ON THE PERIODICITY THEOREM FOR COMPLEX VECTOR BUNDLES

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Introduction

The periodicity theorem for the infinite unitary group [3] can be interpreted as a statement about complex vector bundles. As such it describes the relation between vector bundles over X and $X \times S^2$, where X is a compact (1) space and S^2 is the 2-sphere. This relation is most succinctly expressed by the formula

$$K(X \times S^2) \cong K(X) \otimes K(S^2),$$

where $K(X)$ is the Grothendieck group (2) of complex vector bundles over X. The general theory of these K-groups, as developed in [1], has found many applications in topology and related fields. Since the periodicity theorem is the foundation stone of all this theory it seems desirable to have an elementary proof of it, and it is the purpose of this paper to present such a proof.

Our proof will be strictly elementary. To emphasize this fact we have made the paper entirely self-contained, assuming only basic facts from algebra and topology. In particular we do not assume any knowledge of vector bundles or K-theory. We hope that, by doing this, we have made the paper intelligible to analysts who may be unacquainted with the theory of vector bundles but may be interested in the applications of K-theory to the index problem for elliptic operators [2]. We should point out in fact that our new proof of the periodicity theorem arose out of an attempt to understand the topological significance of elliptic boundary conditions. This aspect of the matter will be taken up in a subsequent paper.(3) In fact for the application to boundary problems we need not only the periodicity theorem but also some more precise results that occur in the course of our present proof.

(1) Compact spaces form the most natural category for our present purposes.

(2) See § 1 for the definition.

(3) See the *Proceedings of the Colloquium on Differential Analysis*, Tata Institute, 1964.

For this reason we have been a little more particular in the statement of some of our results than is necessary for the periodicity theorem itself.

The basic ideas of the proof may be summarized as follows.(1) The vector bundles over S^2 are well-known and are easily determined. If we can carry out this determination in a sufficiently intrinsic manner then it should enable us to determine the bundles on $S^2 \times X$. Now isomorphism classes of m-dimensional vector bundles over S^2 correspond to homotopy classes of maps of the circle S^1 into the general linear group $GL(m, C)$. Moreover the homotopy class of such a map f is determined by the winding number $\omega(f)$ of $\det f$. If we regard S^1 as the unit circle in C and let $f_n = \sum_{-n}^{n} a_k z^k$ be a finite Laurent series approximating f (the a_k being $m \times m$ matrices), then putting $p = z^n f_n$ we have

$$\omega(f) = \omega(f_n) = \omega(p) - nm.$$

$\omega(p)$ is just the number of zeros of the polynomial $\det(p)$ inside the unit circle. For our purposes however it is more significant to observe that

$$\omega(p) = \dim V_p^+,$$

where V_p^+ is a certain vector space intrinsically associated with p. It may be defined in two ways both of which are enlightening. In the first place we can regard p as a homomorphism between free $C[z]$-modules of rank m. Then the cokernel of p is a torsion $C[z]$-module, i.e. a finite-dimensional vector space endowed with an endomorphism T_p. The eigenvalues of T_p do not lie on S^1 (since p is non-singular there) and so we get a decomposition

$$V_p = V_p^+ \oplus V_p^-$$

where V_p^+ corresponds to the eigenvalues of T_p inside S^1 and V_p^- to those outside S^1. Alternatively we can consider the linear system of ordinary differential equations

$$p\left(-i\frac{d}{dx}\right)u = 0.$$

The space of solutions V_p consists of exponential polynominals and decomposes as

$$V_p = V_p^+ \oplus V_p^-,$$

where V_p^+ involves $\exp(i\lambda z)$ with $|\lambda| < 1$ while V_p^- involves those with $|\lambda| > 1$. The first definition brings one close to the work of Grothendieck in algebraic geometry while the second connects up with boundary-value problems as mentioned earlier. In any case V_p^+ is an invariant of p which is a refinement of the winding number $\omega(p)$. If p depends continuously on a parameter space X then the spaces V_p^+ will form a vector bundle over

(1) The terms used here are all defined in the body of the paper.

X. This vector bundle turns out to be a sufficiently good invariant of p so that the relation obtained in this way between vector bundles on $X \times S^2$ and vector bundles on X gives the periodicity theorem.

It should be emphasized that the preceding remarks are made in order to give the reader some insight into the nature of the proof. In fact in our formal development we mention neither modules nor differential equations.

The arrangement of the paper is as follows. In § 1 we define vector bundles, establish a few basic properties and then introduce the groups $K(X)$. The reader who is familiar with vector bundles may skip this section. In § 2 we state the main theorem which in fact is a slight generalization of the periodicity theorem in that $X \times S^2$ is replaced by a suitable fibre bundle with fibre S^2. In essence the additional generality gives what is called the "Thom isomorphism theorem" for line-bundles in K-theory. Since this comes out naturally by our method of proof it seemed reasonable to include it. Also in § 2 we introduce "clutching functions" f and approximate them by *finite Laurent series* f_n. In § 3 we consider *polynomial* clutching functions p and we show how to replace them by essentially equivalent *linear* functions. Then in § 4 we show how to deform any linear clutching function into a standard form. The proof of the main theorem is then given in § 5.

A few words on the general philosophy of this paper may be in order here. In algebraic topology the orthodox method is to replace continuous maps by simplicial approximations, and then use combinatorial methods. When the spaces involved are differentiable manifolds a powerful alternative is to approximate by *differentiable* maps and use differential-geometric techniques. The original proof of the periodicity theorem, using Morse Theory, was of this nature. What we have done here is to use *polynomial* approximation and then apply algebraic techniques. In principle this method is applicable whenever the spaces involved are algebraic varieties. It would be interesting to see this philosophy exploited on other problems.(1)

1. Preliminaries on vector bundles

Let X be a topological space. Then a *complex*(2) *vector bundle* over X is a topological space E endowed with

(i) a continuous map $p: E \to X$ (called the projection),

(ii) a (finite-dimensional) complex vector space structure in each $E_x = p^{-1}(x)$, $x \in X$,

(1) The periodicity theorem for real vector bundles (which is considerably more intricate than the complex case) has recently been dealt with by R. Wood following the general lines of this paper.

(2) The word complex will be omitted from now on, since we shall not be concerned with real vector bundles.

such that E is locally isomorphic to the product of X with a complex vector space. Explicitly this means that, for each $x \in X$, there exists an open set U containing x, an integer n and a homeomorphism $\varphi : p^{-1}(U) \to U \times C^n$ such that

(a) φ commutes with the projections onto U,

(b) for each $x \in U$, φ induces a vector space isomorphism $\varphi_x : E_x \to C^n$. E_x is called the fibre at x. If X is connected then dim E_x is independent of x and is called the dimension of E.

If Y is a subspace of X and E is a vector bundle over X then

$$E|Y = \bigcup_{y \in Y} E_y$$

has a natural vector bundle structure over Y. We call $E|Y$ the *restriction* of E to Y.

A *section* of a vector bundle E is a continuous map $s : X \to E$ with ps = identity. Thus, locally, a section is just the graph of a continuous map of X into a vector space. The space of all sections of E is denoted by $\Gamma(E)$. If E, F are two vector bundles over X then a *homomorphism* of E into F is a continuous map $\varphi : E \to F$ commuting with the projections and inducing vector space homomorphisms $\varphi_x : E_x \to F_x$ for each $x \in X$. The union of all the vector spaces $\mathrm{Hom}(E_x, F_x)$ for $x \in X$ has a natural topology making it into a vector bundle $\mathrm{Hom}(E, F)$, and a section of $\mathrm{Hom}(E, F)$ is then just a homomorphism of E into F. If $\varphi \in \Gamma\, \mathrm{Hom}(E, F)$ is such that φ_x is an isomorphism for all x, then φ^{-1} exists. In fact φ^{-1} is continuous. To see this we work locally so that φ is the graph of a continuous map

$$U \to \mathrm{ISO}(C^n, C^n) = GL(n, C)$$

and observe that the inverse is a continuous map in the topological group $GL(n, C)$. Thus $\varphi^{-1} \in \Gamma\, \mathrm{Hom}(F, E)$ and so φ is an *isomorphism* of vector bundles. The set of all isomorphisms of E onto F will be denoted by $\mathrm{ISO}(E, F)$. A vector bundle is *trivial* if it is isomorphic to $X \times C^n$ for some n.

Natural operations on vector spaces carry over at once to vector bundles. We have already considered $\mathrm{Hom}(E, F)$. In addition we can define the *direct sum* $E \oplus F$, the tensor product $E \otimes F$ and the dual E^*. For example

$$(E \oplus F)_x = E_x \oplus F_x$$

and if E, F are isomorphic over $U \subset X$ to $U \times C^n$, $U \times C^m$ then $(E \oplus F)|U$ is topologized as $U \times (C^n \oplus C^m)$. Canonical isomorphisms also go over to bundles, thus for instance

$$\mathrm{Hom}(E, F) \cong E^* \otimes F.$$

The iterated tensor product $E \otimes E \otimes \ldots \otimes E$ (k times) will be denoted by E^k. If L is a *line-bundle*, i.e. a vector bundle of dimension *one*, we shall write L^{-1} for L^* and L^{-k} for $(L^*)^k$.

This notation is justified by the fact that the (isomorphism classes of) line-bundles over X then form a multiplicative group with L^{-1} as the inverse of L. The unit of this group is the trivial line-bundle $X \times C$ (denoted by 1).

Let $f: Y \to X$ be a continuous map and let E be a vector bundle over X. The *induced bundle* $f^*(E)$ is a vector bundle over Y defined as follows. It is the subspace of $E \times Y$ consisting of pairs (e, y) with $p(e) = f(y)$, the projection and vector space structures of the fibres being the obvious ones. Thus

$$f^*(E)_y = E_{f(y)} \times \{y\}.$$

If E is trivial over $U \subset X$ then $f^*(E)$ is trivial over $f^{-1}(U) \subset Y$. If $\alpha: E \to F$ is a homomorphism of vector bundles over X then this induces in an obvious way a homomorphism

$$f^*(\alpha): f^*(E) \to f^*(F)$$

of vector bundles over Y. Note that, if $f: Y \to X$ is the inclusion of a subspace $Y \subset X$, then $f^*(E) \cong E \mid Y$.

Having given the basic definitions concerning vector bundles we pass now to their homotopy properties.

LEMMA (1.1). *Let Y be a closed subspace of a compact (Hausdorff) space X and let E be a vector bundle over X. Then any section of $E \mid Y$ extends to a section of E.*

Proof. Let s be a section of $E \mid Y$. Now, since a section of a vector bundle is locally the graph of a continuous vector-valued function, we can apply the Tietze extension theorem (1) [4: p. 242] locally and deduce that for each $x \in X$ there exists an open set U containing x and $t \in \Gamma(E \mid U)$ so that t and s coincide on $U \cap Y$. Since X is compact we can then choose a finite open covering $\{U_\alpha\}$ with $t_\alpha \in \Gamma(E \mid U_\alpha)$ coinciding with s on $Y \cap U_\alpha$. Now let $\{\varrho_\alpha\}$ be a partition of unity with support $(\varrho_\alpha) \subset U_\alpha$. Then we get a section s_α of E by defining

$$\begin{aligned} s_\alpha(x) &= \varrho_\alpha(x) t_\alpha(x) \quad \text{if } x \in U_\alpha \\ &= 0 \quad\quad\quad\quad\;\; \text{if } x \notin U_\alpha, \end{aligned}$$

and $\sum_\alpha s_\alpha$ is a section of E extending s as required.

LEMMA (1.2). *Let Y be a closed subspace of a compact space X, E and F two vector bundles over X. Then any isomorphism $s: E \mid Y \to F \mid Y$ extends to an isomorphism $t: E \mid U \to F \mid U$ for some open set U containing Y.*

Proof. s is a section of $\operatorname{Hom}(E, F) \mid Y$. Applying (1.1) we get an extension to a section t of $\operatorname{Hom}(E, F)$. Let U be the subset of X consisting of points x for which t_x is an isomorphism. Then, since $GL(n, C)$ is open in $\operatorname{End}(C^n)$, U is open and contains Y.

(1) In fact for the main results of this paper we only need the Tietze extension theorem in quite simple cases where its proof is trivial.

PROPOSITION (1.3). *Let Y be a compact space, $f_t: Y \to X$ a homotopy* $(0 \leqslant t \leqslant 1)$ *and E a vector bundle over X. Then*

$$f_0^* E \cong f_1^* E.$$

Proof. If I denotes the unit interval, let $f: Y \times I \to X$ be the homotopy, so that $f(y, t) = f_t(y)$ and let $\pi: Y \times I \to Y$ be the projection. Now apply (1.2) to the bundles $f^* E$, $\pi^* f_t^* E$ and the subspace $Y \times \{t\}$ of $Y \times I$, on which there is an obvious isomorphism s. By the compactness of Y we deduce that $f^* E$, and $\pi^* f_t^* E$ are isomorphic in some strip $Y \times \delta t$ where δt denotes a neighbourhood of $\{t\}$ in I. Hence the isomorphism class of $f_t^* E$ is a locally constant function of t. Since I is connected this implies it is constant, whence

$$f_0^* E \cong f_1^* E.$$

A *projection operator* P for a vector bundle E is an endomorphism with $P^2 = P$. Then PE and $(1-P)E$ inherit from E a topology, a projection and vector space structures in the fibres. To see that they are locally trivial choose, for each $x \in X$, local sections $s_1, \ldots, s_n$ of E such that the s_i $(i \leqslant r)$ form a base of $P_x E_x$ and the s_i $(i > r)$ form a base for $(1-P)_x E_x$. Then for a sufficiently small neighbourhood U of x we have a vector bundle isomorphism

$$\varphi: U \times C^n \to E|U$$

given by $$(y \times (a_1, \ldots, a_n)) = \sum_{i=1}^{r} a_i P_y s_i(y) + \sum_{i=r+1}^{n} a_i (1-P)_y s_i(y).$$

This establishes

LEMMA (1.4). *If P is a projection operator for the vector bundle E, then PE and $(1-P)E$ have an induced vector bundle structure and*

$$E = PE \oplus (1-P) E.$$

We turn next to the question of metrics in vector bundles. If E is a complex vector bundle we can consider the vector bundle Herm(E) whose fibre at x consists of all hermitian forms in E_x. A *metric* on E is defined as a section of Herm(E) which is positive definite for each $x \in X$. Since the space of positive definite Hermitian forms is a convex set, the existence of a metric in E over a compact space X follows from the existence of partitions of unity. Moreover, any two metrics in E are homotopic, in fact they can be joined by a linear homotopy.

Vector bundles are frequently constructed by a glueing or clutching construction which we shall now describe. Let

$$X = X_1 \cup X_2, \quad A = X_1 \cap X_2,$$

all the spaces being compact. Assume that E_i is a vector bundle over X_i and that

$\varphi: E_1|A \to E_2|A$ is an isomorphism. Then we define the vector bundle $E_1 \cup_\varphi E_2$ on X as follows. As a topological space $E_1 \cup_\varphi E_2$ is the quotient of the disjoint sum $E_1 + E_2$ by the equivalence relation which identifies $e_1 \in E_1|A$ with $\varphi(e_1) \in E_2|A$. Identifying X with the corresponding quotient of $X_1 + X_2$ we obtain a natural projection $p: E_1 \cup_\varphi E_2 \to X$, and $p^{-1}(x)$ has a natural vector space structure. It remains to show that $E_1 \cup_\varphi E_2$ is locally trivial. Since $E_1 \cup_\varphi E_2 | X - A = (E_1 | X_1 - A) + (E_2 | X_2 - A)$ the local triviality at points $x \notin A$ follows from that of E_1 and E_2. Let therefore $a \in A$ and let V_1 be a closed neighbourhood of a in X_1 over which E_1 is trivial, so that we have an isomorphism

$$\theta_1: E_1 | V_1 \to V_1 \times C^n.$$

Restricting to A we get an isomorphism

$$\theta_1^A: E_1 | V_1 \cap A \to (V_1 \cap A) \times C^n.$$

Let

$$\theta_2^A: E_2 | V_1 \cap A \to (V_1 \cap A) \times C^n$$

be the isomorphism corresponding to θ_1^A under φ. By (1.2) this can be extended to an isomorphism

$$\theta_2: E_2 | V_2 \to V_2 \times C^n$$

where V_2 is a neighbourhood of a in X_2. The pair θ_1, θ_2 then defines in an obvious way an isomorphism

$$\theta_1 \cup_\varphi \theta_2: E_1 \cup_\varphi E_2 | V_1 \cup V_2 \to (V_1 \cup V_2) \times C^n,$$

establishing the local triviality of $E_1 \cup_\varphi E_2$.

Elementary properties of this construction are the following.

(1.5). *If E is a bundle over X and $E_i = E | X_i$, then the identity defines an isomorphism $1_A: E_1 | A \to E_2 | A$, and*

$$E_1 \cup_{1_A} E_2 \cong E.$$

(1.6) *If $\beta_i: E_i \to E_i'$ are isomorphisms on X_i and $\varphi' \beta_1 = \beta_2 \varphi$, then*

$$E_1 \cup_\varphi E_2 \cong E_1' \cup_{\varphi'} E_2'.$$

(1.7). *If (E_i, φ) and (E_i', φ') are two "clutching data" on the X_i, then*

$$(E_1 \cup_\varphi E_2) \oplus (E_1' \cup_{\varphi'} E_2') \cong E_1' \oplus E_1' \bigcup_{\varphi \otimes \varphi'} E_2 \oplus E_2',$$

$$(E_1 \cup_\varphi E_2) \otimes (E_1' \cup_{\varphi'} E_2') \cong E_1 \otimes E_1' \bigcup_{\varphi \otimes \varphi'} E_2 \otimes E_2',$$

$$(E_1 \cup_\varphi E_2)^* \cong E_1^* \cup_{(\varphi^*)^{-1}} E_2^*.$$

Moreover, we also have

LEMMA (1.8). *The isomorphism class of $E_1 \cup_\varphi E_2$ depends only on the homotopy class of the isomorphism $\varphi: E_1|A \to E_2|A$.*

Proof. A homotopy of isomorphisms $E_1|A \to E_2|A$ means an isomorphism

$$\Phi: \pi^* E_1 | A \times I \to \pi^* E_2 | A \times I,$$

where I is the unit interval and $\pi: X \times I \to X$ is the projection. Let

$$f_t: X \to X \times I$$

be defined by $f_t(x) = x \times \{t\}$ and denote by

$$\varphi_t: E_1|A \to E_2|A$$

the isomorphism induced from Φ by f_t. Then

$$E_1 \cup_{\varphi_t} E_2 \cong f_t^*(\pi^* E_1 \cup_\Phi \pi^* E_2).$$

Since f_0 and f_1 are homotopic it follows from (2.2) that

$$E_1 \cup_{\varphi_0} E_2 \cong E_1 \cup_{\varphi_1} E_2$$

as required.

We come finally to the definition of the Grothendieck group $K(X)$. Let us recall first the elementary procedure by which an abelian semi-group defines a group. If A is an abelian semi-group we form an abelian group B by taking a generator $[a]$ for each $a \in A$ and relations $[a] = [b] + [c]$ whenever $a = b + c$. The mapping $\theta: A \to B$ given by $\theta(a) = [a]$ is then a homomorphism and it has an obvious "universal property" (i.e. B is the "best possible" group which can be made out of A): if $\varphi: A \to C$ is any homomorphism of A into an abelian group C then there exists a unique homomorphism $\tilde{\varphi}: B \to C$ so that $\varphi = \tilde{\varphi}\theta$. If X is a compact space we take A to be the set of isomorphism classes of vector bundles over X with the operation $\oplus$. The corresponding abelian group B is denoted by $K(X)$. Thus for each vector bundle E over X we get an element $[E]$ of $K(X)$ and any element of $K(X)$ is a linear combination of such elements. The zero dimensional vector bundle gives the zero of $K(X)$. The universal property of $K(X)$ shows in particular that $K(X)$ is the appropriate object to study in problems involving additive integer-valued functions of vector bundles. This explains the relevance of $K(X)$ for example in the index problem for elliptic operators.[1]

The operation $\otimes$ induces a multiplication in $K(X)$ turning it into a commutative ring with $[1]$ as identity. A continuous map $f: Y \to X$ induces a ring homomorphism

$$f^*: K(X) \to K(Y)$$

where

$$f^*[E] = [f^* E].$$

If X is a point then $K(X)$ is naturally isomorphic to the ring of integers.

(1) It was similar considerations which led Grothendieck to define $K(X)$ in the first place in algebraic geometry.

2. Statement of the periodicity theorem

If E is any vector bundle then by deleting the 0-section and dividing out by the action of non-zero scalars we obtain a space $P(E)$ called the *projective bundle* of E. There is a natural map $P(E)\to X$ and the inverse image of $x\in X$ is the complex projective space $P(E_x)$. If we assign to each $y\in P(E_x)$ the one-dimensional subspace of E_x which corresponds to it we obtain a line-bundle over $P(E)$. This line-bundle is denoted by H^*, i.e. its dual is denoted by H. The projection $P(E)\to X$ induces a ring homomorphism $K(X)\to K(P(E))$ so that $K(P(E))$ becomes a $K(X)$-algebra. Our main theorem determines the structure of this algebra in a particular case:

THEOREM (2.1). *Let L be a line-bundle over the compact space X, H the line-bundle over $P(L\oplus 1)$ defined above. Then, as a $K(X)$-algebra, $K(P(L\oplus 1))$ is generated by $[H]$ subject to the single relation*

$$([H]-[1])([L][H]-[1])=0.$$

If X is a point, so that $P(L\oplus 1)$ is a projective line or 2-sphere S^2, (2.1) implies that $K(S^2)$ is a free abelian group generated by $[1]$ and $[H]$ and that $([H]-[1])^2=0$. Hence (2.1), in the case when L is trivial, can be rephrased as follows:

COROLLARY (2.2). *Let $\pi_1:X\times S^2\to X$, $\pi_2:X\times S^2\to S^2$, denote the projections. Then the homomorphism*

$$f:K(X)\otimes_Z K(S^2)\to K(X\times S^2)$$

defined by

$$f(a\otimes b)=\pi_1^*(a)\pi_2^*(b)$$

is a ring isomorphism.

This corollary is the periodicity theorem proper.

For any x there is a natural embedding $L_x\to P(L\oplus 1)_x$ given by

$$y\to(y\oplus 1)$$

which exhibits $P(L\oplus 1)_x$ as the compactification of L_x obtained by adding the "point at infinity". In this way we get an embedding of L in $P=P(L\oplus 1)$, so that P is the compactification of L obtained by adding the "section at infinity". Now let us choose, once and for all, a definite metric in L and let $S\subset L$ be the unit circle bundle in this metric. We identify L with a subspace of P so that

$$P=P^0\cup P^\infty,\quad S=P^0\cap P^\infty,$$

where P^0 is the closed disc bundle interior to S (i.e. containing the 0-section) and P^∞ is the closed disc bundle exterior to S (i.e. containing the ∞-section). The projections $S\to X$, $P^0\to X$, $P^\infty\to X$ will be denoted by π, π_0, π_∞ respectively.

Suppose now that E^0, E^∞ are two vector bundles over X and that $f \in \mathrm{ISO}(\pi^* E^0, \pi^* E^\infty)$. Then we can form the vector bundle

$$\pi_0^* E^0 \cup_f \pi_\infty^* E^\infty$$

over P. We shall denote this bundle for brevity by (E^0, f, E^∞) and we shall say that f is a clutching function for (E^0, E^∞). That the most general bundle on P is of this form is shown by the following lemma:

LEMMA (2.3). *Let E be any vector bundle over P and let E^0, E^∞ be the vector bundles over X induced by the* 0*-section and* ∞*-section respectively. Then there exists $f \in \mathrm{ISO}(\pi^* E^0, \pi^* E^\infty)$ such that*

$$E \cong (E^0, f, E^\infty),$$

the isomorphism being the obvious one on the 0*-section and the* ∞*-section. Moreover F is uniquely determined, up to homotopy, by these properties.*

Proof. Let $s_0 : X \to P^0$ be the 0-section. Then $s_0 \pi_0$ is homotopic to the identity map of P^0, and so by (1.3) we have an isomorphism

$$f_0 : E \mid P^0 \to \pi_0^* E^0.$$

Two different choices of f_0 differ by an automorphism α of $\pi_0^* E^0$, and any such α is homotopic to the automorphism $\pi_0^* \alpha^0$ where α^0 is the automorphism of E^0 obtained by restricting α to the 0-section. It follows that we can choose f_0 to be the obvious one on the 0-section and that this determines it uniquely up to homotopy. The same remarks apply to $E \mid P^\infty$ and the lemma then follows, taking

$$f = f_\infty f_0^{-1}.$$

Remark. If F is a vector bundle over X then (1.5) shows that $(F, 1, F)$ is the vector bundle over P induced from F by the projection $P \to X$. Written as an equation in $K(P)$ this statement reads

$$[(F, 1, F)] = [F][1],$$

where $[1]$ is the identity of the ring $K(P)$ and $[F][1]$ is module multiplication of $K(X)$ on $K(P)$.

When L is the trivial line-bundle $X \times C^1$, S is the trivial circle bundle $X \times S^1$ so that points of S are represented by pairs (x, z) with $x \in X$ and $z \in C$ with $|z| = 1$. Thus z is a function on S, so also is z^{-1} and we can consider functions on S which are finite Laurent series in z:

$$\sum_{k=-n}^{n} a_k(x) z^k.$$

When L is not trivial we want to introduce a notation which will enable us to deal conveniently with the corresponding expressions. To do this we observe that the inclusion $S \to L$ defines, in a rather tautologous way, a section of $\pi^*(L)$. *We shall denote this section by* z. If $L=1$ then a section of $\pi^*(L)$ is just a function on S and the section z is precisely the function described above. The reader who is primarily interested in Corollary (2.2) may throughout think of this special case and regard z as a function. To obtain the more general Theorem (2.1) however we have to consider z as a section. The only complications introduced by this are notational, since we have to identify all the various bundles which occur.

Using the canonical isomorphisms

$$\pi^*(L) \cong \pi^* \operatorname{Hom}(1, L)$$

we may also regard z as a section of $\pi^* \operatorname{Hom}(1, L)$ and, as such, it has an inverse z^{-1} which is a section of

$$\pi^* \operatorname{Hom}(L, 1) \cong \pi^*(L^{-1}).$$

More generally, for any integer k, we may regard z^k as a section of $\pi^* L^k$. If now $a_k \in \Gamma(L^{-k})$ then

$$\pi^*(a_k) \otimes z^k \in \Gamma \pi^*(1),$$

i.e. it is a function on S. For simplicity of notation we write $a_k z^k$ instead of $\pi^*(a_k) \otimes z^k$. Thus we have given a meaning to the finite Fourier series

$$f = \sum_{-n}^{n} a_k z^k :$$

if $a_k \in \Gamma(L^{-k})$ then f is a function on S. Finally suppose that E^0, E^∞ are two vector bundles on X and that

$$a_k \in \Gamma \operatorname{Hom}(L^k \otimes E^0, E^\infty);$$

then

$$a_k z^k \in \Gamma \operatorname{Hom}(\pi^* E^0, \pi^* E^\infty),$$

where again we have replaced $\pi^*(a_k) \otimes z^k$ by $a_k z^k$. A finite sum

$$f = \sum_{-n}^{n} a_k z^k \in \Gamma \operatorname{Hom}(\pi^* E^0, \pi^* E^\infty)$$

with the a_k as above will be called a finite Laurent series for (E^0, E^∞). If $f \in \operatorname{ISO}(\pi^* E^0, \pi^* F^\infty)$ then it defines a clutching function and we call this a *Laurent clutching function* for (E^0, E^∞).

The simplest Laurent clutching function is z itself—taking $E^0 = 1$, $E^\infty = L$. We shall now identify the bundle $(1, z, L)$ on P defined by this clutching function. We recall first that the line-bundle H^* over P is defined as a sub-bundle of $\pi^*(L \oplus 1)$. For each $y \in P(L \oplus 1)_x$ H_y^* is a subspace of $(L \oplus 1)_x$ and

$$H_y^* = L_x \oplus 0 \Leftrightarrow y = \infty,$$

$$H_y^* = 0 \oplus 1_x \Leftrightarrow y = 0.$$

Thus the composition $$H^* \to \pi^*(L \oplus 1) \to \pi^*(1),$$

induced by the projection $L \oplus 1 \to 1$, defines an isomorphism

$$f_0 : H^* | P^0 \to \pi_0^*(1),$$

and the composition $$H^* \to \pi^*(L \oplus 1) \to \pi^*(L)$$

induced by the projection $L \oplus 1 \to L$, defines an isomorphism

$$f_\infty : H^* | P^\infty \to \pi_0^*(L).$$

Hence $$f = f_\infty f_0^{-1} : \pi^*(1) \to \pi^*(L)$$

is a clutching function for H^*. Clearly, for $y \in S_x$, $f(y)$ is the isomorphism whose graph is H_y^*. Since H_y^* is the subspace of(1) $L_x \oplus 1_x$ spanned by $y \oplus 1$ ($y \in S_x \subset L_x$, $1 \in C$) we see that f is precisely our tautologous section z. Thus

$$H^* \cong (1, z, L). \tag{2.4}$$

From (2.4) and (1.7) we deduce, for any integer k,

$$H^k \cong (1, z^{-k}, L^{-k}). \tag{2.5}$$

Suppose now that $f \in \Gamma \operatorname{Hom}(\pi^* E^0, \pi^* E^\infty)$ is any section, then we can define its Fourier coefficients

$$a_k \in \Gamma \operatorname{Hom}(L^k \otimes E^0, E^\infty)$$

by(2)

$$a_k(x) = \frac{1}{2\pi i} \int_{S_x} f_x z_x^{-k-1} dz_x.$$

Here f_x and z_x denote the restrictions of f, z to S_x and dz_x is therefore a differential on S_x with coefficients in L_x. Let s_n be the partial sums

$$s_n = \sum_{-n}^{n} a_k z^k$$

and define the Cesaro means $$f_n = \frac{1}{n} \sum_{0}^{n} s_k.$$

The proof of Fejer's theorem [5; § 13.32] on $(C, 1)$ summability of Fourier series extends immediately to the present more general case and gives

LEMMA (2.6). *Let f be any clutching function for (E^0, E^∞), f_n the sequence of Cesaro*

(1) The symbol 1 may cause the reader some confusion here since it denotes the trivial line-bundle and also the complex number 1.

(2) Here again we omit the $\otimes$ sign.

means of the Fourier series of f. Then f_n converges uniformly to f and hence is a Laurent clutching function for all sufficiently large n.

Remark. The uniformity can be defined by using metrics in E^0 and E^∞, but does not of course depend on the choice of metrics.

3. Linearization

By a *polynomial clutching function* we shall mean a Laurent clutching function without negative powers of z. In this section we shall describe a linearization procedure for such functions.

Thus let
$$p=\sum_{k=0}^{n} a_k z^k$$
be a polynomial clutching function of degree $\leqslant n$ for (E^0, E^∞). Consider the homomorphism
$$\mathcal{L}^n(\mathrm{p}):\pi^*(\sum_{k=0}^{n} L^k\otimes E^0)\to\pi^*(E^\infty\oplus\sum_{k=1}^{n} L^k\otimes E^0)$$
given by the matrix
$$\mathcal{L}^n(p)=\begin{pmatrix} a_0 & a_1 & \cdot & \cdot & \cdot & a_n \\ -z & 1 & & & & \\ & -z & 1 & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & -z & 1 \end{pmatrix}$$

It is clear that $\mathcal{L}^n(p)$ is *linear* in z. Now define the sequence $p_r(z)$ inductively by $p_0=p$' $zp_{r+1}(z)=p_r(z)-p_r(0)$. Then we have the following matrix identity

$$\mathcal{L}^n(p)=$$
$$\begin{pmatrix} 1 & p_1 & p_2 & \cdot & \cdot & \cdot & p_n \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & \cdot & & & \\ & & & & \cdot & & \\ & & & & & \cdot & \\ & & & & & & 1 \end{pmatrix}\begin{pmatrix} p & & & & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & \cdot & & & \\ & & & & \cdot & & \\ & & & & & \cdot & \\ & & & & & & 1 \end{pmatrix}\begin{pmatrix} 1 & & & & & \\ -z & 1 & & & & \\ & -z & \cdot & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & -z & 1 \end{pmatrix}$$

or more briefly
$$\mathcal{L}^n(p)=(1+N_1)(p\oplus 1)(1+N_2) \tag{3.1}$$
with N_1, N_2 nilpotent.

Since $1+tN$ with $0\leqslant t\leqslant 1$ gives a homotopy of isomorphisms, if N is nilpotent, it follows from (3.1) and (1.8) that we have

PROPOSITION (3.2). *$\mathcal{L}^n(p)$ and $p\oplus 1$ define isomorphic bundles on P, i.e.,*

$$(E^0, p, E^\infty)\oplus\left(\sum_{k=1}^{n} L^k\otimes E^0, 1, \sum_{k=1}^{n} L^k\otimes E^0\right)\cong\left(\sum_{k=0}^{n} L^k\otimes E^0, \mathcal{L}^n(p), E^\infty\oplus\sum_{k=1}^{n} L^k\times E^0\right).$$

Remark. The definition of $\mathcal{L}^n(p)$ is of course modelled on the way one passes from an ordinary differential equation of order n to a system of first order equations.

For brevity we now write $\mathcal{L}^n(E^0, p, E^\infty)$ for the bundle

$$\left(\sum_{k=0}^{n} L^k\otimes E^0, \mathcal{L}^n(p), E^\infty\oplus\sum_{k=1}^{n} L^k\otimes E^0\right).$$

LEMMA (3.3). *Let p be a polynomial clutching function of degree $\leqslant n$ for (E^0, E^∞). Then*

$$\mathcal{L}^{n+1}(E_0, p, E^\infty)\cong\mathcal{L}^n(E^0, p, E^\infty)\oplus(L^{n+1}\otimes E^0, 1, L^{n+1}\otimes E^0), \tag{3.4}$$

$$\mathcal{L}^{n+1}(L^{-1}\otimes E^0, zp, E^\infty)\cong\mathcal{L}^n(E^0, p, E^\infty)\oplus(L^{-1}\otimes E^0, z, E^0). \tag{3.5}$$

Proof. We have

$$\mathcal{L}^{n+1}(p)=\begin{pmatrix}\mathcal{L}^n(p) & 0\\ 0\ldots -z & 1\end{pmatrix}.$$

Multiplying z on the bottom line by a real parameter t with $0\leqslant t\leqslant 1$ then gives a homotopy from $\mathcal{L}^{n+1}(p)$ to $\mathcal{L}^n(p)\oplus 1$ and so (3.4) follows using (1.8). Similarly in

$$\mathcal{L}^{n+1}(zp)=\begin{pmatrix} 0 & a_0 & a_1 & \cdot & \cdot & \cdot & a_n\\ -z & 1 & & & & & \\ & -z & 1 & & & & \\ & & \cdot & \cdot & & & \\ & & & \cdot & \cdot & & \\ & & & & \cdot & \cdot & \\ & & & & & -z & 1\end{pmatrix}$$

we multiply 1 on the second row by t and get a homotopy from $\mathcal{L}^{n+1}(zp)$ to $\mathcal{L}^n(p)\oplus -z$. Using (1.8) and (1.6) (with $E_1=E_1'=L^{-1}\otimes E^0$, $E_2=E_2'=E^0$, $\beta_1=1$, $\beta_2=-1$, $\varphi=z$, $\varphi'=-z$) we deduce (3.5).

We shall now establish a simple algebraic formula in $K(P)$. For convenience we write $[E^0, p, E^\infty]$ for the element $[(E^0, p, E^\infty)]$ in $K(P)$.

PROPOSITION (3.6). *For any polynomial clutching function p for (E^0, E^∞) we have the identity*

$$([E^0, p, E^\infty]-[E^0, 1, E^0])([L][H]-[1])=0$$

Proof. From (3.5) and (3.2) we deduce

$$(L^{-1}\otimes E^0, zp, E^\infty)\oplus(\sum_{k=0}^{n} L^k\otimes E^0, 1, \sum_{k=0}^{n} L^k\otimes E_0)$$
$$\cong (E^0, p, E^\infty)\oplus(\sum_{k=1}^{n} L^k\otimes E^0, 1, \sum_{k=1}^{n} L^k\otimes E^0)\oplus(L^{-1}\otimes E^0, z, E^0).$$

Using (1.7) and (2.5) and passing to $K(P)$ this gives:

$$[L^{-1}][H^{-1}][E^0, p, E^\infty]+[E^0, 1, E^0]=[E^0, p, E^\infty]+[L^{-1}][H^{-1}][E^0, 1, E^0]$$

from which the required result follows.

Putting $E^0=1$, $p=z$, $E^\infty=L$ in (3.6) and using (2.4) we obtain the formula:

$$([H]-[1])([L][H]-[1])=0 \tag{3.7}$$

which is part of the assertion of our main theorem (2.1).

4. Linear clutching functions

We begin by reviewing some elementary facts about linear transformations. Suppose T is an endomorphism of a finite-dimensional vector space E, and let S be a circle in the complex plane which does not pass through any eigenvalue of T. Then

$$Q=\frac{1}{2\pi i}\int_S (z-T)^{-1}dz$$

is a projection operator in E which commutes with T. The decomposition

$$E=E_+\oplus E_-, \quad E_+=QE, \quad E_-=(1-Q)E$$

is therefore invariant under T, so that we can write

$$T=T_+\oplus T_-.$$

Then T_+ has all eigenvalues *inside* S while T_- has all eigenvalues *outside* S. This is just the spectral decomposition of T corresponding to the two components of the complement of S.

We shall now extend these results to vector bundles, but first we make a remark on notation. So far z and hence $p(z)$ have been sections over S. However, they extend in a natural way to sections over the whole of L. It will also be convenient to include the ∞-section of P in certain statements. Thus, if we assert that $p(z)=az+b$ is an isomorphism outside S, we shall take this to include the statement that a is an isomorphism.

PROPOSITION (4.1). *Let p be a linear clutching function for E^0, E^∞ and define endomorphisms Q^0, Q^∞ of E^0, E^∞ by putting*

$$Q_x^0=\frac{1}{2\pi i}\int_{S_x} p_x^{-1}\,dp_x, \quad Q_x^\infty=\frac{1}{2\pi i}\int_{S_x} dp_x p_x^{-1}.$$

Then Q^0 and Q^∞ are projection operators and

$$pQ^0 = Q^\infty p.$$

Write $E^i_+ = Q^i E^i$, $E^i_- = (1-Q^i)E^i$ ($i=0, \infty$) so that $E^i = E^i_+ \oplus E^i_-$. Then p is compatible with these decompositions so that

$$p = p_+ \oplus p_-.$$

Moreover, p_+ is an isomorphism outside S, and p_- is an isomorphism inside S.

Proof. In view of (1.4) it will be sufficient to verify all statements point-wise for each $x \in X$. In other words, we may suppose X is a point, $L = C$ and z is just a complex number. Now since $p(z)$ is an isomorphism for $|z| = 1$ we can find a real number α with $\alpha > 1$ so that $p(\alpha): E^0 \to E^\infty$ is an isomorphism. For simplicity of calculation we shall identify E^0 and E^∞ by this isomorphism. Next we consider the conformal transformation

$$w = \frac{1-\alpha z}{z-\alpha}$$

which preserves the unit circle and its inside. Substituting for z we find (since we have taken $p(\alpha) = 1$)

$$p(z) = \frac{w-T}{w+\alpha},$$

where $T \in \operatorname{End} E^0$. Hence

$$\begin{aligned} Q^0 &= \frac{1}{2\pi i}\int_{|z|=1} p^{-1}dp \\ &= \frac{1}{2\pi i}\int_{|w|=1} [-(w+\alpha)^{-1}dw + (w-T)^{-1}dw] \\ &= \frac{1}{2\pi i}\int_{|w|=1} (w-T)^{-1}dw \quad \text{since } |\alpha| > 1 \\ &= Q^\infty \qquad\qquad \text{similarly.} \end{aligned}$$

All the statements in the proposition now follow from what we have asserted above in connection with a linear transformation T.

COROLLARY (4.2). *Let p be as in (4.1) and write*

$$p_+ = a_+ z + b_+, \quad p_- = a_- z + b_-.$$

Then putting $p^t = p^t_+ + p^t_-$ where

$$p^t_+ = a_+ z + t b_+, \quad p^t_- = t a_- z + b_- \quad 0 \leqslant t \leqslant 1$$

we obtain a homotopy of linear clutching functions connecting p with $a_+z\oplus b_-$. Thus

$$(E^0, p, E^\infty)\cong(E_+^0, z, L\otimes E_+^0)\oplus(E_-^0, 1, E_-^0).$$

Proof. The last part of (4.1) implies that p_+^t and p_-^t are isomorphisms over S for all t with $0\leqslant t\leqslant 1$. Thus p^t is a linear clutching function as stated. Hence by (1.8)

$$(E^0, p, E^\infty)\cong(E^0, p^1, E^\infty)\cong(E_+^0, a_+z, E_+^\infty)\oplus(E_-^0, b_-, E_-^\infty).$$

Since $a_+:L\otimes E_+^0\to E_+^\infty$, $b_-:E_-^0\to E_-^\infty$ are necessarily isomorphisms we can use (1.6) and deduce that

$$(E_+^0, a_+z, E_+^\infty)\cong(E_+^0, z, L\otimes E_+^0)$$
$$(E_-^0, b_-, E_-^\infty)\cong(E_-^0, 1, E_-^0)$$

from which the conclusion follows.

If p is a polynomial clutching function of degree $\leqslant n$ for (E^0, E^∞) then $\mathfrak{L}^n(p)$ is a linear clutching function for (V^0, V^∞) where

$$V^0=\sum_{k=0}^n L^k\otimes E^0,\ V^\infty=E^\infty\oplus\sum_{k=1}^n L^k\otimes E^0.$$

Hence it defines a decomposition

$$V^0=V_+^0\oplus V_-^0$$

as in (4.1). To express the dependence of V_+^0 on p, n we write

$$V_+^0=V_n(E^0, p, E^\infty).$$

Note that this is a vector bundle on X. If p_t is a homotopy of polynomial clutching functions of degree $\leqslant n$ it follows by constructing V_n over $X\times I$ and using (1.3) that

$$V_n(E^0, p_0, E^\infty)\cong V_n(E^0, p_1, E^\infty). \tag{4.3}$$

Hence from the homotopies used in proving (3.4) and (3.5) we obtain

$$V_{n+1}(E^0, p, E^\infty)\cong V_n(E^0, p, E^\infty), \tag{4.4}$$

$$V_{n+1}(L^{-1}\otimes E^0, zp, E^\infty)\cong V_n(E^0, p, E^\infty)\oplus(L^{-1}\otimes E^0), \tag{4.5}$$

or equivalently (using (1.7))

$$V_{n+1}(E^0, zp, L\otimes E^\infty)\cong L\otimes V_n(E^0, p, E^\infty)\oplus E^0.$$

Finally from (3.2), (4.2) and the remark following Lemma (2.3) we obtain the following equation in $K(P)$

$$[E^0,p,E^\infty]+\{\sum_{k=1}^n[L^k\otimes E^0]\}[1]=[V_n(E^0,p,E^\infty)][H^{-1}]+\{\sum_{k=0}^n[L^k\otimes E^0]-[V_n(E^0,p,E^\infty)]\}[1]$$

and hence the vital formula

$$[E^0,p,E^\infty]=V_n(E^0,p,E^\infty)]([H^{-1}]-[1])+[E^0][1]. \tag{4.6}$$

Remark. $V_n(E^0, p, E^\infty)$ is the vector space denoted by V_p^+ in the introduction and (4.6) shows that $[V_p^+]\in K(X)$ completely determines $[E^0, p, E^\infty]\in K(P)$. Bearing in mind the relation with ordinary differential equations mentioned in the introduction analysts may care to ponder over the significance of (4.6).

5. Proof of Theorem (2.1)

Let t be an indeterminate. Then because of (3.7) the mapping $t\to[H]$ induces a $K(X)$-algebra homomorphism

$$\mu: K(X)[t]/(t-1)([L]t-1)\to K(P).$$

To prove theorem (2.1) we have to show that μ is an isomorphism, and we shall do this by explicitly constructing an inverse.

First let f be any clutching function for (E^0, E^∞). Let f_n be the sequence of Cesaro means of its Fourier series and put $p_n = z^n f_n$. Then, if n is sufficiently large, (2.6) asserts that p_n is a polynomial clutching function (of degree $\leqslant 2n$) for $(E^0, L^n\otimes E^\infty)$. Motivated by (4.6) we define

$$\nu_n(f)\in K(X)[t]/(t-1)([L]t-1)$$

by the formula:

$$\nu_n(f) = [V_{2n}(E^0, p_n, L^n\otimes E^\infty)](t^{n-1}-t^n)+[E^0]t^n. \tag{5.1}$$

Now, for sufficiently large n, the linear segment joining p_{n+1} and zp_n provides a homotopy of polynomial clutching functions of degree $\leqslant 2(n+1)$. Hence by (4.3)

$$\begin{aligned} V_{2n+2}(E^0, p_{n+1}, L^{n+1}\otimes E^\infty) &\cong V_{2n+2}(E^0, zp_n, L^{n+1}\otimes E^\infty) \\ &\cong V_{2n+1}(E^0, zp_n, L^{n+1}\otimes E^\infty) && \text{by (4.4)} \\ &\cong L\otimes V_{2n}(E^0, p_n, L^n\otimes E^\infty)\oplus E^0 && \text{by (4.5).} \end{aligned}$$

Hence

$$\nu_{n+1}(f) = \{[L][V_{2n}(E^0, p_n, L^n\otimes E^\infty)]+[E^0]\}(t^n - t^{n+1})+[E^0]t^{n+1} = \nu_n(f)$$

since

$$(t-1)([L]t-1)=0.$$

Thus $\nu_n(f)$, for large n, is independent of n and so depends only on f. We write it as $\nu(f)$. If now g is sufficiently close to f and n is sufficiently large then the linear segment joining f_n and g_n provides a homotopy of polynomial clutching functions of degree $\leqslant 2n$ and hence by (4.3)

$$\nu(f)=\nu_n(f)=\nu_n(g)=\nu(g).$$

Thus $\nu(f)$ is a locally constant function of f and hence depends only on the homotopy class of f. Hence if E is any vector bundle over P and f a clutching function defining E, as in (2.3), we can define

$$\nu(E)=\nu(f),$$

and $\nu(E)$ will depend only on the isomorphism class of E. Since $\nu(E)$ is clearly additive for $\oplus$ it induces a group homomorphism

$$\nu: K(P) \to K(X)[t]/(t-1)([L]t-1).$$

In fact it is clear from its definition that this is a $K(X)$-module homomorphism.

We shall now check that $\mu\nu$ is the identity of $K(P)$. In fact with the above notation we have

$$\begin{aligned}
\mu\nu[E] &= \mu\{[V_{2n}(E^0, p_n, L^n \otimes E^\infty)](t^{n-1}-t^n) + [E^0]t^n\} \\
&= [V_{2n}(E^0, p_n, L^n \otimes E^\infty)]([H]^{n-1}-[H^n]) + [E^0][H]^n \\
&= [E^0, p_n, L^n \otimes E^\infty][H]^n \quad && \text{by (4.6)} \\
&= [E^0, f_n, E^\infty] && \text{by (1.7) and (2.5)} \\
&= [E^0, f, E^\infty] && \text{by (1.8)} \\
&= [E] && \text{by definition of } f.
\end{aligned}$$

Since $K(P)$ is additively generated by elements of the form $[E]$ this proves that $\mu\nu$ is the identity.

Finally we have to show that $\nu\mu$ is the identity of $K(X)[t]/(t-1)([L]t-1)$ Since $\nu\mu$ is a homomorphism of $K(X)$-modules it will be sufficient to check that $\nu\mu(t^n) = t^n$ for all $n \geq 0$. But

$$\begin{aligned}
\nu\mu(t^n) &= \nu[H^n] \\
&= \nu[1, z^{-n}, L^{-n}] && \text{by (2.5)} \\
&= [V_{2n}(1,1,1)](t^{n-1}-t^n) + [1]t^n && \text{from (5.1)} \\
&= t^n && \text{since } V_{2n}(1,1,1) = 0.
\end{aligned}$$

This completes the proof of Theorem (2.1).

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