2. An immediate consequence of this proposition is the existence of a constant $c > 0$ with the property that if $\zeta \in L_G$, $z \in K_\alpha \cup \bigcup_{j \in A} D_j(\alpha)$ then

$$|z - \zeta| > c.$$

The method of proof of Proposition 4.1 now gives

PROPOSITION 5.1. (i) *If* $z \in \Delta$, $g \in G$ *then*

$$F_s(g(z)) \leqslant \mu(g)^{|s-\delta|} F_s(z).$$

(ii)   *There is* $c_1 > 0$ *so that if* $z \in K_\alpha$ *then*

$$0 < F_s(z) \leqslant c_1.$$

(iii)   *There is* $c_2 > 0$ *so that if* $z \in D_j(\alpha)$ *then*

$$0 < F_s(z) < c_2(1 - |z|^2)^s.$$

Our main object now is to supplement this in the cusps.

PROPOSITION 5.2. *There is* $c_3 > 0$ *so that if* $z \in D_k^*$, $s < 2$,

$$0 < F_s(z) < c_3(1 - |z|^2)^{2\delta - s - 1}.$$

Proof. We shall work on **H** with $p_k = \infty$ to prove this proposition. On **H** the function corresponding to h is

$$h_0(z, w) = |z - \bar{w}|^2/(4 \cdot \operatorname{Im}(z) \cdot \operatorname{Im}(w)) \quad (z, w \in \mathbf{H}).$$

We shall prove the proposition first when $s = \delta$. Equation (3.6) becomes

$$F_\delta(z) = \lim_{j \to \infty} M(s_j)^{-1} \sum_{g \in G} k(h_0(i, g(\beta))) \cdot h_0(z, g(\beta))^{-s_j},$$

where

$$M(s) = \sum_{g \in G} k(h_0(i, g(\beta))) \cdot h_0(i, g(\beta))^{-s}.$$

Suppose the group fixing ∞ is generated by $z \mapsto z + \lambda$ ($\lambda > 0$). We shall consider, for $s > \delta$, the series

$$U(z, s) = \sum_{g \in G} k(h_0(i, g(\beta))) \cdot h_0(z, g(\beta))^{-s}$$

$$= \sum_{g \in G_\infty \backslash G} \sum_{m=-\infty}^{\infty} k(h_0(i + m\lambda, g(\beta))) \cdot h_0(z + m\lambda, g(\beta))^{-s}.$$

If we fix $A > 0$ there is a finite number of cosets of $G_\infty \backslash G$ so that, if g does not belong to one of these then

$$\operatorname{Im}(g(\beta)) < A^{-1}.$$

We will choose that representative g of a coset $G_\infty g$ so that

$$-\lambda/2 < \operatorname{Re}(g(\beta)) \leqslant \lambda/2.$$

Also we shall suppose that

$$-\lambda/2 < \mathrm{Re}(z) \le \lambda/2,$$

and that, for some fixed $d > 0$,

$$\mathrm{Im}(z) > d.$$

From the definition of h_0 it follows that, if $m \neq 0$,

$$h_0(z+m\lambda, g(\beta))^{-s} \le 4^s \mathrm{Im}(g(\beta))^s \mathrm{Im}(z)^s (\mathrm{Im}(z)^2 + (|m|-1)^2\lambda^2)^{-s},$$

and that,

$$h_0(z, g(\beta))^{-s} \le 4^s \mathrm{Im}(g(\beta))^s \mathrm{Im}(z)^{-s}.$$

Further

$$h_0(i+m\lambda, g(\beta)) \le \mathrm{Im}(g(\beta))^{-1}((1+\mathrm{Im}(g(\beta))^2 + (|m|+1)^2\lambda^2).$$

By [9; p. 88] $\{\mathrm{Im}(g(\beta))\}$ is, for fixed β, bounded above. It follows that there is $c_4 > 0$, depending only on β and so that

$$h_0(i+m\lambda, g(\beta)) \le c_4 h_0(i, g(\beta))(m^2+1).$$

Choose $\varepsilon > 0$, $\varepsilon \le 1$. Recalling the properties of k as described in section 3 it follows that we can find A so that, if g is not in the finite exceptional set of cosets $G_\infty \backslash G$

$$k(h_0(i+m\lambda, g(\beta)) \le (c_4(m^2+1))^\varepsilon k(h_0(i, g(\beta))).$$

Thus, if g avoids one of the exceptional set of cosets,

$$\sum_{m=-\infty}^{+\infty} k(h_0(i+m\lambda, g(\beta))) h_0(z+m\lambda, g(\beta))^{-s}$$
$$\le c_4^\varepsilon 4^s h(h_0(i, g(\beta)) \mathrm{Im}(g(\beta))^s (\mathrm{Im}(z)^{-s} + \mathrm{Im}(z)^s R(\mathrm{Im}(z))),$$

where

$$R(y) = \sum_{m \neq 0} (m^2+1)^\varepsilon (y^2 + (|m|-1)^2\lambda^2)^{-s}.$$

It is easy to see that $R(y)$ is bounded by

$$2 \int_0^\infty (x^2+1)^\varepsilon (y^2 + (x-2)^2)^{-s} dx.$$

Thus if $s \ge \delta > 1/2$ (which we may assume by [3]), $s \le 5$, $0 < \varepsilon \le 1$ we see that there is c_5 depending only on δ, d so that if $y > d$

$$R(y) \le c_5 y^{1+2\varepsilon-2s}.$$

Gathering together our results we obtain, for some c_6

$$U(z, s) \leqslant Q(z, s) + c_6 \operatorname{Im}(z)^{1+2\varepsilon-\delta} \sum_{g \in G_\infty \backslash G} k(h_0(i, g(\beta))) \operatorname{Im}(g(\beta))^s,$$

where $Q(z, s)$ is the sum over the set of excluded cosets. c_6 does not depend on ε. Using the estimate

$$\operatorname{Im}(g(\beta)) \leqslant c_7 h_0(i, g(\beta))^{-1},$$

where

$$c_7 = \sup(|i + g(\beta)|^2),$$

and expanding the final sum to be over all $g \in G$ we obtain

$$U(z, s) \leqslant Q(z, s) + c_6 c_7^s \operatorname{Im}(z)^{1+2\varepsilon-\delta} M(s).$$

As $Q(z, s)$ is the sum over a finite number of cosets it converges in $s > 1/2$. Then as

$$F_\delta(z) = \lim U(z, s_j)/M(s_j),$$

we obtain, if $\operatorname{Im}(z) > d$,

$$F_\delta(z) \leqslant c_6 c_7^\delta \operatorname{Im}(z)^{1+2\varepsilon-\delta}.$$

The constants are independent of ε and this shows that there is $c_8 > 0$ so that

$$F_\delta(z) \leqslant c_8 \operatorname{Im}(z)^{1-\delta}.$$

This proves the proposition if $s = \delta$.

Recall that on Δ we defined

$$F_s(z) = \int P(z, \eta)^s d\mu(\eta).$$

Assume that $p_k = 1$. Using the map

$$A: \mathbf{H} \to \Delta; \quad w \mapsto (w - i)/(w + i),$$

we can transfer this definition to $\mathbf{H}$. Let, for $w \in \mathbf{H}$, $x \in \mathbf{R}$

$$P_0(w, x) = \operatorname{Im}(w)/|w - x|^2,$$

$$P_0(w, \infty) = \operatorname{Im}(w),$$

be the Poisson kernel on $\mathbf{H}$. Then we find

$$F_s(A(w)) = \int P_0(w, x)^s (x^2 + 1)^s d\mu(A(x)).$$

Let

$$d\mu_0(x) = (1+x^2)^\delta d\mu(A(x)).$$

As F_δ is automorphic μ_0 is periodic with period λ. As μ is a probability measure μ_0 has finite mass on each period. By abuse of language we shall write $F_s(w)$ for $F_s(A(w))$. Let $\mu_0(\infty)$ be the mass at ∞. Then

$$F_s(w) = \mu_0(\infty) \operatorname{Im}(w)^s + \sum_{m=-\infty}^{+\infty} \int_0^\lambda P_0(w, x+m\lambda)^s ((x+m\lambda)^2 + 1)^{s-\delta} d\mu_0(x).$$

By comparison with the corresponding integral we deduce that if $\operatorname{Im}(w) > d$ then, if $2 \geqslant s > \delta > 1/2$ there is $c_9 > 0$ so that

$$\sum_{m=-\infty}^{+\infty} P_0(w, x+m\lambda)^s ((x+m\lambda)^2 + 1)^{s-\delta} \leqslant c_9 \operatorname{Im}(w)^{1+s-2\delta}.$$

But, also

$$F_s(w) \geqslant \mu_0(\infty) \operatorname{Im}(w)^s.$$

As we have already shown that the proposition is true if $s = \delta$ we deduce that

$$\mu_0(\infty) = 0.$$

Thus

$$F_s(w) \leqslant \lambda c_9 \operatorname{Im}(w)^{1+s-2\delta}.$$

This gives the proposition as stated when we reinterpret it on Δ.

Now we can proceed as in section 4. We have, if,

$$s+t+u > 1, \quad s+t > u, \quad t+u > s, \quad u+s > t,$$

that

$$2^{-2u} D(s, t, u) I_{s+t-u}(\mu) \leqslant \sum_{g \in G} \int_D F_s(g(z)) F_t(g(z)) P(g(z), \eta)^u d\sigma(z),$$

where as before η is a fixed point of $(\bar{D} \cap S)^\circ$. Now let

$$W(z, \eta) = \sum_{g \in G} \mu(g)^{s-\delta+t-\delta} P(g(z), \eta)^u.$$

Then by Proposition 5.1 (i), if $s, t > \delta$,

$$2^{-2u} D(s, t, u) I_{s+t-u}(\mu) \leqslant \int_D F_s(z) F_t(z) W(z, \eta) d\sigma(z). \tag{5.1}$$

It is convenient to specialize now. We choose

$$s = t = \delta, \quad u > \delta > 1/2.$$

Now we require bounds on $W(z, \eta)$.

PROPOSITION 5.3. (i) *There is a constant* $c_{10} > 0$ *so that if*

$$z \in K_\alpha \cup \bigcup_{j=1}^{M} D_j(\alpha)$$

then

$$W(z, \eta) \leqslant c_{10}(P(z, \eta)^u + \Sigma_{g \in G} \mu(g)^{-u}(1 - |z|^2)^u).$$

(ii) *There is a constant* $c_{11} > 0$ *so that, if*

$$z \in \bigcup_{k=1}^{N} D_k^*$$

then

$$W(z, \eta) \leqslant c_{11}(\Sigma_{g \in G} \mu(g)^{-u}(1 - |z|^2)^{u-1}).$$

Proof. (i) is essentially proved in section 4. (ii) is long and we postpone it to the end of this section.

From Proposition 5.3 the right hand side of (5.1) is finite if

$$2\delta + u > 1$$
$$2\delta > u > \delta$$
$$u > 2 - 2\delta.$$

Just as at the end of section 4 we deduce from these that

$$d(L_G) \geqslant \min(\delta, 4\delta - 2).$$

As

$$d(L_G) \leqslant \delta$$

we have now proved

THEOREM 5.1. *If* G *has parabolic elements and* $\delta \geqslant 2/3$ *then*

$$d(L_G) = \delta(G).$$

We shall now complete the proof of Proposition 5.3.

Proof of Proposition 5.3 (ii). This proof is rather similar to that of Proposition 5.2. We prove the corresponding statement for **H**.

The series to be considered becomes, by abuse of notation

$$W(z, \eta) = \sum_{g \in G} P_0(g(z), \eta)^u,$$

where now $z \in \mathbf{H}$, $\eta \in \mathbf{R}$ and ∞ is a parabolic fixed point of G. Suppose that G_∞ is generated by $z \mapsto z + \lambda (\lambda > 0)$. Let $z = x + iy$. Then clearly W is a periodic function of x with period λ. Thus W can be expanded as a Fourier series

$$W(z, \eta) = \sum_{n=-\infty}^{+\infty} w_n(y, \eta, u) \cdot \exp(2\pi i n x/\lambda).$$

An easy calculation shows that

$$\lambda \cdot w_n(y, \eta, u) = \sum_{g \in G \setminus G_\infty} \exp(2\pi i g^{-1}(\eta)/\lambda) \left| (g^{-1})'(\eta) \right|^u \cdot v_n(y, u),$$

where

$$v_n(y, u) = \pi^u \sqrt{y} (n/\lambda)^{u-1/2} K_{u-1/2}(2\pi n y/\lambda).$$

We assume that $\mathrm{Im}(z) > d$ for some fixed $d > 0$. By the asymptotic expansion of the Bessel function it follows that there are constants $c_{12}, c_{13} > 0$ so that if $n \neq 0$ then

$$\left| v_n(y, u) \right| < c_{12} \exp(-c_{13} n y)$$

(see [12; p. 374]). On the other hand

$$v_0(y, u) = B(1/2, u - 1/2) y^{1-u}.$$

Now let

$$T(\eta, u) = \sum_{g \in G_\infty \setminus G} \left| g'(\eta) \right|^u.$$

Combining all our results to date shows that, for a suitable constant $c_{14} > 0$

$$W(z, \eta) \leqslant c_{14}(1 + \mathrm{Im}(z)^{1-u}) T(\eta, u). \tag{5.2}$$

We must now obtain bounds for $T(\eta, u)$. To do this let us consider a single coset $G_\infty g$. Suppose that g can be represented as a matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad (c > 0, ad - bc = 1).$$

As η is an ordinary point there is a constant $c_{15} > 0$ so that

$$\left| \eta - g^{-1}(\infty) \right| > c_{15}.$$

From this it follows that there is a constant $c_{16} > 0$ so that

$$|g'(\eta)| < c_{16}(c^2 + d^2)^{-1}. \tag{5.3}$$

It is a classical fact (see [9: p. 88]) that there is a constant $c_{17} > 0$ so that, for all $g \in G$

$$c^2 + d^2 > c_{17}. \tag{5.4}$$

Let

$$\pi = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}.$$

Then an easy calculation shows

$$\mu(\pi^n g) = (c^2 + d^2)(n\lambda + \tau)^2 + (c^2 + d^2) + (c^2 + d^2)^{-1}, \tag{5.5}$$

where

$$\tau = (ac + bd)/(c^2 + d^2).$$

From (5.4) and (5.5) it follows that there is a constant $c_{18} > 0$ so that

$$\sum_n \mu(\pi^n g)^{-u} > c_{18}(c^2 + d^2)^{-u}.$$

Combining this with (5.3) shows that

$$T(\eta, u) < (c_{16}^u/c_{18}) \sum_{g \in G} \mu(g)^{-u}.$$

This and (5.2) together constitute proposition 5.3 (ii) which is thereby proved.

§ 6. Some estimates

In section 3 we introduced the function

$$f(z, w; s) = \sum_{g \in G} h(z, g(w))^{-s}. \tag{6.1}$$

This function will play an important role in the rest of this paper. In order to use it we need various estimates. Let us assume henceforth that $s \leq 2$; this is for convenience only. We shall continue to use the fundamental domain introduced in section 2; we shall also use the notations introduced there. Fix $\alpha \leq \pi/2$. Let

$$D_H = K_\alpha \cup \bigcup_{j=1}^{M} D_j(\alpha),$$

and

$$D_G = \bigcup_{k=1}^{N} D_k^*.$$

The object of this section is to prove the following proposition. The methods of proof are much the same as those of Propositions 5.2 and 5.3.

PROPOSITION 6.1. *Suppose s is such that (6.1) converges. Then there are constants $c_1, c_2, \ldots$ so that the following statements hold.*

(i) $f(z, w; s) \geqslant c_0 (1 - |z|^2)^s (1 - |w|^2)^s \sum_{g \in G} \mu(g)^{-s}$.

(ii) *If $z, w \in D_H$ then*

$$f(z, w; s) \leqslant c_2 (1 - |z|^2)^s (1 - |w|^2)^s \left(\sum_{g \in G} \mu(g)^{-s} + (1 - |z||w|)^{-2s} \right).$$

In particular if $w \in K_\alpha$

$$f(z, w; s) \leqslant c_3 (1 - |z|^2)^s (1 - |w|^2)^s \sum_{g \in G} \mu(g)^{-s}.$$

(iii) *If $z \in D_H$, $w \in D_C$ then*

$$f(z, w; s) \leqslant c_4 (1 - |z|^2)^s (1 - |w|^2)^{s-1} \sum_{g \in G} \mu(g)^{-s}.$$

(iv) *If $z, w \in D_k^*$ then*

$$f(z, w; s) \leqslant c_5 (1 - |z|^2)^{s-1} (1 - |w|^2)^{s-1} \left(\sum_{g \in G} \mu(g)^{-s} + \max(1 - |z|^2, 1 - |w|^2)^{1-2s} \right).$$

(v) *If $z \in D_j^*$, $w \in D_k^*$, $j \neq k$, then*

$$f(z, w; s) \leqslant c_6 (1 - |z|^2)^{s-1} (1 - |w|^2)^{s-1} \sum_{g \in G} \mu(g)^{-s}.$$

Note that the inequality in (v) is not symmetric. This is because it is not the best possible result. The other inequalities are best possible of their kind although this will not be proved now.

Proof. From the definition of the function h and the series (6.1) for f we obtain the following expansion

$$f(z, w; s) = (1 - |z|^2)^s (1 - |w|^2)^s \sum_{g \in G} |g'(w)|^s / |1 - \bar{z} g(w)|^{2s}. \tag{6.2}$$

Consider a particular group element $g \in G$. Suppose that g is represented by the matrix $\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} (|a|^2 - |b|^2 = 1)$. Then

$$|g'(w)| = |\bar{b}w + \bar{a}|^{-2} \geqslant 1/(2(|a|^2 + |b|^2)) = \mu(g)^{-1}. \tag{6.3}$$

Also

$$|1 - \bar{z} \cdot g(w)|^2 \leqslant 4. \tag{6.4}$$

(6.2), (6.3) and (6.4) together imply (i) which is now proved.

Suppose now that $w \in D_H$. As

$$|g'(w)| = |a|^{-2}|1 - zg^{-1}(0)|^{-2}, \tag{6.5}$$

and as D_H is a positive distance from the limit set it follows that there is a constant $c_7 > 0$ so that

$$|g'(w)| \leqslant c_7/\mu(g). \tag{6.6}$$

Suppose now that z, $w \in D_H$. It follows from the description of D_H given in section 2 that there is a finite subset R of G and a constant $c_8 > 0$ so that for every element of G except perhaps one element of R we have

$$|g(w) - z| > c_8. \tag{6.7}$$

There is then a constant $c_9 > 0$ so that for all g such that (6.7) is true

$$|1 - g(w)\bar{z}| > c_9. \tag{6.8}$$

On the other hand there is $c_{10} > 0$ so that for the one possible exceptional case

$$|1 - g(w)\bar{z}| > c_{10}(1 - |z||w|). \tag{6.9}$$

Substituting (6.6), (6.8) and (6.9) into (6.2) gives (ii). It is obvious that the second assertion of (ii) is a direct consequence of the first.

It is easy to check that, if $\gamma \in \mathrm{Con}(\Delta)$, z, $w \in \Delta$ then

$$h(\gamma(z), \gamma(w)) = h(z, w),$$

that is, h is a point-pair invariant. From this it follows that

$$f(z, w; s) = f(w, z; s). \tag{6.10}$$

Now we shall prove (iii). Suppose that $z \in D_H$ and that $w \in D_k^*$ for some k. Then by the results of section 2 there is $c_{11} > 0$ so that

$$|1 - g(w)\bar{z}| > c_{11}. \tag{6.11}$$

Consequently, if η is an ordinary point in S there is $c_{12} > 0$ so that

$$|g'(w)|(1 - |w|^2)/|1 - g(w)\bar{z}|^2 \leqslant c_{12}P(w, \eta). \tag{6.12}$$

The conclusion follows from Proposition 5.3, equations (6.2) and (6.12).

Now let us prove (iv). We look at the corresponding series on $\mathbf{H}$ which is (see the proof of Proposition 5.2)

$$\sum_{g \in G} \operatorname{Im}(z)^s \operatorname{Im}(g(w))^s / |\bar{z} - g(w)|^{2s}. \tag{6.13}$$

We are abusing notations here but it should cause no confusion. Let us suppose that ∞ is the parabolic fixed point in question and that the parabolic subgroup of G fixing ∞ is generated by $z \mapsto z + 1$. We may suppose that

$$D_k^* = \{z \colon |\operatorname{Re}(z)| < 1/2, \quad \operatorname{Im}(z) > d\}. \tag{6.14}$$

Recall that $s \geqslant \delta(G) > 1/2$.

The function represented by the series (6.13), which we shall also call $f(z, w; s)$, is clearly periodic in both z and w. Thus there is an expansion of the form

$$f(z, w; s) = \sum_{m, n} c_{m, n}(\operatorname{Im}(z), \operatorname{Im}(w), s) e^{2\pi i(m \operatorname{Re}(z) + n \operatorname{Re}(w))}. \tag{6.15}$$

The Fourier coefficients, $c_{m, n}$, can be easily calculated.

$$\begin{aligned}
c_{m, n}(x, y, s) = {} & \delta_{m, -n} (\operatorname{Im}(z) \operatorname{Im}(w))^s \operatorname{Im}(z + w)^{1 - 2s} D(m \operatorname{Im}(z + w), s) \\
& + \sum_{G_\infty \backslash G / G_\infty} \frac{e^{2\pi i(m g^{-1}(\infty) + n g(\infty))}}{|c(g)|^{2s}} \\
& \times (\operatorname{Im}(z) \operatorname{Im}(w))^{1 - s} D(m \operatorname{Im}(z), n \operatorname{Im}(w), 1/c(g)^2 \operatorname{Im}(z) \operatorname{Im}(w)), s).
\end{aligned} \tag{6.16}$$

In this formula δ is the Kronecker symbol. $c(g)$ is defined as follows. If $g \in G$ take a representative matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad (c > 0, ad - bc = 1).$$

$c(g)$ is the "c" in this matrix. In the second term the summation is taken over all double cosets except the one to which the identity belongs. The two functions D are defined as follows.

$$D(y, s) = \int_{-\infty}^{+\infty} \exp(-2\pi ixy)(1 + x^2)^{-s} dx. \tag{6.17}$$

$$D(x, y, u, s) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{e^{-2\pi i(px + qy)}}{((pq + 1 + u)^2 + (p - q)^2)^s} \, dp \, dq. \tag{6.18}$$

By partial integration one can show that, for any integers M, $N \geqslant 0$, as $s \geqslant \delta(G)$ there are constants c_{13}, $c_{14} > 0$ (c_{14} independent of u) so that

$$D(y, s) \leqslant c_{13} |y|^{-N} |s|^{N}, \tag{6.19}$$

$$D(x, y, u, s) \leqslant c_{14} |x|^{-M} |y|^{-N} |s|^{M+N}. \tag{6.20}$$

Combining (6.16) to (6.20) we obtain, if we also assume that $s \leqslant 2$

$$f(z, w; s) \leqslant c_{15} \left(\frac{\operatorname{Im}(z)^s \operatorname{Im}(w)^s}{\operatorname{Im}(z+w)^{2s-1}} + (\operatorname{Im}(z) \operatorname{Im}(w))^{1-s} \sum |c(g)|^{-2s} \right).$$

The technique used in the proof Proposition 5.3 shows that for a suitable constant $c_{16} > 0$

$$\sum_{g \in G_\infty \backslash G / G_\infty} c(g)^{-2s} \leqslant c_{16} \sum_{g \in G} \mu(g)^{-s}. \tag{6.21}$$

Combining the last two inequalities gives the assertion of the proposition.

This only leaves (v) to be proved. We continue to work on **H**. We do not use the full power of our assumptions but merely assume that $z \in D_k^*$ (as defined by (6.14)) and $\operatorname{Im}(w) < d$. We require an asymptotic estimate for $\operatorname{Im}(w)$ small. Applying (6.15) to (6.20) we obtain

$$f(z, w; s) \leqslant c_{17} \left(\frac{\operatorname{Im}(z)^s \operatorname{Im}(w)^s}{\operatorname{Im}(z+w)^{2s-1}} + (\operatorname{Im}(z) \operatorname{Im}(w))^{1-s} \operatorname{Im}(w)^{-2} \sum |c(g)|^{-2s} \right).$$

Reinterpretating this on **H** gives the assertion. This completes the proof of the proposition.

§ 7. An analysis of F

In this section we shall only consider groups of the second kind for which $\delta(G) > 1/2$. In this case, by the results of sections 3, 4, 5, $F = F_\delta$ is in $L^2(G \backslash \Delta)$ and is an eigenfunction of the Laplace-Beltrami operator. In [10] A. Selberg showed how eigenfunctions of the Laplace-Beltrami operator could also be characterised as eigenfunctions of certain integral operators. The integral operators in question are those with a kernel of the form

$$\sum_{g \in G} k(x, g(y)),$$

where k is a point-pair invariant. He gives a method of calculating the eigenvalue.

We may apply the considerations to the kernel $f(z, w; s)$ and the eigenfunction $F(w)$. Using Propositions 4.1, 5.1, 5.2 and 6.1 the following integral converges absolutely.

$$\int_D f(z, w; s) F(w) \, d\sigma(w).$$

Carrying out the prescription of [10] we obtain

$$\int_D f(z, w; s) F(w) d\sigma(w) = C(s) \Gamma(s - \delta) \Gamma(s - (1 - \delta)) F(z), \qquad (7.1)$$

where

$$C(s) = 4^{s-1} \Gamma(1/2) \Gamma(s - 1/2) / \Gamma(2s - 1) \Gamma(s).$$

All we need to know about $C(s)$ is that it is an entire function.

Equation (7.1) contains a great deal of information. For instance Proposition 6.1 gives an upper bound for f. Substituting this into (7.1) with z fixed shows that there is a constant $c_1 > 0$ so that, for $2 \geqslant s > \delta(G)$

$$\sum_{g \in G} \mu(g)^{-s} \geqslant c_1 / (s - \delta(G)). \qquad (7.2)$$

If we use Proposition 6.1 (i) we see that there is a positive function Y on $\Delta \times \Delta$ so that

$$f(z, w; s) \geqslant Y(z, w) / (s - \delta(G)). \qquad (7.3)$$

On the other hand Proposition 6.1 (i) gives a lower bound for f. Substituting this into (7.1) shows that there is a constant $c_2 > 0$ so that

$$\sum_{g \in G} \mu(g)^{-s} \leqslant c_2 / (s - \delta(G)). \qquad (7.4)$$

Using the rest of Proposition 6.1 shows that there is a positive function Z on $\Delta \times \Delta$ so that

$$f(z, w; s) \leqslant Z(z, w) / (s - \delta(G)). \qquad (7.5)$$

The inequalities (7.2) to (7.5) are rather striking. They show that, if we regard f as an analytic function of s it behaves almost as if it has a pole at $s = \delta(G)$. In particular in the considerations of section 3 it appears that in this case (i.e. $\delta(G) > 1/2$) that the introduction of the function k was unnecessary. Thus the results of section 3 can be simplified. In particular, from equation (3.6) we now see that F can be defined as follows. There is a sequence $s_j \to \delta(G)$ so that (up to a constant factor), for some fixed w,

$$F(z) = \lim_j (s_j - \delta(G)) f(z, w; s_j). \qquad (7.6)$$

The arguments that we use at each point show even that given a sequence $s_j'' \to \delta(G)$ there is a subsequence s_j' on which the limit (7.6) exists.

Suppose $s_j^{(1)}$ and $s_j^{(2)}$ are two sequences decreasing to $\delta(G)$ on which the limit on the right hand side of (7.6) exists. Let F_1 and F_2 be the respective limit functions.

Taking the limit of (7.1) along the sequences $s_j^{(1)}$ and $s_j^{(2)}$ (this is justified by our estimates and the Lebesgue dominated convergence theorem) we obtain

$$\int_D F_i(z)\, F_j(z)\, d\sigma(z) = C(\delta)\, \Gamma(2\delta - 1)\, F_j(w). \tag{7.7}$$

Recall that w is a fixed point. From (7.7) it follows that

$$\int_D (F_1(z) - F_2(z))^2\, d\sigma(z) = 0,$$

and so

$$F_1(z) = F_2(z) \quad \text{(a.e.).}$$

By equation (3.5) F_1 and F_2 are real-analytic functions of z. Hence

$$F_1(z) = F_2(z).$$

We have now shown that given any sequence s_j decreasing to $\delta(G)$ there is a subsequence s_j' so that

$$\lim_j (s_j' - \delta(G))\, f(z, w; s_j') = F_1(z).$$

From this it follows that

$$\lim_{s \to \delta(G)} (s - \delta(G))\, f(z, w; s),$$

exists and is equal to $F_1(z)$. This is true for each fixed value of w. Thus there is a function $G(z, w)$ on $\Delta \times \Delta$ so that

$$\lim_{s \to \delta(G)} (s - \delta(G))\, f(z, w; s) = G(z, w). \tag{7.8}$$

Clearly G is an eigenfunction of the Laplace-Beltrami operator in both variables. Indeed, from (7.8) we have

$$G(z, w) = G(w, z). \tag{7.9}$$

Also G is automorphic under G in both variables. Let us state these results formally.

THEOREM 7.1. *There is a strictly positive function* $G(z, w)$ *on* $\Delta \times \Delta$ *so that*

$$\lim_{s \to \delta(G)} (s - \delta(G))\, f(z, w; s) = G(z, w).$$

COROLLARY. *If G is of the second kind then*

$$\delta(G) < 1.$$

Proof. If $\delta(G) = 1$ then it would follow that $f(z, w; s)$ diverged at 1. But it is a classical fact (see [9: p. 178]) that f converges at $s = 1$. Hence $\delta(G) < 1$.

This corollary is due to A. F. Beardon (see [4]). His method, which involves a close analysis of the action of G on Δ, is completely different from our own. The assertion of the theorem, that the series diverges at its exponent of convergence is a much finer result than any classical method is capable of producing.

We shall resume the investigation into the nature of F. On passing to the limit in (7.1) we now obtain

$$\int_D G(z, w) F(w) \, d\sigma(w) = C(\delta) \, \Gamma(2\delta - 1) \, F(z). \tag{7.10}$$

From Proposition 6.1 we see that there are constants c_3, c_4 so that

$$G(z, z) \leqslant c_3 (1 - |z|^2)^{2\delta(G)}, \quad (z \in D_H), \tag{7.11}$$

$$G(z, z) \leqslant c_4 (1 - |z|^2)^{2\delta(G) - 2}, \quad (z \in D_C), \tag{7.12}$$

From these it follows

$$\int_D G(z, z) \, d\sigma(z) < \infty. \tag{7.13}$$

Now let E_δ be the Hilbert subspace of $L^2(G \backslash \Delta)$ of eigenfunctions of the Laplace-Beltrami operators with eigenvalue $-\delta(G)(1 - \delta(G))$. As $G(z, w)$ as a function of z is in E_δ it follows that the map

$$f \mapsto \int_D G(\cdot, w) f(w) \, d\sigma(w); \ L^2(G \backslash \Delta) \to L^2(G \backslash \Delta), \tag{7.14}$$

maps $L^2(G \backslash \Delta)$ into E_δ. By (7.10) this map on E_δ is a constant multiple of the identity. By (7.13) it has finite trace. Thus E_δ is a finite dimensional space.

Let us now go back to the methods of section 3. Recall that there we introduced the functions F only after having found a certain measure. Now we may redefine this measure and investigate it somewhat more closely.

Let w, v be fixed points in Δ. Then define

$$\mu_{s, w, v} = f(w, v; s)^{-1} \sum_{g \in G} h(w, g(v))^{-s} \delta_{g(v)}. \tag{7.15}$$

We find that if s_j is a sequence decreasing to $\delta(G)$ so that $\mu_{s_j, 0, v}$ converges weakly to μ_v^* say then

$$\int P(z, \eta)^\delta d\mu_v^*(\eta) = \lim_j f(z, v; s_j)/f(0, v; s_j) = G(z, v)/G(0, v). \tag{7.16}$$

The function on the right hand side here does not depend on the sequence s_j. As we have already remarked during the proof of Proposition 4.1 μ_v^* is completely determined by the left hand side, and hence the right hand side of (7.16). Thus all such weak limits of $\mu_{s,0,v}$ are the same. On the other hand for any sequence s_j' decreasing to $\delta(G)$ there is, by Helly's theorem, a subsequence s_j along which $\mu_{s,0,v}$ converges weakly. Consequently $\mu_{s,0,v}$ converges weakly to μ_v^* as s decreases to $\delta(G)$.

LEMMA 7.1. *Let v_1, $v_2 \in \Delta$. Then $\mu_{v_1}^*$ is absolutely continuous with respect to $\mu_{v_2}^*$.*

Proof. Let φ be a continuous function on $\bar{\Delta}$. Then φ is uniformly continuous. Choose $\varepsilon > 0$. There is $r > 0$ so that, if $|z_1 - z_2| < r$ then $|\varphi(z_1) - \varphi(z_2)| < \varepsilon$.

Now we find a finite subset T of G so that, if $g \notin G \setminus T$,

$$|g(v_1) - g(v_2)| < r.$$

Recall the inequality

$$h(z_1, z_2) h(z_2, z_3) \geqslant h(z_1, z_3).$$

As

$$\int \varphi(z) \, d\mu_{s,0,v}(z) = f(0, v; s)^{-1} \sum_{g \in G} h(0, g(v))^{-s} \varphi(g(v)),$$

we see that

$$\left| \int \varphi(z) \, d\mu_{s,0,v_1}(z) \right| = \left| f(0, v_1; s)^{-1} \sum_{g \in G} h(0, g(v_1))^{-s} \varphi(g(v_1)) \right|$$

$$\leqslant f(0, v_1; s)^{-1} \sum_{g \in G} h(0, g(v_1))^{-s} |\varphi(g(v_1)) - \varphi(g(v_2))|$$

$$+ f(0, v_1; s)^{-1} h(v_1, v_2)^s \sum_{g \in G} h(0, g(v_2))^{-s} |\varphi(g(v_2))|.$$

Let us examine the first term as $s \to \delta(G)$. As $f(0, v_1; s) \to \infty$ as $s \to \delta(G)$ it follows that the first term is

$$f(0, v_1; s)^{-1} \sum_{g \in G \setminus T} h(0, g(v_1))^{-1} |\varphi(g(v_1)) - \varphi(g(v_2))| + o(1)$$

$$\leqslant \varepsilon f(0, v_1; s)^{-1} \sum_{g \in G \setminus T} h(0, g(v_1))^{-s} + o(1) = \varepsilon + o(1).$$

Let φ be a positive function. Then from the inequality above we obtain as $s \to \delta(G)$

$$\int \varphi(z) \, d\mu_{v_1}^*(z) \leqslant h(v_1, v_2)^\delta \left(G(0, v_2) / G(0, v_1) \right) \int \varphi(z) \, d\mu_{v_2}^*(z) + \varepsilon.$$

ε was an arbitrary positive number. So we obtain the above inequality with $\varepsilon = 0$. This proves the lemma.

From the lemma and the Radon-Nikodym theorem there is $q \in L^1(\mu_{v_1}^*) = L^1(\mu_{v_2}^*)$ so that $\mu_{v_1}^* = q \cdot \mu_{v_2}^*$. If $g \in G$ then, as we showed during the proof of Proposition 4.1

$$\mu_{v_j}^* \circ g = |g'|^{\delta} \mu_{v_j}^*.$$

Hence

$$q \circ g = q.$$

Suppose $A_1, ..., A_K$ are disjoint subsets of L_G so that

$$\mu_{v_1}^*(A_j) > 0 \quad (1 \leqslant j \leqslant K),$$

and so that the A_j are invariant under G. Let μ_j be the restriction of $\mu_{v_1}^*$ to A_j. Then, as A_j is invariant under G, for any $g \in G$ we have

$$\mu_j \circ g = |g'|^{\delta} \mu_j.$$

Thus

$$f_j(z) = \int P(z, \eta)^{\delta} d\mu_j(\eta), \qquad (7.17)$$

is a member of E_{δ}. On the other hand, the measures μ_j are linearly independent as the sets A_j are disjoint. So, as the integral on the right hand side of (7.17) determines μ_j completely it follows that the f_j are linearly independent.

We have shown, however, that E_{δ} is a finite dimensional space. Thus there is a maximal finite set $B_1, ..., B_L$ of disjoint G-invariant subsets of positive $\mu_{v_1}^*$-measure (at least up to sets of zero $\mu_{v_1}^*$-mass). Clearly every G-invariant subset of L_G is, up to a set of $\mu_{v_1}^*$ measure zero, a union of some of the B_j.

Let m_j $(1 \leqslant j \leqslant L)$ be the restriction of $\mu_{v_1}^*$ to B_j. As $B_1, ..., B_L$ is a maximal set we have, up to a set of $\mu_{v_1}^*$ measure zero,

$$L_G = \bigcup_{j=1}^{L} B_j. \qquad (7.18)$$

From this it follows at once that

$$\mu_{v_1}^* = \sum_{j=1}^{L} m_j. \qquad (7.19)$$

Recall that we had shown that

$$\mu_{v_1}^* = q \mu_{v_2}^*.$$

Likewise there is a function q_0 so that

$$\mu_{v_2}^* = q_0 \mu_{v_1}^*. \tag{7.20}$$

Clearly $\mu_{v_1}^*$-almost everywhere

$$q \cdot q_0 = 1.$$

On the other hand both q and q_0 are G-invariant. Let I_j be the characteristic function of B_j. Then there are constants $a(j)$, $a_0(j)$ $(1 \leqslant j \leqslant L)$ so that

$$q = \sum_{j=1}^{L} a(j) I_j,$$

and

$$q_0 = \sum_{j=1}^{L} a_0(j) I_j.$$

As $q \cdot q_0 = 1$ we have that

$$a(j) a_0(j) = 1.$$

As $\mu_{v_j}^*$ is a probability measure we have that the $a(j)$, $a_0(j)$ are finite. As both $\mu_{v_1}^*$ and $\mu_{v_2}^*$ are positive measures $a(j)$ and $a_0(j)$ are both positive. We conclude that

$$0 < a_0(j) < \infty. \tag{7.21}$$

From (7.18), (7.19) and (7.20) we conclude

$$\mu_{v_2}^* = \sum_{j=1}^{L} a_0(j) m_j. \tag{7.22}$$

We can summarise what we have just shown in the following way. For $1 \leqslant j \leqslant L$ define

$$F_j(z) = \int P(z, \eta)^\delta dm_j(\eta). \tag{7.23}$$

This is a strictly positive element of E_δ. Let $v \in \Delta$. We have then shown that there are L strictly positive functions $\varphi_j(v)$ so that, in view of (7.16),

$$G(z, v) = \sum_{j=1}^{L} F_j(z) \varphi_j(v). \tag{7.24}$$

We shall now deduce that $L = 1$. Suppose that $L > 1$. By (7.10), applied to $F_j(z)$ we see that

$$\int_D G(z, v) F_j(v) d\sigma(v) = C(\delta) \Gamma(2\delta - 1) F_j(z). \tag{7.25}$$

If we let

$$b_k = \int_D F_j(v)\,\varphi_k(v)\,d\sigma(v),\tag{7.26}$$

then from (7.24) and (7.25) we find

$$\sum_{k=1}^{L} b_k\,F_k(z) = C(\delta)\,\Gamma(2\delta-1)\,F_j(z).\tag{7.27}$$

As F_j and φ_k are both strictly positive, from (7.26),

$$b_k > 0.\tag{7.28}$$

(7.28) implies that (7.27) is a non-trivial linear relation between the F_j. This is impossible for suppose we had a relation of the form

$$\sum_{k=1}^{L} c_k \cdot F_\kappa(z) = 0.$$

Let

$$M = \sum_{k=1}^{L} c_k \cdot m_k,$$

which is a signed measure so that

$$\int P(z,\eta)^\delta dM(\eta) = 0.$$

During the proof of Proposition 4.1 we remarked that this implied that $M=0$. From the definition of the m_k this shows that, for all k,

$$c_k = 0.$$

Thus $L=1$. This completes our investigations; the conclusions are expressed in the next theorem.

THEOREM 7.2. *Let G be a finitely generated Fuchsian group of the second kind with $\delta(G) > 1/2$. Then there is one, and only one, probability measure μ supported on L_G so that, for every $g \in G$,*

$$\mu \circ g = |g'|^\delta \mu.$$

The space E_δ of square-integrable eigenfunctions of the Laplace-Beltrami operator with eigenvalue $-\delta(G)(1-\delta(G))$ is one-dimensional and is generated by

$$F(z) = \int P(z,\eta)^{\delta(G)} d\mu(\eta).$$

There is a constant γ so that if $G(z, w)$ is the function appearing in Theorem 7.1

$$G(z, w) = \gamma F(z) F(w).$$

If Q is a subset of L_G which is G-invariant up to a set of μ-measure zero then

$$\mu(Q) = 0 \text{ or } 1.$$

We shall conclude this paper by a few general remarks. If we carry out the construction of μ in the case of a finitely generated group of the first kind we see that it is the ordinary Lebesgue measure on S. The eigenfunction is a constant function. The fact that as $s \to 1$ $(s-1)f(z, w; s)$ converges to a constant corresponds to Tsuji's "fundamental theorem" ([11]). The classification of G-invariant sets corresponds to the "easy" ergodic theorem ([9: p. 321]). This paper gives a new proof of these classical theorems.

The results of this paper admit of extensions in two different ways. Firstly, it is very desirable that the relation between the measure that we have constructed and the distribution of orbits of points under the group be clarified and sharpened. It appears that even for groups of the second kind it would be profitable to study the spectral decomposition of the Laplace operator. I hope to deal with this topic later but the reader should consult, on this point, the very interesting papers of Elstrodt [7].

References

[1]. AKAZA, T., Local property of the singular sets of some Kleinian groups. *Tôhoku Math. J.*, 25 (1973), 1–22.

[2]. BEARDON, A. F., The Hausdorff dimension of singular sets of properly discontinuous groups. *Amer. J. Math.*, 88 (1966), 722–736.

[3]. —— The exponent of convergence of Poincaré series. *Proc. London Math. Soc.*, (3) 18 (1968), 461–483.

[4]. —— Inequalities for certain Fuchsian groups. *Acta Math.*, 127 (1971), 221–258.

[5]. —— Kleinian groups with parabolic elements. Preprint.

[6]. CARLESON, L., *Selected problems on exceptional sets*. Van Nostrand Mathematical Studies, 14 (1967).

[7]. ELSTRODT, J., Die Resolvente zum Eigenwertproblem der automorphen Formen in der hyperbolischen Ebene. Teil I. *Math. Ann.*, 203 (1973), 295–330. Teil II. *Math. Z.*, 132 (1973), 99–134. Teil III. *Math. Ann.*, 208 (1973), 99–132.

[8]. GREENBERG, L., Fundamental polygons for Fuchsian groups. *J. Analyse Math.*, 18 (1967) 99–105.

[9]. LEHNER, J., Discontinuous groups and automorphic functions. *Amer. Math. Soc.*, 1964.

[10]. SELBERG, A., Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series. *J. Ind. Math. Soc.*, 20 (1956), 47–87.

[11]. TSUJI, M., *Potential theory in modern function theory*. Maruzen, 1959.

[12]. WHITTAKER, E. T. & WATSON, G. N., *Modern Analysis*. Cambridge, 1946.