ON THE NON-LINEAR COHOMOLOGY OF LIE EQUATIONS. I

BY

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Introduction

The infinitesimal transformations of a Lie pseudogroup, acting on a manifold X, are solutions of a linear partial differential equation R_k which is a Lie equation in the tangent bundle T of X; the space $R_{\infty,x}$ of formal solutions of R_k at a point $x \in X$ is a topological Lie algebra and, if the pseudogroup is transitive, it is a transitive Lie algebra in the sense of Guillemin-Sternberg [13].

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Using the theory of Lie equations elaborated by Malgrange, Kumpera and the second-named author (see [22], [19] and [18]) and the results of Guillemin and Sternberg on transitive Lie algebras (see [13] and [12]), the first-named author initiated, in preceding papers [8], [9] and [10], a program (announced in [7]) of investigating the relationship between Lie equations and transitive Lie algebras in order to show in what way certain properties of a formally transitive and formally integrable analytic Lie equation R_k depend only on the transitive Lie algebra $R_{\infty,x}$ of formal solutions of R_k at $x \in X$ and to what extent the classical theory of finite-dimensional Lie groups and their Lie algebras can be generalized to Lie equations and transitive Lie algebras. In [10] it was shown, in particular, that the graded Lie algebra $H^*(R_k)_x = \oplus_{j\geqslant 0} H^j(R_k)_x$ of linear Spencer(1) cohomology at $x \in X$ of an analytic Lie equation depends, up to an isomorphism, only on the topological Lie algebra $R_{\infty,x}$. On identifying two graded Lie algebras of cohomology which are isomorphic, there is associated to every transitive Lie algebra L a graded Lie algebra $H^*(L) = \oplus_{j\geqslant 0} H^j(L)$ of linear Spencer cohomology with the following properties:

(i) *the graded Lie algebra $H^*(L)$ depends only on the isomorphism class of L as a topological Lie algebra;*

(ii) *a graded Lie algebra $H^*(L, I) = \oplus_{j\geqslant 0} H^j(L, I)$ of linear Spencer cohomology can be defined for a closed ideal I of L such that $H^*(L, L) = H^*(L)$ and it depends only on the isomorphism class of (L, I) as a pair of topological Lie algebras;*

(iii) *to each exact sequence*

$$0 \longrightarrow I \longrightarrow L \xrightarrow{\phi} L'' \longrightarrow 0$$

where I is an ideal of L and $\phi: L \to L''$ is a continuous homomorphism of transitive Lie algebras, there corresponds an exact sequence of linear cohomology

$$\dots \longrightarrow H^j(L, I) \longrightarrow H^j(L) \xrightarrow{H^j(\phi)} H^j(L'') \xrightarrow{\partial^{\#}} H^{j+1}(L, I) \longrightarrow \dots .$$

One of the purposes of the present paper is to extend these results to the non-linear Spencer cohomology $\tilde{H}^1(R_k)$ of a formally integrable Lie equation R_k. In general, the notion of a structure associated to a Lie equation can be defined as well as the notions of equivalence and integrability of such structures. Then $\tilde{H}^1(R_k)_x$ is the set of equivalence classes of germs at $x \in X$ of formally integrable R_k-structures; it is a set with distinguished element 0 and we say that it vanishes if it is equal to 0. We write $\tilde{H}^1(R_k) = 0$ if $\tilde{H}^1(R_k)_x = 0$ for all $x \in X$. The vanishing of $\tilde{H}^1(R_k)$ expresses that the integrability problem for R_k-

(1) Despite the misgivings of the second author, we employ a terminology adopted in preceding papers.

structure is solvable, namely that an R_k-structure which satisfies the requisite compatibility conditions is in fact an integrable R_k-structure. We now list most of the known results about the integrability problem and the Spencer cohomology of Lie equations.

(I) If R_k is of finite type, that is if there is an integer $l_0 \geqslant 0$ such that R_{k+l} is isomorphic to R_{k+l_0} for all $l \geqslant l_0$, where R_{k+l} is the l-th prolongation of the equation R_k, then $H^j(R_k)=0$ for $j>0$ and the integrability problem for R_k is solved. This is a consequence of Frobenius' theorem.

(II) If R_k is analytic with respect to a real-analytic structure on X, then, in the category of analytic manifolds and mappings, we have $H^j(R_k)=0$ for $j>0$ and $\tilde{H}^1(R_k)=0$. This result is a consequence of the Cartan-Kähler theorem.

(III) If R_k is elliptic and is either analytic with respect to a real-analytic structure on X or formally transitive, then $H^j(R_k)=0$ for $j>0$ and $\tilde{H}^1(R_k)=0$. The vanishing of $\tilde{H}^1(R_k)$ for equations R_k which are elliptic and analytic was proved by Malgrange [19], generalizing an earlier theorem of Newlander-Nirenberg which asserts the solvability of the integrability problem for complex-analytic structure. In [9] it is shown that Malgrange's result implies that a formally transitive, elliptic Lie equation is analytic with respect to a real-analytic structure on X.

(IV) The integrability problem for flat Lie pseudogroups has been studied and, in a context different from the present one, partial results have recently been obtained by Buttin-Molino [2] and Pollack [20]. For example, let $\mathfrak{g} \subset \mathfrak{gl}(n, \mathbf{R})$ be a Lie subalgebra. If $(x^1, \dots, x^n)$ are the standard coordinates on $\mathbf{R}^n$ and $\xi = \sum_{j=1}^n \xi^j \partial/\partial x^j$ is a vector field on an open subset U of $\mathbf{R}^n$, the differential equation $(\partial\xi^j(x)/\partial x^k) \in \mathfrak{g}$ for all $x \in U$ is a flat Lie equation $R_1(\mathfrak{g})$ of order 1.

(V) Guillemin and Sternberg [15] have given an example, based on H. Lewy's counterexample to the local solvability of partial differential equations, which shows that the integrability problem is not always solvable.

We say that two non-linear cohomologies are isomorphic if they are connected by a bijective mapping sending 0 into 0, and we shall identify two cohomologies if there is an isomorphism of cohomology between them. In the case of a formally transitive and formally integrable analytic Lie equation R_k on a connected manifold X, the cohomology $\tilde{H}^1(R_k)_x$ is then independent of the point $x \in X$ and we show that its vanishing depends only on the transitive Lie algebra $R_{\infty,x}$. We associate to every transitive Lie algebra L a non-linear cohomology $\tilde{H}^1(L)$ with the following properties:

(i) *The cohomology $\tilde{H}^1(L)$ depends only on the isomorphism class of L as a topological Lie algebra.*

(ii) *A non-linear cohomology $\tilde{H}^1(L, I)$ can be defined for a closed ideal I of L such that $\tilde{H}^1(L, L) = \tilde{H}^1(L)$ and it depends only on the isomorphism class of (L, I) as a pair of topological Lie algebras.*

(iii) *Let $\phi: L \to L''$ be an epimorphism of transitive Lie algebras and $I \subset L$, $I'' \subset L''$ be closed ideals of L and L'' such that $\phi(I) = I''$; let I' be the closed ideal of L which is the kernel of ϕ: $I \to I''$. If $\tilde{H}^1(L, I') = 0$ and $\tilde{H}^1(L'', I'') = 0$, then $\tilde{H}^1(L, I) = 0$; if $\phi: I \to I''$ is an isomorphism, we have an isomorphism of cohomology*

$$\tilde{H}^1(L, I) \to \tilde{H}^1(L'', I'').$$

In particular, if J is the kernel of ϕ and $\tilde{H}^1(L, J) = 0$, $\tilde{H}^1(L'') = 0$, then $\tilde{H}^1(L) = 0$.

(iv) *Let $R_q^{\#}$ be a formally transitive and formally integrable Lie equation on a manifold Y and let $y \in Y$. If the transitive Lie algebras L and $R_{\infty, y}^{\#}$ are isomorphic, then we have a bijective mapping*

$$\tilde{H}^1(L) \to \tilde{H}^1(R_q^{\#})_y.$$

This last property together with the third fundamental theorem (Theorem 7.1) reduces the computation of the non-linear Spencer cohomology of formally transitive Lie equations to the case of analytic equations.

The systematic study of transitive Lie algebras, a program which was initiated by Guillemin and Sternberg in their paper [13], resulted in the fundamental paper [12] of Guillemin in which a Jordan-Hölder decomposition is constructed for a closed ideal of a transitive Lie algebra. This decomposition is an outgrowth of a program outlined by Guillemin in the introduction of [12] which is motivated by the integrability problem. Our results (see § 10) reduce the integrability problem to the vanishing of the non-linear cohomology of the quotients of successive ideals in Jordan-Hölder decompositions. In particular, consider the following three conjectures:

I. *Let L be a transitive Lie algebra and I a non-abelian minimal closed ideal of L. Then $H^j(L, I) = 0$ for $j > 0$ and $\tilde{H}^1(L, I) = 0$.*

II. *Let L be a transitive Lie algebra and I a closed ideal of L. Let*

$$I = I_0 \supset I_1 \supset \ldots \supset I_k = 0$$

be a Jordan-Hölder sequence for (L, I), that is, a nested sequence of closed ideals of L such that, for each j, where $0 \leqslant j \leqslant k-1$, either I_j/I_{j+1} is abelian or there are no closed ideals of L properly contained between I_j and I_{j+1}. If for each j for which I_j/I_{j+1} is abelian, where $0 \leqslant j \leqslant k-1$, we have $H^1(L/I_{j+1}, I_j/I_{j+1}) = 0$, then $H^1(L, I) = 0$ and $\tilde{H}^1(L, I) = 0$.

III. *Let L be a transitive Lie algebra and I a closed ideal of L. If there exist a fundamental subalgebra L^0 of L, closed subalgebras A, B of L such that A is abelian and*

$$L = L^0 + A + B,$$

$$[A, B] = 0, \ [B, I] = 0,$$

then $H^j(L, I) = 0$ *for* $j > 0$ *and* $\tilde{H}^1(L, I) = 0$.

We prove (Theorems 13.1 and 13.2) that I implies II and III and we outline a proof of I which is based on Guillemin's structure theorem for a non-abelian minimal closed ideal of a transitive Lie algebra (Theorem 2 of [12]), on the classification of infinite-dimensional simple real transitive Lie algebras, the Newlander-Nirenberg theorem, and on theorems of [10] and § 10 of this paper. Conjecture II implies that *the solvability of the integrability problem for formally transitive and formally integrable Lie equations is reduced to the local solvability of overdetermined systems of linear partial differential equations*. We have the following consequence of III (see § 13):

Assume that X is connected. Let $R_k \subset J_k(T)$ be a formally transitive and formally integrable Lie equation and $N_k \subset R_k$ a formally integrable Lie equation such that $N_{\infty,a}$ is a closed ideal of $R_{\infty,a}$ for all $a \in X$. Let $x \in X$; if there is a fundamental subalgebra L^0 of $R_{\infty,x}$ and an abelian subalgebra A of $R_{\infty,x}$ such that

$$R_{\infty,x} = L^0 \oplus A,$$

then

$$H^j(N_k)_a = 0, \ H^j(R_k)_a = 0, \ \tilde{H}^1(N_k)_a = 0, \ \tilde{H}^1(R_k)_a = 0$$

for $j > 0$ and all $a \in X$.

In particular, III implies that the integrability problem is solved for all Lie pseudogroups acting on $\mathbf{R}^n$ which contain the translations, a fortiori for all flat pseudogroups.

We now give a brief summary of the contents of this paper. In § 1 we recall certain facts from the formal theory of linear partial differential equations, the constructions of the "naive" linear Spencer operator D, of various brackets and Lie algebras arising from the study of jets of vector fields; we also give the fundamental formulas relating the operator D to these objects. The corresponding non-linear theory is described in § 2, namely the operations of jet bundles of diffeomorphisms on jets of vector fields, the non-linear Spencer complexes, the fundamental formulas involving the "naive" Spencer operators $\mathcal{D}$ and $\bar{\mathcal{D}}$ and the facts from the formal theory of non-linear differential equations which are used in Chapter II. Although much of § 1 and § 2 is a reorganization of known material, mainly from [19] and [18], with the purpose of fixing notation and terminology which we use throughout the paper, new results required in the sequel are also proved. In particular, in § 2 we examine the relationship between the structure of affine bundle and the structure of groupoid which certain jet bundles of diffeomorphisms possess, using the methods developed in [4] and expressing the relationship in terms of the operations of these bundles on jets of vector fields. We usually do not prove facts whose proofs are readily found in [19] or [18]. In § 3 we begin by recalling results of [6] concerning fibrations and the naive operator D which we complement by Lemma 3.1. The remainder of the section is devoted to the construction and properties of a generalization of the naive operator D (see Proposition 3.1) which is required in § 5 in order to define the structure equation of an extension of the classical Cartan fundamental form. In the next section, § 4, a non-linear complex for a bundle of Lie groups is defined in terms of the Maurer-Cartan form and the exactness of the complex, a consequence of Frobenius' theorem, is used at a crucial point in the proof of the basic Theorem 9.1. In the following section, § 5, the extended Cartan fundamental form mentioned above is defined on the bundle of $(k+1)$-jets of diffeomorphisms $X \to X$ and takes its values in the bundle of k-jets of vector fields; it is related to the form on this jet bundle described in [11] and its restriction to the bundle of $(k+1)$-jets with fixed source (bundle of frames of order $k+1$) is the classical fundamental form of Cartan. The structure equation for the classical fundamental form follows directly from the Cartan structure equation for the extended form. The naive non-linear operator $\mathcal{D}$ has a natural definition in terms of the extended Cartan form. Finally, the connection between the theory of Lie equations of Spencer and Malgrange and the work of Guillemin and Sternberg [14] is clarified. In § 6, the last section of Chapter I, using the extended Cartan form, we show how a surjective submersion ϱ of X onto another differentiable manifold Y induces a projection of the non-linear $\mathcal{D}$-complex, restricted to sheaves of jets of ϱ-projectable sections, onto another complex which is a non-linear analogue of the complex occurring in [6], whose linear operators are the exterior differential along the fibers of ϱ followed by a projection. The latter non-linear complex is related to the complex of § 4. The essential purpose of this section is to construct a finite form of the linear theory developed in [6]; its results are crucial in proving the main theorems of § 9.

In Chapter I we have considered arbitrary vector fields and diffeomorphisms; in Chapter II we consider vector fields and diffeomorphisms which satisfy respectively linear and (in general) non-linear partial differential equations, namely so-called Lie equations. In § 7 we begin by defining a linear Lie equation R_k (of order k) for vector fields (infinitesimal form) and a corresponding non-linear Lie equation P_k (finite form). Next, under the assumption that the prolonged equations R_{k+l} are vector bundles for $l \geq 0$, the two non-linear Spencer cohomologies of P_k or R_k are defined in terms of the naive complexes corresponding to the operators $\mathcal{D}$ and $\bar{\mathcal{D}}$ and are shown to be isomorphic; hence they are identified and denoted by $\tilde{H}^1(R_k)$, where $\tilde{H}^1(R_k) = \bigcup_{x \in X} \tilde{H}^1(R_k)_x$. If R_k is formally integrable, $\tilde{H}^1(R_k)$ is also isomorphic to the cohomology defined in terms of the sophisticated

Spencer complex corresponding to the operator $\hat{\mathcal{D}}$. If two formally transitive and formally integrable Lie equations are transformed one into the other by a section of a jet bundle, it is shown that the corresponding cohomologies are connected by a bijective mapping (Proposition 7.9); from the third fundamental theorem (see [9] and Theorem 7.1), we deduce that the computation of the cohomology $\tilde{H}^1(R_k)$ of a formally transitive and formally integrable Lie equation R_k is reducible to the case where R_k is analytic. The next section, § 8, contains a proof, based on Frobenius' theorem, that the non-linear cohomology of a certain multifoliate Lie equation(¹) vanishes; this fact is an essential step in the proof of Theorem 9.1. The results of § 6–§ 8 are used in § 9 to prove non-linear analogues (finite forms) of certain results of the linear theory of [6]. Theorem 9.1 establishes the key fact that, if R_k is a formally integrable ϱ-projectable Lie equation on X satisfying the conditions (I) and (II) of § 9, then its non-linear cohomology is isomorphic to the non-linear cohomology defined in terms of ϱ-projectable sections. Under the same hypotheses an exact sequence of non-linear cohomology is constructed (Proposition 9.1) relating the cohomology of R_k to the cohomology of a Lie equation R''_{k_1} on Y and to the cohomology of a kernel Lie equation $\bar{R}_k$ on X. This sequence has the disadvantage that the equation $\bar{R}_k$ is in general not formally integrable. Under additional assumptions one can modify this exact sequence and replace the cohomology of $\bar{R}_k$ by the cohomology of the formally integrable Lie equation R'_{m_0} obtained from $\bar{R}_k$ by the technique of [5] or [6] (see Theorem 9.2). Finally Theorem 9.3 gives more precise results when R'_{m_0} vanishes; in particular, the cohomology $\tilde{H}^1(R_k)_a$ of R_k at $a \in X$ is isomorphic to the cohomology $\tilde{H}^1(R''_{k_1})_{\varrho(a)}$ of R''_{k_1} at $\varrho(a)$. In § 10 the results of § 9, combined with results and techniques of [10] (and [9]), enable us to associate to every transitive Lie algebra L a non-linear cohomology $\tilde{H}^1(L)$ with the properties briefly described above. In § 11 we examine the structure of abelian Lie equations and prove Conjecture III in the case where I is abelian (Theorem 11.5); the proof is based on the theorem of Ehrenpreis-Malgrange, which asserts the local solvability of differential operators with constant coefficients (see Theorem 11.2). The stability under classical prolongation of the hypotheses of Conjecture III is established in § 12, and we remark that under prolongation the subalgebra B, even if it is assumed initially to be zero, reappears and contains a subalgebra corresponding to transformations along the fibers of a principal bundle and the transitive Lie algebra L corresponds to a closed ideal of a transitive Lie algebra. Thus in studying the cohomology of transitive Lie algebras, one is necessarily led

(¹) The multifoliate Lie equation considered here is of a slightly different nature from that of the ones defined in [17], which correspond to flat pseudogroups and are of the type $R_1(\mathfrak{g})$ for appropriate Lie algebras $\mathfrak{g}$.

into examining the cohomology of closed ideals of transitive Lie algebras. The results of the final section of the paper, § 13, have been described above.

We conclude this introduction with some short remarks on notation, terminology and background. For the definitions and properties of fibered manifolds and jet bundles as affine bundles, we refer the reader to [4]. Notation and terminology are the same as in the papers [8], [9], [10], and essentially the same as in [19]. However, it is perhaps worthwhile to explain one piece of notation which might be confusing. Namely, if E, F, G are finite-dimensional vector spaces, we always identify $E^*\otimes F$ with Hom (E, F) and, if $u\in E^*\otimes F$, $v\in F^*\otimes G$, we denote by $v\circ u$ the element of $E^*\otimes G$ defined by composition.

Chapter I. Differential equations, fibrations and Cartan forms

1. Linear differential equations and vector fields

Let X be a differentiable manifold of dimension n and class C^∞ whose tangent bundle we denote by $T=T_X$. We write O_X for the sheaf of real-valued, differentiable functions on X. If E is a fibered manifold over X, we denote by $\mathcal{E}$ the sheaf of sections of E, and by E_x (resp. $\mathcal{E}_x$) the fiber of E (resp. the stalk of $\mathcal{E}$) at $x\in X$; sometimes, however, we write $E(x)$ for the fiber E_x of E at $x\in X$. The bundle of vertical tangent vectors of E will be denoted by $V(E)=T(E/X)$. We denote by $J_k(E)$ the fibered manifold of k-jets of sections of E, by $j_k\colon \mathcal{E}\to J_k(\mathcal{E})$ the differential operator of order k which sends a section s of E over a neighborhood of $x\in X$ into the k-jet $j_k(s)$ of this section, and by $\pi_k\colon J_{k+l}(E)\to J_k(E)$ and $\pi\colon J_k(E)\to X$ the natural projections sending $j_{k+l}(s)(x)$ into $j_k(s)(x)$ and $j_k(s)(x)$ into its source x respectively. The natural injection

$$\lambda_l\colon J_{k+l}(E)\to J_l(J_k(E)),$$

which sends $j_{k+l}(s)(x)$ into $j_l(j_k(s))(x)$, where s is a section of E over a neighborhood of $x\in X$, is a monomorphism of fibered manifolds. If F is another fibered manifold over X and $\varphi\colon E\to F$ is a morphism of fibered manifolds over X, then

$$J_k(\varphi)\colon J_k(E)\to J_k(F)$$

is the morphism of fibered manifolds over X sending $j_k(s)(x)$ into $j_k(\varphi\circ s)(x)$ (see [4]). We shall always suppose that the fibers of a vector bundle are of the same dimension.

If E is a vector bundle over X, we have the exact sequence of vector bundles

$$0\longrightarrow S^kT^*\otimes E\xrightarrow{\varepsilon} J_k(E)\xrightarrow{\pi_{k-1}} J_{k-1}(E)\longrightarrow 0 \tag{1.1}$$

which yields the exact sequence

$$0 \longrightarrow T^* \otimes J_{k-1}(E) \xrightarrow{\varepsilon} J_1(J_{k-1}(E)) \xrightarrow{\pi_0} J_{k-1}(E) \longrightarrow 0.$$

We define a first-order differential operator

$$D\colon J_k(\mathcal{E}) \to \mathcal{T}^* \otimes J_{k-1}(\mathcal{E})$$

by the formula

$$\varepsilon Du = j_1(\pi_{k-1}u) - \lambda_1 u, \quad u \in J_k(\mathcal{E}), \tag{1.2}$$

and obtain the Spencer complex, which is an exact sequence,

$$\begin{aligned} 0 \longrightarrow \mathcal{E} \xrightarrow{j_k} J_k(\mathcal{E}) \xrightarrow{D} \mathcal{T}^* \otimes J_{k-1}(\mathcal{E}) \xrightarrow{D} \wedge^2 \mathcal{T}^* \otimes J_{k-2}(\mathcal{E}) \xrightarrow{D} \\ \cdots \longrightarrow \wedge^n \mathcal{T}^* \otimes J_{k-n}(\mathcal{E}) \longrightarrow 0, \end{aligned} \tag{1.3}$$

where $J_k(E) = 0$ for $k < 0$, by setting

$$D(\omega \wedge u) = d\omega \wedge \pi_{k-1}u + (-1)^j \omega \wedge Du \tag{1.4}$$

for $\omega \in \wedge^j \mathcal{T}^*$, $u \in \wedge \mathcal{T}^* \otimes J_k(\mathcal{E})$. Then

$$\langle \xi \wedge \eta, Du \rangle = \xi \mathbin{\overline{\wedge}} Du(\eta) - \eta \mathbin{\overline{\wedge}} Du(\xi) - \pi_{k-1}u([\xi, \eta]) \tag{1.5}$$

for $u \in \mathcal{T}^* \otimes J_k(\mathcal{E})$ and all $\xi, \eta \in \mathcal{T}$.

Lemma 1.1 (see [5], Proposition 6). *If F is a vector bundle over X and $\varphi\colon E \to F$ is a morphism of vector bundles, the diagram*

$$\begin{array}{ccc} \wedge^j \mathcal{T}^* \otimes J_k(\mathcal{E}) & \xrightarrow{D} & \wedge^{j+1} \mathcal{T}^* \otimes J_{k-1}(\mathcal{E}) \\ \downarrow{\scriptstyle \mathrm{id} \otimes J_k(\varphi)} & & \downarrow{\scriptstyle \mathrm{id} \otimes J_{k-1}(\varphi)} \\ \wedge^j \mathcal{T}^* \otimes J_k(\mathcal{F}) & \xrightarrow{D} & \wedge^{j+1} \mathcal{T}^* \otimes J_{k-1}(\mathcal{F}) \end{array}$$

is commutative.

Proof. In virtue of (1.4), it suffices to show that the diagram is commutative for $j = 0$. From the diagram

$$\begin{array}{ccccc} J_k(\mathcal{E}) & \xrightarrow{D} & \mathcal{T}^* \otimes J_{k-1}(\mathcal{E}) & \xrightarrow{\varepsilon} & J_1(J_{k-1}(\mathcal{E})) \\ \downarrow{\scriptstyle J_k(\varphi)} & & \downarrow{\scriptstyle \mathrm{id} \otimes J_{k-1}(\varphi)} & & \downarrow{\scriptstyle J_1(J_{k-1}(\varphi))} \\ J_k(\mathcal{F}) & \xrightarrow{D} & \mathcal{T}^* \otimes J_{k-1}(\mathcal{F}) & \xrightarrow{\varepsilon} & J_1(J_{k-1}(\mathcal{F})), \end{array}$$

whose right-hand square is commutative (see [3]) and whose mappings ε are monomorphisms of vector bundles, we see that it is sufficient to show that its outer rectangle commutes. By (1.2), we are now reduced to verifying that the diagrams

$$\begin{array}{ccccc} J_k(\mathcal{E}) & \xrightarrow{\pi_{k-1}} & J_{k-1}(\mathcal{E}) & \xrightarrow{j_1} & J_1(J_{k-1}(\mathcal{E})) \\ \downarrow{\scriptstyle J_k(\varphi)} & & \downarrow{\scriptstyle J_{k-1}(\varphi)} & & \downarrow{\scriptstyle J_1(J_{k-1}(\varphi))} \\ J_k(\mathcal{F}) & \xrightarrow{\pi_{k-1}} & J_{k-1}(\mathcal{F}) & \xrightarrow{j_1} & J_1(J_{k-1}(\mathcal{F})) \end{array}$$

and

$$\begin{array}{ccc} J_k(E) & \xrightarrow{\lambda_1} & J_1(J_{k-1}(E)) \\ \downarrow{\scriptstyle J_k(\varphi)} & & \downarrow{\scriptstyle J_1(J_{k-1}(\varphi))} \\ J_k(F) & \xrightarrow{\lambda_1} & J_1(J_{k-1}(F)) \end{array}$$

are commutative; however this last fact follows immediately from the definitions of the maps involved.

By (1.4), the restriction of $-D$ to $\wedge^j \mathcal{T}^* \otimes \varepsilon(S^k \mathcal{T}^* \otimes \mathcal{E})$ is O_X-linear and therefore comes from a morphism

$$\delta\colon \wedge^j T^* \otimes S^k T^* \otimes E \to \wedge^{j+1} T^* \otimes S^{k-1} T^* \otimes E$$

of vector bundles, and we obtain an exact sequence of vector bundles for $k > 0$

$$0 \longrightarrow S^k T^* \otimes E \xrightarrow{\delta} T^* \otimes S^{k-1} T^* \otimes E \xrightarrow{\delta} \wedge^2 T^* \otimes S^{k-2} T^* \otimes E \xrightarrow{\delta}$$
$$\cdots \longrightarrow \wedge^n T^* \otimes S^{k-n} T^* \otimes E \longrightarrow 0, \tag{1.6}$$

where

$$\delta(\omega \wedge u) = (-1)^j \omega \wedge \delta u$$

for $\omega \in \wedge^j T^*$, $u \in \wedge T^* \otimes S^m T^* \otimes E$ (see [3], [21]).

A vector sub-bundle $R_k \subset J_k(E)$ is a *linear differential equation* of order k on E. A *solution* of R_k over an open set $U \subset X$ is a section s of E over U such that $j_k(s)$ is a section of R_k, and we denote by $\mathrm{Sol}\,(R_k)$ the sheaf of solutions of R_k, namely the sub-sheaf of $\mathcal{E}$ of elements s satisfying $j_k(s) \in \mathcal{R}_k$. For $l \geq 0$, we associate to R_k its l-th prolongation $(R_k)_{+l} \subset J_{k+l}(E)$ with possibly varying fiber, namely

$$(R_k)_{+l} = J_{k+l}(E) \cap J_l(R_k),$$

which we often denote by R_{k+l} when no confusion arises. Here we have identified $J_{k+l}(E)$ with a sub-bundle of $J_l(J_k(E))$ by means of λ_l. We set

$$R_\infty = \varprojlim R_{k+l}.$$

Recall that, if $(R_k)_{+l}$ is a vector bundle, then the m-th prolongation of $(R_k)_{+l}$ is equal to $(R_k)_{+(l+m)}$.

The following lemma is part of Proposition 5.1 of [3] and its proof will be omitted.

LEMMA 1.2. *Let $R_k \subset J_k(E)$ be a differential equation. For $l \geq 1$, let $R_l' \subset J_l(R_k)$ be the image of R_{k+l} under the map λ_l: $J_{k+l}(E) \to J_l(J_k(E))$. If R_{k+1} is a vector bundle, then*

$$(R_1')_{+l} = R_{l+1}'$$

for all $l \geq 0$.

Let $R_k \subset J_k(E)$ be a differential equation. If, for each $l \geq 0$, R_{k+l} is a vector bundle and the projection π_{k+l}: $R_{k+l+1} \to R_{k+l}$ is surjective, we say that R_k is *formally integrable*. We say that R_k is *integrable* if, for all $l \geq 0$ and $u \in R_{k+l,x}$ with $x \in X$, there exists a section s of E over a neighborhood of x which is a solution of R_k such that $j_{k+l}(s)(x) = u$. If X is endowed with the structure of an analytic manifold and E is an analytic vector bundle and if R_k is an analytic, formally integrable differential equation on E then, according to Theorem 7.1 of [3] or the appendix of [19], R_k is integrable. Let $\mathcal{R}_{k+l} = (\mathcal{R}_k)_{+l}$ be the sheaf of sections of R_{k+l} (which determines R_{k+l} if the latter is a bundle). An element u of $J_{k+l+1}(\mathcal{T})$ belongs to $\mathcal{R}_{k+l+1}$ if and only if $\pi_{k+l} u \in \mathcal{R}_{k+l}$ and $Du \in \mathcal{T}^* \otimes \mathcal{R}_{k+l}$. By restriction of (1.3), we obtain the Spencer complex

$$0 \longrightarrow \mathcal{R}_m \xrightarrow{D} \mathcal{T}^* \otimes \mathcal{R}_{m-1} \xrightarrow{D} \wedge^2 \mathcal{T}^* \otimes \mathcal{R}_{m-2} \xrightarrow{D} \cdots \longrightarrow \wedge^n \mathcal{T}^* \otimes \mathcal{R}_{m-n} \longrightarrow 0, \quad (1.7)$$

where $R_m = J_m(E)$ if $m < k$. The cohomology of (1.7) at $\wedge^j \mathcal{T}^* \otimes \mathcal{R}_{m-j}$ will be denoted by $H^j(R_k)_{m-j}$. Moreover, let $g_m \subset S^m T^* \otimes E$ be the sub-bundle with possibly varying fiber such that the sequence

$$0 \longrightarrow g_m \xrightarrow{\varepsilon} R_m \xrightarrow{\pi_{m-1}} R_{m-1}$$

is exact; then (1.6) gives by restriction a complex

$$0 \longrightarrow g_m \xrightarrow{\delta} T^* \otimes g_{m-1} \xrightarrow{\delta} \wedge^2 T^* \otimes g_{m-2} \xrightarrow{\delta} \cdots \longrightarrow \wedge^n T^* \otimes g_{m-n} \longrightarrow 0, \quad (1.8)$$

whose cohomology at $\wedge^j T^* \otimes g_{m-j}$ we denote by $H^{m-j,j}(g_k)$. We say that g_k is r-acyclic if $H^{k+l,j}(g_k) = 0$ for $l \geq 0$ and $0 \leq j \leq r$, and we remark that g_k is always 1-acyclic if $k \geq 1$. We say that g_k is *involutive* if g_k is n-acyclic. There exists an integer $k_0 \geq k$, which depends only on n, k and rank E such that g_{k_0} is involutive.

If the l-th prolongation R_{k+l} of R_k is a vector bundle for $l \geq 0$ and if the mappings π_m: $R_{m+1} \to R_m$ are of constant rank for $m \geq k$, there exists an integer $m_1 \geq k$ such that

π_m: $H^j(R_k)_{m+1} \to H^j(R_k)_m$ is an isomorphism for $m \geqslant m_1$. Then $H^j(R_k)_m$ is independent of m for $m \geqslant m_1$ and we denote $H^j(R_k)_m$ with $m \geqslant m_1$ by $H^j(R_k)$, the j-th Spencer cohomology group of R_k; the group $H^0(R_k)$ is the sheaf of solutions of R_k. Here, as in the sequel, we always identify two cohomology groups if they are isomorphic (see [3], [5]).

We next turn to the consideration of vector fields and their brackets (see [18], [19]). Let Δ be the diagonal of $X \times X$ and let pr_1, pr_2 denote the projections of $X \times X$ onto the first and second factor respectively. A sheaf on X (resp. on Δ) will always be identified with its inverse image by pr_1: $\Delta \to X$ (resp. with its direct image by $\Delta \to X \times X$). Consider now the tangent bundle T of X, and identify $J_k(\mathcal{T})$ with the sheaf of vector fields on $X \times X$ which are pr_1-vertical, modulo those which vanish to order k on Δ. We call *diagonal* the vector fields on $X \times X$ which are pr_1-projectable and tangent to Δ, and we denote by $\tilde{J}_k(\mathcal{T})$ the sheaf of diagonal vector fields modulo those which vanish to order k on Δ. The vector bundle over X corresponding to $\tilde{J}_k(\mathcal{T})$ will be denoted by $\tilde{J}_k(T)$. The mapping which sends a diagonal vector field on $X \times X$ into its pr_1-vertical component yields, by passage to the quotient, a vector bundle isomorphism

$$\nu\colon \tilde{J}_k(T) \to J_k(T).$$

In the sequel it will be convenient to identify $\tilde{J}_0(T)$ with T. The sheaf $\check{J}_k(\mathcal{T})$ of vector fields on $X \times X$ which are pr_1-projectable modulo those vanishing to order k on Δ corresponds to a vector bundle $\check{J}_k(T)$ over X which is the sum of $J_k(T)$ and $\tilde{J}_k(T)$, where

$$J_k^0(T) = \{\xi \in J_k(T) \mid \pi_0 \xi = 0\} = J_k(T) \cap \tilde{J}_k(T).$$

We denote by π_k: $\check{J}_{k+l}(T) \to \check{J}_k(T)$ the natural projection. The projection pr_1 gives the exact sequence

$$0 \to J_k(T) \to \check{J}_k(T) \to T \to 0 \tag{1.9}$$

which enables us to identify T^* with a sub-bundle of $\check{J}_k(T)^*$. The injection $J_k(T) \to \check{J}_k(T)$ gives, by passage to the quotient, an isomorphism

$$J_k(T)/J_k^0(T) \simeq \check{J}_k(T)/\tilde{J}_k(T).$$

Since the kernel of the projection π_0: $J_k(T) \to J_0(T)$ is $J_k^0(T)$, we obtain an exact sequence

$$0 \to \tilde{J}_k(T) \to \check{J}_k(T) \to J_0(T) \to 0, \tag{1.10}$$

which gives, by duality, an injection $J_0(T)^* \to \check{J}_k(T)^*$; we shall identify $J_0(T)^*$ with its image under this mapping.

The bracket of vector fields on $X \times X$ gives, by restriction and passage to the quotient, a bracket

$$J_k(T) \times_X J_k(T) \to J_{k-1}(T) \tag{1.11}$$

which is defined fiber by fiber in the following way: if ξ, η are sections of T over a neighborhood of $x \in X$, then $[j_k(\xi)(x), j_k(\eta)(x)] = j_{k-1}([\xi, \eta])(x)$. It also gives the brackets

$$\tilde{J}_k(\mathcal{T}) \times_X \tilde{J}_k(\mathcal{T}) \to \tilde{J}_k(\mathcal{T}); \tag{1.12}$$

$$\check{J}_k(\mathcal{T}) \times_X \check{J}_k(\mathcal{T}) \to \check{J}_{k-1}(\mathcal{T}); \tag{1.13}$$

$$\tilde{J}_{k+1}(\mathcal{T}) \times_X J_k(\mathcal{T}) \to J_k(\mathcal{T}). \tag{1.14}$$

We note in particular that $\tilde{J}_k(\mathcal{T})$ is a sheaf of Lie algebras. If $\xi, \eta \in J_{k+1}(\mathcal{T})$ and $\tilde{\xi} = \nu^{-1}\xi$, $\tilde{\eta} = \nu^{-1}\eta$, then

$$\mathcal{L}(\tilde{\xi})\pi_k \eta = [\xi, \eta] + \tilde{\xi} \,\overline{\wedge}\, D\eta, \tag{1.15}$$

and

$$\mathcal{L}(\tilde{\xi})\pi_k \eta = \nu[\pi_k \xi, \pi_k \tilde{\eta}] + \tilde{\eta} \,\overline{\wedge}\, D\xi. \tag{1.16}$$

Write

$$\check{J}_\infty(T) = \varprojlim \check{J}_k(T), \qquad \check{J}_\infty(T)^* = \varinjlim \check{J}_k(T)^*,$$

and define similarly $J_\infty(T)$, $\tilde{J}_\infty(T)$, $J_\infty(T)^*$, $\tilde{J}_\infty(T)^*$. Then $\check{J}_\infty(\mathcal{T})$ is a sheaf of Lie algebras and $J_\infty(\mathcal{T})$, $\tilde{J}_\infty(\mathcal{T})$ are sub-sheaves of Lie algebras.

Following Malgrange [19], we next define a bracket on $\wedge \check{J}_\infty(\mathcal{T})^* \otimes \check{J}_\infty(\mathcal{T})$. First, from the bracket on $\check{J}_\infty(\mathcal{T})$ we obtain, by duality, an exterior differential d on $\wedge \check{J}_\infty(\mathcal{T})^*$ which is defined as follows. For $f \in \mathcal{O}_X = \wedge^0 \check{J}_\infty(\mathcal{T})^*$, we define df to be the usual differential of f which is identified with its image in $\check{J}_\infty(\mathcal{T})^*$. For $\alpha \in \check{J}_\infty(\mathcal{T})^*$, we define $d\alpha$ by the familiar formula

$$\langle \xi \wedge \eta, d\alpha \rangle = \mathcal{L}(\xi)\langle \eta, \alpha \rangle - \mathcal{L}(\eta)\langle \xi, \alpha \rangle - \langle [\xi, \eta], \alpha \rangle,$$

where $\xi, \eta \in \check{J}_\infty(\mathcal{T})$, and extend this operation as a derivation of degree 1 of $\wedge \check{J}_\infty(\mathcal{T})^*$. We see, by a classical calculation, that $d^2 = 0$. The natural injection $\mathrm{pr}_1^*\colon \wedge \mathcal{T}^* \to \wedge \check{J}_\infty(\mathcal{T})^*$ commutes with d, and hence the identification of $\wedge \mathcal{T}^*$ with its image under pr_1^* is justified. For $u = \alpha \otimes \xi \in \wedge^p \check{J}_\infty(\mathcal{T})^* \otimes \check{J}_\infty(\mathcal{T})$, $\beta \in \check{J}_\infty(\mathcal{T})^*$, we define a derivation $i(u)$ of degree $p-1$ of $\wedge \check{J}_\infty(\mathcal{T})^*$ by $i(u)\beta = \alpha \wedge i(\xi)\beta$, where $i(\xi)$ is the derivation of $\wedge \check{J}_\infty(\mathcal{T})^*$ of degree -1, interior product with ξ, and extend this operation to arbitrary u by linearity. For $u \in \wedge^p \check{J}_\infty(\mathcal{T})^* \otimes \check{J}_\infty(\mathcal{T})$, we define the Lie derivative $\mathcal{L}(u)$ by the formula

$$\mathcal{L}(u) = [i(u), d] = i(u) \cdot d - (-1)^{p-1} d \cdot i(u);$$

if $u = \alpha \otimes \xi$ and $\beta \in \wedge \check{J}_\infty(\mathcal{T})^*$, then

$$\mathcal{L}(\alpha \otimes \xi)\beta = \alpha \wedge \mathcal{L}(\xi)\beta + (-1)^p d\alpha \wedge i(\xi)\beta. \tag{1.17}$$

For $u = \alpha \otimes \xi \in \wedge^p \check{J}_\infty(\mathcal{T})^* \otimes \check{J}_\infty(\mathcal{T})$, $v = \beta \otimes \eta \in \wedge^q \check{J}_\infty(\mathcal{T})^* \otimes \check{J}_\infty(\mathcal{T})$, we define $[u, v]$ by the formula for the Nijenhuis bracket (see [18] and [19]), namely

$$[\alpha\otimes\xi, \beta\otimes\eta] = (\alpha\wedge\beta)\otimes[\xi, \eta] + \mathcal{L}(\alpha\otimes\xi)\beta\otimes\eta - (-1)^{pq}\mathcal{L}(\beta\otimes\eta)\alpha\otimes\xi, \tag{1.18}$$

and extend this definition to arbitrary u, v by bilinearity. Then $[\xi, u] = \mathcal{L}(\xi)u$, for $\xi \in \tilde{J}_\infty(\mathcal{T})$, $u \in \wedge^p\tilde{J}_\infty(\mathcal{T})^*\otimes\tilde{J}_\infty(\mathcal{T})$; if $v \in \wedge^q\tilde{J}_\infty(\mathcal{T})^*\otimes\tilde{J}_\infty(\mathcal{T})$ and $w \in \wedge\tilde{J}_\infty(\mathcal{T})^*\otimes\tilde{J}_\infty(\mathcal{T})$, then

$$\mathcal{L}[u, v] = [\mathcal{L}(u), \mathcal{L}(v)] = \mathcal{L}(u)\circ\mathcal{L}(v) - (-1)^{pq}\mathcal{L}(v)\circ\mathcal{L}(u)$$

and Jacobi's identity holds:

$$[u, [v, w]] = [[u, v], w] + (-1)^{pq}[v, [u, w]].$$

Thus $\wedge\tilde{J}_\infty(\mathcal{T})^*\otimes\tilde{J}_\infty(\mathcal{T})$ is a sheaf of graded Lie algebras.

We obtain by restriction brackets on $\wedge J_0(\mathcal{T})^*\otimes\tilde{J}_\infty(\mathcal{T})$, $\wedge\mathcal{T}^*\otimes\tilde{J}_\infty(\mathcal{T})$ and $\wedge\mathcal{T}^*\otimes J_\infty(\mathcal{T})$, which, for $k \geq 0$, by passage to the quotient, induce on $\wedge J_0(\mathcal{T})^*\otimes\tilde{J}_{k+1}(\mathcal{T})$, $\wedge\mathcal{T}^*\otimes\tilde{J}_k(\mathcal{T})$ structures of graded Lie algebras and a bracket

$$(\wedge^p T^*\otimes J_{k+1}(T))\otimes(\wedge^q T^*\otimes J_{k+1}(T)) \to \wedge^{p+q}T^*\otimes J_k(T). \tag{1.19}$$

In order to verify that $\wedge J_0(\mathcal{T})^*\otimes\tilde{J}_\infty(\mathcal{T})$ is a sub-sheaf of graded Lie algebras of $\wedge\tilde{J}_\infty(\mathcal{T})^*\otimes\tilde{J}_\infty(\mathcal{T})$, it is sufficient, in view of (1.17) and (1.18), to verify that

$$\mathcal{L}(\alpha\otimes\xi)\beta = \alpha\wedge\mathcal{L}(\xi)\beta, \quad \mathcal{L}(\xi)\beta \in \wedge^q J_0(\mathcal{T})^*$$

for $\xi \in \tilde{J}_\infty(\mathcal{T})$, $\alpha \in \wedge^p J_0(\mathcal{T})^*$, $\beta \in \wedge^q J_0(\mathcal{T})^*$. These assertions follow from the fact that $J_0(T)^*$ is the annihilator of $\tilde{J}_\infty(T)$ (a consequence of (1.10) by duality). Furthermore, $\mathcal{L}(\xi)\beta$ depends only on β and the projection of ξ in $\tilde{J}_1(\mathcal{T})$ (see [19]). Hence, for $k \geq 1$, $\wedge J_0(\mathcal{T})^*\otimes\tilde{J}_k(\mathcal{T})$ is a sheaf of graded Lie algebras, a quotient of the preceding. Since d preserves $\wedge\mathcal{T}^*$ and, if $\xi \in \tilde{J}_\infty(\mathcal{T})$, the restriction of $i(\xi)$ to $\wedge\mathcal{T}^*$ is the usual derivation $i(\xi_0)$ of $\wedge\mathcal{T}^*$, where ξ_0 is the projection of ξ in $\tilde{J}_0(\mathcal{T}) = \mathcal{T}$, we see that $\mathcal{L}(\xi)\beta$ is the usual Lie derivative of $\beta \in \wedge\mathcal{T}^*$ along $\xi_0 \in \mathcal{T}$. Hence $\wedge\mathcal{T}^*\otimes\tilde{J}_\infty(\mathcal{T})$ is a sub-sheaf of graded Lie algebras of $\wedge\tilde{J}_\infty(\mathcal{T})^*\otimes\tilde{J}_\infty(\mathcal{T})$ according to (1.17) and (1.18), and, for $k \geq 0$, $\wedge\mathcal{T}^*\otimes\tilde{J}_k(\mathcal{T})$ is a sheaf of graded Lie algebras, a quotient of the preceding. Finally, since T^* is the annihilator of $J_\infty(T)$ (a consequence of (1.9) by duality), formula (1.18) induces a bracket on $\wedge\mathcal{T}^*\otimes J_\infty(\mathcal{T})$, defined fiber by fiber by the formula

$$[\alpha\otimes\xi, \beta\otimes\eta] = (\alpha\wedge\beta)\otimes[\xi, \eta],$$

where $\alpha, \beta \in \wedge T^*$, $\xi, \eta \in J_\infty(T)$, and a quotient bracket (1.19) defined by the same formula with $\xi, \eta \in J_{k+1}(T)$.

In [22] and [18], another bracket on $\wedge\mathcal{T}^*\otimes\tilde{J}_\infty(\mathcal{T})$ is introduced; it can be obtained by transport of the bracket on $\wedge J_0(\mathcal{T})^*\otimes\tilde{J}_\infty(\mathcal{T})$. Namely, one defines

$$[u, v] = (\nu^*\otimes\mathrm{id})[(\nu^{*-1}\otimes\mathrm{id})u, (\nu^{*-1}\otimes\mathrm{id})v]$$

for $u, v \in \wedge\mathcal{T}^*\otimes\tilde{J}_\infty(\mathcal{T})$; this bracket does not coincide with the bracket on $\wedge\mathcal{T}^*\otimes\tilde{J}_\infty(\mathcal{T})$

obtained above by restriction from the bracket on $\wedge \tilde{J}_\infty(\mathcal{T})^* \otimes \tilde{J}_\infty(\mathcal{T})$, but is related to it (see formula (3.13.2) of [19], and [18], p. 115). However, in this paper, we shall not use these brackets on $\wedge \mathcal{T}^* \otimes \tilde{J}_\infty(\mathcal{T})$.

We note that, if $u \in \wedge^p J_0(\mathcal{T})^* \otimes \tilde{J}_k(\mathcal{T})$ (resp. $\wedge^p T^* \otimes J_k(T)$), $v \in \wedge^q J_0(\mathcal{T})^* \otimes \tilde{J}_k(\mathcal{T})$ (resp. $\wedge^q T^* \otimes J_k(T)$), where $k \geq 1$, and if u and v satisfy $\pi_0 u = 0$ and $\pi_0 v = 0$, then $\pi_0[u, v] = 0$. This can be seen from (1.18).

We list two formulas which are direct consequences of the definition of the brackets and which will be used in the sequel:

$$\langle \zeta_1 \wedge \zeta_2, [u, v] \rangle = [\zeta_1 \barwedge u, \zeta_2 \barwedge v] - [\zeta_2 \barwedge u, \zeta_1 \barwedge v] \tag{1.20}$$

where $u, v \in T^* \otimes J_k(T)$ and $\zeta_1, \zeta_2 \in T$;

$$\begin{aligned} \langle \zeta_1 \wedge \zeta_2, [u, v] \rangle = {} & [\zeta_1 \barwedge u, \zeta_2 \barwedge v] - [\zeta_2 \barwedge u, \zeta_1 \barwedge v] - (\mathcal{L}(\zeta_1 \barwedge u)\zeta_2) \barwedge v \\ & + (\mathcal{L}(\zeta_2 \barwedge u)\zeta_1) \barwedge v - (\mathcal{L}(\zeta_1 \barwedge v)\zeta_2) \barwedge u + (\mathcal{L}(\zeta_2 \barwedge v)\zeta_1) \barwedge u \end{aligned} \tag{1.21}$$

where $u, v \in J_0(\mathcal{T})^* \otimes \tilde{J}_k(\mathcal{T})$ and $\zeta_1, \zeta_2 \in J_0(\mathcal{T})$. A formula analogous to (1.21) holds for $u, v \in \mathcal{T}^* \otimes \tilde{J}_k(\mathcal{T})$ and $\zeta_1, \zeta_2 \in \mathcal{T}$, namely formula (3.3) of [9].

We shall identify $S^k J_0(T)^* \otimes J_0(T)$ with the kernels of the projections $\pi_{k-1}: \tilde{J}_k(T) \to \tilde{J}_{k-1}(T)$ and $\pi_{k-1}: J_k(T) \to J_{k-1}(T)$; this identification will not lead to difficulties when we have to consider diagonal automorphisms of $X \times X$. Then $-D$ gives by restriction a morphism of vector bundles

$$\wedge^j T^* \otimes S^k J_0(T)^* \otimes J_0(T) \to \wedge^{j+1} T^* \otimes S^{k-1} J_0(T)^* \otimes J_0(T)$$

which we shall denote by δ. Denote by $\tilde{\nu}$ the isomorphism

$$\nu^* \otimes \nu \colon \wedge J_0(T)^* \otimes \tilde{J}_k(T) \to \wedge T^* \otimes J_k(T)$$

and by $\bar{D}$ the differential operator

$$\tilde{\nu}^{-1} \circ D \circ \tilde{\nu} \colon \wedge J_0(\mathcal{T})^* \otimes \tilde{J}_k(\mathcal{T}) \to \wedge J_0(\mathcal{T})^* \otimes \tilde{J}_{k-1}(\mathcal{T}).$$

Then $-\bar{D}$ gives by restriction the morphism of vector bundles

$$\bar{\delta} \colon \wedge^j J_0(T)^* \otimes S^k J_0(T)^* \otimes J_0(T) \to \wedge^{j+1} J_0(T)^* \otimes S^{k-1} J_0(T)^* \otimes J_0(T).$$

Consider the sheaf $\mathcal{H}$ of vector fields on $X \times X$ which are pr_1-projectable and pr_2-vertical modulo those which vanish to infinite order on Δ. Then $\mathcal{H}$ is a sub-sheaf of $\check{J}_\infty(\mathcal{T})$ and

$$\check{J}_\infty(\mathcal{T}) = \mathcal{H} \oplus J_\infty(\mathcal{T}), \qquad \check{J}_\infty(\mathcal{T}) = \mathcal{H} \oplus \tilde{J}_\infty(\mathcal{T}).$$

The two projections of $\check{J}_\infty(\mathcal{T})$ onto $\mathcal{H}$ parallel to $J_\infty(\mathcal{T})$ and $\tilde{J}_\infty(\mathcal{T})$ respectively, by the exactness of (1.9) and (1.10), are determined by maps $T \to \check{J}_\infty(T)$, $J_0(T) \to \check{J}_\infty(T)$, and therefore by sections χ of $T^* \otimes \check{J}_\infty(T)$ and $\bar{\chi}$ of $J_0(T)^* \otimes \check{J}_\infty(T)$ respectively. In fact

$$\bar{\chi} = (\nu^{*-1} \otimes \mathrm{id})\chi$$

and

$$\chi \circ \pi_0 = \mathrm{id} - \nu : \check{J}_\infty(T) \to \check{J}_\infty(T),$$

$$\bar{\chi} \circ \pi_0 = \nu^{-1} - \mathrm{id} : J_\infty(T) \to \check{J}_\infty(T).$$

We shall also denote by χ and $\bar{\chi}$ the sections of $T^* \otimes \check{J}_k(T)$ and $J_0(T)^* \otimes \check{J}_k(T)$ corresponding to $\pi_k \circ \chi$ and $\pi_k \circ \bar{\chi}$ respectively. We have the formulas (see [19])

$$Du = [\chi, u], \quad \text{for} \quad u \in \wedge \mathcal{T}^* \otimes J_\infty(\mathcal{T}); \tag{1.22}$$

$$\bar{D}u = [\bar{\chi}, u], \quad \text{for} \quad u \in \wedge J_0(\mathcal{T})^* \otimes \check{J}_\infty(\mathcal{T}). \tag{1.23}$$

Set

$$B_k^p = \wedge^p J_0(T)^* \otimes \check{J}_k(T) / \bar{\delta}(\wedge^{p-1} J_0(T)^* \otimes S^{k+1} J_0(T)^* \otimes J_0(T))$$

and $B_k = \oplus_p B_k^p$. We remark that $\mathcal{B}_k$ is a sheaf of graded Lie algebras for the bracket which is the quotient of that on $\wedge J_0(\mathcal{T})^* \otimes \check{J}_k(\mathcal{T})$, and $\bar{D}$ induces a differential operator

$$\hat{D} : \mathcal{B}_k^p \to \mathcal{B}_k^{p+1}.$$

The "sophisticated" Spencer complex

$$0 \longrightarrow \mathcal{T} \xrightarrow{\tilde{j}_k} \mathcal{B}_k^0 \xrightarrow{\hat{D}} \mathcal{B}_k^1 \xrightarrow{\hat{D}} \cdots \longrightarrow \mathcal{B}_k^n \longrightarrow 0, \tag{1.24}$$

where $\tilde{j}_k = \nu^{-1} \circ j_k$, is acyclic.

The differential operators D, $\bar{D}$, $\hat{D}$ are compatible with the corresponding brackets, namely for $u \in \wedge^p \mathcal{T}^* \otimes J_k(\mathcal{T})$, $v \in \wedge^q \mathcal{T}^* \otimes J_k(\mathcal{T})$, $\bar{u} = (\nu^{*-1} \otimes \nu^{-1})u$, $\bar{v} = (\nu^{*-1} \otimes \nu^{-1})v$, we have, if $k \geqslant 2$,

$$D[u, v] = [Du, \pi_{k-1} v] + (-1)^p [\pi_{k-1} u, Dv]; \tag{1.25}$$

$$\bar{D}[\bar{u}, \bar{v}] = [\bar{D}\bar{u}, \pi_{k-1}\bar{v}] + (-1)^p [\pi_{k-1}\bar{u}, \bar{D}\bar{v}]; \tag{1.26}$$

and, for $u \in \mathcal{B}_k^p$, $v \in \mathcal{B}_k^q$, if $k \geqslant 1$,

$$\hat{D}[u, v] = [\hat{D}u, v] + (-1)^p [u, \hat{D}v]. \tag{1.27}$$

Thus $\mathcal{B}_k$ is a differential graded Lie algebra for each $k \geqslant 1$. These formulas are direct consequences of (1.22) and (1.23) by use of the Jacobi identity.

For $u \in J_0(\mathcal{T})^* \otimes \check{J}_1(\mathcal{T})$, we set $\tilde{u} = (\nu^* \otimes \mathrm{id})u \in \mathcal{T}^* \otimes \check{J}_1(\mathcal{T})$, $u_0 = \pi_0 u \in J_0(\mathcal{T})^* \otimes \mathcal{T}$ and $\tilde{u}_0 = \pi_0 \tilde{u} \in \mathcal{T}^* \otimes \mathcal{T}$.

LEMMA 1.3. *Let u be a section of $J_0(T)^* \otimes \check{J}_1(T)$. Then*

$$\bar{D}u - \tfrac{1}{2}\pi_0[u, u] = 0 \tag{1.28}$$

if and only if, for all $\zeta_1, \zeta_2 \in J_0(\mathcal{T})$, $\tilde{\zeta}_1 = \nu^{-1}\zeta_1$, $\tilde{\zeta}_2 = \nu^{-1}\zeta_2$, *we have*

$$[\tilde{\zeta}_1 - \tilde{u}_0(\tilde{\zeta}_1), \tilde{\zeta}_2 - \tilde{u}_0(\tilde{\zeta}_2)] = (\mathrm{id} - \tilde{u}_0)\{[\zeta_1, \tilde{\zeta}_2] - \nu^{-1}(\mathcal{L}(u(\zeta_1))\,\zeta_2 - \mathcal{L}(u(\zeta_2))\,\zeta_1)\}. \qquad (1.29)$$

Proof. By (1.5) (with $k=1$) and (1.21) for $\zeta_1, \zeta_2 \in J_0(\mathcal{T})$, $\tilde{\zeta}_1 = \nu^{-1}\zeta_1$, $\tilde{\zeta}_2 = \nu^{-1}\zeta_2$, we have

$$\langle \zeta_1 \wedge \zeta_2, \bar{D}u - \tfrac{1}{2}\pi_0[u,u]\rangle = \zeta_1 \barwedge \bar{D}u(\zeta_2) - \zeta_2 \barwedge \bar{D}u(\zeta_1) - \tilde{u}_0([\tilde{\zeta}_1, \tilde{\zeta}_2]) - [\tilde{u}_0(\tilde{\zeta}_1), \tilde{u}_0(\tilde{\zeta}_2)]$$
$$+ (\mathcal{L}(u(\zeta_1))\,\zeta_2) \barwedge u_0 - (\mathcal{L}(u(\zeta_2))\,\zeta_1) \barwedge u_0. \qquad (1.30)$$

Next, using (1.16), we have

$$[\tilde{\zeta}_1 - \tilde{u}_0(\tilde{\zeta}_1), \tilde{\zeta}_2 - \tilde{u}_0(\tilde{\zeta}_2)] = [\zeta_1, \tilde{\zeta}_2] + [\tilde{u}_0(\tilde{\zeta}_1), \tilde{u}_0(\tilde{\zeta}_2)]$$
$$+ \nu^{-1}\{\mathcal{L}(u(\zeta_2))\,\zeta_1 - \mathcal{L}(u(\zeta_1))\,\zeta_2\} - \zeta_1 \barwedge \bar{D}u(\zeta_2) + \zeta_2 \barwedge \bar{D}u(\zeta_1).$$

Substituting from this last identity into (1.30), we obtain

$$\langle \zeta_1 \wedge \zeta_2, \bar{D}u - \tfrac{1}{2}\pi_0[u,u]\rangle = -[\tilde{\zeta}_1 - \tilde{u}_0(\tilde{\zeta}_1), \tilde{\zeta}_2 - \tilde{u}_0(\tilde{\zeta}_2)]$$
$$+ [\tilde{\zeta}_1, \tilde{\zeta}_2] - \tilde{u}_0[\tilde{\zeta}_1, \tilde{\zeta}_2] - \{\mathcal{L}(u(\zeta_1))\,\zeta_2 \barwedge (\nu^{-1} - u_0) - \mathcal{L}(u(\zeta_2))\,\zeta_1 \barwedge (\nu^{-1} - u_0)\},$$

and so the vanishing of the left-hand side of (1.30) is equivalent to (1.29).

Following Malgrange [19], we set $X^3 = X \times X \times X$, let $\mathrm{pr}_i\colon X^3 \to X$ be the projection on the i-th factor ($i = 1, 2, 3$), and $\mathrm{pr}_{ij} = (\mathrm{pr}_i, \mathrm{pr}_j)\colon X^3 \to X \times X$ be the projection onto the product of the i-th and the j-th factors. We denote by $\mathfrak{J}^{(l,k)}$ the ideal of O_{X^3} generated by $\mathrm{pr}_{12}^*\mathfrak{J}^{(l)} + \mathrm{pr}_{23}^*\mathfrak{J}^{(k)}$, where $\mathfrak{J}$ is the ideal of functions of $O_{X\times X}$ which vanish on the diagonal Δ of $X \times X$. The support of the sheaf $O_{X^3}/\mathfrak{J}^{(l+1,k+1)}$ is the diagonal Δ_2 of X^3. A sheaf on X (resp. on Δ_2) will be identified with its inverse image by $\mathrm{pr}_1\colon \Delta_2 \to X$ (resp. with its direct image by $\Delta_2 \to X^3$). We identify $J_l(J_k(\mathcal{T}))$ with the sheaf of vector fields on X^3 which are pr_{12}-vertical, modulo $\mathfrak{J}^{(l+1,k+1)}$. A vector field ξ on X^3 will be called bidiagonal if it is tangent to $\mathrm{pr}_{23}^{-1}(\Delta)$ and pr_{12}-projectable with $\mathrm{pr}_{12*}(\xi)$ diagonal on $X \times X$. We denote by $\tilde{J}_{(l,k)}(\mathcal{T})$ the sheaf of bidiagonal vector fields on X^3, modulo $\mathfrak{J}^{(l+1,k+1)}$, and by $\tilde{J}_{(l,k)}(T)$ the corresponding vector bundle over X. The mapping which sends a bidiagonal vector field on X^3 into its pr_{12}-vertical component yields, by passage to the quotient, a vector bundle isomorphism

$$\nu\colon \tilde{J}_{(l,k)}(T) \to J_l(J_k(T)).$$

We identify $J_l(\tilde{J}_k(\mathcal{T}))$ with the sheaf of vector fields ξ on X^3 which are tangent to $\mathrm{pr}_{23}^{-1}(\Delta)$ and are pr_{12}-projectable with $\mathrm{pr}_{12*}(\xi)$ pr_1-vertical on $X \times X$, modulo $\mathfrak{J}^{(l+1,k+1)}$.

The bracket of vector fields on X^3 gives, by restriction and passage to the quotient, brackets

$$\tilde{J}_{(l,k)}(\mathcal{T}) \times_X \tilde{J}_{(l,k)}(\mathcal{T}) \to \tilde{J}_{(l,k)}(\mathcal{T}), \tag{1.31}$$

$$J_l(J_k(T)) \times_X J_l(J_k(T)) \to J_l(J_{k-1}(T)), \tag{1.32}$$

$$J_l(\tilde{J}_k(T)) \times_X J_l(\tilde{J}_k(T)) \to J_{l-1}(\tilde{J}_k(T)). \tag{1.33}$$

The brackets (1.32) and (1.33) are defined fiber by fiber in the following way: if ξ, η are sections of $J_k(T)$ over a neighborhood of $x \in X$ and $\tilde{\xi} = \nu^{-1}\xi$, $\tilde{\eta} = \nu^{-1}\eta$, then

$$[j_l(\xi)(x), j_l(\eta)(x)] = j_l([\xi, \eta])(x), \tag{1.34}$$

$$[j_l(\tilde{\xi})(x), j_l(\tilde{\eta})(x)] = j_{l-1}([\tilde{\xi}, \tilde{\eta}])(x), \tag{1.35}$$

where $[\xi, \eta]$ and $[\tilde{\xi}, \tilde{\eta}]$ are defined in terms of the brackets (1.11) and (1.12) respectively. For $k \geqslant 1$, the diagram

$$\begin{tikzcd} J_{k+l}(T) \times_X J_{k+l}(T) \arrow[r] \arrow[d, "(\lambda_l, \lambda_l)"] & J_{k+l-1}(T) \arrow[d, "\lambda_l"] \\ J_l(J_k(T)) \times_X J_l(J_k(T)) \arrow[r] & J_l(J_{k-1}(T)), \end{tikzcd} \tag{1.36}$$

whose horizontal arrows are given by (1.11) and (1.32), is commutative. If

$$\tilde{\lambda}_l \colon J_{k+l}(T) \to J_l(\tilde{J}_k(T))$$

is the composition

$$J_{k+l}(T) \xrightarrow{\lambda_l} J_l(J_k(T)) \xrightarrow{J_l(\nu^{-1})} J_l(\tilde{J}_k(T)),$$

the diagram

$$\begin{tikzcd} J_{k+l}(T) \times_X J_{k+l}(T) \arrow[r] \arrow[d, "(\tilde{\lambda}_l, \tilde{\lambda}_l)"] & J_{k+l-1}(T) \arrow[d, "\tilde{\lambda}_{l-1}"] \\ J_l(\tilde{J}_k(T)) \times_X J_l(\tilde{J}_k(T)) \arrow[r] & J_{l-1}(\tilde{J}_k(T)), \end{tikzcd} \tag{1.37}$$

whose horizontal arrows are given by (1.11) and (1.33), is commutative. With the bracket (1.31), $\tilde{J}_{(l,k)}(\mathcal{T})$ is a sheaf of Lie algebras. If

$$\tilde{\lambda}_l \colon \tilde{J}_{k+l}(T) \to \tilde{J}_{(l,k)}(T)$$

is the canonical injection equal to $\nu^{-1} \circ \lambda_l \circ \nu$, the corresponding sheaf map is a morphism of sheaves of Lie algebras. The mapping $\tilde{j}_l \colon \tilde{J}_k(\mathcal{T}) \to \tilde{J}_{(l,k)}(\mathcal{T})$ defined by $\nu^{-1} \circ j_l \circ \nu$ is also a morphism of sheaves of Lie algebras.

LEMMA 1.4. *Let R_k, N_k, S_k be formally integrable differential equations in $J_k(T)$. If $[R_{k+1}, N_{k+1}] \subset S_k$, then for all $l \geqslant 0$*

$$[R_{k+l+1}, N_{k+l+1}] \subset S_{k+l} \tag{1.38}$$

and

$$[\mathrm{Sol}\,(R_k), \mathrm{Sol}\,(N_k)] \subset \mathrm{Sol}\,(S_k). \tag{1.39}$$

Proof. The bracket (1.32) induces a bracket

$$J_l(R_{k+1}) \times_X J_l(N_{k+1}) \to J_l(S_k);$$

since we have $R_{k+l+1} = (R_{k+1})_{+l}$, $N_{k+l+1} = (N_{k+1})_{+l}$ and $S_{k+l} = (S_k)_{+l}$, from the commutativity of (1.36) we deduce (1.38). If ξ is a solution of R_k and η is a solution of N_k, then

$$[j_{k+1}(\xi), j_{k+1}(\eta)] = j_k([\xi, \eta]),$$

and so $[\xi, \eta]$ is a solution of S_k.

LEMMA 1.5. *Let R_k, N_k be formally integrable differential equations in $J_k(T)$. Then, if $\tilde{R}_{k+l} = \nu^{-1} R_{k+l}$ for $l \geqslant 0$, the following assertions are equivalent:*

(a) $[R_{k+1}, N_{k+1}] \subset N_k$;

(b) $[R_{k+l+1}, N_{k+l+1}] \subset N_{k+l}$, *for all* $l \geqslant 0$;

(c) $[\tilde{\mathcal{R}}_{k+1}, \mathcal{N}_k] \subset \mathcal{N}_k$;

(d) $[\tilde{\mathcal{R}}_{k+l+1}, \mathcal{N}_{k+l}] \subset \mathcal{N}_{k+l}$, *for all* $l \geqslant 0$.

Proof. The equivalence of (a) and (b) follows from Lemma 1.4. Since $\pi_{k+l}: N_{k+l+1} \to N_{k+l}$ is surjective, the equivalence of (a) and (c) or of (b) and (d) is deduced from (1.15).

2. Jets of transformations

Consider $E = X \times X$ as a bundle over X via the projection pr_1 and identify a map f: $X \to X$ with its graph $\bar{f}: X \to X \times X$ and the k-jet $j_k(f)(x)$ of f at x with the k-jet $j_k(\bar{f})(x)$ of $\bar{f}$ at x. In accordance with the usual terminology, we call $f(x)$ the target of $j_k(f)(x)$. If $F = j_k(f)(b)$, $G = j_k(g)(a) \in J_k(E)$, where $f: X \to X$, $g: X \to X$ are maps defined on neighborhoods of b and a respectively satisfying $g(a) = b$, then $F \cdot G = j_k(f \circ g)(a)$ is a well-defined element of $J_k(E)$.

Let Q_k be the open fibered submanifold of $J_k(E)$ of jets of order k of local diffeomorphisms $X \to X$; in fact, $Q_k = \pi_1^{-1} Q_1$, for $k \geqslant 1$. We consider, unless it is explicitly stated to the contrary, Q_k to be a bundle over X via the projection "source" $\pi: Q_k \to X$. The multiplication on $J_k(E)$ defined above determines a structure of differentiable groupoid on Q_k. Let $Q_k(a)$ (resp. $Q_k(a, b)$) be the set of jets of order k of Q_k with source a (resp. with source a and target b).

Consider mappings $F: X\times X\to X\times X$ of the form (f^0, f), where $f: X\times X\to X$ and f^0: $X\to X$ is defined by $f^0(x)=f(x, x)$ for $x\in X$. These mappings preserve Δ and are pr_1-projectable; moreover, we shall say that F is *diagonal* if, in addition, for each $x\in X$ the germ at x of the mapping $x'\mapsto f(x, x')$ is invertible. To the diagonal mapping F, we associate the section of Q_k whose value at x is equal to the jet of order k at x of $x'\mapsto f(x, x')$. Two diagonal mappings F and G determine the same section of Q_k if and only if F and G have the same principal part of order k, that is to say if they coincide on Δ together with their partial derivatives of orders not exceeding k. We shall regard a section of Q_k as the principal part of a diagonal mapping; such a section $F=(f^0, f)$ is invertible if and only if f^0 is invertible. We denote by $\tilde{Q}_k$ the sub-sheaf of Q_k of invertible (*étales*) elements of Q_k. Let Aut (X) be the sheaf of local diffeomorphisms $X\to X$; if $f\in\mathrm{Aut}\,(X)$, $j_k(f)$ is the principal part of order k of the germ of diagonal mapping $(x, x')\mapsto(f(x), f(x'))$ (see [19]).

Let $Q_{(l,k)}$ be the bundle of jets of order l of sections of $\tilde{Q}_k$. The composition of jets assigns to it a structure of groupoid and we denote by $\tilde{Q}_{(l,k)}$ the sheaf of invertible (*étales*) sections of $Q_{(l,k)}$. The mapping j_l: $\tilde{Q}_k\to\tilde{Q}_{(l,k)}$ induced by j_l: $Q_k\to J_l(Q_k)$ is a homomorphism of groupoids; the natural inclusion λ_l: $Q_{k+l}\to Q_{(l,k)}$ is a homomorphism of groupoids.

The action of diagonal automorphisms of $X\times X$ on vector fields gives, by passage to the quotient, for each section F of $\tilde{Q}_{k+1}$ the following mappings:

$$F: J_k(T)_a\to J_k(T)_b, \tag{2.1}$$

$$F: \tilde{J}_{k+1}(T)_a\to \tilde{J}_{k+1}(T)_b, \tag{2.2}$$

$$F: \tilde{J}_k(T)_a\to \tilde{J}_k(T)_b, \tag{2.3}$$

where $a\in X$ and $b=$target $F(a)$. The mapping (2.1) depends only on $F(a)$, while the mappings (2.2) and (2.3) depend only on $j_1(F)(a)\in Q_{(1,k+1)}$. Thus (2.2) gives us a mapping

$$Q_{(1,k)}\times_X \tilde{J}_k(T)\to\tilde{J}_k(T)$$

sending (H,ξ) into $H(\xi)$; if $F\in Q_{k+1}(a, b)$, then the mapping $\lambda_1 F$: $\tilde{J}_k(T)_a\to\tilde{J}_k(T)_b$ is given by (see [19], formula (6.2))

$$\lambda_1 F(\xi)=\nu^{-1}F(\nu\xi), \qquad \xi\in\tilde{J}_k(T)_a. \tag{2.4}$$

However the restriction of (2.2) to $J^0_{k+1}(T)=J_{k+1}(T)\cap\tilde{J}_{k+1}(T)$ does not depend on the 1-jet of $F\in\tilde{Q}_{k+1}$ but only on $F(a)$; thus we have a mapping

$$Q_k\times_X J^0_k(T)\to J^0_k(T).$$

We have a canonical section I_k of Q_k over X sending $x\in X$ into $I_k(x)$, the k-jet of the identity mapping of X at x. If F_t is a one-parameter family of sections of $\tilde{Q}_k$, with $F_0=I_k$, then $\xi=dF_t/dt|_{t=0}\in\Gamma(X, \tilde{J}_k(T))$ where the sections of $\tilde{J}_k(T)$ are regarded as diagonal vector fields on $X \times X$; every section of $\tilde{J}_k(T)$ can be written in this way locally. We can also regard $\xi(x)$ as a vector tangent to $Q_k(x)$ at $I_k(x)$; hence we have an isomorphism

$$\tilde{J}_k(T)_x \xrightarrow{\sim} V_{I_k(x)}(Q_k)$$

which enables us to identify these two spaces. In fact, if f_t is a one-parameter family of local diffeomorphisms of X defined on a neighborhood of x with $f_0 = \mathrm{id}$, and $\xi = df_t/dt\,|_{t=0}$ is its infinitesimal generator, then the image of $\tilde{j}_k(\xi)(x)$ under this map is the tangent vector $dj_k(f_t)(x)/dt\,|_{t=0}$ to Q_k.

If $G \in Q_k(a, b)$, then the mapping $Q_k(b) \to Q_k(a)$ sending F into $F \cdot G$ is a bijection. Therefore we obtain an isomorphism

$$T_F(Q_k(b)) \to T_{F\cdot G}(Q_k(a))$$

or

$$V_F(Q_k) \to V_{F\cdot G}(Q_k)$$

sending ξ into $\xi \cdot G$. Taking $F = I_k(b)$, we obtain the isomorphism

$$\tilde{J}_k(T)_b \to V_G(Q_k(a)).$$

If $\xi \in \Gamma(X, \tilde{J}_k(T))$, the vector field $\tau_k(\xi)$ on Q_k whose value at $F \in Q_k$ is $\xi(b) \cdot F$, where $b =$ target F, is clearly invariant under this right action of Q_k. Moreover τ_k is a morphism of Lie algebras from $\Gamma(X, \tilde{J}_k(T))$ to the algebra of vector fields on Q_k.

Let G be a section of $\tilde{Q}_k$. For $a \in X$ the map $Q_k(a) \to Q_k(a)$ sending F into $G(b) \cdot F$, where $F \in Q_k(a)$ and $b =$ target F, is an automorphism of $Q_k(a)$. Therefore we obtain mappings

$$T_F(Q_k) \to T_{G(b)\cdot F}(Q_k),$$
$$V_F(Q_k) \to V_{G(b)\cdot F}(Q_k),$$

sending ξ into $G\xi$; this left action on $V(Q_k)$ commutes with the right action defined above. These mappings depend only on $H = j_1(G)(b) \in Q_{(1,k)}$ and we write $H\xi = G\xi$, for $\xi \in T_F(Q_k)$. Taking $F = I_k(a)$, we obtain the isomorphism

$$\tilde{J}_k(T)_a \to V_{G(a)}(Q_k),$$

which depends only on the 1-jet of G at a, and a mapping

$$Q_{(1,k)} \times_X \tilde{J}_k(T) \to V(Q_k)$$

sending (H, ξ) into $H\xi$. The isomorphism (2.2) is given by

$$F(\xi) = F \cdot \xi \cdot F(a)^{-1} \tag{2.5}$$

for $F \in \tilde{Q}_k$, $\xi \in \tilde{J}_k(T)_a$. We therefore obtain, by (2.4), the formula

$$\lambda_1 F(\xi) = \lambda_1 F \cdot \xi \cdot \pi_k F^{-1} = \nu^{-1} F(\nu\xi) \tag{2.6}$$

for $F \in Q_{k+1}(a)$, $\xi \in \tilde{J}_k(T)_a$ (see [19]).

For $x \in X$, we have an isomorphism

$$\tilde{J}_{(l,k)}(T)_x \longrightarrow V_{j_l(I_k)(x)}(Q_{(l,k)}) \tag{2.7}$$

which enables us to identify these two spaces. In fact, if F_t is a one-parameter family of sections of $\tilde{Q}_k$ over a neighborhood U of x, with $F_0 = I_k$, then $\xi = dF_t/dt|_{t=0} \in \Gamma(U, \tilde{J}_k(T))$, and (2.7) sends $\tilde{j}_l(\xi)(x)$ into the tangent vector $dj_l(F_t)(x)/dt|_{t=0}$ to $Q_{(l,k)}$. If $F \in Q_{(l,k)}$ and $F_0 = \pi_0 F \in Q_k$, with source $F_0 = a$, target $F_0 = b$, the mapping $Q_{(l,k)}(b) \to Q_{(l,k)}(a)$ sending G into $G \cdot F$ is a bijection. Therefore we obtain an isomorphism

$$T_G(Q_{(l,k)}(b)) \to T_{G\cdot F}(Q_{(l,k)}(a))$$

or

$$V_G(Q_{(l,k)}) \to V_{G\cdot F}(Q_{(l,k)}).$$

Taking $G = j_l(I_k)(b)$, we obtain the isomorphism

$$\tilde{J}_{(l,k)}(T)_b \to V_F(Q_{(l,k)}).$$

If $F \in Q_{k+l}$ with $b = \text{target } F$, it is easily seen that the diagram

$$\begin{array}{ccc} \tilde{J}_{k+l}(T)_b & \longrightarrow & V_F(Q_{k+l}) \\ \Big\downarrow{\scriptstyle \tilde{\lambda}_l} & & \Big\downarrow{\scriptstyle \lambda_{l*}} \\ \tilde{J}_{(l,k)}(T)_b & \longrightarrow & V_{\lambda_l F}(Q_{(l,k)}), \end{array} \tag{2.8}$$

whose horizontal arrows are given by the right action of F on Q_{k+l} and of $\lambda_l F$ on $Q_{(l,k)}$ respectively, commutes. If $\xi \in \Gamma(X, \tilde{J}_{(l,k)}(T))$, the vector field $\tau_{(l,k)}(\xi)$ on $Q_{(l,k)}$ whose value at $F \in Q_{(l,k)}$ is $\xi(b) \cdot F$, where $b = \text{target } \pi_0 F$, is clearly invariant under the right action of $Q_{(l,k)}$. Moreover, $\tau_{(l,k)}$ is a morphism of Lie algebras from $\Gamma(X, \tilde{J}_{(l,k)}(T))$ to the algebra of vector fields on $Q_{(l,k)}$.

If $F \in J_k(E)$ and $f: X \to X$ is a mapping defined on a neighborhood of $x \in X$ such that $F = j_k(f)(x)$, we denote by

$$F_*: T_x \longrightarrow T_{\pi_{k-1}F}(J_{k-1}(E))$$

the map $j_{k-1}(\tilde{f})_*$; in fact, F_* depends only on F and determines F uniquely. If $k=1$, then $F_*: T_x \to T_{\tilde{f}(x)}(E) = T_x \times T_{f(x)}$ is the graph of the map $f = f_*: T_x \to T_{f(x)}$, the differential of f at x, which is given by (2.2) when f is a local diffeomorphism. The map $F: J_0(T)_x \to J_0(T)_{f(x)}$ sending ξ into $F\xi = \nu(f\nu^{-1}\xi)$ is the map (2.1) when $k=0$ and f is a local diffeomorphism. We remark that $F \in Q_1$ if and only if $f: T_x \to T_{f(x)}$ is invertible.

According to Proposition 5.1 of [4], $J_k(E)$ is an affine bundle over $J_{k-1}(E)$ whose associated vector bundle over $J_{k-1}(E)$ is induced from the vector bundle

$$S^kT^* \otimes_E V(E) = (\mathrm{pr}_1^{-1}S^kT^*) \otimes_E (\mathrm{pr}_2^{-1}T)$$

over E, since $V(E)$ can be identified with $\mathrm{pr}_2^{-1}T$. If $k=1$ and $j_1(f)(x) \in J_1(E)$, $u \in T_x^* \otimes T_{f(x)}$, then

$$(j_1(f)(x)+u)_* \xi = (\xi, f\xi + u(\xi)), \quad \xi \in T_x. \tag{2.9}$$

Hence $j_1(f)(x)+u$ belongs to Q_1 if and only if $f+u$: $T_x \to T_{f(x)}$ is invertible.

We now examine the compatibility relations between the multiplication on $J_k(E)$ and the operations on $J_k(E)$ given by its structure of affine bundle over $J_{k-1}(E)$. We assume that $k \geqslant 1$.

PROPOSITION 2.1. *Let F, $G \in J_k(E)$ where source $G=a$, target $G=b=$source F, target $F=c$.*

(i) *If $u \in S^kT_a^* \otimes T_b$, then*

$$F \cdot (G+u) = F \cdot G + (\mathrm{id} \otimes \nu^{-1} \circ \pi_1 F \circ \nu) u, \tag{2.10}$$

where $(\mathrm{id} \otimes \nu^{-1} \circ \pi_1 F \circ \nu) u \in S^kT_a^* \otimes T_c$.

(ii) *If $u \in S^kT_b^* \otimes T_c$, then*

$$(F+u) \cdot G = F \cdot G + (\nu^* \circ \pi_1 G \circ \nu^{*-1} \otimes \mathrm{id}) u, \tag{2.11}$$

where $(\nu^* \circ \pi_1 G \circ \nu^{*-1} \otimes \mathrm{id}) u \in S^kT_a^* \otimes T_c$.

(iii) *Let $u \in S^kT_a^* \otimes T_b$ and assume that $G \in Q_k$. If $k>1$, then $G+u \in Q_k$ and*

$$(G+u)^{-1} = G^{-1} - (\nu^* \circ \pi_1 G^{-1} \circ \nu^{*-1} \otimes \nu^{-1} \circ \pi_1 G^{-1} \circ \nu) u, \tag{2.12}$$

where $(\nu^* \circ \pi_1 G^{-1} \circ \nu^{*-1} \otimes \nu^{-1} \circ \pi_1 G^{-1} \circ \nu) u \in S^kT_b^* \otimes T_a$. If $k=1$, then $G+u \in Q_1$ if and only if $G + \nu \circ u \circ \nu^{-1}$: $J_0(T)_a \to J_0(T)_b$ is invertible; if this condition holds then

$$(G+u)^{-1} = G^{-1} - [\nu^* \circ G^{-1} \circ \nu^{*-1} \otimes \nu^{-1} \circ (G + \nu \circ u \circ \nu^{-1})^{-1} \circ \nu] u, \tag{2.13}$$

where $[\nu^* \circ G^{-1} \circ \nu^{*-1} \otimes \nu^{-1} \circ (G + \nu \circ u \circ \nu^{-1})^{-1} \circ \nu] u \in T_b^* \otimes T_a$.

Proof. (i) Let f: $X \to X$ be a map such that $j_k(f)(b)=F$. Consider the morphism of fibered manifolds $J_k(f)$: $J_k(E) \to J_k(E)$ over X sending H into $j_k(f)(x) \cdot H$, with $x=$ target H; in fact, $J_k(f)$ is the k-th prolongation of the map $\mathrm{id} \times f$: $E \to E$ over X, and $J_k(f)H = F \cdot H$ when target $H=b$. Hence, by Proposition 5.6 of [4], $J_k(f)$ is a morphism of affine bundles over $J_{k-1}(f)$: $J_{k-1}(E) \to J_{k-1}(E)$ whose associated morphism of vector bundles is induced by the endomorphism $\mathrm{id} \otimes f$ of $(\mathrm{pr}_1^{-1}S^kT^*) \otimes_E (\mathrm{pr}_2^{-1}T)$ over the map $\mathrm{id} \times f$. Therefore

$$J_k(f)(G+u) = J_k(f)G + (\mathrm{id} \otimes f) u$$

which gives us (2.10).

(ii) It is sufficient to verify (2.11) when $u \in S^k T_b^* \otimes T_c$ is of the form $\varepsilon^{-1} j_k(h)(b) \otimes \xi$ where h is a local real-valued function on X satisfying $j_{k-1}(h)(b)=0$. Let $F=j_k(f)(b)$, $G=j_k(g)(a)$, where f, g are local maps $X \to X$; let $\tilde{f}\colon U \times \mathbf{R} \to X$ be a one-parameter family of maps of a neighborhood U of b into X such that $\tilde{f}(x,0)=f(x)$, for $x \in U$, and $d\tilde{f}(b,t)/dt\,|_{t=0}=\xi$. Then according to Lemma 5.1 and Proposition 5.1 of [4]

$$F+u = j_k(\tilde{f}(x, h(x)))(b)$$

and

$$(F+u)\cdot G = j_k(\tilde{f}(g(x), h(g(x))))(a) = j_k(f \circ g)(a) + \varepsilon^{-1} j_k(h \circ g)(a) \otimes \xi,$$

since the local map $\varphi = \tilde{f} \circ (g \times \mathrm{id})\colon X \times \mathbf{R} \to X$ is a one-parameter family of maps $X \to X$ defined on a neighborhood of a such that $\varphi(x,0)=f(g(x))$ and $d\varphi(a,t)/dt\,|_{t=0}=\xi$. Since $\varepsilon^{-1} j_k(h \circ g)(a) = g^* \varepsilon^{-1} j_k(h)(b)$, we obtain (2.11).

(iii) We suppose first that $k>1$. We have by (2.10)

$$I_k(a) = (G+u)^{-1}\cdot(G+u) = (G+u)^{-1}\cdot G + (\mathrm{id} \otimes \nu^{-1} \circ \pi_1 G^{-1} \circ \nu)\, u.$$

Hence

$$(G+u)^{-1} = [I_k(a) - (\mathrm{id} \otimes \nu^{-1} \circ \pi_1 G^{-1} \circ \nu)\, u]\cdot G^{-1},$$

and therefore, by (2.11), we obtain the formula (2.12). We now consider the case $k=1$; then $G+u\colon J_0(T)_a \to J_0(T)_b$ is given, according to (2.9), by

$$(G+u)\xi = (G + \nu \circ u \circ \nu^{-1})\xi$$

for $\xi \in J_0(T)_a$. Hence $G+u \in Q_1$ if and only if this map is invertible. Assume that this is the case; the mapping

$$(G+u)^{-1}\colon J_0(T)_b \to J_0(T)_a$$

is given by

$$(G+u)^{-1} = (G + \nu \circ u \circ \nu^{-1})^{-1}. \tag{2.14}$$

By (2.10),

$$I_1(a) = (G+u)^{-1}\cdot(G+u) = (G+u)^{-1}\cdot G + (\mathrm{id} \otimes \nu^{-1} \circ (G+u)^{-1} \circ \nu)\, u$$

and hence

$$(G+u)^{-1} = [I_1(a) - (\mathrm{id} \otimes \nu^{-1} \circ (G+u)^{-1} \circ \nu)\, u]\cdot G^{-1} = G^{-1} - (\nu^* \circ G^{-1} \circ \nu^{*-1} \otimes \nu^{-1} \circ (G+u)^{-1} \circ \nu)\, u,$$

by (2.11). Substituting into this formula from (2.14), we obtain (2.13).

Assume that $k \geqslant 0$. By Proposition 5.1 of [4], $J_1(Q_k)$ is an affine bundle over Q_k whose associated vector bundle is $T^* \otimes_{Q_k} V(Q_k)$, and $Q_{(1,k)}$ is an open submanifold of $J_1(Q_k)$. Identifying Q_0 with $E = X \times X$, then $J_1(\pi_0)\colon J_1(Q_k) \to J_1(E)$ is the map sending $j_1(F)(x)$ into $j_1(\pi_0 F)(x)$, where F is a section of Q_k over a neighborhood of x. If $f = \pi_0 F$, then $j_1(F)(x)$ belongs to $Q_{(1,k)}$ if and only if $j_1(f)(x) \in Q_1$, that is if $f\colon T_x \to T_{f(x)}$ is invertible. Thus $Q_{(1,0)} = Q_1$ and $Q_{(1,k)} = J_1(\pi_0)^{-1} Q_1$. If $F \in Q_k$, with source $F = a$, target $F = b$, let

$$T_a^* \otimes \tilde{J}_k(T)_b \to T_a^* \otimes V_F(Q_k)$$
$$u \mapsto uF$$

be the isomorphism sending $\alpha \otimes \eta$ into $\alpha \otimes (\eta F)$. If $H \in J_1(Q_k)$ with $\pi_0 H = F$ and $u \in T_a^* \otimes \tilde{J}_k(T)_b$, then the affine bundle structure of $J_1(Q_k)$ over Q_k gives an element $H + uF$ of $J_1(Q_k)$ with $\pi_0(H+uF) = F$. We examine the compatibility relations between the structure of affine bundle of $J_1(Q_k)$ over Q_k and the structure of groupoid of $Q_{(1,k)}$.

PROPOSITION 2.2. *Let $F \in Q_k$ with source $F = a$, target $F = b$.*

(i) *Let $H \in Q_{(1,k)}$ with $\pi_0 H = F$ and $J_1(\pi_0) H = j_1(f)(a)$, where f is a local diffeomorphism of X defined on a neighborhood of a. If $u \in T_a^* \otimes \tilde{J}_k(T)_b$, then $H + uF$ belongs to $Q_{(1,k)}$ if and only if $f + \pi_0 u$: $T_a \to T_b$ is invertible.*

(ii) *Let $H \in Q_{(1,k)}$ with $\pi_0 H = F$, and $u \in T_a^* \otimes \tilde{J}_k(T)_b$. If $H + uF \in Q_{(1,k)}$, we have*

$$(H+uF)(\xi) = H(\xi) + (\pi_0 \xi) \mathbin{\overline{\wedge}} u \tag{2.15}$$

for all $\xi \in \tilde{J}_k(T)_a$.

(iii) *If $H_1, H \in Q_{(1,k)}$ with $\pi_0 H_1 = \pi_0 H = F$, then $H_1 = H$ if and only if $H_1(\xi) = H(\xi)$ for all $\xi \in \tilde{J}_k(T)_a$.*

(iv) *If $F_1 \in Q_k$ with source $F_1 = b$, target $F_1 = c$ and $H_1, H \in Q_{(1,k)}$ with $\pi_0 H_1 = F_1$, $\pi_0 H = F$, then*

$$H_1 \cdot (H + uF) = H_1 \cdot H + [(\mathrm{id} \otimes H_1) u] F_1 \cdot F$$

for $u \in T_a^* \otimes \tilde{J}_k(T)_b$ such that $H + uF \in Q_{(1,k)}$, where in the second term of the right member H_1 is the map $\tilde{J}_k(T)_b \to \tilde{J}_k(T)_c$; furthermore

$$(H_1 + vF_1) \cdot H = H_1 \cdot H + [(f \otimes \mathrm{id}) v] F_1 \cdot F$$

for $v \in T_b^* \otimes \tilde{J}_k(T)_c$ such that $H_1 + vF_1 \in Q_{(1,k)}$.

Proof. (i) We have $H + uF \in Q_{(1,k)}$ if and only if $J_1(\pi_0)(H+uF) \in Q_1$. By Proposition 5.4 of [4], $J_1(\pi_0)$: $J_1(Q_k) \to J_1(E)$ is a morphism of affine bundles over π_0: $Q_k \to E$ whose associated morphism of vector bundles $T^* \otimes_{Q_k} V(Q_k) \to \mathrm{pr}_1^{-1} T^* \otimes_E \mathrm{pr}_2^{-1} T$ sends uF into $(\mathrm{id} \otimes \pi_{0*})(uF) = \pi_0 u$, if $u \in T_a^* \otimes \tilde{J}_k(T)_b$. Therefore

$$J_1(\pi_0)(H+uF) = J_1(\pi_0) H + (\mathrm{id} \otimes \pi_{0*}) uF = j_1(f)(a) + \pi_0 u,$$

and $H + uF \in Q_{(1,k)}$ if and only if $j_1(f)(a) + \pi_0 u \in Q_1$, from which we deduce (i).

(ii) If $H_1 = H + uF \in Q_{(1,k)}$ then, by formula (2.4) of [8], we have for $\xi \in \tilde{J}_k(T)_a$,

$$H_1(\xi) - H(\xi) = (\pi_0 \xi) \mathbin{\overline{\wedge}} ((H_1 - H) F^{-1}) = (\pi_0 \xi) \mathbin{\overline{\wedge}} u,$$

since $H_1 - H = uF \in T_a^* \otimes V_F(Q_k)$, from which the identity (2.15) follows.

(iii) We can write $H_1 = H + uF$ for a suitable $u \in T^*_a \otimes \tilde{J}_k(T)_b$. If $H_1(\xi) = H(\xi)$ for all $\xi \in \tilde{J}_k(T)_a$, we conclude from (2.15) that $u = 0$ and hence $H_1 = H$.

(iv) If $u \in T^*_a \otimes \tilde{J}_k(T)_b$ and $H + uF \in Q_{(1,k)}$, then for $\xi \in \tilde{J}_k(T)_a$ we have by (2.15)

$$\begin{aligned} H_1((H+uF)(\xi)) &= H_1(H(\xi) + \pi_0(\xi) \mathbin{\bar\wedge} u) \\ &= H_1(H(\xi)) + H_1(\pi_0(\xi) \mathbin{\bar\wedge} u) = (H_1 \cdot H + [(\mathrm{id} \otimes H_1)u] F_1 \cdot F)(\xi). \end{aligned}$$

If $v \in T^*_b \otimes \tilde{J}_k(T)_c$ and $H_1 + vF_1 \in Q_{(1,k)}$, then for $\xi \in \tilde{J}_k(T)_a$ we have by (2.15)

$$\begin{aligned} (H_1 + vF_1)(H(\xi)) &= H_1(H(\xi)) + (\pi_0 H(\xi)) \mathbin{\bar\wedge} v \\ &= H_1(H(\xi)) + (\pi_0 \xi) \mathbin{\bar\wedge} [(f \otimes \mathrm{id}) v] = (H_1 \cdot H + [(f \otimes \mathrm{id}) v] F_1 \cdot F)(\xi). \end{aligned}$$

From these two identities and (iii) we deduce the formulas of (iv).

Assume that $k \geq 1$. Let $v(Q_k)$ be the sub-bundle of vectors of $V(Q_k)$ whose projection in $V(Q_{k-1})$ vanishes. Then, for $a \in X$, we see that $v_{I_k(a)}(Q_k)$ is identified with $S^k J_0(T)^*_a \otimes J_0(T)_a$ when we identify $V_{I_k(a)}(Q_k)$ with $\tilde{J}_k(T)_a$. The structure of affine bundle of $J_k(E)$ over $J_{k-1}(E)$ gives us an isomorphism for $G \in Q_k$, with source $G = a$, target $G = b$,

$$\mu(G)\colon S^k T^*_a \otimes T_b \to v_G(Q_k)$$

sending u into $d(G + tu)/dt\,|_{t=0}$, where $t \in \mathbf{R}$. One verifies easily that, for $a \in X$, the diagram

$$\begin{tikzcd} S^k T^*_a \otimes T_a \arrow[r, "\mu(I_k(a))"] \arrow[dr, "\nu^{*-1} \otimes \nu"'] & v_{I_k(a)}(Q_k) \\ & S^k J_0(T)^*_a \otimes J_0(T)_a \arrow[u] \end{tikzcd} \tag{2.16}$$

is commutative, where the vertical arrow is the natural identification. If $G \in Q_k$, with source $G = a$, target $G = b$, then the diagram

$$\begin{tikzcd} S^k T^*_a \otimes T_b \arrow[r, "\mu(G)"] & V_G(Q_k) \\ S^k J_0(T)^*_b \otimes J_0(T)_b \arrow[r] \arrow[u, "\nu^* \circ \pi_1 G \otimes \nu^{-1}"] & \tilde{J}_k(T)_b \arrow[u, "G"'] \end{tikzcd} \tag{2.17}$$

is commutative. Indeed, if $u \in S^k T^*_a \otimes T_b$, we have by (2.11)

$$(\mu(G)u)G^{-1} = \frac{d}{dt}(G+tu)\cdot G^{-1}|_{t=0} = \frac{d}{dt}(I_k(b)+t(\nu^*\circ\pi_1 G^{-1}\circ\nu^{*-1}\otimes\mathrm{id})u)|_{t=0}$$
$$= \mu(I_k(b))(\nu^*\circ\pi_1 G^{-1}\circ\nu^{*-1}\otimes\mathrm{id})u,$$

and so the commutativity of (2.17) follows from that of (2.16).

PROPOSITION 2.3. *Assume that* $k\geqslant 0$.

(i) *Let* $F\in Q_{k+1}$, *with source* $F=a$, *target* $F=b$, *and* $u\in S^{k+1}T_a^*\otimes T_b$; *if* $F+u\in Q_{k+1}$, *then,*

$$\lambda_1(F+u) = \lambda_1 F + vG, \quad \textit{for } k\geqslant 1, \tag{2.18}$$

where $G=\pi_k F$ *and* $v=(\mathrm{id}\otimes\pi_1 G^{-1}\circ\nu^{*-1}\otimes\nu)\delta u\in T_a^*\otimes S^k J_0(T)_b^*\otimes J_0(T)_b$, *and for* $\xi\in J_k(T)_a$

$$(F+u)\xi = F\xi + (\pi_0\xi)\,\overline{\wedge}\,(\nu^{*-1}\otimes\pi_1 G^{-1}\circ\nu^{*-1}\otimes\nu)\delta u, \quad \textit{for } k\geqslant 1, \tag{2.19}$$

$$(F+u)\xi = F\xi + \xi\,\overline{\wedge}\,(\nu^{*-1}\otimes\nu)u, \quad \textit{for } k=0. \tag{2.20}$$

(ii) *If* $F_1, F\in Q_{k+1}$, *with source* $F_1=$ *source* $F=a$, *target* $F_1=$ *target* $F=b$, *then* $F_1=F$ *if and only if* $F_1\xi=F\xi$ *for all* $\xi\in J_k(T)_a$.

Proof. (i) First assume that $k\geqslant 1$. According to Proposition 5.6 of [4], $\lambda_1\colon J_{k+1}(E)\to J_1(J_k(E))$ is a morphism of affine bundles over $J_k(E)$ and

$$\lambda_1(F+u) = \lambda_1 F + (\mathrm{id}\otimes\mu(G))\delta u$$

for $F\in Q_{k+1}$, with source $F=a$, target $F=b$, and $u\in S^{k+1}T_a^*\otimes T_b$, where $G=\pi_k F$ and $\delta u\in T_a^*\otimes S^k T_a^*\otimes T_b$. Now (2.18) follows from the commutativity of (2.17) and, using (2.4), we see that (2.19) is a direct consequence of (2.18) and (2.15). For $k=0$, by (2.4) we deduce (2.20) from (2.9).

(ii) Assume that F_1, $F\in Q_{k+1}$ satisfy $F_1\xi=F\xi$ for all $\xi\in J_k(T)_a$. We prove that $F_1=F$ by induction on k. Let $k\geqslant 0$ and suppose that, if $k\geqslant 1$, our assertion holds for $k-1$. If $k\geqslant 1$, we have $\pi_{k-1}F_1=\pi_{k-1}F$ by our induction hypothesis. Therefore we can always write $F_1=F+u$, with $u\in S^{k+1}T_a^*\otimes T_b$. From (2.20) if $k=0$ and (2.19) if $k\geqslant 1$, we conclude that $u=0$ and that $F_1=F$.

For $k\geqslant 0$, let Q_{k+1}^k be the bundle of the $G\in Q_{k+1}$ satisfying $\pi_k G=I_k(a)$, where $a=$ source G. Assume that $k\geqslant 1$. The bundle Q_{k+1}^k is an affine bundle over X whose associated vector bundle is $S^{k+1}T^*\otimes T$; it possesses a canonical section I_{k+1}, which induces a bijection

$$Q_{k+1}^k \to S^{k+1}T^*\otimes T$$

sending $G\in Q_{k+1}^k(a)$ into $G-I_{k+1}(a)$. Composing this mapping with

$$\nu^{*-1}\otimes\nu\colon S^{k+1}T^*\otimes T\to S^{k+1}J_0(T)^*\otimes J_0(T),$$

we obtain a bijection

$$\partial\colon Q^k_{k+1} \to S^{k+1}J_0(T)^* \otimes J_0(T)$$

which is an isomorphism of bundles of Lie groups over X, by Proposition 2.1, (i) and (ii). For $G \in Q^k_{k+1}(a)$, we have

$$G_* - I_{k+1}(a)_*\colon T_a \to v_{I_k(a)}(Q_k)$$

and

$$G_* - I_{k+1}(a)_* = \delta\partial G, \tag{2.21}$$

by the definition of ∂ (see [4], § 5).

For $k \geqslant 0$, let $Q^0_{(1,k)}$ be the set of the $G \in Q_{(1,k)}$ which project in Q_k onto I_k. The bundle $Q^0_{(1,k)}$ has a canonical section $j_1(I_k) = \lambda_1(I_{k+1})$ over X and we therefore obtain the injection

$$\partial\colon Q^0_{(1,k)} \to T^* \otimes \tilde{J}_k(T)$$

sending $H \in Q^0_{(1,k)}(a)$ into $H - j_1(I_k)(a)$, whose image is, by Proposition 2.2, (i),

$$(T^* \otimes \tilde{J}_k(T))^\wedge = \{u \in T^* \otimes \tilde{J}_k(T) \mid \mathrm{id} + \pi_0 u\colon T \to T \text{ is invertible}\}.$$

By (2.18), for $k \geqslant 1$ the diagram

$$\begin{array}{ccc} Q^k_{k+1} & \xrightarrow{\lambda_1} & Q^0_{(1,k)} \\ \Big\downarrow{\scriptstyle\partial} & & \Big\downarrow{\scriptstyle\partial} \\ S^{k+1}J_0(T)^* \otimes J_0(T) & \xrightarrow{\delta} & T^* \otimes \tilde{J}_k(T) \end{array} \tag{2.22}$$

is commutative. For $k = 0$, $\lambda_1\colon Q^0_1 \to Q^0_{(1,0)}$ is a bijection and we define $\partial\colon Q^0_1 \to J_0(T)^* \otimes J_0(T)$ so that the diagram (2.22) is commutative, where $\delta = \nu^* \otimes \nu^{-1}$. Then (2.21) holds with $k = 0$.

We now list fundamental formulas which will be used in the sequel (see [19], [18]). We have the following non-linear Spencer complex, a finite form of the initial portion of (1.3) (with T replacing E and $k+1$ replacing k):

$$\operatorname{Aut}(X) \xrightarrow{j_{k+1}} \tilde{\mathcal{Q}}_{k+1} \xrightarrow{\mathcal{D}} (\mathcal{T}^* \otimes J_k(\mathcal{T}))^\wedge \xrightarrow{\mathcal{D}_1} \wedge^2 \mathcal{T}^* \otimes J_{k-1}(\mathcal{T}) \tag{2.23}$$

which is exact, where (finite form of (1.22))

$$\mathcal{D}F = \chi - F^{-1}(\chi) \in \mathcal{T}^* \otimes \tilde{J}_k(\mathcal{T}), \tag{2.24}$$

$$\mathcal{D}F = F^{-1}(\nu) - \nu \in \tilde{J}_k(\mathcal{T})^* \otimes J_k(\mathcal{T}), \tag{2.25}$$

for $F \in \tilde{\mathcal{Q}}_{k+1}$, and

$$\mathcal{D}_1 u = Du - \tfrac{1}{2}[u, u], \quad u \in \mathcal{T}^* \otimes J_k(\mathcal{T}), \tag{2.26}$$

and

$$(T^* \otimes J_k(T))^\wedge = \{u \in T^* \otimes J_k(T) \mid \nu + \pi_0 u\colon T \to J_0(T) \text{ is invertible}\}.$$

We have (finite form of (1.2))

$$\partial[(\lambda_1 F)^{-1}\cdot j_1(\pi_k F)] = (\mathrm{id}\otimes\nu^{-1})\mathcal{D}F, \qquad F\in\tilde{Q}_{k+1}, \tag{2.27}$$

and hence also

$$(\pi_0\xi)\,\overline{\wedge}\,\mathcal{D}F = \nu((\lambda_1 F)^{-1}\cdot\pi_k F\cdot\xi-\xi), \qquad \xi\in\tilde{J}_k(\mathcal{T}),\ F\in\tilde{Q}_{k+1}. \tag{2.28}$$

If $G\in Q^k_{k+1}$, then since $j_1(I_k)=\lambda_1 I_{k+1}$, we have by (2.27)

$$\partial(\lambda_1 G)^{-1} = (\mathrm{id}\otimes\nu^{-1})\mathcal{D}G,$$

where $\mathcal{D}G\in\mathcal{T}^*\otimes J^0_k(\mathcal{T})$ if $k\geqslant 1$, and by (2.22)

$$\partial(\lambda_1 G)^{-1} = \partial\lambda_1 G^{-1} = \delta\partial G^{-1}.$$

By Proposition 2.1, (iii), we therefore have for $G\in Q^k_{k+1}$

$$\mathcal{D}G = -\delta g, \qquad \text{if } k\geqslant 1, \tag{2.29}$$

$$\mathcal{D}G = -(\mathrm{id}+g)^{-1}\circ g\circ\nu = [(\mathrm{id}+g)^{-1}-\mathrm{id}]\circ\nu, \qquad \text{if } k=0, \tag{2.30}$$

where $g=\partial G$. If $u\in(\mathcal{T}^*\otimes J_k(\mathcal{T}))_y$, $F\in\tilde{Q}_{k+1,x}$ with $(\pi_0 F)(x)=y$, we define

$$u^F = F^{-1}(u)+\mathcal{D}F. \tag{2.31}$$

This right operation of $\tilde{Q}_{k+1}$ on $\mathcal{T}^*\otimes J_k(\mathcal{T})$ conserves $(\mathcal{T}^*\otimes J_k(\mathcal{T}))^\wedge$.

LEMMA 2.1. *Let $F\in\tilde{Q}_{k+1}$, u_1, $u_2\in(\mathcal{T}^*\otimes J_k(\mathcal{T}))^\wedge$. Then $u_2=u_1^F$ if and only if*

$$\lambda_1 F\cdot(j_1(I_k)+\nu^{-1}\circ u_2) = j_1(\pi_k F)+(\nu^{-1}\circ u_1\circ f)\pi_k F$$

as elements of $Q_{(1,k)}$, where $f=\pi_0 F$ and $(\nu^{-1}\circ u_1\circ f)(a)\in T^*_a\otimes\tilde{J}_k(T)_{f(a)}$, $[(\nu^{-1}\circ u_1\circ f)\pi_k F](a)\in T^*_a\otimes V_{\pi_k F(a)}(Q_k)$.

Proof. It follows from (2.28) that $u_2=u_1^F$ if and only if we have, for all $\xi\in\tilde{J}_k(\mathcal{T})$,

$$u_2(\pi_0\xi) = \nu((\lambda_1 F)^{-1}\cdot\pi_k F\cdot\xi-\xi)+F^{-1}((u_1\circ f)(\pi_0\xi)),$$

i.e., by (2.6) and (2.5),

$$\begin{aligned}\xi+\nu^{-1}\cdot u_2(\pi_0\xi) &= \lambda_1 F^{-1}\cdot\pi_k F\cdot\xi\cdot\pi_k F^{-1}\cdot\pi_k F+\lambda_1 F^{-1}((\nu^{-1}\circ u_1\circ f)(\pi_0\xi))\\ &= \lambda_1 F^{-1}(\pi_k F\cdot\xi\cdot\pi_k F^{-1}+(\nu^{-1}\circ u_1\circ f)(\pi_0\xi))\\ &= \lambda_1 F^{-1}(j_1(\pi_k F)(\xi)+(\nu^{-1}\circ u_1\circ f)(\pi_0\xi)).\end{aligned}$$

According to (2.15), this equation is equivalent to

$$(j_1(I_k)+\nu^{-1}\circ u_2)(\xi) = \lambda_1 F^{-1}((j_1(\pi_k F)+(\nu^{-1}\circ u_1\circ f)\pi_k F)(\xi));$$

hence, by Proposition 2.2, (iii), the equation $u_2=u_1^F$ holds if and only if

$$j_1(I_k)+\nu^{-1}\circ u_2 = \lambda_1 F^{-1}\cdot(j_1(\pi_k F)+(\nu^{-1}\circ u_1\circ f)\pi_k F),$$

which implies the desired assertion.

Next, the non-linear Spencer complex

$$\mathrm{Aut}\,(X) \xrightarrow{j_{k+1}} \tilde{Q}_{k+1} \xrightarrow{\tilde{\mathcal{D}}} (J_0(\mathcal{T})^* \otimes \tilde{J}_k(\mathcal{T}))^\wedge \xrightarrow{\tilde{\mathcal{D}}_1} \wedge^2 J_0(\mathcal{T})^* \otimes \tilde{J}_{k-1}(\mathcal{T}) \tag{2.32}$$

is exact, where (finite form of (1.23))

$$\tilde{\mathcal{D}}F = \bar{\chi} - F^{-1}(\bar{\chi}) \in J_0(\mathcal{T})^* \otimes \tilde{J}_k(\mathcal{T}), \tag{2.33}$$

$$\tilde{\mathcal{D}}F = \nu^{-1} - F^{-1}(\nu^{-1}) \in J_k(\mathcal{T})^* \otimes \tilde{J}_k(\mathcal{T}), \tag{2.34}$$

for $F \in \tilde{Q}_{k+1}$, and

$$\tilde{\mathcal{D}}_1 u = \tilde{D}u - \tfrac{1}{2}\pi_{k-1}[u, u], \qquad u \in J_0(\mathcal{T})^* \otimes \tilde{J}_k(\mathcal{T}), \tag{2.35}$$

and

$$(J_0(T)^* \otimes \tilde{J}_k(T))^\wedge = \{u \in J_0(T)^* \otimes \tilde{J}_k(T) \,|\, \nu^{-1} - \pi_0 u \colon J_0(T) \to T \text{ is invertible}\}.$$

The analogues of (2.27)–(2.30) are:

$$\partial[(j_1(\pi_k F))^{-1} \cdot (\lambda_1 F)] = -(\nu^* \otimes \mathrm{id})\tilde{\mathcal{D}}F, \qquad F \in \tilde{Q}_{k+1}; \tag{2.36}$$

$$(\pi_0 \nu \xi) \,\bar{\wedge}\, \tilde{\mathcal{D}}F = \xi - \pi_k F^{-1} \cdot (\lambda_1 F) \cdot \xi, \qquad \xi \in \tilde{J}_k(\mathcal{T}),\ F \in \tilde{Q}_{k+1}; \tag{2.37}$$

$$\tilde{\mathcal{D}}G = -\delta g, \qquad \text{if } k \geqslant 1, \tag{2.38}$$

$$\tilde{\mathcal{D}}G = -(\mathrm{id} \otimes \nu^{-1}) g, \qquad \text{if } k = 0, \tag{2.39}$$

where $g = \partial G$, for $G \in Q^k_{k+1}$. If $u \in (J_0(\mathcal{T})^* \otimes \tilde{J}_k(\mathcal{T}))_y$, $F \in \tilde{Q}_{k+1,x}$ with $(\pi_0 F)(x) = y$, we define

$$u^F = F^{-1}(u) + \tilde{\mathcal{D}}F. \tag{2.40}$$

This right operation of $F \in \tilde{Q}_{k+1}$ on $J_0(\mathcal{T})^* \otimes \tilde{J}_k(\mathcal{T})$ conserves $(J_0(\mathcal{T})^* \otimes \tilde{J}_k(\mathcal{T}))^\wedge$ and the action of F^{-1} on $\wedge J_0(\mathcal{T})^* \otimes \tilde{J}_k(\mathcal{T})$ depends only on $\pi_k F$ if $k \geqslant 1$; hence, if $k \geqslant 1$,

$$u^F = (\pi_k F)^{-1}(u) + \tilde{\mathcal{D}}F.$$

We have the following important identities whose analogues are also valid for the operators $\mathcal{D}$, $\mathcal{D}_1$ and $\wedge \mathcal{T}^* \otimes J_k(\mathcal{T})$; if $F \in \tilde{Q}_{k+1,x}$, $G \in \tilde{Q}_{k+1,z}$ with $(\pi_0 F)(x) = y$ and $(\pi_0 G)(z) = x$, then

$$\tilde{\mathcal{D}}(F \cdot G) = G^{-1}(\tilde{\mathcal{D}}F) + \tilde{\mathcal{D}}G, \tag{2.41}$$

$$u^{FG} = (u^F)^G, \tag{2.42}$$

and

$$\tilde{\mathcal{D}}_1 u^F = F^{-1}(\tilde{\mathcal{D}}_1 u), \tag{2.43}$$

for $u \in (J_0(\mathcal{T})^* \otimes \tilde{J}_k(\mathcal{T}))_y$.

There is a canonical bijection

$$(T^* \otimes J_k(T))^\wedge \to (J_0(T)^* \otimes \tilde{J}_k(T))^\wedge \tag{2.44}$$

sending the element $u \in (T^* \otimes J_k(T))^\wedge$ into $\tilde{u} \in (J_0(T)^* \otimes \tilde{J}_k(T))^\wedge$, which is defined as follows. Let $u \in (T^* \otimes J_k(T))^\wedge$; then

$$\nu + u\circ\pi_0 \colon \tilde{J}_k(T) \to J_k(T)$$

is invertible since

$$\mathrm{id} + \nu^{-1}\circ u\circ\pi_0 \colon \tilde{J}_k(T) \to \tilde{J}_k(T)$$

is invertible by Proposition 2.2, (i), with $H = j_1 I_k = \lambda_1 I_{k+1}$ and u replaced by $\nu^{-1}\circ u$. Let $\tilde{u}$ be the element of $J_k(T)^*\otimes\tilde{J}_k(T)$ which is defined by $\nu^{-1} - \tilde{u} = (\nu + u\circ\pi_0)^{-1}$; we have

$$\mathrm{id} = (\nu^{-1} - \tilde{u})\circ(\nu + u\circ\pi_0) = \mathrm{id} - \tilde{u}\circ\nu + (\nu^{-1} - \tilde{u})\circ u\circ\pi_0$$

and hence

$$\tilde{u}\circ\nu = (\nu^{-1} - \tilde{u})\circ u\circ\pi_0 \colon \tilde{J}_k(T) \to \tilde{J}_k(T).$$

Since $u\circ\pi_0$ vanishes on $J_k^0(T)\subset\tilde{J}_k(T)$ and ν is the identity on $J_k^0(T)$, we conclude that $\tilde{u}$ vanishes on $J_k^0(T)$; hence $\tilde{u} = \bar{u}\circ\pi_0$ where $\bar{u}\in(J_0(T)^*\otimes\tilde{J}_k(T))^\wedge$ and $\bar{u}$ is the image of u in (2.44). Since $T^*\subset\tilde{J}_k(T)^*$, $J_0(T)^*\subset J_k(T)^*$, we can drop π_0 and define $\bar{u}$ by

$$\nu^{-1} - \bar{u} = (\nu + u)^{-1} \colon J_k(T) \to \tilde{J}_k(T).$$

LEMMA 2.2 *The following assertions are true for the mapping* (2.44):

(i) *Let R_k be a sub-bundle of $J_k(T)$ and $\tilde{R}_k = \nu^{-1}R_k$, and let $u\in(T^*\otimes J_k(T))^\wedge$. Then $u\in T^*\otimes R_k$ if and only if $\bar{u}\in J_0(T)^*\otimes\tilde{R}_k$.*

(ii) *If $u\in(\mathcal{T}^*\otimes J_k(\mathcal{T}))^\wedge$, then*

$$\overline{u^F} = \bar{u}^F, \qquad \textit{for } F\in\tilde{Q}_{k+1}.$$

(iii) *We have*

$$\overline{\mathfrak{D}F} = \tilde{\mathfrak{D}}F, \qquad \textit{for } F\in\tilde{Q}_{k+1}.$$

(iv) *If $u\in(\mathcal{T}^*\otimes J_k(\mathcal{T}))^\wedge$, then $\mathfrak{D}_1 u = 0$ if and only if $\tilde{\mathfrak{D}}_1\bar{u} = 0$.*

Proof. We have

$$\mathrm{id} = (\nu^{-1} - \bar{u})\circ(\nu + u) = \mathrm{id} - \bar{u}\circ(\nu + u) + \nu^{-1}\circ u,$$

and hence

$$\nu^{-1}\circ u = \bar{u}\circ(\nu + u) \colon \tilde{J}_k(T) \to \tilde{J}_k(T). \tag{2.45}$$

Similarly

$$\mathrm{id} = (\nu + u)\circ(\nu^{-1} - \bar{u}) = \mathrm{id} - \nu\circ\bar{u} + u\circ(\nu^{-1} - \bar{u}),$$

and hence

$$\nu\circ\bar{u} = u\circ(\nu^{-1} - \bar{u}) \colon J_k(T) \to J_k(T). \tag{2.46}$$

Assertion (i) follows immediately from (2.45) and (2.46).

If $u\in(\mathcal{T}^*\otimes J_k(\mathcal{T}))^\wedge$, $F\in\tilde{Q}_{k+1}$, we have by (2.25)

$$\nu + u^F = F^{-1}(\nu + u) = F^{-1}\circ(\nu + u)\circ F,$$

and hence

$$\begin{aligned}\nu^{-1} - \overline{u^F} &= (\nu + u^F)^{-1} = F^{-1}\circ(\nu + u)^{-1}\circ F = F^{-1}\circ(\nu^{-1} - \bar{u})\circ F \\ &= \nu^{-1} - (\nu^{-1} - F^{-1}(\nu^{-1})) - F^{-1}(\bar{u}) = \nu^{-1} - (\tilde{\mathfrak{D}}F + F^{-1}(\bar{u})) = \nu^{-1} - \bar{u}^F\end{aligned}$$

by (2.34), that is to say, (ii) holds. Taking $u=0$ in (ii), we obtain (iii). Finally let $u\in(\mathcal{T}^*\otimes J_k(\mathcal{T}))^\wedge$; then, by the exactness of (2.23), $\mathcal{D}_1 u=0$ if and only if $u=\mathcal{D}F$, $F\in\tilde{Q}_{k+1}$ and hence by (iii), $\bar{u}=\bar{\mathcal{D}}F$, which, by the exactness of (2.32), is equivalent to $\bar{\mathcal{D}}_1\bar{u}=0$.

LEMMA 2.3. *Let u be a section of $J_0(T)^*\otimes T$ over X and $f\colon X\to X$ a mapping. Let F be the section $j_1(f)-f\circ u\circ\nu$ of $J_1(E)$. Then:*

(i) $F\xi=\nu(f(\nu^{-1}-u)\xi)$, *for* $\xi\in J_0(T)$;

(ii) *F is a section of $\tilde{Q}_1$ if and only if $\nu^{-1}-u\colon J_0(T)\to T$ is invertible and f is an immersion;*

(iii) *if F is a section of $\tilde{Q}_1$, we have $\bar{\mathcal{D}}F=u$.*

Proof. According to Proposition 2.1, (i), we have

$$F=j_1(f)\cdot(I_1-u\circ\nu), \tag{2.47}$$

so (i) holds since $I_1-u\circ\nu\colon J_0(T)\to J_0(T)$ is equal to $\mathrm{id}-\nu\circ u$. Hence F is a section of $\tilde{Q}_1$ if and only if $f\circ(\nu^{-1}-u)\colon J_0(T)\to T$ is invertible, and so we obtain (ii). Applying $\bar{\mathcal{D}}$ to (2.47) we obtain, by (2.41) and (2.39),

$$\bar{\mathcal{D}}F=\bar{\mathcal{D}}(I_1-u\circ\nu)=u.$$

Finally let $\hat{B}^1_k$ be the set of the $u\in B^1_k$ whose projection $\pi_0 u$ in $J_0(T)^*\otimes T$ satisfies the condition that $\nu^{-1}-\pi_0 u\colon J_0(T)\to T$ is invertible. The operator $\bar{\mathcal{D}}\colon\tilde{Q}_{k+1}\to J_0(\mathcal{T})^*\otimes\tilde{J}_k(\mathcal{T})$ induces a differential operator $\hat{\mathcal{D}}\colon\tilde{Q}_k\to\hat{\mathcal{B}}^1_k$ for $k\geqslant 1$ and, for $u\in\mathcal{B}^1_k$, let $\hat{\mathcal{D}}_1 u=\hat{D}u-\tfrac{1}{2}[u,u]\in\mathcal{B}^2_k$. We thus obtain the "sophisticated" version of (2.32)

$$\mathrm{Aut}\,(X)\xrightarrow{j_k}\tilde{Q}_k\xrightarrow{\hat{\mathcal{D}}}\hat{\mathcal{B}}^1_k\xrightarrow{\hat{\mathcal{D}}_1}\mathcal{B}^2_k \tag{2.48}$$

which is an exact sequence for $k\geqslant 1$. Let $F\in\tilde{Q}_k$ where $k\geqslant 1$; the action of F^{-1} on $\wedge J_0(\mathcal{T})^*\otimes\tilde{J}_k(\mathcal{T})$ gives by passage to the quotient an action on $\mathcal{B}_k$ and we define

$$u^F=F^{-1}(u)+\hat{\mathcal{D}}F, \tag{2.49}$$

for $u\in\mathcal{B}^1_{k,y}$, $F\in\tilde{Q}_{k,x}$ with $(\pi_0 F)(x)=y$. This right action of $\tilde{Q}_k$ on $\mathcal{B}^1_k$ conserves $\hat{\mathcal{B}}^1_k$ and the analogues of formulas (2.41)–(2.43) hold for the operators $\hat{\mathcal{D}}$, $\hat{\mathcal{D}}_1$ and the sheaf $\mathcal{B}_k$.

We conclude this section by recalling the definition and some properties of a non-linear partial differential equation. A (non-linear) partial differential equation P_k (of order k) in $J_k(E)$ is a fibered submanifold of $\pi\colon J_k(E)\to X$. The l-th prolongation of P_k is the subset of $J_{k+l}(E)$,

$$(P_k)_{+l}=\lambda_l^{-1}(J_l(P_k)\cap\lambda_l(J_{k+l}(E))),$$

where λ_l is the injection $J_{k+l}(E)\to J_l(J_k(E))$. A solution of P_k is a mapping $f\colon U\to X$ defined on an open set $U \subset X$ and satisfying $j_k(f)(x) \in P_k$ for all $x \in U$; then $j_{k+l}(f)$ is a section of $(P_k)_{+l}$ over U, for all $l \geqslant 0$. The mapping $\pi_{k+l}: J_{k+l+1}(E) \to J_{k+l}(E)$ induces a mapping $\pi_{k+l}: (P_k)_{+(l+1)} \to (P_k)_{+l}$. Following [4], we say that P_k is *formally integrable* if for all $l \geqslant 0$, $\pi: (P_k)_{+l} \to X$ is a fibered submanifold of $\pi: J_{k+l}(E) \to X$ and $\pi_{k+l}: (P_k)_{+(l+1)} \to (P_k)_{+l}$ is a fibered submanifold of $\pi_{k+l}: J_{k+l+1}(E)_{|(P_k)_{+l}} \to (P_k)_{+l}$. According to Proposition 7.1 of [4], if P_k is formally integrable, then $\pi_{k+l}: (P_k)_{+(l+1)} \to (P_k)_{+l}$ is an affine sub-bundle of $\pi_{k+l}: J_{k+l+1}(E)_{|(P_k)_{+l}} \to (P_k)_{+l}$. We say that P_k is *integrable* if, for all $l \geqslant 0$ and $p \in (P_k)_{+l,x}$, there exists a solution f of P_k on a neighborhood of x such that $j_{k+l}(f)(x) = p$. If X is endowed with a structure of an analytic manifold and P_k is an analytic, formally integrable differential equation in $J_k(E)$ then, according to Theorem 9.1 of [4] or the appendix of [19], it is integrable.

If $P_k \subset Q_k$ and $k \geqslant 1$, then a solution of P_k is necessarily a local immersion $X \to X$; furthermore, if $\tilde{J}_l(P_k) = J_l(P_k) \cap Q_{(l,k)}$, we have

$$(P_k)_{+l} = \lambda_l^{-1}(\tilde{J}_l(P_k) \cap \lambda_l(Q_{k+l})),$$

where λ_l is the mapping $Q_{k+l} \to Q_{(l,k)}$.

3. Jet bundles and fibrations

Let Y be a differentiable manifold, whose tangent bundle we denote by T_Y, and let $\varrho: X \to Y$ be a surjective submersion, $V = T(X/Y)$ the integrable sub-bundle of $T = T_X$ of vectors tangent to the fibers of ϱ. If $\varrho = \varrho_*: T \to T_Y$ is the differential of ϱ, then

$$0 \longrightarrow V \longrightarrow T \xrightarrow{\varrho} \varrho^{-1}T_Y \longrightarrow 0$$

is an exact sequence of vector bundles over X. Let E and F be fibered manifolds over X and Y respectively and $\varphi: E \to F$ a morphism of fibered manifolds over ϱ. We denote by $\mathfrak{F}$, $\mathfrak{F}_X$ the sheaves of sections of F over Y and of $\varrho^{-1}F$ over X respectively. We say that a section s of E over $U \subset X$ is φ-projectable if $\varphi s(a) = \varphi s(b)$ for $a, b \in U$ whenever $\varrho(a) = \varrho(b)$. Then the section φs of F over $\varrho(U)$, which sends $y \in \varrho(U)$ into $\varphi s(a)$ where $a \in U$, $\varrho(a) = y$, is well-defined. We denote by $\mathcal{E}_\varphi$ the sheaf of sections of E which are φ-projectable and by $J_k(E; \varphi) \subset J_k(E)$ the set of k-jets of sections of $\mathcal{E}_\varphi$. If $\varphi: E \to F$ has constant rank, $J_k(E; \varphi)$ is a bundle and if, moreover, E, F are vector bundles and φ is a morphism of vector bundles, it is a vector bundle; the sheaf of solutions of $J_k(E; \varphi)$ is $\mathcal{E}_\varphi$. If $J_k(F; Y)$ is the bundle of k-jets of sections of F over Y, we have a mapping

$$\varphi: J_k(E; \varphi) \to J_k(F; Y). \tag{3.1}$$

We now assume that E, F are vector bundles and that $\varphi: E \to F$ is a morphism of vector bundles. If K is the kernel of $\varphi: E \to \varrho^{-1}F$ and if this mapping is surjective, then $J_k(K)$ is the kernel of the mapping (3.1).

Let $F_i^{i+j}(\varrho)$ be the sub-bundle $\wedge^j T^* \otimes \varrho^*(\wedge^i T_Y^*)$ of $\wedge^{i+j} T^*$ for $j \geq 0$; we set $F_i^{i+j}(\varrho) = F_{i+j}^{i+j}(\varrho)$ for $j<0$. Then $F_{i+1}^{i+j}(\varrho) \subset F_i^{i+j}(\varrho)$ and $F_0^j(\varrho) = \wedge^j T^*$. We define, for $j \geq 0$,

$$F_i^{i+j}(J_k(E); \varphi) = \{u \in \wedge^{i+j} T^* \otimes J_k(E; \varphi) \,|\, (\mathrm{id} \otimes \varphi)\, u \in F_i^{i+j}(\varrho) \otimes_X J_k(F; Y)\}$$

and, for $j<0$, we set

$$F_i^{i+j}(J_k(E); \varphi) = \wedge^{i+j} T^* \otimes J_k(K).$$

Then

$$F_0^j(J_k(E); \varphi) = \wedge^j T^* \otimes J_k(E; \varphi),$$

$$F_{i+1}^{i+j}(J_k(E); \varphi) \subset F_i^{i+j}(J_k(E); \varphi),$$

and

$$F_i^{i+j}(\varrho) \otimes J_k(E; \varphi) \subset F_i^{i+j}(J_k(E); \varphi).$$

We suppose henceforth that $\varphi: E \to \varrho^{-1}F$ is surjective. Then the sequence

$$0 \longrightarrow F_{i+1}^{i+j}(J_k(E); \varphi) \longrightarrow F_i^{i+j}(J_k(E); \varphi) \xrightarrow{\varphi} \wedge^j V^* \otimes_X (\wedge^i T_Y^* \otimes J_k(F; Y)) \longrightarrow 0$$

of vector bundles is exact for $j \geq 0$, where φ sends $u \in F_i^{i+j}(J_k(E); \varphi)$ into the element φu defined by the formula

$$(\varphi u)(\xi_1 \wedge \ldots \wedge \xi_j \otimes \bar{\eta}_1 \wedge \ldots \wedge \bar{\eta}_i) = \varphi(u(\xi_1 \wedge \ldots \wedge \xi_j \wedge \eta_1 \wedge \ldots \wedge \eta_i))$$

where $\xi_1, \ldots, \xi_j \in V$, $\eta_1, \ldots, \eta_i \in T$ and $\bar{\eta}_l = \varrho(\eta_l) \in T_Y$ for $1 \leq l \leq i$. In particular, we have the exact sequence

$$0 \longrightarrow \wedge^i T^* \otimes J_k(K) \longrightarrow F_i^i(J_k(E); \varphi) \xrightarrow{\varphi} \varrho^{-1}(\wedge^i T_Y^* \otimes J_k(F; Y)) \longrightarrow 0$$

and

$$F_i^i(J_k(E); \varphi) = \wedge^i T^* \otimes J_k(K) + \varrho^*(\wedge^i T_Y^*) \otimes J_k(E; \varphi).$$

We denote by $(\wedge^i \mathcal{T}^* \otimes J_k(\mathcal{E}; \varphi))_\varphi$ the sheaf of φ-projectable sections of $F_i^i(J_k(E); \varphi)$; we then have the mapping

$$\varphi: (\wedge^i \mathcal{T}^* \otimes J_k(\mathcal{E}; \varphi))_\varphi \to \wedge^i \mathcal{T}_Y^* \otimes J_k(\mathcal{F}; Y).$$

According to Proposition 3 of [6],

$$D(F_i^{i+j}(J_k(\mathcal{E}); \varphi)) \subset F_i^{i+j+1}(J_{k-1}(\mathcal{E}); \varphi),$$

and so, in particular,

$$D(J_k(\mathcal{E}; \varphi)) \subset \mathcal{T}^* \otimes J_{k-1}(\mathcal{E}; \varphi).$$

For $k \geq 1$, the sub-bundle $J_k(E; \varphi)$ of $J_k(E)$ is in fact a formally integrable equation whose

l-th prolongation is $J_{k+l}(E; \varphi)$ (see [6], Corollary 3). Furthermore, if

$$d_{X/Y}\colon \wedge^j \mathcal{V}^* \otimes \mathfrak{F}_X \to \wedge^{j+1} \mathcal{V}^* \otimes \mathfrak{F}_X$$

is the exterior derivative along the fibers of $\varrho\colon X \to Y$, then the diagram

$$\begin{tikzcd} F_i^{i+j}(J_k(\mathcal{E}); \varphi) \arrow[r, "D"] \arrow[d, "\varphi"] & F_i^{i+j+1}(J_k(\mathcal{E}); \varphi) \arrow[d, "\varphi"] \\ \wedge^j \mathcal{V}^* \otimes (\wedge^i \mathcal{T}_Y^* \otimes J_k(\mathfrak{F}; Y))_X \arrow[r, "\pi_{k-1} \cdot d_{X/Y}"] & \wedge^{j+1} \mathcal{V}^* \otimes (\wedge^i \mathcal{T}_Y^* \otimes J_{k-1}(\mathfrak{F}; Y))_X \end{tikzcd}$$

commutes. In particular, we have the commutative diagram

$$\begin{tikzcd} \wedge^i \mathcal{T}^* \otimes J_k(\mathcal{E}; \varphi) \arrow[r, "D"] \arrow[d, "\varphi"] & \wedge^{i+1} \mathcal{T}^* \otimes J_{k-1}(\mathcal{E}; \varphi) \arrow[d, "\varphi"] \\ \wedge^i \mathcal{V}^* \otimes J_k(\mathfrak{F}; Y)_X \arrow[r, "\pi_{k-1} \cdot d_{X/Y}"] & \wedge^{i+1} \mathcal{V}^* \otimes J_{k-1}(\mathfrak{F}; Y)_X. \end{tikzcd} \tag{3.2}$$

The following lemma complements Proposition 4 of [6] and will not be used in this paper.

LEMMA 3.1. *Assume that* $\varphi\colon E \to \varrho^{-1}F$ *is surjective and let* $u \in J_k(\mathcal{E}; \varphi)$; *then* $u \in J_k(\mathcal{E}; \varphi)_\varphi$ *if and only if* $Du \in (\mathcal{T}^* \otimes J_{k-1}(\mathcal{E}; \varphi))_\varphi$.

Proof. By Proposition 4, (ii) of [6] we know that, if $u \in J_k(\mathcal{E}; \varphi)_\varphi$, then $Du \in (\mathcal{T}^* \otimes J_{k-1}(\mathcal{E}; \varphi))_\varphi$. We now prove the converse. We have the following commutative diagram:

$$\begin{tikzcd} F_1^1(J_{k-1}(E); \varphi) \arrow[r, "\varphi"] \arrow[d, "\varepsilon"] & T_Y^* \otimes J_{k-1}(F; Y) \arrow[d, "\varepsilon"] \\ J_1(J_{k-1}(E; \varphi); \varphi) \arrow[r, "\varphi"] & J_1(J_{k-1}(F; Y); Y) \\ J_k(E; \varphi) \arrow[r, "\varphi"] \arrow[u, "\lambda_1"] & J_k(F; Y) \arrow[u, "\lambda_1"] \end{tikzcd}$$

all of whose vertical arrows are injections. The mappings ε, λ_1 in the left column are respectively the restrictions of the mappings

$$\varepsilon\colon T^* \otimes J_{k-1}(E; \varphi) \to J_1(J_{k-1}(E; \varphi)), \qquad \lambda_1\colon J_k(E) \to J_1(J_{k-1}(E)).$$

We remark that the commutativity of the upper square of the diagram follows from the fact that, if s is a φ-projectable section of $J_{k-1}(E)$ and f is the pull-back to X of a function on Y, then $f\cdot s$ is also φ-projectable. Let $u\in J_k(\mathcal{E};\varphi)$ and suppose that $Du\in(\mathcal{T}^*\otimes J_{k-1}(\mathcal{E};\varphi))_\varphi$. By the commutativity of the lower square of the diagram, u is φ-projectable if and only if $\lambda_1 u$ is φ-projectable. By Proposition 4, (i) of [6], we know that $\pi_{k-1}u\in J_{k-1}(\mathcal{E};\varphi)_\varphi$ and hence $j_1(\pi_{k-1}u)\in J_1(J_{k-1}(\mathcal{E};\varphi))_\varphi$. Finally, we infer from formula (1.2) that $\lambda_1 u$ is φ-projectable if and only if εDu is φ-projectable and the φ-projectability of εDu follows from the commutativity of the upper square of the diagram.

LEMMA 3.2. *Let $x\in X$ with $y=\varrho(x)$. All linear maps*

$$D_x\colon J_k(\mathcal{F};\,Y)_{X,x}\to T_x^*\otimes J_k(F;\,Y)_y$$

satisfying the following two conditions are equal:

(i) *for $s\in\mathcal{F}_y$,*

$$D_x(j_k(s)\circ\varrho)=0;$$

(ii) *for $f\in O_{X,x}$, $u\in J_k(\mathcal{F};\,Y)_{X,x}$,*

$$D_x(fu)=(df\otimes\pi_{k-1}u)(x)+f(x)\,D_xu.$$

Proof. Suppose that D_x, D_x' are two such maps satisfying these conditions. Then for $s\in\mathcal{F}_y$, we have by (i)

$$(D_x-D_x')(j_k(s)\circ\varrho)=0.$$

By (ii), for $f\in O_{X,x}$, $u\in J_k(\mathcal{F};\,Y)_{X,x}$,

$$(D_x-D_x')(fu)=f(x)(D_x-D_x')u.$$

Since $J_k(\mathcal{F};\,Y)_{X,x}$ is generated as an $O_{X,x}$-module by the elements of the form $j_k(s)\circ\varrho$, with $s\in\mathcal{F}_y$, these two relations imply that $D_x-D_x'=0$.

We now construct a generalization of the differential operator D of § 1.

PROPOSITION 3.1. *There exists a unique linear, first-order differential operator*

$$D\colon J_k(\mathcal{F};\,Y)_X\to\mathcal{T}^*\otimes J_{k-1}(\mathcal{F};\,Y)_X \tag{3.3}$$

satisfying one of the following equivalent conditions:

(i) *For all sections s of F over Y,*

$$D(j_k(s)\circ\varrho)=0 \tag{3.4}$$

and

$$D(fu)=df\otimes\pi_{k-1}u+fDu, \tag{3.5}$$

for $f\in O_X$, $u\in J_k(\mathcal{F};\,Y)_X$.

(ii) *If $E=\varrho^{-1}F$ and $\varphi\colon E\to\varrho^{-1}F$ is the identity map, the diagram*

$$\begin{array}{ccc} J_k(\mathcal{E};\varphi) & \xrightarrow{D} & \mathcal{T}^*\otimes J_{k-1}(\mathcal{E};\varphi) \\ \Big\downarrow \varphi & & \Big\downarrow \mathrm{id}\otimes\varphi \\ J_k(\mathfrak{F};Y)_X & \xrightarrow{D} & \mathcal{T}^*\otimes J_{k-1}(\mathfrak{F};Y)_X \end{array} \tag{3.6}$$

commutes.

(iii) *If* $u\in J_k(\mathfrak{F};Y)_{X,x}$ *and* $u(x)=j_k(s)(\varrho(x))$ *for* $s\in\mathfrak{F}_{\varrho(x)}$,

$$(\varepsilon Du)(x) = j_1(\pi_{k-1}u)(x) - j_1(j_{k-1}(s)\circ\varrho)(x). \tag{3.7}$$

(iv) *If* τ *is any section of* $\varrho\colon X\to Y$ *defined on a neighborhood of* $y\in Y$ *and* $\tau(y)=x$, *then for all sections* u *of* $J_k(\mathfrak{F};Y)_X$ *over a neighborhood of* x *and* $\xi\in T_x$,

$$\langle\xi, Du\rangle = \langle\xi-\tau_*\varrho_*\xi,\, d_{X/Y}\pi_{k-1}u\rangle + \langle\varrho_*\xi,\, D(u\circ\tau)\rangle, \tag{3.8}$$

as elements of $J_{k-1}(F;Y)_y$, *where the operator*

$$D\colon J_k(\mathfrak{F};Y)\to\mathcal{T}^*_Y\otimes J_{k-1}(\mathfrak{F};Y) \tag{3.9}$$

on the right-hand side is the one defined in § 1.

(v) *If* τ *is any section of* $\varrho\colon X\to Y$ *then, for all sections* u *of* $J_k(\mathfrak{F};Y)_X$,

$$(\tau^*\otimes\mathrm{id})(Du)\circ\tau = D(u\circ\tau) \tag{3.10}$$

as sections of $T^*_Y\otimes J_k(F;Y)$ *over* Y, *where* $\tau^*\colon T^*_{\tau(y)}\to T^*_{Y,y}$, *for* $y\in Y$, *and the operator* D *on the right-hand side is the one defined in* § 1, *namely* (3.9), *and*

$$(Du)_{|V} = \pi_{k-1}\cdot d_{X/Y}u. \tag{3.11}$$

Proof. If D is a linear operator (3.3) we define, for $x\in X$,

$$D_x\colon J_k(\mathfrak{F};Y)_{X,x}\to T^*_x\otimes J_k(F;Y)_{\varrho(x)}$$

by setting $D_x u=(Du)(x)$, for $u\in J_k(\mathfrak{F};Y)_{X,x}$; then D satisfies the conditions (i) if and only if the operator D_x satisfies the conditions of Lemma 3.2 for all $x\in X$. In particular, this permits us to deduce from Lemma 3.2 the uniqueness of an operator D satisfying the conditions (i). We begin by proving the existence of an operator D satisfying (i). Let $E=\varrho^{-1}F$ and $\varphi\colon E\to\varrho^{-1}F$ be the identity map; then

$$\varphi\colon J_m(E;\varphi)\to\varrho^{-1}J_m(F;Y)$$

is an isomorphism and sends $j_m(s\circ\varrho)(x)$ into $j_m(s)(\varrho(x))$, where $x\in X$ and s is a section of F over a neighborhood of $\varrho(x)$. Therefore there exists a unique map (3.3) such that the diagram (3.6) is commutative; it remains to verify that this operator satisfies (i). If s is a section of F over Y, we have

$$D(j_k(s)\circ\varrho) = (\mathrm{id}\otimes\varphi)\,D\varphi^{-1}(j_k(s)\circ\varrho) = (\mathrm{id}\otimes\varphi)\,Dj_k(s\circ\varrho) = 0,$$

and for $f \in \mathcal{O}_X$, $u \in J_k(\mathfrak{F};\, Y)_X$, by (1.4),

$$D(fu) = (\mathrm{id} \otimes \varphi)\, D\varphi^{-1}(fu) = (\mathrm{id} \otimes \varphi)\, D(f\varphi^{-1}u)$$
$$= (\mathrm{id} \otimes \varphi)\,(df \otimes \pi_{k-1}\varphi^{-1}u + fD(\varphi^{-1}u)) = df \otimes \pi_{k-1}u + f(\mathrm{id} \otimes \varphi)\, D(\varphi^{-1}u) = df \otimes \pi_{k-1}u + fDu,$$

which gives us the existence of an operator D satisfying (i) and shows that (i) and (ii) are equivalent.

Let $x \in X$, τ a section of $\varrho\colon X \to Y$ defined on a neighborhood of $y \in Y$ with $\tau(y) = x$, and let

$$D'_x,\ D''_x\colon J_k(\mathfrak{F};\, Y)_{X,x} \to T^*_x \otimes J_k(F;\, Y)_y$$

be the mappings defined by setting $\varepsilon D'_x u$ equal to the right-hand side of (3.7) and $\langle \xi, D''_x u\rangle$ equal to the right-hand side of (3.8), for $u \in J_k(\mathfrak{F};\, Y)_{X,x}$ and $\xi \in T_x$, with $u(x) = j_k(s)(y)$ and $s \in \mathfrak{F}_y$. We now show that these mappings satisfy the conditions of Lemma 3.2, from which it follows by Lemma 3.2 that (3.7) and (3.8) hold and that assertions (i)–(iv) are equivalent. If $s \in \mathfrak{F}_y$,

$$\varepsilon D'_x(j_k(s) \circ \varrho) = j_1(\pi_{k-1} j_k(s) \circ \varrho)(x) - j_1(j_{k-1}(s) \circ \varrho)(x) = 0$$

and, for $\xi \in T_x$,

$$\langle \xi, D''_x(j_k(s) \circ \varrho)\rangle = \langle \varrho_* \xi, D(j_k(s))\rangle = 0,$$

since $d_{X/Y}(j_k(s) \circ \varrho) = 0$ and $j_k(s) \circ \varrho \circ \tau = j_k(s)$. If $f \in \mathcal{O}_{X,x}$, $u \in J_k(\mathfrak{F};\, Y)_{X,x}$ with $u(x) = j_k(s)(y)$ where $s \in \mathfrak{F}_y$, we have $(fu)(x) = f(x)\, j_k(s)(y)$ and

$$\begin{aligned} \varepsilon D'_x(fu) &= j_1(\pi_{k-1} fu)(x) - j_1(f(x)\, j_{k-1}(s) \circ \varrho)(x) \\ &= j_1((f - f(x))\pi_{k-1}u)(x) + f(x)\, j_1(\pi_{k-1}u)(x) - f(x)\, j_1(j_{k-1}(s) \circ \varrho)(x) \\ &= \varepsilon(df \otimes \pi_{k-1}u)(x) + f(x)\, \varepsilon D'_x u. \end{aligned}$$

On the other hand for $\xi \in T_x$, since $(fu) \circ \tau = \tau^* f \cdot (u \circ \tau)$ and $\xi - \tau_* \varrho_* \xi \in V_x$, we have by (1.4):

$$\begin{aligned} \langle \xi, D''_x(fu)\rangle &= \langle \xi - \tau_*\varrho_*\xi,\, d_{X/Y}(f\pi_{k-1}u)\rangle + \langle \varrho_*\xi,\, D(\tau^*f \cdot (u \circ \tau))\rangle \\ &= \langle \xi - \tau_*\varrho_*\xi,\, d_{X/Y} f \otimes \pi_{k-1}u\rangle + \langle \xi - \tau_*\varrho_*\xi,\, f d_{X/Y}\, \pi_{k-1}u\rangle \\ &\qquad + \langle \varrho_*\xi,\, d\tau^* f \otimes \pi_{k-1}(u \circ \tau)\rangle + \langle \varrho_*\xi,\, \tau^*f \cdot D(u \circ \tau)\rangle \\ &= \langle \xi - \tau_*\varrho_*\xi,\, df\rangle \pi_{k-1}u(x) + f(x)\langle \xi - \tau_*\varrho_*\xi,\, d_{X/Y}\, \pi_{k-1}u\rangle \\ &\qquad + \langle \varrho_*\xi,\, \tau^* df\rangle \pi_{k-1}u(\tau(y)) + (\tau^*f)(y)\langle \varrho_*\xi,\, D(u \circ \tau)\rangle \\ &= \langle \xi,\, (df \otimes \pi_{k-1}u)(x)\rangle - \langle \tau_*\varrho_*\xi,\, df\rangle \pi_{k-1}u(x) \\ &\qquad + f(x)\langle \xi - \tau_*\varrho_*\xi,\, d_{X/Y}\pi_{k-1}u\rangle + \langle \tau_*\varrho_*\xi,\, df\rangle \pi_{k-1}u(x) + f(x)\langle \varrho_*\xi,\, D(u \circ \tau)\rangle \\ &= \langle \xi,\, (df \otimes \pi_{k-1}u)(x)\rangle + f(x)\langle \xi, D''_x u\rangle. \end{aligned}$$

Thus D'_x and D''_x satisfy the conditions of Lemma 3.2.

To complete the proof of the proposition, we now show that (v) implies (iv) and then that (i) implies (v). Let τ be a section of $\varrho\colon X \to Y$ defined on a neighborhood of $y \in Y$ and $x = \tau(y)$. Assume that D satisfies (v) and let $\xi \in T_x$; then $\xi - \tau_*\varrho_*\xi \in V_x$ and, by (3.10) and (3.11), if $u \in J_k(\mathfrak{F};\, Y)_X$,

$$\begin{aligned}\langle \xi, Du\rangle &= \langle \xi - \tau_*\varrho_*\xi, Du\rangle + \langle \tau_*\varrho_*\xi, Du\rangle \\ &= \langle \xi - \tau_*\varrho_*\xi, \pi_{k-1} d_{X/Y}\, u\rangle + \langle \varrho_*\xi, (\tau^*\otimes \mathrm{id})\, Du(\tau(y))\rangle \\ &= \langle \xi - \tau_*\varrho_*\xi, d_{X/Y}\pi_{k-1} u\rangle + \langle \varrho_*\xi, D(u\circ\tau)\rangle\end{aligned}$$

and thus (iv) holds. Finally, to show that (i) implies (v), we take $u = j_k(s)\circ\varrho$ in (3.10) and (3.11), where s is a section of F over Y; then both sides of each of these equations vanish by (3.4) and the facts that $Dj_k(s) = 0$, $d_{X/Y}(j_k(s)\circ\varrho) = 0$. If $f \in \mathcal{O}_X$ and $u \in J_k(\mathfrak{F};\, Y)_X$ then, by (1.4) and (3.5),

$$\begin{aligned}&(\tau^*\otimes\mathrm{id})\,(D(fu))\circ\tau - D((fu)\circ\tau) \\ &\qquad = (\tau^*\otimes\mathrm{id})\,(df\otimes\pi_{k-1}u)\circ\tau + (f\circ\tau)\,(\tau^*\otimes\mathrm{id})\,(Du)\circ\tau - d\tau^* f\otimes\pi_{k-1}u\circ\tau - \tau^* f\cdot D(u\circ\tau) \\ &\qquad = (f\circ\tau)\,[(\tau^*\otimes\mathrm{id})\,(Du)\circ\tau - D(u\circ\tau)].\end{aligned}$$

Similarly, if $\xi \in V$, we have by (3.5),

$$\begin{aligned}\langle \xi, D(fu) - \pi_{k-1} d_{X/Y}(fu)\rangle &= \langle \xi, df\otimes\pi_{k-1}u + fDu - d_{X/Y} f\otimes\pi_{k-1}u - f\cdot\pi_{k-1} d_{X/Y}\, u\rangle \\ &= \langle \xi, f(Du - \pi_{k-1} d_{X/Y}\, u)\rangle.\end{aligned}$$

Since $J_k(\mathfrak{F};\, Y)_{X,x}$ is generated as an $\mathcal{O}_{X,x}$-module by the elements $j_k(s)\circ\varrho$, where $s \in \mathfrak{F}_{\varrho(x)}$, for all $x \in X$, we obtain the identities (3.10) and (3.11).

We now define

$$D\colon \wedge^i \mathcal{T}^*\otimes J_k(\mathfrak{F};\, Y)_X \to \wedge^{i+1}\mathcal{T}^*\otimes J_{k-1}(\mathfrak{F};\, Y)_X$$

by setting

$$D(\alpha\otimes u) = d\alpha\otimes\pi_{k-1}u + (-1)^i \alpha\wedge Du$$

for $\alpha \in \wedge^i\mathcal{T}^*$, $u \in J_k(\mathfrak{F};\, Y)_X$; this is a well-defined operator because of (3.5). The operator

$$D\colon \wedge\mathcal{T}^*\otimes J_k(\mathfrak{F};\, Y)_X \to \wedge\mathcal{T}^*\otimes J_{k-1}(\mathfrak{F};\, Y)_X \tag{3.12}$$

satisfies

$$D(\alpha\wedge u) = d\alpha\wedge\pi_{k-1}u + (-1)^i\alpha\wedge Du, \tag{3.13}$$

for $\alpha \in \wedge^i\mathcal{T}^*$, $u \in \wedge\mathcal{T}^*\otimes J_k(\mathfrak{F};\, Y)_X$, and

$$\langle \xi\wedge\eta, Du\rangle = \xi \mathbin{\overline{\wedge}} D\langle \eta, u\rangle - \eta \mathbin{\overline{\wedge}} D\langle \xi, u\rangle - \pi_{k-1}\langle [\xi,\eta], u\rangle, \tag{3.14}$$

for $\xi, \eta \in \mathcal{T}$, $u \in \mathcal{T}^*\otimes J_k(\mathfrak{F};\, Y)_X$. Since $D^2 = 0$, as is easily seen, we obtain a complex

$$0 \longrightarrow \varrho^{-1}\mathfrak{F} \xrightarrow{j_k} J_k(\mathfrak{F};\, Y)_X \xrightarrow{D} \mathcal{T}^* \otimes J_{k-1}(\mathfrak{F};\, Y)_X \xrightarrow{D} \cdots \longrightarrow \wedge^n \mathcal{T}^* \otimes J_{k-n}(\mathfrak{F};\, Y) \longrightarrow 0$$

where the map j_k is induced from $j_k\colon \mathfrak{F} \to J_k(\mathfrak{F};\, Y)$ by ϱ. This complex is not exact at $\wedge^i \mathcal{T}^* \otimes J_{k-i}(\mathfrak{F};\, Y)$ for $i \geqslant 0$; however, the corresponding complex with $k = \infty$ is exact.

If $E = \varrho^{-1}F$ and $\varphi\colon E \to \varrho^{-1}F$ is the identity mapping, it follows from (3.6), (1.4) and (3.13) that the diagram

$$\begin{array}{ccc} \wedge^i \mathcal{T}^* \otimes J_k(\mathcal{E};\, \varphi) & \xrightarrow{D} & \wedge^{i+1} \mathcal{T}^* \otimes J_{k-1}(\mathcal{E};\, \varphi) \\ \Big\downarrow {\scriptstyle \mathrm{id} \otimes \varphi} & & \Big\downarrow {\scriptstyle \mathrm{id} \otimes \varphi} \\ \wedge^i \mathcal{T}^* \otimes J_k(\mathfrak{F};\, Y)_X & \xrightarrow{D} & \wedge^{i+1} \mathcal{T}^* \otimes J_{k-1}(\mathfrak{F};\, Y)_X \end{array} \tag{3.15}$$

is commutative, where the vertical arrows are isomorphisms, generalizing assertion (ii) of Proposition 3.1.

Let $i\colon V \to T$ denote the natural inclusion. Combining diagrams (3.2) and (3.15), we obtain the commutative diagram

$$\begin{array}{ccc} \wedge^i \mathcal{T}^* \otimes J_k(\mathfrak{F};\, Y)_X & \xrightarrow{D} & \wedge^{i+1} \mathcal{T}^* \otimes J_{k-1}(\mathfrak{F};\, Y)_X \\ \Big\downarrow {\scriptstyle i^* \otimes \mathrm{id}} & & \Big\downarrow {\scriptstyle i^* \otimes \mathrm{id}} \\ \wedge^i \mathcal{V}^* \otimes J_k(\mathfrak{F};\, Y)_X & \xrightarrow{\pi_{k-1} \cdot d_{X/Y}} & \wedge^{i+1} \mathcal{V}^* \otimes J_{k-1}(\mathfrak{F};\, Y)_X \end{array} \tag{3.16}$$

which generalizes relation (3.11).

If τ is a section of $\varrho\colon X \to Y$ and $y \in Y$, and if u is a section of $\wedge^i \mathcal{T}^* \otimes J_k(\mathfrak{F};\, Y)_X$ over a neighborhood of $x = \tau(y)$, let $\tau^* u$ be the section of $\wedge^i T_Y^* \otimes J_k(F;\, Y)$ over a neighborhood of y defined by

$$(\tau^* u)(a) = (\tau^* \otimes \mathrm{id})\, u(\tau(a)), \qquad \text{for } a \in Y, \tag{3.17}$$

where τ^* on the right-hand side is the map

$$\tau^*\colon\ \wedge^i T^*_{\tau(a)} \to \wedge^i T^*_{Y,a}.$$

Then, by (3.10), (3.13) and (1.4), we see that

$$\tau^* D u = D \tau^* u, \tag{3.18}$$

where the operator D on the right-hand side is the one defined in § 1, namely (3.9). The relation (3.18) generalizes (3.10).

We now give a construction of the operator (3.12) similar to the one given by Malgrange [19] for the Spencer operator D of § 1. Let $\Delta_{X,Y}$ be the subset $X \times_Y Y$ of $X \times Y$. Let

pr_1: $X \times Y \to X$ be the projection onto the first factor. We shall identify a sheaf on X (resp. on $\Delta_{X,Y}$) with its inverse image by pr_1: $\Delta_{X,Y} \to X$ (resp. with its direct image by the inclusion $\Delta_{X,Y} \to X \times Y$). Let $\mathcal{J}_Y^{k+1}$ be the sub-sheaf of $\mathcal{O}_{Y\times Y}$ of functions which vanish to order k on the diagonal Δ_Y of $Y \times Y$. Let $\mathcal{J}_{X,Y}^{k+1}$ be the inverse image of this sheaf by $\varrho \times \mathrm{id}$: $X \times Y \to Y \times Y$. If $\mathbf{1}_Y$ is the trivial line bundle over Y, we see that $\mathcal{O}_{X\times Y}/\mathcal{J}_{X,Y}^{k+1}$ is the sheaf of sections of $\varrho^{-1}J_k(\mathbf{1}_Y; Y)$ over X. Furthermore

$$J_k(\mathcal{F}; Y)_X = (\mathcal{O}_{X\times Y}/\mathcal{J}_{X,Y}^{k+1}) \otimes_{\mathrm{pr}_2^{-1}\mathcal{O}_Y} \mathrm{pr}_2^{-1}\mathcal{F}, \tag{3.19}$$

where pr_2: $X \times Y \to Y$ is the projection onto the second factor. Lifting differential forms on X to $X \times Y$ by pr_1^*, we may regard elements of

$$\wedge \mathcal{T}^* \otimes_{\mathcal{O}_X} (\mathcal{O}_{X\times Y}/\mathcal{J}_{X,Y}^{k+1})$$

as germs of differential forms on $X \times Y$ modulo $\mathcal{J}_{X,Y}^{k+1}$. The exterior differential operator on $X \times Y$ with respect to the first factor X gives by passage to the quotient a map

$$D\colon \wedge \mathcal{T}^* \otimes_{\mathcal{O}_X} (\mathcal{O}_{X\times Y}/\mathcal{J}_{X,Y}^{k+1}) \to \wedge \mathcal{T}^* \otimes_{\mathcal{O}_X} (\mathcal{O}_{X\times Y}/\mathcal{J}_{X,Y}^{k}). \tag{3.20}$$

Since D is $\mathrm{pr}_2^{-1}\mathcal{O}_Y$-linear, by applying the functor

$$\otimes_{\mathrm{pr}_2^{-1}\mathcal{O}_Y} \mathrm{pr}_2^{-1}\mathcal{F}$$

to (3.20) and using (3.19), we obtain an operator

$$D\colon \wedge \mathcal{T}^* \otimes J_k(\mathcal{F}; Y)_X \to \wedge \mathcal{T}^* \otimes J_{k-1}(\mathcal{F}; Y)_X$$

which is none other than our operator (3.12), as it is easily seen that it satisfies conditions (i) of Proposition 3.1 and (3.13).

Finally, the operator (3.20), or more generally (3.12), is easily written in terms of local coordinates. For simplicity of notation, we shall consider only the case

$$D\colon \mathcal{O}_{X\times Y}/\mathcal{J}_{X,Y}^{k+1} \to \mathcal{T}^* \otimes_{\mathcal{O}_X} (\mathcal{O}_{X\times Y}/\mathcal{J}_{X,Y}^{k}).$$

We introduce on X the local coordinate (v, y), where $v = (v^1, \ldots, v^q)$ is the coordinate along the fiber of ϱ: $X \to Y$ and $y = (y^1, \ldots, y^m)$ is a local coordinate for Y. If u represents a germ of $\mathcal{O}_{X\times Y}/\mathcal{J}_{X,Y}^{k+1}$, we have in the usual multi-index notation,

$$u = \sum_{|\alpha|\leqslant k} a_\alpha(v, y) \frac{(y'-y)^\alpha}{\alpha!} \quad \left(\mathrm{mod}\ \mathcal{J}_{X,Y}^{k+1}\right),$$

where $\alpha = (\alpha_1, \ldots, \alpha_m)$, $(y'-y)^\alpha = (y'^1 - y^1)^{\alpha_1} \ldots (y'^m - y^m)^{\alpha_m}$, $\alpha! = (\alpha_1!)(\alpha_2!) \ldots (\alpha_m!)$, $|\alpha| = \alpha_1 + \ldots + \alpha_m$, and (y, y') are respectively the coordinates along the first and second factors of $Y \times Y$. Then we have

$$Du = \sum_{|\alpha| \leqslant k-1} \left\{ \sum_{i=1}^{q} dv^i \otimes \frac{\partial a_\alpha}{\partial v^i} + \sum_{j=1}^{m} dy^j \otimes \left(\frac{\partial a_\alpha}{\partial y^j} - a_{\alpha+1_j} \right) \right\} \frac{(y'-y)^\alpha}{\alpha!} \pmod{\mathcal{J}_{X,Y}^{k+1}},$$

where 1_j denotes the multi-index with 1 in the j-th position and 0 elsewhere. This formula should be compared with (3.8).

4. A complex associated with Lie groups

Let G be a bundle of Lie groups over Y; the multiplication map $G \times_Y G \to G$ is a morphism of fibered manifolds over Y. Let $T(G/Y)$ denote the bundle of vectors tangent to the fibers of $G \to Y$. If $g \in G_y$, the mappings $G_y \to G_y$ sending h into $g \cdot h$ and $h \cdot g$ respectively induce isomorphisms $T_h(G_y) \to T_{gh}(G_y)$, $T_h(G_y) \to T_{hg}(G_y)$ sending ξ into $g \cdot \xi$ and $\xi \cdot g$ respectively for all $h \in G_y$. Let I be the section of G over Y sending $y \in Y$ into the identity element $I(y)$ of the group G_y. The Lie algebra $\mathfrak{g}$ of G is the vector bundle over Y whose fiber $\mathfrak{g}_y$ at $y \in Y$ is $T_{I(y)}(G/Y) = T_{I(y)}(G_y)$. If $\xi \in \mathfrak{g}_y$ and $g \in G_y$, then we write

$$\mathrm{Ad}\, g \cdot \xi = g \cdot \xi \cdot g^{-1}.$$

The bracket on $\mathfrak{g}$ is a morphism of vector bundles over Y

$$\mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g},$$

which, when restricted to the fiber $\mathfrak{g}_y$, is the usual bracket defined in terms of left-invariant vector fields on G_y. The Maurer-Cartan form of G

$$\omega \colon T(G/Y) \to \mathfrak{g}$$

is defined by

$$\langle \xi, \omega \rangle = g^{-1} \cdot \xi, \qquad \text{for } \xi \in T_g(G/Y);$$

if $y \in Y$ and $g \in G_y$, its restriction to $T_g(G/Y) = T_g(G_y)$ is the left-invariant Maurer-Cartan form of the Lie group G_y with values in $\mathfrak{g}_y$.

We define a bracket

$$(T^*(X/Y) \otimes_X \mathfrak{g}) \otimes (T^*(X/Y) \otimes_X \mathfrak{g}) \to \wedge^2 T^*(X/Y) \otimes_X \mathfrak{g} \tag{4.1}$$

by the formula

$$[\alpha \otimes \xi, \beta \otimes \eta] = (\alpha \wedge \beta) \otimes [\xi, \eta],$$

for $\alpha, \beta \in T^*(X/Y)$, $\xi, \eta \in \mathfrak{g}$. Then the Maurer-Cartan form of G satisfies the equation

$$d_{G/Y}\omega + \tfrac{1}{2}[\omega, \omega] = 0, \tag{4.2}$$

where the bracket is given by (4.1) with X replaced by G.

For $i \geqslant 1$, let $\wedge^i \mathcal{V}^* \otimes_X \mathfrak{g}$ denote the sheaf of sections of $\wedge^i V^* \otimes_X \mathfrak{g}$. We introduce the differential operator

$$\mathcal{D}_{X/Y}\colon \mathcal{G}_X \to \mathcal{V}^* \otimes_X \mathfrak{g},$$

which sends $\phi \in \mathcal{G}_X$ into $\phi^*\omega \in \mathcal{V}^* \otimes_X \mathfrak{g}$. If ϕ is a section of $\mathcal{G}_X$ over $U \subset X$, then

$$\langle \xi, \mathcal{D}_{X/Y}\phi\rangle = \langle \phi_*\xi, \omega\rangle = \phi(a)^{-1}\phi_*\xi, \tag{4.3}$$

for all $\xi \in V_a$, $a \in U$, where $\phi_*\xi \in T_{\phi(a)}(G/Y)$ and $\phi(a)^{-1}\colon T_{\phi(a)}(G/Y) \to \mathfrak{g}$. Hence

$$\mathcal{D}_{X/Y}\phi = \phi^*\omega = 0 \text{ if and only if } \phi_*\xi = 0 \text{ for all } \xi \in V. \tag{4.4}$$

We have, for $\phi \in \mathcal{G}_X$,

$$d_{X/Y}(\mathcal{D}_{X/Y}\phi) = d_{X/Y}(\phi^*\omega) = \phi^*(d_{G/Y}\omega) = -\tfrac{1}{2}\phi^*[\omega, \omega] = -\tfrac{1}{2}[\phi^*\omega, \phi^*\omega]$$

by (4.2), i.e.,

$$d_{X/Y}(\mathcal{D}_{X/Y}\phi) + \tfrac{1}{2}[\mathcal{D}_{X/Y}\phi, \mathcal{D}_{X/Y}\phi] = 0. \tag{4.5}$$

Therefore defining

$$\mathcal{D}_{1,X/Y}\colon \mathcal{V}^* \otimes_X \mathfrak{g} \to \wedge^2\mathcal{V}^* \otimes_X \mathfrak{g}$$

by the formula

$$\mathcal{D}_{1,X/Y}v = d_{X/Y}v + \tfrac{1}{2}[v, v], \qquad v \in \mathcal{V}^* \otimes_X \mathfrak{g},$$

we obtain the complex

$$I \longrightarrow \mathcal{G} \xrightarrow{\varrho^{-1}} \mathcal{G}_X \xrightarrow{\mathcal{D}_{X/Y}} \mathcal{V}^* \otimes_X \mathfrak{g} \xrightarrow{\mathcal{D}_{1,X/Y}} \wedge^2\mathcal{V}^* \otimes_X \mathfrak{g}. \tag{4.6}$$

This complex is clearly exact at $\mathcal{G}_X$ in view of (4.4).

If $u \in \wedge V_x^* \otimes \mathfrak{g}_y$, where $x \in X$ and $\varrho(x) = y$, and $g \in G_y$, we define

$$g(u) = (\mathrm{id} \otimes \mathrm{Ad}\, g)u.$$

If ϕ, ψ are sections of $\mathcal{G}_X$ over an open set $U \subset X$, we obtain a section $\phi\cdot\psi$ of $\mathcal{G}_X$ over U by setting

$$(\phi\cdot\psi)(a) = \phi(a)\cdot\psi(a), \qquad a \in U.$$

Then

$$\mathcal{D}_{X/Y}(\phi\cdot\psi) = \psi^{-1}(\mathcal{D}_{X/Y}\phi) + \mathcal{D}_{X/Y}\psi. \tag{4.7}$$

Indeed, if $\xi \in V_a$, $a \in U$,

$$(\phi\cdot\psi)_*\xi = \phi_*\xi\cdot\psi(a) + \phi(a)\cdot\psi_*\xi$$

and so, according to (4.3),

$$\begin{aligned}\langle \xi, \mathcal{D}_{X/Y}(\phi\cdot\psi)\rangle &= (\phi(a)\cdot\psi(a))^{-1}(\phi_*\xi\cdot\psi(a) + \phi(a)\cdot\psi_*\xi)\\ &= \psi(a)^{-1}\cdot\phi(a)^{-1}\cdot\phi_*\xi\cdot\psi(a) + \psi(a)^{-1}\cdot\psi_*\xi = \psi(a)^{-1}(\langle \xi, \mathcal{D}_{X/Y}\phi\rangle) + \langle \xi, \mathcal{D}_{X/Y}\psi\rangle\end{aligned}$$

which gives (4.7). Replacing ψ in (4.7) by ϕ^{-1}, we obtain

$$\mathcal{D}_{X/Y}\phi^{-1} = -\phi(\mathcal{D}_{X/Y}\phi). \tag{4.8}$$

If $u \in \mathcal{V}^* \otimes_X \mathfrak{g}$, $\phi \in \mathcal{G}_X$, we define

$$u^\phi = \phi^{-1}(u) + \mathcal{D}_{X/Y}\phi. \tag{4.9}$$

Then, if $\psi \in \mathcal{G}_X$, we have by (4.7)

$$u^{\phi\cdot\psi} = \psi^{-1}(\phi^{-1}(u)) + \mathcal{D}_{X/Y}(\phi\cdot\psi) = \psi^{-1}(\phi^{-1}(u) + \mathcal{D}_{X/Y}\,\phi) + \mathcal{D}_{X/Y}\,\psi,$$

i.e.,

$$u^{\phi\cdot\psi} = (u^{\phi})^{\psi}. \tag{4.10}$$

We have

$$\mathcal{D}_{1,\,X/Y}u^{\phi} = \phi^{-1}(\mathcal{D}_{1,\,X/Y}u), \quad \text{for } u \in \mathcal{V}^* \otimes \mathfrak{g},\ \phi \in \mathcal{G}_X. \tag{4.11}$$

To establish (4.11), we first make the following digression.

Let ϕ be a section of $\mathcal{G}_X$ over a neighborhood U of a point $a \in X$ and let x_t be a curve in U with $x_0 = a$ and $\varrho(x_t) = \varrho(a) = y$; set $dx_t/dt\,|_{t=0} = \xi \in V_a$. For simplicity, we write $\phi_t = \phi(x_t)$; then, for $\zeta \in \mathfrak{g}_y$, we have the formula

$$\frac{d}{dt}\operatorname{Ad}\phi_t\cdot\zeta\,|_{t=0} = \operatorname{Ad}\phi(a)\cdot([\langle\xi, \mathcal{D}_{X/Y}\phi\rangle, \zeta]). \tag{4.12}$$

In fact, we have

$$\operatorname{Ad}\phi_t\cdot\zeta = \phi_0\cdot\{\phi_0^{-1}\cdot\phi_t\cdot\zeta\cdot\phi_t^{-1}\cdot\phi_0\}\cdot\phi_0^{-1} = \operatorname{Ad}\phi_0\cdot\operatorname{Ad}(\phi_0^{-1}\cdot\phi_t)\cdot\zeta$$

and hence

$$\frac{d}{dt}\operatorname{Ad}\phi_t\cdot\zeta\,|_{t=0} = \operatorname{Ad}\phi_0\cdot\operatorname{ad}\left(\frac{d}{dt}\phi_0^{-1}\cdot\phi_t|_{t=0}\right)\cdot\zeta = \operatorname{Ad}\phi_0\cdot\left(\left[\frac{d}{dt}\phi_0^{-1}\cdot\phi_t|_{t=0}, \zeta\right]\right),$$

since the differential of Ad at the identity of the group G_y is equal to ad (see [16], p. 118). Since, by (4.3),

$$\frac{d}{dt}\phi^{-1}(a)\cdot\phi_t|_{t=0} = \phi^{-1}(a)\cdot\phi_*\xi = \langle\xi, \mathcal{D}_{X/Y}\,\phi\rangle,$$

we obtain (4.12).

Next, if $\xi \in V_a$ and ζ is a section of $\varrho^{-1}\mathfrak{g}$ and ϕ is a section of $\mathcal{G}_X$ over a neighborhood U of a, we have

$$\xi\cdot(\operatorname{Ad}\phi\cdot\zeta) = \operatorname{Ad}\phi(a)\,(\xi\cdot\zeta) + \operatorname{Ad}\phi(a)\,([\langle\xi, \mathcal{D}_{X/Y}\phi\rangle, \zeta(a)]). \tag{4.13}$$

For let x_t be a curve in U as above with $dx_t/dt\,|_{t=0} = \xi$, and write $\zeta_t = \zeta(x_t)$, $\phi_t = \phi(x_t)$; then

$$\xi\cdot(\operatorname{Ad}\phi\cdot\zeta) = \frac{d}{dt}(\operatorname{Ad}\phi_t\cdot\zeta_t)|_{t=0} = \operatorname{Ad}\phi(a)\cdot(\xi\cdot\zeta) + \frac{d}{dt}\operatorname{Ad}\phi_t\cdot\zeta(a)|_{t=0}$$

and we obtain (4.13) by substitution from (4.12).

We obtain, for $u \in \mathcal{V}^* \otimes_X \mathfrak{g}$, $\phi \in \mathcal{G}_X$,

$$d_{X/Y}\,\phi^{-1}(u) - \phi^{-1}(d_{X/Y}\,u) = \phi^{-1}([\mathcal{D}_{X/Y}\,\phi^{-1}, u]). \tag{4.14}$$

In fact, let $\xi, \eta \in \mathcal{V}$; then

$$\langle \xi \wedge \eta, d_{X/Y}\phi^{-1}(u) - \phi^{-1}(d_{X/Y}u)\rangle$$
$$= \xi \cdot \langle \eta, \phi^{-1}(u)\rangle - \eta \cdot \langle \xi, \phi^{-1}(u)\rangle - \langle [\xi, \eta], \phi^{-1}(u)\rangle - \mathrm{Ad}\,\phi^{-1}(\xi \cdot \langle \eta, u\rangle - \eta \cdot \langle \xi, u\rangle - \langle [\xi, \eta], u\rangle)$$
$$= \xi \cdot (\mathrm{Ad}\,\phi^{-1}\langle \eta, u\rangle) - \eta \cdot (\mathrm{Ad}\,\phi^{-1}\langle \xi, u\rangle) - \mathrm{Ad}\,\phi^{-1}(\xi \cdot \langle \eta, u\rangle) + \mathrm{Ad}\,\phi^{-1}(\eta \cdot \langle \xi, u\rangle),$$

since $\mathrm{Ad}\,\phi^{-1}\langle [\xi, \eta], u\rangle = \langle [\xi, \eta], \phi^{-1}(u)\rangle$ and the two terms of this form cancel. By (4.13), with ϕ replaced by ϕ^{-1} and with ζ replaced by $\langle \eta, u\rangle$, $\langle \xi, u\rangle$, we obtain

$$\langle \xi \wedge \eta, d_{X/Y}\phi^{-1}(u) - \phi^{-1}(d_{X/Y}u)\rangle = \mathrm{Ad}\,\phi^{-1}([\langle \xi, \mathcal{D}_{X/Y}\phi^{-1}\rangle, \langle \eta, u\rangle] - [\langle \eta, \mathcal{D}_{X/Y}\phi^{-1}\rangle, \langle \xi, u\rangle])$$
$$= \mathrm{Ad}\,\phi^{-1}(\langle \xi \wedge \eta, [\mathcal{D}_{X/Y}\phi^{-1}, u]\rangle)$$

and this is (4.14).

We now prove (4.11). In fact, for $u \in \mathcal{V}^* \otimes_X \mathfrak{g}$, $\phi \in \mathcal{G}_X$, we have, using (4.5),

$$\mathcal{D}_{1,X/Y}u^{\phi} = \mathcal{D}_{1,X/Y}(\phi^{-1}(u) + \mathcal{D}_{X/Y}\phi)$$
$$= d_{X/Y}\phi^{-1}(u) + \tfrac{1}{2}[\phi^{-1}(u), \phi^{-1}(u)] + [\mathcal{D}_{X/Y}\phi, \phi^{-1}(u)]$$
$$= \phi^{-1}(d_{X/Y}u + \tfrac{1}{2}[u, u]) + \phi^{-1}([\phi(\mathcal{D}_{X/Y}\phi), u] + [\mathcal{D}_{X/Y}\phi^{-1}, u])$$

by (4.14). Since $\phi(\mathcal{D}_{X/Y}\phi) = -\mathcal{D}_{X/Y}\phi^{-1}$ by (4.8), we obtain (4.11).

PROPOSITION 4.1. *The complex* (4.6) *is exact. Moreover, suppose that there is given a section v of $V^* \otimes_X \mathfrak{g}$ over a neighborhood U of a point $x_0 \in X$ satisfying $\mathcal{D}_{1,X/Y}v = 0$, a local section s: $Y \to X$ mapping $\varrho(U)$ into U such that $s(\varrho(x_0)) = x_0$, and a local section ϕ_0: $Y \to G$ defined on $\varrho(U)$. Then there are a neighborhood $U' \subset U$ of x_0 and a unique section ϕ: $U' \to \mathcal{G}_X$ satisfying $\mathcal{D}_{X/Y}\phi = v$ and $\phi(s(y)) = \phi_0(y)$, for all $y \in \varrho(U')$. If $v(x_0) = 0$ and $\phi_0 = I$, then $j_1(\phi)(x_0) = j_1(I \circ \varrho)(x_0)$.*

Proof. Consider the fibered manifold $G_X = X \times_Y G$ over Y; let pr_1: $G_X \to X$, pr_2: $G_X \to G$ be the projections onto the first and second factors respectively, which are morphisms of fibered manifolds over Y. Let v be a local section of $V^* \otimes_X \mathfrak{g}$ over X; set

$$\tilde{\omega} = \mathrm{pr}_2^*\,\omega : T(G_X/Y) \to \mathfrak{g},$$
$$\tilde{v} = \mathrm{pr}_1^*\,v : T(G_X/Y) \to \mathfrak{g}.$$

Let ϕ be a local section of G_X over X; if $\tilde{\phi}$: $X \to X \times_Y G$ is the graph of ϕ, which sends $x \in X$ into $(x, \phi(x))$, then $\mathrm{pr}_1 \circ \tilde{\phi} = \mathrm{id}$, $\mathrm{pr}_2 \circ \tilde{\phi} = \phi$, and hence

$$v - \phi^*\omega = \tilde{\phi}^*(\mathrm{pr}_1^* v - \mathrm{pr}_2^*\omega) = \tilde{\phi}^*(\tilde{v} - \tilde{\omega}). \tag{4.15}$$

Therefore $\phi^*\omega = v$ if and only if $\tilde{\phi}^*(\tilde{v} - \tilde{\omega}) = 0$ where

$$\tilde{v} - \tilde{\omega}: T(G_X/Y) \to \mathfrak{g}. \tag{4.16}$$

Let K be the kernel of pr_{1*}: $T(G_X/Y) \to T(X/Y)$.

LEMMA 4.1. *Let v be a section of $V^* \otimes_X \mathfrak{g}$ over X. Then* $\ker(\tilde{v}-\tilde{\omega})$ *is a distribution on G_X such that*

$$K \oplus \ker(\tilde{v}-\tilde{\omega}) = T(G_X/Y); \tag{4.17}$$

if $\mathcal{D}_{1,X/Y}v = 0$, it is integrable.

Proof. If y is the image in Y of $g \in G$, then ω: $T_g(G/Y) \to \mathfrak{g}_y$ is an isomorphism, so $\tilde{v}-\tilde{\omega}$: $T_z(G_X) \to \mathfrak{g}_y$ is surjective and $\tilde{\omega}$: $K_z \to \mathfrak{g}_y$ is an isomorphism, for all $z \in G_X$ whose projection in Y is y. Since $\tilde{v}_{|K} = 0$, it follows that $\ker(\tilde{v}-\tilde{\omega})$ is a sub-bundle of $T(G_X/Y)$ of rank equal to $\dim X - \dim Y$ and $K \cap \ker(\tilde{v}-\tilde{\omega}) = 0$. By a dimension argument, we see that (4.17) holds. Next, we have

$$d_{G_X/Y}\tilde{\omega} + \tfrac{1}{2}[\tilde{\omega}, \tilde{\omega}] = 0$$

and, if $\mathcal{D}_{1,X/Y}v = 0$, we have also

$$d_{G_X/Y}\tilde{v} + \tfrac{1}{2}[\tilde{v}, \tilde{v}] = 0,$$

where the brackets are given by (4.1) with X replaced by G_X. Hence

$$d_{G_X/Y}(\tilde{v}-\tilde{\omega}) = \tfrac{1}{2}([\tilde{\omega}, \tilde{\omega}] - [\tilde{v}, \tilde{v}]) = -\tfrac{1}{2}[\tilde{v}-\tilde{\omega}, \tilde{v}+\tilde{\omega}].$$

Let ξ, η be sections of $\ker(\tilde{v}-\tilde{\omega})$ over G_X. Then

$$\begin{aligned}\langle [\xi,\eta], \tilde{v}-\tilde{\omega}\rangle &= \xi\cdot\langle \eta, \tilde{v}-\tilde{\omega}\rangle - \eta\cdot\langle \xi, \tilde{v}-\tilde{\omega}\rangle - \langle \xi\wedge\eta, d_{G_X/Y}(\tilde{v}-\tilde{\omega})\rangle \\ &= -\langle \xi\wedge\eta, d_{G_X/Y}(\tilde{v}-\tilde{\omega})\rangle = \tfrac{1}{2}([(\tilde{v}-\tilde{\omega})(\xi), (\tilde{v}+\tilde{\omega})(\eta)] - [(\tilde{v}-\tilde{\omega})(\eta), (\tilde{v}+\tilde{\omega})(\xi)]) = 0.\end{aligned}$$

Hence $[\xi, \eta]$ is a section of $\ker(\tilde{v}-\tilde{\omega})$, i.e., $\ker(\tilde{v}-\tilde{\omega})$ is an integrable distribution.

Let us return to the proof of Proposition 4.1, and let v, s, ϕ_0 be as described in the proposition. Since $\ker(\tilde{v}-\tilde{\omega})$ is an integrable distribution, Frobenius' theorem asserts that, through each point of $U \times_Y G$ lying over $y \in \varrho(U) \subset Y$, there passes a leaf of the corresponding foliation lying in $U_y \times G_y$. Because of (4.17), if U is replaced by a possibly smaller neighborhood U' which, for simplicity, we again denote by U, then there exists a morphism of fibered manifolds $\tilde{\phi}$: $X \to G_X$ over Y defined on U, which is a section of the fibered manifold pr_1: $G_X \to X$ and therefore the graph of a map ϕ: $U \to G$, such that $\tilde{\phi}(U_y)$ is the leaf of the foliation containing the point $(s(y), \phi_0(y))$, for all $y \in \varrho(U)$. Then $\tilde{\phi}^*(\tilde{v}-\tilde{\omega}) = 0$ and hence, by (4.15), $\phi^*\omega = v$. If $\phi_0 = I$, then $\phi(x_0) = I(\varrho(x_0))$ and the equality $j_1(\phi)(x_0) = j_1(I\circ\varrho)(x_0)$ is equivalent to

$$\phi_*\xi = (I\circ\varrho)_*\xi, \qquad \text{for all } \xi \in T_{x_0}. \tag{4.18}$$

We write $T_{x_0} = V_{x_0} \oplus H_{x_0}$, where $H_{x_0} = s_* T_{Y,\varrho(x_0)}$. Suppose that $v(x_0) = 0$. If $\xi \in V_{x_0}$, then $\phi_*\xi = 0$ by (4.4), and $(I\circ\varrho)_*\xi = I_*\varrho_*\xi = 0$. If $\xi \in H_{x_0}$, then $\xi = s_*\zeta$, with $\zeta = \varrho_*\xi \in T_{Y,\varrho(x_0)}$, and

$$\phi_*\xi = \phi_* s_*\zeta = I_*\zeta = I_*\varrho_*\xi = (I\circ\varrho)_*\xi.$$

Thus (4.18) holds under our assumptions on ϕ_0 and $v(x_0)$ and we obtain the desired equality.

5. Cartan fundamental forms

As in § 2, we regard $E = X \times X$ as a bundle over X via the projection pr_1. We begin by recalling the definition of the fundamental form on $J_{k+1}(E)$ given in [11], namely the mapping

$$\sigma: T(J_{k+1}(E)) \to V(J_k(E)),$$

which is a morphism of vector bundles over $\pi_k: J_{k+1}(E) \to J_k(E)$. If $F \in J_{k+1}(E)$, $\xi \in T_F(J_{k+1}(E))$, the form σ is defined by the formula

$$\langle \xi, \sigma \rangle = \pi_{k*}\xi - F_* \pi_* \xi, \tag{5.1}$$

where $F_*: T_x \to T_{\pi_k F}(J_k(E))$ and $\langle \xi, \sigma \rangle \in V_{\pi_k F}(J_k(E))$. If u is a section of $J_{k+1}(E)$ over X, then $u^*\sigma$ is the $V(J_k(E))$-valued 1-form on X defined by

$$\langle \xi, u^*\sigma \rangle = \langle u_* \xi, \sigma \rangle, \quad \text{for } \xi \in T.$$

Then, according to Propositions 1.1 and 1.2 of [11], a section u of $J_{k+1}(E)$ over $U \subset X$ satisfies $u^*\sigma = 0$ if and only if it is equal to $j_{k+1}(s)$, where $s = \pi_0 u$.

The Cartan fundamental form on Q_{k+1} with values in $J_k(T)$ is the mapping

$$\omega: T(Q_{k+1}) \to J_k(T),$$

which is a morphism of vector bundles over $\pi: Q_{k+1} \to X$ defined by

$$\langle \xi, \omega \rangle = \nu(\lambda_1 F)^{-1} \langle \xi, \sigma \rangle \tag{5.2}$$

for $F \in Q_{k+1}$, $\xi \in T_F(Q_{k+1})$. In fact, $\langle \xi, \omega \rangle$ belongs to $J_k(T)_a$ if $\xi \in T_F(Q_{k+1})$, where $a = \pi F$. If $F = j_{k+1}(f)(a)$, where f is a local diffeomorphism of X defined on a neighborhood of a, the mapping

$$(\lambda_1 F)^{-1} \cdot F_*: T_a \to T_{I_k(a)}(Q_k)$$

sends η into $j_k(f)^{-1} \cdot j_k(f)_* \eta = I_{k*}\eta$. Therefore, by (5.1) and (5.2),

$$\langle \xi, \omega \rangle = \nu((\lambda_1 F)^{-1} \pi_{k*} \xi - I_{k*} \pi_* \xi). \tag{5.3}$$

The restriction

$$\omega_V: V(Q_{k+1}) \to J_k(T)$$

of ω to $V(Q_{k+1})$ is given by

$$\langle \xi, \omega_V \rangle = \nu(\lambda_1 F)^{-1} \pi_{k*} \xi \tag{5.4}$$

for $F \in Q_{k+1}$, $\xi \in V_F(Q_{k+1})$. The further restriction of ω or of ω_V to the fiber $Q_{k+1}(a)$, the "bundle of frames of order $k+1$ with source a", is the fundamental form of Cartan on the principal bundle $Q_{k+1}(a)$ with values in $J_k(T)_a$ (see [14]).

If F is a section of Q_{k+1} over X, then $F^*\omega$ is the $J_k(T)$-valued 1-form on X defined by

$$\langle \xi, F^*\omega\rangle = \langle F_*\xi, \omega\rangle, \qquad \text{for } \xi \in T,$$

which we shall also consider as a section of $T^*\otimes J_k(T)$ over X.

PROPOSITION 5.1. *The fundamental form ω on Q_{k+1} has the following properties:*

(i) *If $\xi \in \tilde{J}_{k+1}(T)_b$, $G \in Q_{k+1}$, with target $G = b$,*

$$\langle \xi G, \omega\rangle = G^{-1}(\pi_k \nu \xi). \tag{5.5}$$

(ii) *If F is a section of Q_{k+1} over $U \subset X$, then $F^*\omega = 0$ if and only if $F = j_{k+1}(f)$, where $f\colon U \to X$ is an immersion. If F is a section of $\tilde{Q}_{k+1}$, then*

$$\mathcal{D}F = F^*\omega. \tag{5.6}$$

(iii) *If F is a section of $\tilde{Q}_{k+1}$ over $U \subset X$, then*

$$\langle F\cdot\xi\cdot G, \omega\rangle - \langle \xi\cdot G, \omega\rangle = G^{-1}((\pi_0\xi) \wedge \mathcal{D}F), \tag{5.7}$$

for $\xi \in \tilde{J}_{k+1}(T)_b$, $G \in Q_{k+1}$ with target $G = b \in U$; furthermore

$$\langle F\xi, \omega\rangle = \langle \xi, \omega\rangle, \tag{5.8}$$

for all $\xi \in T_G(Q_{k+1})$, $G \in Q_{k+1}$ with targets lying in U, if and only if $F = j_{k+1}(f)$, where $f\colon U \to X$ is an immersion.

Remark. Let $a \in X$ and Ω be the restriction of ω or of ω_V to $Q_{k+1}(a)$. Some of the assertions of Proposition 5.1 are related to properties of Ω given in [14]. Namely, the equivariance of Ω corresponds to (5.5). Furthermore, if h is a diffeomorphism of $Q_{k+1}(a)$, then the operator $h \mapsto h^*\Omega - \Omega$ is connected to $\mathcal{D}$ by formula (5.7) and the conditions of [14] for the vanishing of $h^*\Omega - \Omega$ are analogous to the second part of (iii).

Proof of Proposition 5.1. (i) We have $\xi G \in V_G(Q_{k+1})$ and

$$\langle \xi G, \omega\rangle = \nu(\lambda_1 G)^{-1}\pi_{k*}(\xi G) = \nu(\lambda_1 G)^{-1}\cdot \pi_k\xi\cdot\pi_k G = G^{-1}(\nu\pi_k\xi),$$

according to (2.6).

(ii) We have $F^*\omega = 0$ if and only if $F^*\sigma = 0$. From the properties of σ, it follows that the latter condition is equivalent to $F = j_{k+1}(f)$, where $f\colon U \to X$ is an immersion, because $\pi_1 \circ F$ is a section of Q_1. If F is a section of $\tilde{Q}_{k+1}$ over U and $a \in U$, then $\lambda_1 F(a)^{-1}\cdot j_1(\pi_k F)(a) \in Q^0_{(1,k)}$, and we have by (5.3), for $\xi \in T_a$,

$$\begin{aligned}
\nu^{-1}\langle F_*\xi, \omega\rangle &= ((\lambda_1 F(a))^{-1}\pi_{k*}F_* - I_{k*})\xi \\
&= (\lambda_1 F(a)^{-1}\cdot j_1(\pi_k F)(a)_* - j_1(I_k)(a)_*)\xi \\
&= ((\lambda_1 F(a)^{-1}\cdot j_1(\pi_k F)(a))_* - j_1(I_k)(a)_*)\xi \\
&= \xi \wedge \partial(\lambda_1 F(a)^{-1}\cdot j_1(\pi_k F)(a)) = \nu^{-1}(\xi \wedge \mathcal{D}F)
\end{aligned}$$

according to (2.27).

(iii) By (i), (5.3) and (2.6), we have

$$\begin{aligned}\langle F\cdot\xi\cdot G,\omega\rangle-\langle\xi\cdot G,\omega\rangle &= \nu(\lambda_1(F(b)\cdot G)^{-1}\pi_{k*}(F\cdot\xi\cdot G)-G^{-1}(\nu\pi_k\xi))\\ &= \nu(\lambda_1 G^{-1}\cdot\lambda_1 F(b)^{-1}\cdot\pi_k F\cdot\pi_k\xi\cdot\pi_k G-G^{-1}(\nu\pi_k\xi))\\ &= G^{-1}(\nu(\lambda_1 F(b)^{-1}\cdot\pi_k F\cdot\pi_k\xi-\pi_k\xi)) = G^{-1}((\pi_0\xi)\,\overline{\wedge}\,\mathcal{D}F)\end{aligned}$$

according to (2.28). If (5.8) holds for all $\xi\in V_G(Q_{k+1})$, then by (5.7),

$$G^{-1}(\eta\,\overline{\wedge}\,\mathcal{D}F)=0,\quad \text{for all } \eta\in T_b,\ b=\text{target } G,$$

and $(\mathcal{D}F)(b)=0$; hence if (5.8) holds for all $\xi\in V_G(Q_{k+1})$, $G\in Q_{k+1}$ whose targets lie in U, then $\mathcal{D}F=0$ and $F=j_{k+1}(f)$, where f is a section of Aut (X) over U. Conversely, if $f\colon U\to X$ is an immersion, $\xi\in T_G(Q_{k+1})$ and $G\in Q_{k+1}$ with target lying in U, then by (5.3)

$$\begin{aligned}\langle j_{k+1}(f)\xi,\omega\rangle &= \nu(\lambda_1(j_{k+1}(f)(b)\cdot G)^{-1}j_k(f)\pi_{k*}\xi-I_{k*}\pi_* j_{k+1}(f)\cdot\xi)\\ &= \nu((\lambda_1 G)^{-1}\lambda_1(j_{k+1}(f)(b))^{-1}j_k(f)\pi_{k*}-I_{k*}\pi_*)\xi\\ &= \nu((\lambda_1 G)^{-1}j_k(f)^{-1}j_k(f)\pi_{k*}-I_{k*}\pi_*)\xi=\langle\xi,\omega\rangle.\end{aligned}$$

If ξ is a vertical vector field on Q_{k+1} which is the infinitesimal generator of a one-parameter family of diffeomorphisms Φ_t of Q_{k+1} defined on an open set $W\subset Q_{k+1}$ and satisfying $\pi\circ\Phi_t=\pi$, $\Phi_0=\mathrm{id}$, we define the Lie derivative $\mathcal{L}(\xi)\omega$ of ω along ξ, which is a section of $T^*(Q_{k+1})\otimes_{Q_{k+1}}J_k(T)$ over W, by the formula

$$\langle\zeta,\mathcal{L}(\xi)\,\omega\rangle=\frac{d}{dt}\langle\Phi_{t*}\zeta,\omega\rangle|_{t=0},$$

for $\zeta\in T_G(Q_{k+1})$, $G\in W$. We set

$$\xi\cdot u=\langle\xi, d_{Q_{k+1}/X}u\rangle$$

for $u\in J_k(\mathcal{T})_{Q_{k+1}}$. Then, if ζ is a vector field on Q_{k+1}, the usual type of formula holds, namely

$$\langle\zeta,\mathcal{L}(\xi)\omega\rangle=\xi\cdot\langle\zeta,\omega\rangle-\langle[\xi,\zeta],\omega\rangle. \tag{5.9}$$

Now let $\bar\xi$ be a vector field on an open set $U\subset X$ and write $\bar\xi_{k+1}=\tau_{k+1}(\tilde{j}_{k+1}(\bar\xi))$, that is

$$\bar\xi_{k+1}(G)=\tilde{j}_{k+1}(\bar\xi)(b)\,G\in V_G(Q_{k+1}),$$

for $G\in Q_{k+1}$ with target $G=b\in U$. From Proposition 5.1, (i), we see that

$$\langle\bar\xi_{k+1}(G),\omega\rangle=G^{-1}(j_k(\bar\xi)(b)). \tag{5.10}$$

If $\bar\xi$ is the infinitesimal generator of a one-parameter family of diffeomorphisms f_t of X defined on $U'\subset U$, with $f_0=\mathrm{id}$, then by Proposition 5.1, (iii),

$$\langle\zeta,\mathcal{L}(\bar\xi_{k+1})\,\omega\rangle=\frac{d}{dt}\langle j_{k+1}(f_t)\cdot\zeta,\omega\rangle|_{t=0}=\frac{d}{dt}\langle\zeta,\omega\rangle|_{t=0}=0,$$

for $\zeta \in T_G(Q_{k+1})$, with target $G \in U'$; hence

$$\mathfrak{L}(\bar{\xi}_{k+1})\omega = 0. \tag{5.11}$$

Next, we define brackets

$$(T^*(Q_{k+1}) \otimes_{Q_{k+1}} J_k(T)) \otimes (T^*(Q_{k+1}) \otimes_{Q_{k+1}} J_k(T)) \to \wedge^2 T^*(Q_{k+1}) \otimes_{Q_{k+1}} J_{k-1}(T),$$

$$(V^*(Q_{k+1}) \otimes_{Q_{k+1}} J_k(T)) \otimes (V^*(Q_{k+1}) \otimes_{Q_{k+1}} J_k(T)) \to \wedge^2 V^*(Q_{k+1}) \otimes_{Q_{k+1}} J_{k-1}(T),$$

by the formula

$$[\alpha \otimes \xi, \beta \otimes \eta] = (\alpha \wedge \beta) \otimes [\xi, \eta],$$

for $\alpha, \beta \in T^*(Q_{k+1})$ or $V^*(Q_{k+1})$, $\xi, \eta \in J_k(T)$. Regarding ω as a section of $T^*(Q_{k+1}) \otimes_{Q_{k+1}} J_k(T)$ and ω_V as a section of $V^*(Q_{k+1}) \otimes_{Q_{k+1}} J_k(T)$ over Q_{k+1}, we thus obtain sections $[\omega, \omega]$ of $\wedge^2 T^*(Q_{k+1}) \otimes_{Q_{k+1}} J_{k-1}(T)$ and $[\omega_V, \omega_V]$ of $\wedge^2 V^*(Q_{k+1}) \otimes_{Q_{k+1}} J_{k-1}(T)$ over Q_{k+1} satisfying

$$[\omega, \omega]_{|\wedge^2 V(Q_{k+1})} = [\omega_V, \omega_V].$$

Taking $X = Q_{k+1}$, $Y = X$, $F = T$ and $\varrho = \pi: Q_{k+1} \to X$ in § 3, we obtain a section $D\omega$ of $\wedge^2 T^*(Q_{k+1}) \otimes_{Q_{k+1}} J_{k-1}(T)$ satisfying

$$(D\omega)_{|\wedge^2 V(Q_{k+1})} = \pi_{k-1} \cdot d_{Q_{k+1}/X} \omega_V \tag{5.12}$$

by the commutativity of diagram (3.16).

PROPOSITION 5.2. *The fundamental form ω on Q_{k+1} regarded as a section of $T^*(Q_{k+1}) \otimes_{Q_{k+1}} J_k(T)$ over Q_{k+1} satisfies the Cartan structure equation*

$$D\omega - \tfrac{1}{2}[\omega, \omega] = 0. \tag{5.13}$$

The form ω_V regarded as a section of $V^*(Q_{k+1}) \otimes_{Q_{k+1}} J_k(T)$ over Q_{k+1} satisfies the Cartan structure equation

$$\pi_{k-1} \cdot d_{Q_{k+1}/X} \omega_V - \tfrac{1}{2}[\omega_V, \omega_V] = 0. \tag{5.14}$$

Remark. Formula (5.14) is given in [14].

Proof of Proposition 5.2. We show first that $D\omega - \frac{1}{2}[\omega, \omega]$ vanishes on $\wedge^2 V(Q_{k+1})$; the proof is similar to that of the formula (5.14) given in [14]. Let $\bar{\xi}, \bar{\eta}$ be vector fields on an open set $U \subset X$, and let $\bar{\xi}_{k+1} = \tau_{k+1}(\tilde{j}_{k+1}(\bar{\xi}))$, $\bar{\eta}_{k+1} = \tau_{k+1}(\tilde{j}_{k+1}(\bar{\eta}))$. Then

$$[\bar{\xi}_{k+1}, \bar{\eta}_{k+1}] = \tau_{k+1}([\tilde{j}_{k+1}(\bar{\xi}), \tilde{j}_{k+1}(\bar{\eta})]) = \tau_{k+1}(\tilde{j}_{k+1}([\bar{\xi}, \bar{\eta}])) = [\bar{\xi}, \bar{\eta}]_{k+1}. \tag{5.15}$$

We have by (5.12), (5.9) and (5.11),

$$\begin{aligned}\langle \bar{\xi}_{k+1} \wedge \bar{\eta}_{k+1}, D\omega \rangle &= \langle \bar{\xi}_{k+1} \wedge \bar{\eta}_{k+1}, \pi_{k-1} \cdot d_{Q_{k+1}/X} \omega_V \rangle \\ &= \pi_{k-1}(\bar{\xi}_{k+1} \cdot \langle \bar{\eta}_{k+1}, \omega \rangle - \bar{\eta}_{k+1} \cdot \langle \bar{\xi}_{k+1}, \omega \rangle - \langle [\bar{\xi}_{k+1}, \bar{\eta}_{k+1}], \omega \rangle) \\ &= \pi_{k-1}(\langle [\bar{\xi}_{k+1}, \bar{\eta}_{k+1}], \omega \rangle - \langle [\bar{\eta}_{k+1}, \bar{\xi}_{k+1}], \omega \rangle - \langle [\bar{\xi}_{k+1}, \bar{\eta}_{k+1}], \omega \rangle) = \pi_{k-1} \langle [\bar{\xi}_{k+1}, \bar{\eta}_{k+1}], \omega \rangle.\end{aligned}$$

It follows from (5.10) and (5.15) that, for $G \in Q_{k+1}$ with target $G = b \in U$,

$$\begin{aligned}\langle \bar{\xi}_{k+1} \wedge \bar{\eta}_{k+1}, \tfrac{1}{2}[\omega, \omega] \rangle (G) &= \langle \bar{\xi}_{k+1} \wedge \bar{\eta}_{k+1}, \tfrac{1}{2}[\omega_V, \omega_V] \rangle (G) \\ &= [\langle \bar{\xi}_{k+1}, \omega \rangle, \langle \bar{\eta}_{k+1}, \omega \rangle] = [G^{-1}(j_k(\bar{\xi})(b)), G^{-1}(j_k(\bar{\eta})(b))] = \pi_k G^{-1}([j_k(\bar{\xi})(b), j_k(\bar{\eta})(b)]) \\ &= \pi_k G^{-1}(j_{k-1}([\bar{\xi}, \bar{\eta}])(b)) = \pi_{k-1}(G^{-1}(j_k([\bar{\xi}, \bar{\eta}])(b))) = \pi_{k-1} \langle [\bar{\xi}, \bar{\eta}]_{k+1}, \omega \rangle (G) \\ &= \pi_{k-1} \langle [\bar{\xi}_{k+1}, \bar{\eta}_{k+1}], \omega \rangle (G).\end{aligned}$$

Therefore

$$\langle \bar{\xi}_{k+1} \wedge \bar{\eta}_{k+1}, D\omega - \tfrac{1}{2}[\omega, \omega] \rangle = 0. \tag{5.16}$$

Since $V(Q_{k+1})$ is generated by vector fields of the form $\bar{\xi}_{k+1}$, this proves that $D\omega - \frac{1}{2}[\omega, \omega]$ vanishes on $\wedge^2 V(Q_{k+1})$.

Next, let τ be a section of $\pi_{k+1}: Q_{k+2} \to Q_{k+1}$ defined on an open subset W of Q_{k+1} and let $\bar{\zeta}$ be a vector field on πW; we define a vector field ζ on W by the formula

$$\zeta(G) = \tau(G)_* \bar{\zeta}(\pi G), \quad G \in W.$$

Then $\pi_* \zeta(G) = \bar{\zeta}(\pi G)$ and, by (5.1),

$$\langle \zeta, \sigma \rangle (G) = \pi_{k*} \tau(G)_* \bar{\zeta}(\pi G) - G_* \pi_* \zeta = G_* \bar{\zeta}(\pi G) - G_* \pi_* \zeta = 0,$$

so

$$\langle \zeta, \omega \rangle = 0. \tag{5.17}$$

Assume that the mapping "target": $Q_{k+1} \to X$ sends W into U. Then by (5.9), (5.17) and (5.11),

$$\langle [\bar{\xi}_{k+1}, \zeta], \omega \rangle = -\langle \zeta, \mathfrak{L}(\bar{\xi}_{k+1}) \omega \rangle = 0. \tag{5.18}$$

We have by (3.14)

$$\langle \bar{\xi}_{k+1} \wedge \zeta, D\omega \rangle = \bar{\xi}_{k+1} \mathbin{\bar{\wedge}} D\langle \zeta, \omega \rangle - \zeta \mathbin{\bar{\wedge}} D\langle \bar{\xi}_{k+1}, \omega \rangle - \pi_{k-1} \langle [\bar{\xi}_{k+1}, \zeta], \omega \rangle \tag{5.19}$$

where the first and last terms on the right-hand side vanish in view of (5.17) and (5.18) respectively. Now let $G \in W$ and let g be a local diffeomorphism of X defined on a neighborhood of $a \in X$ such that $\tau(G) = j_{k+2}(g)(a)$; by (3.8) we have

$$\begin{aligned}\zeta(G) \mathbin{\bar{\wedge}} D\langle \bar{\xi}_{k+1}, \omega \rangle &= \langle \zeta(G) - j_{k+1}(g)_* \pi_* \zeta(G), \pi_{k-1} \cdot d_{Q_{k+1}/X} \langle \bar{\xi}_{k+1}, \omega \rangle \rangle \\ &\quad + \langle \pi_* \zeta(G), D(\langle \bar{\xi}_{k+1}, \omega \rangle \circ j_{k+1}(g)) \rangle.\end{aligned}$$

The first term on the right-hand side of this equation vanishes since

$$\zeta(G) = \tau(G)_* \bar{\zeta}(\pi G) = j_{k+1}(g)_* \pi_* \zeta(G).$$

We now examine the second term; we have by (5.10)

$$(\langle \bar{\xi}_{k+1}, \omega \rangle \circ j_{k+1}(g))(x) = j_{k+1}(g)(x)^{-1} \cdot (j_k(\bar{\xi})(g(x))) = j_k(\bar{\xi}')(x),$$

where $\bar{\xi}'$ is the vector field on X given by

$$\bar{\xi}'(x) = g_* \xi(g^{-1}(x)).$$

Hence

$$D(\langle \bar{\xi}_{k+1}, \omega \rangle \circ j_{k+1}(g)) = Dj_k(\bar{\xi}') = 0$$

and it follows that

$$\zeta \mathbin{\bar{\wedge}} D\langle \bar{\xi}_{k+1}, \omega \rangle = 0;$$

therefore, by (5.19),

$$\langle \bar{\xi}_{k+1} \wedge \zeta, D\omega \rangle = 0.$$

By (5.17),

$$\langle \bar{\xi}_{k+1} \wedge \zeta, \tfrac{1}{2}[\omega, \omega] \rangle = [\langle \bar{\xi}_{k+1}, \omega \rangle, \langle \zeta, \omega \rangle] = 0,$$

and so

$$\langle \bar{\xi}_{k+1} \wedge \zeta, D\omega - \tfrac{1}{2}[\omega, \omega] \rangle = 0. \tag{5.20}$$

Finally, let $\bar{\zeta}'$ be another vector field on πW and ζ' the vector field on W given by

$$\zeta'(G) = \tau(G)_* \bar{\zeta}'(\pi G), \quad G \in W.$$

Let $G \in W$ and assume that $\tau(G) = j_{k+2}(g)(a)$; then

$$\begin{aligned}\langle \zeta \wedge \zeta', D\omega \rangle (G) &= \langle j_{k+1}(g)_* \bar{\zeta}(a) \wedge j_{k+1}(g)_* \bar{\zeta}'(a), D\omega \rangle \\ &= \langle j_{k+1}(g)_* (\bar{\zeta} \wedge \bar{\zeta}'), D\omega \rangle (G) = \langle \bar{\zeta} \wedge \bar{\zeta}', j_{k+1}(g)^* D\omega \rangle (a) \\ &= \langle \bar{\zeta} \wedge \bar{\zeta}', D(j_{k+1}(g)^* \omega) \rangle (a) = 0\end{aligned}$$

by (3.18) and Proposition 5.1, (ii). By (5.17)

$$\langle \zeta \wedge \zeta', \tfrac{1}{2}[\omega, \omega] \rangle = [\omega(\zeta), \omega(\zeta')] = 0$$

and so

$$\langle \zeta \wedge \zeta', D\omega - \tfrac{1}{2}[\omega, \omega] \rangle = 0. \tag{5.21}$$

Since $T(Q_{k+1})$ is generated by vector fields of the type $\bar{\xi}_{k+1}$ and ζ, we deduce (5.13) from (5.16), (5.20) and (5.21). Formula (5.14) is a consequence of (5.12).

From (5.13) we derive the identity

$$\mathcal{D}_1 \mathcal{D} F = D \mathcal{D} F - \tfrac{1}{2}[\mathcal{D} F, \mathcal{D} F] = 0, \quad \text{for } F \in \tilde{Q}_{k+1}$$

(see § 2). Indeed, if F is a section of $\tilde{Q}_{k+1}$, then by Proposition 5.1, (ii), and (3.18)

$$\begin{aligned}D\mathcal{D}F - \tfrac{1}{2}[\mathcal{D}F, \mathcal{D}F] &= DF^*\omega - \tfrac{1}{2}[F^*\omega, F^*\omega] = F^*D\omega - \tfrac{1}{2}F^*[\omega, \omega] \\ &= F^*(D\omega - \tfrac{1}{2}[\omega, \omega]) = 0.\end{aligned}$$

The form ω_V on Q_{k+1} is the natural generalization of the Maurer-Cartan form on a Lie group. In fact, let

$$Q^0_{k+1} = \{F \in Q_{k+1} \mid \pi_0 F = I_0(a), a = \pi F\}.$$

The fiber $Q^0_{k+1}(a)$ of Q^0_{k+1} over $a \in X$ is equal to $Q_{k+1}(a, a)$. Thus Q^0_{k+1} is a bundle of Lie groups over X and $J^0_{k+1}(T)_a$ is identified with the Lie algebra $V_{I_{k+1}(a)}(Q^0_{k+1})$ when we identify $\tilde{J}_{k+1}(T)_a$ with $V_{I_{k+1}(a)}(Q_{k+1})$. The bracket

$$J^0_{k+1}(T) \otimes J^0_{k+1}(T) \to J^0_{k+1}(T), \tag{5.22}$$

which is obtained from the bracket on $\tilde{J}_{k+1}(\mathcal{T})$, gives a structure of Lie algebra on the vector bundle $J^0_{k+1}(T)$ over X. If $\xi \in J^0_{k+1}(T)_a$, the vector field $\tau_{k+1}(\xi)$ on $Q^0_{k+1}(a)$, whose value at $F \in Q^0_{k+1}(a)$ is $\xi \cdot F$, is a right-invariant vector field on this Lie group. Since the mapping τ_{k+1} from $\Gamma(X, \tilde{J}_{k+1}(T))$ to the Lie algebra of vector fields on Q_{k+1} is a morphism of Lie algebras, we can identify the Lie algebra $J^0_{k+1}(T)_a$ with the Lie algebra of *right*-invariant vector fields on $Q^0_{k+1}(a)$. Therefore the natural identification

$$J^0_{k+1}(T)_a \to V_{I_{k+1}(a)}(Q^0_{k+1}) \tag{5.23}$$

is an *anti-isomorphism* of Lie algebras. Using this identification, we regard the Maurer-Cartan form of Q^0_{k+1} of § 4 as a mapping

$$\omega^0 \colon V(Q^0_{k+1}) \to J^0_{k+1}(T);$$

equation (4.2) becomes

$$d_{Q^0_{k+1}/X}\,\omega^0 - \tfrac{1}{2}[\omega^0, \omega^0] = 0, \tag{5.24}$$

where the bracket is given by (4.1) with $X = Q^0_{k+1}$, $Y = X$ and $\mathfrak{g} = J^0_{k+1}(T)$ considered as a Lie algebra with the bracket (5.22). The restriction of ω_V to Q^0_{k+1} is equal to the composition of the Maurer-Cartan form ω^0 of Q^0_{k+1} and the projection π_k of $J^0_{k+1}(T)$ onto $J^0_k(T)$.

6. Jets of projectable vector fields and transformations

Consider the mapping $\varrho\colon T \to \varrho^{-1}T_Y$, whose kernel is V; taking $E = T$, $F = T_Y$, $\varphi = \varrho$ in (3.1), we obtain a projection

$$\varrho\colon J_k(T; \varrho) \to J_k(T_Y; Y).$$

We note that, for $k \geq 1$, the sheaf of solutions of $J_k(T; \varrho)$ is $\mathcal{T}_\varrho$, the sheaf of sections of T which are ϱ-projectable, and that $\pi_0\colon J_k(T; \varrho) \to J_0(T)$ is surjective. We have the exact sequences

$$0 \longrightarrow J_k(V) \longrightarrow J_k(T;\varrho) \xrightarrow{\varrho} \varrho^{-1}J_k(T_Y;Y) \longrightarrow 0, \tag{6.1}$$

$$0 \longrightarrow J_k(\mathfrak{V}) \longrightarrow J_k(\mathfrak{T};\varrho)_\varrho \xrightarrow{\varrho} \varrho^{-1}J_k(\mathfrak{T}_Y;Y) \longrightarrow 0. \tag{6.2}$$

We set

$$\tilde{J}_k(T;\varrho) = \nu^{-1}J_k(T;\varrho),\ \tilde{J}_k(V) = \nu^{-1}J_k(V),$$

and thus obtain a projection

$$\varrho: \tilde{J}_k(T;\varrho) \to \tilde{J}_k(T_Y;Y).$$

We have (see [10])

$$[\tilde{J}_{k+1}(\mathfrak{T};\varrho),\ J_k(\mathfrak{V})] \subset J_k(\mathfrak{V}), \tag{6.3}$$

and conversely if $\xi \in \tilde{J}_{k+1}(\mathfrak{T})$ satisfies $[\xi, J_k(\mathfrak{V})] \subset J_k(\mathfrak{V})$, then $\xi \in \tilde{J}_{k+1}(\mathfrak{T};\varrho)$.

LEMMA 6.1. *Let $R_k \subset J_k(T)$ be a formally integrable differential equation, with $k \geq 1$. Assume that $R_1 = \pi_1 R_k$ is a vector bundle and $R_k \subset (R_1)_{+(k-1)}$. Let $B_k \subset J_k(T)$, $B_{k+1} \subset J_{k+1}(T)$ be differential equations with $B_{k+1} \subset (B_k)_{+1}$. If π_0: $B_{k+1} \to J_0(V)$ is surjective, $\pi_0(B_k) \subset J_0(V)$ and $[R_{k+1}, B_{k+1}] \subset J_k(V)$, then $R_k \subset J_k(T;\varrho)$.*

Remark. If $R_1 = \pi_1 R_k$ is a vector bundle and R_k is integrable, then $R_k \subset (R_1)_{+(k-1)}$.

Proof of Lemma 6.1. Let $\xi \in \mathcal{R}_{k+1}$, $\eta \in \mathcal{B}_{k+1}$; then if $\tilde{\xi} = \nu^{-1}\xi$, by (1.15),

$$\mathfrak{L}(\tilde{\xi})\pi_k\eta = [\tilde{\xi}, \eta] + (\pi_0\tilde{\xi}) \mathbin{\bar{\wedge}} D\eta \in J_k(\mathfrak{V}) + \mathcal{B}_k.$$

Hence since R_1 is a vector bundle and $\pi_0(\mathcal{B}_k) \subset J_0(\mathfrak{V})$, we have

$$\mathfrak{L}(\pi_1\tilde{\xi})\pi_0\eta \in J_0(\mathfrak{V})$$

and $[\tilde{\mathcal{R}}_1, J_0(\mathfrak{V})] \subset J_0(\mathfrak{V})$, where $\tilde{R}_1 = \nu^{-1}R_1$, which implies that $R_1 \subset J_1(T;\varrho)$. As $R_k \subset (R_1)_{+(k-1)}$, we have $R_k \subset (J_1(T;\varrho))_{+(k-1)}$ or $R_k \subset J_k(T;\varrho)$.

The following bracket relations hold:

$$\begin{aligned}
&[J_k(T;\varrho),\ J_k(T;\varrho)] \subset J_{k-1}(T;\varrho),\\
&[\tilde{J}_k(\mathfrak{T};\varrho),\ \tilde{J}_k(\mathfrak{T};\varrho)] \subset \tilde{J}_k(\mathfrak{T};\varrho),\\
&[\tilde{J}_{k+1}(\mathfrak{T};\varrho),\ J_k(\mathfrak{T};\varrho)] \subset J_k(\mathfrak{T};\varrho).
\end{aligned} \tag{6.4}$$

If $\xi, \eta \in J_k(T;\varrho)$, then

$$\varrho[\xi,\eta] = [\varrho\xi, \varrho\eta], \tag{6.5}$$

which implies that

$$[J_k(T;\varrho),\ J_k(V)] \subset J_{k-1}(V). \tag{6.6}$$

If $\tilde{\xi}, \tilde{\eta} \in \tilde{J}_k(\mathfrak{T};\varrho)_\varrho$, then $[\tilde{\xi}, \tilde{\eta}] \in \tilde{J}_k(\mathfrak{T};\varrho)_\varrho$ and

$$\varrho[\tilde{\xi},\tilde{\eta}] = [\varrho\tilde{\xi}, \varrho\tilde{\eta}]. \tag{6.7}$$

Moreover, if $\eta' = \nu\pi_{k-1}\tilde{\eta}$, then $[\tilde{\xi}, \eta'] \in J_{k-1}(\mathcal{T}; \varrho)_\varrho$ and

$$\varrho[\tilde{\xi}, \eta'] = [\varrho\tilde{\xi}, \varrho\eta']. \tag{6.8}$$

Let $u \in F^{i_1+j_1}_{i_1}(J_k(T); \varrho)$, $v \in F^{i_2+j_2}_{i_2}(J_k(T); \varrho)$; then it can be verified, by use of (6.6), that

$$[u, v] \in F^{i_1+i_2+j_1+j_2}_{i_1+i_2}(J_{k-1}(T); \varrho).$$

In particular, we have for $u \in F^i_i(J_k(T); \varrho)$, $v \in F^j_j(J_k(T); \varrho)$,

$$[u, v] \in F^{i+j}_{i+j}(J_{k-1}(T); \varrho) \quad \text{and} \quad \varrho[u, v] = [\varrho u, \varrho v], \tag{6.9}$$

where ϱ is the mapping

$$\varrho: F^p_p(J_m(T); \varrho) \to \wedge^p T^*_Y \otimes J_m(T_Y; Y),$$

with $p = i$ or j and $m = k$, or $p = i+j$ and $m = k-1$, and where the brackets are given by (1.19). From (6.9) it follows that if $u \in (\wedge^i \mathcal{T}^* \otimes J_k(\mathcal{T}; \varrho))_\varrho$, $v \in (\wedge^j \mathcal{T}^* \otimes J_k(\mathcal{T}; \varrho))_\varrho$, then $[u, v] \in (\wedge^{i+j} \mathcal{T}^* \otimes J_{k-1}(\mathcal{T}; \varrho))_\varrho$ and

$$\varrho[u, v] = [\varrho u, \varrho v]. \tag{6.10}$$

We have the bracket

$$(\wedge^i V^* \otimes_X J_k(T_Y; Y)) \otimes (\wedge^j V^* \otimes_X J_k(T_Y; Y)) \to \wedge^{i+j} V^* \otimes_X J_{k-1}(T_Y; Y) \tag{6.11}$$

defined by the formula

$$[\alpha \otimes \xi, \beta \otimes \eta] = (\alpha \wedge \beta) \otimes [\xi, \eta],$$

for $\alpha \in \wedge^i V^*$, $\beta \in \wedge^j V^*$, ξ, $\eta \in J_k(T_Y; Y)$. If $u \in \wedge^i T^* \otimes J_k(T; \varrho)$, $v \in \wedge^j T^* \otimes J_k(T; \varrho)$, then by (6.5)

$$\varrho[u, v] = [\varrho u, \varrho v], \tag{6.12}$$

where ϱ is the mapping

$$\varrho: \wedge T^* \otimes J_m(T; \varrho) \to \wedge V^* \otimes_X J_m(T_Y; Y),$$

with $m = k$ or $k-1$.

Writing $J_0(T_Y) = J_0(T_Y; Y)$ and

$$(\wedge^i J_0(\mathcal{T})^* \otimes \tilde{J}_k(\mathcal{T}; \varrho))_\varrho = (\nu^{*-1} \otimes \nu^{-1})(\wedge^i \mathcal{T}^* \otimes J_k(\mathcal{T}; \varrho))_\varrho,$$

we have the mapping

$$\varrho: (\wedge^i J_0(\mathcal{T})^* \otimes \tilde{J}_k(\mathcal{T}; \varrho))_\varrho \to \wedge^i J_0(\mathcal{T}_Y)^* \otimes \tilde{J}_k(\mathcal{T}_Y; Y).$$

If $u \in (\wedge^i J_0(\mathcal{T})^* \otimes \tilde{J}_k(\mathcal{T}; \varrho))_\varrho$, $v \in (\wedge^j J_0(\mathcal{T})^* \otimes \tilde{J}_k(\mathcal{T}; \varrho))_\varrho$, then $[u, v] \in (\wedge^{i+j} J_0(\mathcal{T})^* \otimes \tilde{J}_k(\mathcal{T}; \varrho))_\varrho$ and

$$\varrho[u, v] = [\varrho u, \varrho v]. \tag{6.13}$$

Let $Q_k(\varrho)$ be the bundle of invertible jets of order k of ϱ-projectable mappings $X \to X$ (i.e., which induce mappings $Y \to Y$). The automorphisms of X which are solutions of $Q_k(\varrho)$, $k \geq 1$, regarded as a differential equation in $J_k(E)$, where $E = X \times X$ is viewed as a bundle over X via pr_1, are the ϱ-projectable automorphisms of X. Let

$$\varrho \colon Q_k(\varrho) \to Q_k(Y) \tag{6.14}$$

be the natural projection of $Q_k(\varrho)$ onto the bundle $Q_k(Y)$ of invertible jets of order k of mappings $Y \to Y$; it is a homomorphism of groupoids over $\varrho\colon X \to Y$. The sub-bundle $Q_k(V)$ of $Q_k(\varrho)$ of jets, whose image by ϱ in $Q_k(Y)$ is equal to the jet of order k of the identity mapping $Y \to Y$, is a sub-groupoid of $Q_k(\varrho)$.

Let $\tilde{\mathcal{Q}}_k(\varrho)$ be the sub-sheaf of $\mathcal{Q}_k(\varrho)$ of invertible elements and let $\tilde{\mathcal{Q}}_k(\varrho)_\varrho$ be its sub-sheaf of ϱ-projectable sections. The mapping $\varrho\colon \mathcal{Q}_k(\varrho)_\varrho \to \mathcal{Q}_k(Y)$ gives by restriction a mapping

$$\varrho\colon \tilde{\mathcal{Q}}_k(\varrho)_\varrho \to \tilde{\mathcal{Q}}_k(Y).$$

We denote by $Q_{(l,k)}(\varrho)$ the bundle of l-jets of sections of $\tilde{\mathcal{Q}}_k(\varrho)_\varrho$; it is a sub-groupoid of $Q_{(l,k)}$. Let $Q_{(l,k)}(Y)$ be the bundle of l-jets of sections of $\tilde{\mathcal{Q}}_k(Y)$. The mapping ϱ: $J_l(Q_k(\varrho); \varrho) \to J_l(Q_k(Y); Y)$ induces a mapping

$$\varrho\colon Q_{(l,k)}(\varrho) \to Q_{(l,k)}(Y);$$

it is a homomorphism of groupoids since (6.14) is, and the diagram

$$\begin{tikzcd} Q_{(l,k)}(\varrho) \arrow[r, "\pi_0"] \arrow[d, "\varrho"] & Q_k(\varrho) \arrow[d, "\varrho"] \\ Q_{(l,k)}(Y) \arrow[r, "\pi_0"] & Q_k(Y) \end{tikzcd}$$

commutes. The inclusion $\lambda_l\colon Q_{k+l}(\varrho) \to Q_{(l,k)}(\varrho)$ induced by $\lambda_l\colon Q_{k+l} \to Q_{(l,k)}$ is a homomorphism of groupoids and the diagram

$$\begin{tikzcd} Q_{k+l}(\varrho) \arrow[r, "\lambda_l"] \arrow[d, "\varrho"] & Q_{(l,k)}(\varrho) \arrow[d, "\varrho"] \\ Q_{k+l}(Y) \arrow[r, "\lambda_l"] & Q_{(l,k)}(Y) \end{tikzcd} \tag{6.15}$$

is commutative.

For $a \in X$, it is easily seen that $\tilde{J}_k(T; \varrho)_a$ is identified with $V_{I_k(a)}(Q_k(\varrho))$ when we identify $\tilde{J}_k(T)_a$ with $V_{I_k(a)}(Q_k)$ and that the diagram

$$\begin{array}{ccc} \tilde{J}_k(T;\varrho)_a & \longrightarrow & V_{I_k(a)}(Q_k(\varrho)) \\ \downarrow \varrho & & \downarrow \varrho_* \\ \tilde{J}_k(T_Y;Y)_{\varrho(a)} & \longrightarrow & T_{I_{Y,k}(\varrho(a))}(Q_k(Y)/Y) \end{array} \tag{6.16}$$

is commutative, where $I_{Y,k}(y)$ is the k-jet of the identity of Y at $y \in Y$. Since $Q_k(\varrho)$ is a groupoid it follows that, for $F \in Q_k(\varrho)$ with target $F = b$ and $\xi \in \tilde{J}_k(T;\varrho)_b$, we have $\xi F \in V_F(Q_k(\varrho))$; furthermore, if G is a section of $\tilde{Q}_k(\varrho)$ over a neighborhood of $b \in X$, for $\xi \in T_F(Q_k(\varrho))$ the mapping $\xi \mapsto G\xi$ induces isomorphisms

$$T_F(Q_k(\varrho)) \to T_{G(b)\cdot F}(Q_k(\varrho)),$$

$$V_F(Q_k(\varrho)) \to V_{G(b)\cdot F}(Q_k(\varrho)).$$

In particular, taking $F = I_k(b)$, we obtain the isomorphism

$$\tilde{J}_k(T;\varrho)_b \to V_{G(b)}(Q_k(\varrho)),$$

which depends only on $H = j_1(G)(b)$ and sends ξ into $H \cdot \xi = G \cdot \xi$; hence we have a corresponding mapping

$$\begin{gathered} Q_{(1,k)}(\varrho) \times_X \tilde{J}_k(T;\varrho) \to V(Q_k(\varrho)) \\ (H, \xi) \mapsto H\xi. \end{gathered} \tag{6.17}$$

From these considerations and (2.5), we conclude that the mapping (2.2) induced by G restricts to give a mapping

$$G\colon \tilde{J}_k(T;\varrho)_b \to \tilde{J}_k(T;\varrho)_c,$$

where $c = \text{target } G(b)$, which in turn determines a mapping

$$\begin{gathered} Q_{(1,k)}(\varrho) \times_X \tilde{J}_k(T;\varrho) \to \tilde{J}_k(T;\varrho) \\ (H, \xi) \mapsto H(\xi). \end{gathered} \tag{6.18}$$

From (2.4) we deduce next that if $F \in Q_k(\varrho)$, $a = \text{source } F$, $b = \text{target } F$, then the mapping (2.1) restricts to give a mapping

$$F\colon J_{k-1}(T;\varrho)_a \to J_{k-1}(T;\varrho)_b.$$

Since the mapping (6.14) is a homomorphism of groupoids we see, by the commutativity of (6.16), that the diagram

$$\begin{tikzcd}
\tilde{J}_k(T;\varrho)_b \arrow[r, "F"] \arrow[d, "\varrho"] & V_F(Q_k(\varrho)) \arrow[d, "\varrho_*"] \\
\tilde{J}_k(T_Y;Y)_{\varrho(b)} \arrow[r, "\phi"] & T_\phi(Q_k(Y)/Y)
\end{tikzcd} \tag{6.19}$$

is commutative, where $\phi = \varrho F \in Q_k(Y)$, with target $\phi = \varrho(b)$, and F, ϕ operate on the right. Since (6.14) is a homomorphism of groupoids, if G is a ϱ-projectable section of $\tilde{Q}_k(\varrho)$ over a neighborhood of b and $\psi = \varrho G$ is the corresponding image section of $\tilde{Q}_k(Y)$ over a neighborhood of $\varrho(b)$, the diagram

$$\begin{tikzcd}
T_F(Q_k(\varrho)) \arrow[r, "G"] \arrow[d, "\varrho_*"] & T_{G(b)\cdot F}(Q_k(\varrho)) \arrow[d, "\varrho_*"] \\
T_\phi(Q_k(Y)) \arrow[r, "\psi"] & T_{\psi(\varrho(b))\cdot\phi}(Q_k(Y))
\end{tikzcd} \tag{6.20}$$

is commutative, where G, ψ operate on the left. From the commutativity of (6.20) and (6.16), it follows that the diagram

$$\begin{tikzcd}
\tilde{J}_k(T;\varrho)_b \arrow[r, "G"] \arrow[d, "\varrho"] & V_{G(b)}(Q_k(\varrho)) \arrow[d, "\varrho_*"] \\
\tilde{J}_k(T_Y;Y)_{\varrho(b)} \arrow[r, "\psi"] & T_{\psi(\varrho(b))}(Q_k(Y)/Y)
\end{tikzcd} \tag{6.21}$$

is also commutative, where G, ψ operate on the right, as is the corresponding diagram

$$\begin{tikzcd}
Q_{(1,k)}(\varrho) \times_X \tilde{J}_k(T;\varrho) \arrow[r] \arrow[d, "\varrho\times\varrho"] & V(Q_k(\varrho)) \arrow[d, "\varrho_*"] \\
Q_{(1,k)}(Y) \times_Y \tilde{J}_k(T_Y;Y) \arrow[r] & T(Q_k(Y)/Y)
\end{tikzcd} \tag{6.22}$$

whose top horizontal arrow is (6.17). From the commutativity of (6.19) and (6.22), it follows by (2.5) that the diagram

$$\begin{tikzcd}
Q_{(1,k)}(\varrho) \times_X \tilde{J}_k(T;\varrho) \arrow[r] \arrow[d, "\varrho\times\varrho"] & \tilde{J}_k(T;\varrho) \arrow[d, "\varrho"] \\
Q_{(1,k)}(Y) \times_Y \tilde{J}_k(T_Y;Y) \arrow[r] & \tilde{J}_k(T_Y;Y),
\end{tikzcd} \tag{6.23}$$

whose horizontal arrows are induced by the mapping (2.2), is commutative. From (6.23), we deduce that $\tilde{Q}_k(\varrho)_\varrho$ operates on $\tilde{J}_k(\mathcal{T};\varrho)_\varrho$ and $\tilde{J}_k(\mathcal{V})$, and that the diagram

$$\begin{array}{ccc} \tilde{Q}_k(\varrho)_\varrho \times_X \tilde{J}_k(\mathcal{T};\varrho)_\varrho & \longrightarrow & \tilde{J}_k(\mathcal{T};\varrho)_\varrho \\ \Big\downarrow{\scriptstyle \varrho\times\varrho} & & \Big\downarrow{\scriptstyle \varrho} \\ \tilde{Q}_k(Y)\times_Y \tilde{J}_k(\mathcal{T}_Y;Y) & \longrightarrow & \tilde{J}_k(\mathcal{T}_Y;Y) \end{array} \tag{6.24}$$

is commutative. From the commutativity of (6.15) and (6.23), we see by (2.4) that the diagram

$$\begin{array}{ccc} Q_{k+1}(\varrho)\times_X J_k(T;\varrho) & \longrightarrow & J_k(T;\varrho) \\ \Big\downarrow{\scriptstyle \varrho\times\varrho} & & \Big\downarrow{\scriptstyle \varrho} \\ Q_{k+1}(Y)\times_Y J_k(T_Y;Y) & \longrightarrow & J_k(T_Y;Y) \end{array} \tag{6.25}$$

is commutative, where the horizontal arrows are induced by the mapping (2.1).

Let

$$Q^k_{k+1}(\varrho)=\{F\in Q_{k+1}(\varrho)\,|\,\pi_k F=I_k(a),\text{ if } a=\text{source } F\},$$

and

$$g_k(T;\varrho)=\{u\in S^kJ_0(T)^*\otimes J_0(T)\,|\,\xi\mathbin{\overline{\wedge}}\delta u\in S^{k-1}J_0(T)^*\otimes J_0(V)\quad\text{for all }\xi\in J_0(V)\}.$$

One verifies easily that

$$\partial\colon Q^k_{k+1}(\varrho)\to g_{k+1}(T;\varrho)$$

is an isomorphism for $k\geqslant 1$.

PROPOSITION 6.1. *Let $a,b\in X$ and $F\in Q_{k+1}(a,b)$.*

(i) *F belongs to $Q_{k+1}(\varrho)$ if and only if $F(J_k(V)_a)=J_k(V)_b$.*

(ii) *F belongs to $Q_{k+1}(V)$ if and only if $\varrho(a)=\varrho(b)$ and $\varrho F=\varrho$ as mappings*

$$J_k(T;\varrho)_a\to J_k(T_Y;Y)_{\varrho(a)}.$$

Proof. (i) If $F\in Q_{k+1}(\varrho)$, then the commutativity of (6.25) implies that $F(J_k(V)_a)=J_k(V)_b$. Conversely, we prove that this last assertion implies that F belongs to $Q_{k+1}(\varrho)$ by induction on k. First, let $k=0$ and $F=j_1(f)(a)$, where f is a local diffeomorphism of X defined on a neighborhood of a; then $F\in Q_1(\varrho)$ if and only if $(\varrho\circ f)_*\xi=0$ for all $\xi\in V_a$. By (2.4), this last statement is equivalent to $\varrho F(\xi)=0$ or $F(\xi)\in J_0(V)_b$ for all $\xi\in J_0(V)_a$. Now assume that $k\geqslant 1$ and that our assertion is valid for $k-1$. Then $\pi_k F\in Q_k(\varrho)$ by our induction hypothesis. There exists $F_1\in Q_{k+1}(\varrho)$ such that $\pi_k F_1=\pi_k F$. Then $G=F_1^{-1}\cdot F\in Q^k_{k+1}(a)$ and by (2.19)

$$(\pi_0\xi) \wedge \tilde{\delta}\partial G = G(\xi) - \xi \in J_k(V)_a$$

for all $\xi \in J_k(V)_a$. Hence $\partial G \in g_{k+1}(T; \varrho)$ and $G \in Q^k_{k+1}(\varrho)$. Therefore $F = F_1 \cdot G \in Q_{k+1}(\varrho)$.

(ii) If $F \in Q_{k+1}(V)$, then $\varrho F = I_{Y,k+1}(\varrho(a))$ and so $\varrho(a) = \varrho(b)$ and we have the equality of the mappings ϱF and ϱ by the commutativity of (6.25). Conversely if $\varrho(a) = \varrho(b)$ and $\varrho F = \varrho$ as mappings $J_k(T; \varrho)_a \to J_k(T_Y; Y)_{\varrho(a)}$, then $F \in Q_{k+1}(\varrho)$ according to (i). Hence if $\phi = \varrho F \in Q_{k+1}(Y)$, then by the commutativity of (6.25), ϕ acts on $J_k(T_Y; Y)_{\varrho(a)}$ as the identity map. By Proposition 2.3, (ii), $\phi = I_{Y,k+1}(\varrho(a))$ and $F \in Q_{k+1}(V)$.

We now give criteria in order that an element $H \in Q_{(1,k)}$ belong to $Q_{(1,k)}(\varrho)$, and we examine the structure of $Q_{(1,k)}(\varrho)$. However, before doing so, we require the following definitions. For $a, b \in X$, let

$$\varrho : T^*_a \otimes \tilde{J}_k(T; \varrho)_b \to V^*_a \otimes \tilde{J}_k(T_Y; Y)_{\varrho(b)} \tag{6.26}$$

be the mapping sending $u \in T^*_a \otimes \tilde{J}_k(T; \varrho)_b$ into the element ϱu defined by

$$(\varrho u)(\xi) = \varrho(u(\xi)), \quad \text{for } \xi \in V_a.$$

Denote by $F_1(T^*_a \otimes \tilde{J}_k(T; \varrho)_b)$ the kernel of (6.26) and let

$$\varrho : F_1(T^*_a \otimes \tilde{J}_k(T; \varrho)_b) \to T^*_{Y,\varrho(a)} \otimes \tilde{J}_k(T_Y; Y)_{\varrho(b)} \tag{6.27}$$

be the mapping defined by setting

$$(\varrho u)(\tilde{\eta}) = \varrho(u(\eta))$$

for $\eta \in T_a$, $\tilde{\eta} = \varrho(\eta) \in T_{Y,\varrho(a)}$. From (6.26), we obtain a similar mapping

$$\varrho : T^*_a \otimes J_k(T; \varrho)_b \to V^*_a \otimes J_k(T_Y; Y)_{\varrho(b)}$$

generalizing the map defined earlier in the case $a = b$.

PROPOSITION 6.2. *Let $H \in Q_{(1,k)}$ with $\pi_0 H = F \in Q_k$, source $F = a$, target $F = b$.*

(i) *$H \in J_1(Q_k(\varrho))$ if and only if $F \in Q_k(\varrho)$ and $H(\tilde{J}_k(T; \varrho)_a) = \tilde{J}_k(T; \varrho)_b$.*

(ii) *H belongs to $Q_{(1,k)}(\varrho)$ if and only if $F \in Q_k(\varrho)$ and*

$$H(\tilde{J}_k(T; \varrho)_a) = \tilde{J}_k(T; \varrho)_b, \quad H(\tilde{J}_k(V)_a) = \tilde{J}_k(V)_b.$$

(iii) *If $H \in Q_{(1,k)}(\varrho)$ and $J_1(\pi_0)H = j_1(f)(a)$, where f is a local diffeomorphism of X defined on a neighborhood of a, and $u \in T^*_a \otimes \tilde{J}_k(T)_b$, then $H + uF$ belongs to $Q_{(1,k)}(\varrho)$ if and only if:*

(a) *$f + \pi_0 u : T_a \to T_b$ is invertible;*

(b) *$u \in F_1(T^*_a \otimes \tilde{J}_k(T; \varrho)_b)$.*

If $H + uF \in Q_{(1,k)}(\varrho)$, then

$$\varrho(H+uF)=\varrho H+(\varrho u)\cdot\varrho F \tag{6.28}$$

as elements of $Q_{(1,k)}(Y)$, *where* $\varrho u\in T^*_{Y,\varrho(a)}\otimes\tilde{J}_k(T_Y;\,Y)_{\varrho(b)}$ *is defined by* (6.27).

Proof. (i) If $H\in J_1(Q_k(\varrho))$, then we write $H=j_1(G)(a)$ for some section G of $\tilde{Q}_k(\varrho)$ over a neighborhood of a, and for $\xi\in\tilde{J}_k(T;\varrho)_a$ we know that $H(\xi)=G(\xi)$ belongs to $\tilde{J}_k(T;\varrho)_b$. Conversely if $F\in Q_k(\varrho)$, let G_0 be a section of $\tilde{Q}_k(\varrho)$ over a neighborhood of a such that $G_0(a)=F$. Then there exists $u\in T^*_a\otimes\tilde{J}_k(T)_b$ such that $H=j_1(G_0)(a)+uF$. Assume now that $H(\tilde{J}_k(T;\varrho)_a)=\tilde{J}_k(T;\varrho)_b$. If $\xi\in\tilde{J}_k(T;\varrho)_a$, then by (2.15)

$$(\pi_0\xi)\,\bar{\wedge}\,u=H(\xi)-j_1(G_0)(a)(\xi)\in\tilde{J}_k(T;\varrho)_b.$$

Since $\pi_0\colon\tilde{J}_k(T;\varrho)\to T$ is surjective, we deduce that $u\in T^*_a\otimes\tilde{J}_k(T;\varrho)_b$ and $uF\in T^*_a\otimes V_F(Q_k(\varrho))$. As $J_1(Q_k(\varrho))$ is an affine sub-bundle of $J_1(Q_k)_{|Q_k(\varrho)}$, it follows that $j_1(G_0)(a)+uF$ belongs to $J_1(Q_k(\varrho))$ or that $H\in J_1(Q_k(\varrho))$.

(ii) If $H=j_1(G)(a)$, where G is a section of $\tilde{Q}_k(\varrho)$ over a neighborhood of a, then $H\in Q_{(1,k)}(\varrho)$ if and only if $(\varrho\circ G)_*\xi_0=0$ for all $\xi_0\in V_a$. Let G_0 be a section of $\tilde{Q}_k(\varrho)_\varrho$ over a neighborhood of a such that $G_0(a)=F$. Since $J_1(Q_k(\varrho))$ is an affine bundle over $Q_k(\varrho)$, there exists $u\in T^*_a\otimes\tilde{J}_k(T;\varrho)_b$ such that $H=j_1(G_0)(a)+uF$, where $uF\in V_F(Q_k(\varrho))$, and

$$G_*(\pi_0\xi)-G_{0*}(\pi_0\xi)=(\pi_0\xi)\,\bar{\wedge}\,uF=((\pi_0\xi)\,\bar{\wedge}\,u)\,F=(G(\xi)-G_0(\xi))\,F$$

for $\xi\in\tilde{J}_k(T)_a$, by (2.15). Therefore, for $\xi\in\tilde{J}_k(V)_a$, by the commutativity of (6.19),

$$(\varrho\circ G)_*(\pi_0\xi)=\varrho_*G_{0*}(\pi_0\xi)+\varrho_*((G(\xi)-G_0(\xi))\,F)=(\varrho\circ G_0)_*(\pi_0\xi)+\varrho(G(\xi)-G_0(\xi))\circ\varrho F.$$

Since G_0 is a section of $\tilde{Q}_k(\varrho)_\varrho$, the first term on the right-hand side vanishes, while $\varrho G_0(\xi)=0$ by the commutativity of (6.24). Hence, we obtain

$$(\varrho\circ G)_*(\pi_0\xi)=\varrho(G(\xi))\cdot\varrho F.$$

Therefore $(\varrho\circ G)_*\xi_0=0$ for all $\xi_0\in V_a$ if and only if $\varrho(G(\xi))=0$ for all $\xi\in\tilde{J}_k(V)_a$, i.e., $H(\tilde{J}_k(V)_a)=\tilde{J}_k(V)_b$. We conclude that $H\in Q_{(1,k)}(\varrho)$ if and only if $H(\tilde{J}_k(V)_a)=\tilde{J}_k(V)_b$; from (i), we now deduce (ii).

(iii) The first part of (iii) follows directly from Proposition 2.2, (i), (2.15) and (ii), since $\pi_0\colon\tilde{J}_k(T;\varrho)\to T$ is surjective. If $H+uF\in Q_{(1,k)}(\varrho)$, by the commutativity of (6.23) and (2.15)

$$\begin{aligned}(\varrho(H+uF))(\eta)&=\varrho((H+uF)(\xi))=\varrho(H(\xi)+(\pi_0\xi)\,\bar{\wedge}\,u)\\&=(\varrho H)(\eta)+(\pi_0\eta)\,\bar{\wedge}\,\varrho u=(\varrho H+\varrho u\cdot\varrho F)(\eta)\end{aligned}$$

for all $\eta\in\tilde{J}_k(T_Y;\,Y)_{\varrho(a)}$ and $\xi\in\tilde{J}_k(T;\varrho)_a$ with $\varrho(\xi)=\eta$. Hence by Proposition 2.2, (iii), we deduce (6.28).

From the map (6.17) and (5.2), we see that the restriction to $Q_{k+1}(\varrho)$ of the Cartan fundamental form ω on $Q_{k+1}(\varrho)$ is a map

$$\omega\colon T(Q_{k+1}(\varrho)) \to J_k(T;\varrho);$$

from Proposition 5.1, (ii), it follows that if $F \in \tilde{Q}_{k+1}(\varrho)$, then $\mathcal{D}F \in \mathcal{T}^* \otimes J_k(\mathcal{T};\varrho)$. Furthermore, if ω_Y is the Cartan fundamental form on $Q_{k+1}(Y)$, the diagram

$$\begin{tikzcd} T(Q_{k+1}(\varrho)) \arrow[r, "\omega"] \arrow[d, "\varrho_*"] & J_k(T;\varrho) \arrow[d, "\varrho"] \\ T(Q_{k+1}(Y)) \arrow[r, "\omega_Y"] & J_k(T_Y;Y) \end{tikzcd} \tag{6.29}$$

commutes. Indeed, if $\xi \in T_F(Q_{k+1}(\varrho))$, $F \in Q_{k+1}(\varrho)$,

$$\begin{aligned} \varrho\langle \xi, \omega\rangle &= \nu\varrho(\lambda_1 F)^{-1}\cdot(\pi_{k*}\xi - F_*\pi_*\xi) = \nu(\lambda_1(\varrho F))^{-1}\cdot\varrho_*(\pi_{k*}\xi - F_*\pi_*\xi) \\ &= \nu(\lambda_1(\varrho F))^{-1}\cdot(\pi_{k*}\varrho_*\xi - (\varrho F)_*\pi_*\xi) = \langle \varrho_*\xi, \omega_Y\rangle \end{aligned}$$

by the commutativity of (6.22) and (6.15).

Definition 6.1. Let $\tilde{Q}_k(Y)_X$ be the sub-sheaf of $Q_k(Y)_X$ whose sections are local mappings $\phi\colon X \to Q_k(Y)$ such that source$\circ\phi = \varrho$ and such that the composition $f = \text{target}\circ\phi\colon X \to Y$ is a submersion.

If $Q_k^0(Y)$ is the sub-bundle of $Q_k(Y)$ composed of the elements F such that $\pi_0 F = I_{Y,0}(y)$, with $y = \text{source } F$, then $Q_k^0(Y)_X$ is the sub-sheaf of $\tilde{Q}_k(Y)_X$ whose sections ϕ satisfy target$\circ\phi = \varrho$.

The injection $Q_k(Y) \to Q_k(Y)_X$ sending ϕ into $\phi\circ\varrho$ induces an injection

$$\tilde{Q}_k(Y) \to \tilde{Q}_k(Y)_X. \tag{6.30}$$

Indeed, if ϕ is a local section of $\tilde{Q}_k(Y)$ over Y, then target$\circ\phi\circ\varrho$ is a submersion. We have the mapping

$$\varrho\colon \tilde{Q}_k(\varrho) \to \tilde{Q}_k(Y)_X \tag{6.31}$$

sending F into ϱF, where $\varrho F = \varrho\circ F$, since target$\circ\varrho F = \varrho\circ\pi_0 F$ is a submersion.

Next, let $\phi \in \tilde{Q}_k(Y)_X$ and let f be a germ of a diffeomorphism $X \to X$ satisfying $\varrho\circ f = \text{target}\circ\phi$; such an f exists by the implicit-function theorem. We define $\phi_f^{-1} \in \tilde{Q}_k(Y)_X$ by the formula

$$\phi_f^{-1}(x) = \phi(f^{-1}(x))^{-1}, \quad x \in X. \tag{6.32}$$

We have

$$\text{target}\circ\phi_f^{-1} = \varrho\circ f^{-1},$$

$$\phi(x)\cdot\phi_f^{-1}(f(x)) = I_{Y,k}(\varrho(f(x))), \quad \phi_f^{-1}(f(x))\cdot\phi(x) = I_{Y,k}(\varrho(x)),$$

for $x \in X$. Finally, if $F \in \tilde{Q}_k(\varrho)$, then $\varrho F^{-1} = \phi_f^{-1}$, where $f = \pi_0 F$ and $\phi = \varrho F$.

We now define

$$\mathcal{D}_{X/Y}\colon \tilde{Q}_{k+1}(Y)_X \to \mathcal{V}^* \otimes J_k(\mathcal{T}_Y;\, Y)_X, \tag{6.33}$$

sending ϕ into

$$\mathcal{D}_{X/Y}\,\phi = (\phi^*\omega_Y)_{|V}.$$

If ϕ is a section of $\tilde{Q}_{k+1}(Y)_X$ over $U \subset X$, then $\phi_*\xi \in T_{\phi(a)}(Q_{k+1}(Y)/Y)$, for $\xi \in V_a$, $a \in U$ and

$$\langle \xi, \mathcal{D}_{X/Y}\,\phi\rangle = \langle \phi_*\xi, \omega_Y\rangle.$$

By (5.3), we have the formula

$$\langle \xi, \mathcal{D}_{X/Y}\,\phi\rangle = \nu(\lambda_1\phi(a))^{-1}\cdot(\pi_k\phi)_*\xi, \tag{6.34}$$

for $\xi \in V_a$, $a \in U$, where $(\pi_k\phi)_*\xi \in T_{\pi_k\phi(a)}(Q_k(Y)/Y)$ and

$$\lambda_1\phi(a)\colon \tilde{J}_k(T_Y;\, Y)_{\varrho(a)} \to T_{\pi_k\phi(a)}(Q_k(Y)/Y)$$

is the left-action of $Q_{(1,k)}(Y)(\varrho(a))$ on $\tilde{J}_k(T_Y;\, Y)_{\varrho(a)}$; therefore

$$\nu(\lambda_1\phi(a))^{-1}\cdot(\pi_k\phi)_*\xi \in J_k(T_Y;\, Y)_{\varrho(a)}.$$

We also have the mapping

$$\mathcal{D}_{X/Y}\colon Q_k^0(Y)_X \to \mathcal{V}^* \otimes J_k^0(\mathcal{T}_Y;\, Y)_X \tag{6.35}$$

defined in § 4 in terms of the Maurer-Cartan form of the bundle of Lie groups $Q_k^0(Y)$ over Y, identifying $J_k^0(T_Y;\, Y)$ with the Lie algebra of $Q_k^0(Y)$ by the maps (5.23) (with X replaced by Y and $k+1$ by k). The restriction of (6.33) to $Q_{k+1}^0(Y)_X$ is equal to the composition of (6.35) (with $k+1$ replacing k) and the projection $\mathrm{id}\otimes\pi_k$ of $\mathcal{V}^*\otimes J_{k+1}^0(\mathcal{T}_Y;\, Y)_X$ onto $\mathcal{V}^*\otimes J_k^0(\mathcal{T}_Y;\, Y)_X$.

LEMMA 6.2. *For $\phi \in \tilde{Q}_{k+1}(Y)_X$, we have $\mathcal{D}_{X/Y}\phi = 0$ if and only if $\pi_k\phi \in \tilde{Q}_k(Y)$.*

Proof. If ϕ is a section of $\tilde{Q}_{k+1}(Y)_X$ over $U \subset X$, then by (6.34), $\mathcal{D}_{X/Y}\,\phi = 0$ if and only if $(\pi_k\phi)_*\xi = 0$ for all $\xi \in V_a$, $a \in U$.

We define

$$\mathcal{D}_{1,X/Y}\colon \mathcal{V}^*\otimes J_k(\mathcal{T}_Y;\, Y)_X \to \wedge^2\mathcal{V}^*\otimes J_{k-1}(\mathcal{T}_Y;\, Y)_X$$

by the formula

$$\mathcal{D}_{1,X/Y}v = \pi_{k-1}\cdot d_{X/Y}v - \tfrac{1}{2}[v, v], \quad v \in \mathcal{V}^*\otimes J_k(\mathcal{T}_Y;\, Y)_X, \tag{6.36}$$

where the bracket is given by (6.11); from the definition of $\mathcal{D}_{X/Y}$ and the Cartan structure equation (5.14), we obtain

$$\mathcal{D}_{1,X/Y}\cdot\mathcal{D}_{X/Y}\phi=0,\quad\text{for } \phi\in\tilde{Q}_{k+1}(Y)_X.$$

Since (5.23) is an anti-isomorphism of Lie algebras, the restriction of $\mathcal{D}_{1,X/Y}$ to $\mathcal{V}^*\otimes J_k^0(\mathcal{T}_Y;Y)_X$ is equal to the composition of the operator

$$\mathcal{D}_{1,X/Y}\colon \mathcal{V}^*\otimes J_k^0(\mathcal{T}_Y;Y)_X\to\wedge^2\mathcal{V}^*\otimes J_k^0(\mathcal{T}_Y;Y)_X$$

of § 4, if we identify $J_k^0(T_Y;Y)$ with the Lie algebra of Q_k^0 by (5.23), and the projection

$$\mathrm{id}\otimes\pi_{k-1}\colon \wedge^2\mathcal{V}^*\otimes J_k^0(\mathcal{T}_Y;Y)_X\to\wedge^2\mathcal{V}^*\otimes J_{k-1}^0(\mathcal{T}_Y;Y)_X.$$

Let $\tilde{Q}_{k+1}(Y)_X^k$ be the sub-sheaf of $\tilde{Q}_{k+1}(Y)_X$ composed of the elements ϕ satisfying $\pi_k\phi\in\tilde{Q}_k(Y)$. Then we have the complex

$$\tilde{Q}_{k+1}(Y)_X^k\longrightarrow\tilde{Q}_{k+1}(Y)_X\xrightarrow{\mathcal{D}_{X/Y}}\mathcal{V}^*\otimes J_k(\mathcal{T}_Y;Y)_X\xrightarrow{\mathcal{D}_{1,X/Y}}\wedge^2\mathcal{V}^*\otimes J_{k-1}(\mathcal{T}_Y;Y)_X \tag{6.37}$$

which, by Lemma 6.2, is exact at $\tilde{Q}_{k+1}(Y)_X$. We also have the following complex, which is obtained from (4.6) by replacing G by $Q_k^0(Y)$ and the Lie algebra $\mathfrak{g}$ by $J_k^0(T_Y;Y)$,

$$Q_k^0(Y)\longrightarrow Q_k^0(Y)_X\xrightarrow{\mathcal{D}_{X/Y}}\mathcal{V}^*\otimes J_k^0(\mathcal{T}_Y;Y)_X\xrightarrow{\mathcal{D}_{1,X/Y}}\wedge^2\mathcal{V}^*\otimes J_k^0(\mathcal{T}_Y;Y)_X \tag{6.38}$$

and which is exact by Proposition 4.1.

PROPOSITION 6.3. *The diagram*

$$\begin{tikzcd}
\tilde{Q}_{k+1}(\varrho) \arrow[r, "\mathcal{D}"] \arrow[d, "\varrho"] & \mathcal{T}^*\otimes J_k(\mathcal{T};\varrho) \arrow[r, "\mathcal{D}_1"] \arrow[d, "\varrho"] & \wedge^2\mathcal{T}^*\otimes J_{k-1}(\mathcal{T};\varrho) \arrow[d, "\varrho"] \\
\tilde{Q}_{k+1}(Y)_X \arrow[r, "\mathcal{D}_{X/Y}"] & \mathcal{V}^*\otimes J_k(\mathcal{T}_Y;Y)_X \arrow[r, "\mathcal{D}_{1,X/Y}"] & \wedge^2\mathcal{V}^*\otimes J_{k-1}(\mathcal{T}_Y;Y)_X
\end{tikzcd} \tag{6.39}$$

commutes.

Proof. If F is a section of $\tilde{Q}_{k+1}(\varrho)$ over a neighborhood of $a\in X$, then for $\xi\in V_a$,

$$\langle\xi,\varrho(\mathcal{D}F)\rangle=\varrho\langle\xi,\mathcal{D}F\rangle=\varrho\langle\xi,F^*\omega\rangle=\varrho\langle F_*\xi,\omega\rangle=\langle\phi_*\xi,\omega_Y\rangle=\langle\xi,\mathcal{D}_{X/Y}\phi\rangle$$

by (5.6) and the commutativity of (6.29). The commutativity of the right-hand square of (6.39) follows from the commutativity of (3.2) and from (6.12).

LEMMA 6.3. *Let ϕ be a section of $\tilde{Q}_{k+1}(Y)_X$ over $U\subset X$; then*

$$(\phi\circ\mathcal{D}_{X/Y}\phi)(a)\in V_a^*\otimes J_k(T_Y;Y)_{\mathrm{target}\,\phi(a)},$$

for $a\in U$, and $\phi\circ\mathcal{D}_{X/Y}\phi$ depends only on $\pi_k\phi$.

Proof. According to (2.6), we have for $a \in U$, $\eta \in \tilde{J}_k(T_Y; Y)_{\mathrm{target}\,\phi(a)}$

$$\phi(a)\,(\nu(\lambda_1\phi(a)^{-1}\cdot\eta\cdot\pi_k\phi(a)) = \nu\eta.$$

Taking in this formula $\eta = (\pi_k\phi)_*\xi\cdot\pi_k\phi(a)^{-1}$, where $\xi \in V_a$, we obtain

$$\langle\xi, (\phi\circ\mathcal{D}_{X/Y}\phi)(a)\rangle = \phi(a)\cdot\nu(\lambda_1\phi(a))^{-1}\cdot(\pi_k\phi)_*\xi = \nu((\pi_k\phi)_*\xi\cdot\pi_k\phi(a)^{-1}).$$

If $F \in \tilde{Q}_{k+1}(\varrho)$, $\phi = \varrho F \in \tilde{Q}_{k+1}(Y)_X$, $f = \pi_0 F$ and $u \in \mathcal{T}^*\otimes J_k(\mathcal{T}; \varrho)$, then

$$\varrho(u^F) = \phi_f^{-1}\circ\varrho(u\circ f) + \mathcal{D}_{X/Y}\phi. \tag{6.40}$$

This formula follows immediately from Proposition 6.3.

LEMMA 6.4. *Let $F \in \tilde{Q}_{k+1}(\varrho)$, $\phi = \varrho F$ and let $u \in \mathcal{T}^*\otimes J_k(\mathcal{T}; \varrho)$. Then $\phi\circ\varrho(u^F)$ depends only on $\pi_k F$.*

This lemma is an immediate consequence of (6.40), (6.32) and Lemma 6.3.

Let f be a germ of a diffeomorphism of X. Then f is ϱ-projectable, that is, is the germ of a ϱ-projectable diffeomorphism, if and only if f preserves V.

LEMMA 6.5. *Let $F \in \tilde{Q}_{k+1}(\varrho)$, $\phi = \varrho F$, $f = \pi_0 F$* and let $u \in \mathcal{T}^*\otimes J_k(\mathcal{T}; \varrho)$. *If f is ϱ-projectable, then $\varrho(u) = 0$ if and only if $\varrho(u^F) = \mathcal{D}_{X/Y}\phi$.*

Proof. By (6.40), we have since f preserves V

$$\varrho(u^F) = \phi_f^{-1}\circ\varrho(u)\circ f + \mathcal{D}_{X/Y}\phi. \tag{6.41}$$

Now $\varrho(u) = 0$ if and only if $\phi_f^{-1}\circ\varrho(u)\circ f = 0$, which is equivalent therefore to $\varrho(u^F) = \mathcal{D}_{X/Y}\phi$.

If $u \in (T^*\otimes J_k(T))^\wedge \cap F_1^1(J_k(T); \varrho)$, it is easily seen that $\varrho u \in (T_Y^*\otimes J_k(T_Y; Y))^\wedge$. Set

$$(\mathcal{T}^*\otimes J_k(\mathcal{T}; \varrho))_\varrho^\wedge = (\mathcal{T}^*\otimes J_k(\mathcal{T}; \varrho))_\varrho \cap (\mathcal{T}^*\otimes J_k(\mathcal{T}))^\wedge.$$

Therefore if $u \in (\mathcal{T}^*\otimes J_k(\mathcal{T}; \varrho))_\varrho^\wedge$, then ϱu belongs to $(\mathcal{T}_Y^*\otimes J_k(\mathcal{T}_Y; Y))^\wedge$.

PROPOSITION 6.4. (i) *Let $F \in \tilde{Q}_{k+1}(\varrho)$; then $F \in \tilde{Q}_{k+1}(\varrho)_\varrho$ if and only if*

$$\mathcal{D}F \in (\mathcal{T}^*\otimes J_k(\mathcal{T}; \varrho))_\varrho.$$

(ii) *Let $u_1, u_2 \in (\mathcal{T}^*\otimes J_k(\mathcal{T}; \varrho))_\varrho^\wedge$ and $F \in \tilde{Q}_{k+1}(\varrho)$. If $u_2 = u_1^F$, then $F \in \tilde{Q}_{k+1}(\varrho)_\varrho$.*

(iii) *We have*

$$\mathcal{D}_1\colon (\mathcal{T}^*\otimes J_k(\mathcal{T}; \varrho))_\varrho \to (\wedge^2\mathcal{T}^*\otimes J_{k-1}(\mathcal{T}; \varrho))_\varrho, \tag{6.42}$$

and the diagram

$$\begin{array}{ccccc} \tilde{Q}_{k+1}(\varrho)_\varrho & \xrightarrow{\mathcal{D}} & (\mathcal{T}^*\otimes J_k(\mathcal{T};\varrho))_\varrho & \xrightarrow{\mathcal{D}_1} & (\wedge^2\mathcal{T}^*\otimes J_{k-1}(\mathcal{T};\varrho))_\varrho \\ \downarrow{\scriptstyle\varrho} & & \downarrow{\scriptstyle\varrho} & & \downarrow{\scriptstyle\varrho} \\ \tilde{Q}_{k+1}(Y) & \xrightarrow{\mathcal{D}} & \mathcal{T}_Y^*\otimes J_k(\mathcal{T}_Y;Y) & \xrightarrow{\mathcal{D}_1} & \wedge^2\mathcal{T}_Y^*\otimes J_{k-1}(\mathcal{T}_Y;Y) \end{array} \tag{6.43}$$

commutes.

(iv) *If* $u\in(\mathcal{T}^*\otimes J_k(\mathcal{T};\varrho))_\varrho$, $F\in\tilde{Q}_{k+1}(\varrho)_\varrho$, *then* $u^F\in(\mathcal{T}^*\otimes J_k(\mathcal{T};\varrho))_\varrho$ *and*

$$\varrho(u^F)=(\varrho u)^{\varrho F}.$$

Proof. We first prove that, if $F\in\tilde{Q}_{k+1}(\varrho)_\varrho$, then $\mathcal{D}F\in(\mathcal{T}^*\otimes J_k(\mathcal{T};\varrho))_\varrho$. Take $F_1\in\tilde{Q}_{k+2}(\varrho)$ with $\pi_{k+1}F_1=F$. Then, since $\varrho F\in\tilde{Q}_{k+1}(Y)$, by Proposition 6.3 and Lemma 6.2

$$\varrho(\mathcal{D}F_1)=\mathcal{D}_{X/Y}(\varrho F_1)=0,$$

so $\mathcal{D}F_1\in F_1^1(J_{k+1}(\mathcal{T});\varrho)$. We have

$$0=\mathcal{D}_1(\mathcal{D}F_1)=D(\mathcal{D}F_1)-\tfrac{1}{2}[\mathcal{D}F_1,\mathcal{D}F_1],$$

where $[\mathcal{D}F_1,\mathcal{D}F_1]\in F_2^2(J_k(\mathcal{T});\varrho)$ by (6.9); hence $D(\mathcal{D}F_1)\in F_2^2(J_k(\mathcal{T});\varrho)$. From Proposition 4, (i) of [6], it follows that $\mathcal{D}F\in(\mathcal{T}^*\otimes J_k(\mathcal{T};\varrho))_\varrho$.

We next prove (ii). Let F be a section of $\tilde{Q}_{k+1}(\varrho)$ over an open set $U\subset X$ and $f=\pi_0F$; let u_1, u_2 be sections of $(\mathcal{T}^*\otimes J_k(\mathcal{T};\varrho))_\varrho^\wedge$ over $f(U)$ and U respectively. If $u_2=u_1^F$, then

$$u_2=\mathcal{D}F+F^{-1}(u_1)=F^{-1}(\nu)-\nu+F^{-1}(u_1)$$

by (2.25). Hence, since $\varrho(\pi_0u_2)=0$ and $\varrho(\pi_0\nu)=0$ as sections of $V^*\otimes_X J_0(T_Y)$, we have

$$0=\varrho(\pi_0(F^{-1}(\nu)-\nu+F^{-1}(u_1))=\varrho((\pi_1F)^{-1}\circ(\nu+\pi_0u_1)\circ f)=(\pi_1\varrho F)^{-1}\varrho((\nu+\pi_0u_1)\circ f)$$

by the commutativity of (6.25). Therefore the composition

$$V_a\xrightarrow{f}T_{f(a)}\xrightarrow{\nu+\pi_0u_1}J_0(T)_{f(a)}\xrightarrow{\varrho}J_0(T_Y)_{\varrho(f(a))}$$

is zero for all $a\in U$. The mapping $\nu+\pi_0u_1\colon T_{f(a)}\to J_0(T)_{f(a)}$ is invertible by hypothesis and maps $V_{f(a)}$ into $J_0(V)_{f(a)}$, since $\varrho(\pi_0u_1)=0$, and so we conclude that $f(V_a)=V_{f(a)}$, that is, f preserves V. Now let us consider the corresponding germs of our sections; if $F\in\tilde{Q}_{k+1}(\varrho)$, u_1, $u_2\in(\mathcal{T}^*\otimes J_k(\mathcal{T};\varrho))_\varrho^\wedge$ satisfy $u_2=u_1^F$, then we have shown that $f=\pi_0F$ is ϱ-projectable. By Lemma 6.5, it follows that $\mathcal{D}_{X/Y}(\varrho F)=0$. From Lemma 6.2, we deduce that $\pi_kF\in\tilde{Q}_k(\varrho)_\varrho$. By Lemma 2.1, we have

$$\lambda_1F=(j_1(\pi_kF)+(\nu^{-1}\circ u_1\circ f)\pi_kF)\cdot(j_1(I_k)+\nu^{-1}\circ u_2)^{-1}.$$

To show that $F\in\tilde{Q}_{k+1}(\varrho)_\varrho$, it suffices by this formula and the commutativity of (6.15) to show that the two elements $j_1(\pi_k F)+(\nu^{-1}\circ u_1\circ f)\pi_k F$ and $j_1(I_k)+\nu^{-1}\circ u_2$ of $Q_{(1,k)}$ belong to $Q_{(1,k)}(\varrho)_\varrho$. First, they belong to $Q_{(1,k)}(\varrho)$ according to the criterion of Proposition 6.2, (iii), since $\varrho(u_1)=0$, f preserves V and $\varrho(u_2)=0$. From (6.28) we conclude that the former belongs to $Q_{(1,k)}(\varrho)_\varrho$ since $\pi_k F\in\tilde{Q}_k(\varrho)_\varrho$ and $u_1\in(\mathcal{T}^*\otimes J_k(\mathcal{T};\varrho))_\varrho$, and the latter belongs to $Q_{(1,k)}(\varrho)_\varrho$ since $u_2\in(\mathcal{T}^*\otimes J_k(\mathcal{T};\varrho))_\varrho$. Hence $F\in\tilde{Q}_{k+1}(\varrho)_\varrho$.

If $F\in\tilde{Q}_{k+1}(\varrho)$ and if $\mathcal{D}F=0^F$ belongs to $(\mathcal{T}^*\otimes J_k(\mathcal{T};\varrho))_\varrho$, (ii) implies that F belongs to $\tilde{Q}_{k+1}(\varrho)_\varrho$, completing the proof of (i).

We now verify (iii). First, (6.42) and the commutativity of the right-hand square of (6.43) are consequences of Proposition 4, (ii) of [6] and (6.10). As for the left-hand square of (6.43), let F be a ϱ-projectable section of $\tilde{Q}_{k+1}(\varrho)$ over an open set $U\subset X$ and $\phi=\varrho F$ be the corresponding image section of $\tilde{Q}_{k+1}(Y)$ over $\varrho U\subset Y$. Then for $\xi\in T_a$, $a\in U$, by (5.6) and the commutativity of (6.29)

$$\begin{aligned}\langle\varrho\xi,\varrho(\mathcal{D}F)\rangle &= \varrho\langle\xi,\mathcal{D}F\rangle=\varrho\langle\xi,F^*\omega\rangle=\varrho\langle F_*\xi,\omega\rangle=\langle\varrho_*F_*\xi,\omega_Y\rangle\\ &=\langle(\varrho\circ F)_*\xi,\omega_Y\rangle=\langle(\phi\circ\varrho)_*\xi,\omega_Y\rangle=\langle\phi_*(\varrho\xi),\omega_Y\rangle=\langle\varrho\xi,\mathcal{D}\phi\rangle,\end{aligned}$$

i.e., $\varrho(\mathcal{D}F)=\mathcal{D}\phi$.

(iv) is an immediate consequence of (i) and (iii).

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