Spherical functions and invariant differential operators on complex Grassmann manifolds

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Abstract

Proofs are given of two theorems of Berezin and Karpelevič, which as far as we know never have been proved correctly. By using eigenfunctions of the Laplace—Beltrami operator it is shown that the spherical functions on a complex Grassmann manifold are given by a determinant of certain hypergeometric functions. By application of this result, it is proved that a certain system of operators, for which explicit expressions are given, generates the algebra of radial parts of invariant differential operators.

KEY WORDS & PHRASES: *Complex Grassmann manifold, spherical function, radial part of an invariant differential operator, hypergeometric function, Jacobi function.*

0. Introduction and motivation

In [1] BEREZIN and KARPELEVIČ gave an explicit expression for the zonal spherical functions on a complex Grassmann manifold. Unfortunately, no proof was given there.

In [9] TAKAHASHI stated the same result, but he also gave a proof. This proof, however, was not correct. It relies upon another result of BEREZIN and KARPELEVIČ (also in [1], unproved), namely that the algebra $\delta(\mathbf{D}_0(G))$ of radial parts of invariant differential operators is generated by a system of operators Δ_i $(i=1,\dots,n)$, for which they could give explicit expressions. This being proved, it is sufficient to find the eigenfunctions of all Δ_i.

Takahashi's error was in the proof that $\delta(\mathbf{D}_0(G))$ is generated by the Δ_i. I'll try to indicate where he went wrong. He proceeded as follows.

Let $G:=SU(n,n+k;\mathbf{C})$, and $\mathfrak{g}=\mathfrak{su}(n,n+k)$ its Lie algebra. Let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$. Let $S(\mathfrak{p})$ be the symmetric algebra over $\mathfrak{p}$, and let $I(\mathfrak{p})$ be the subalgebra consisting of K-invariants. Let λ denote the canonical linear one-to-one mapping of $S(\mathfrak{g})$ onto $\mathbf{D}(G)$. Take $p \in I(\mathfrak{p})$. Then there exists a polynomial q such that $\delta(\lambda(p)) = q(s_1, \ldots, s_n)$+terms of lower order. Define $p' := \delta(\lambda(p)) - q(\Delta_1, \ldots, \Delta_n)$. Then we have *degree* $p' <$ *degree* p. Now, according to Takahashi, the result follows by induction to the degree of p. But nothing guarantees us that p' has a highest order term with constant coefficients, so the induction step is not justified.

In this paper another proof of these two theorems is given, namely by using eigenfunctions of all $\delta(D)$ $(D \in \mathbf{D}_0(G))$ — say Φ — which have a certain convergent series expansion at ∞ in a positive Weyl chamber, instead of spherical functions — say φ — which are eigenfunctions of all $\delta(D)$ being regular in 0. To obtain these Φ, we only need to find the eigenfunctions of $\delta(\Omega)$ (radial part of the Laplace—Beltrami operator) which have the desired series expansion. That such a function is an eigenfunction of all $\delta(D)$ $(D \in \mathbf{D}_0(G))$ is a result of HARISH—CHANDRA [3]. A simpler proof is given by HELGASON [4]. Then we use that a spherical functions φ can be written as a combination of Φ*'s*. This gives us the first theorem of Berezin and Karpelevič. Finally, in the last chapter the second theorem of Berezin and Karpelevič, which states that the algebra $\delta(\mathbf{D}_0(G))$ is generated by the Δ_i $(i = 1, 2, \ldots, n)$, is proved.

1. The group $G = SU(n, n+k; \mathbf{C})$

Let $G = SU(n, n+k; \mathbf{C})$ be the group of all complex $(n+m) \times (n+m)$ matrices with determinant 1 $(m = n+k,\ k \geqq 0)$, which leave invariant the hermitian form:

$$x_1 \bar{x}_1 + x_2 \bar{x}_2 + \ldots + x_n \bar{x}_n - x_{n+1} \bar{x}_{n+1} - \ldots - x_{n+m} \bar{x}_{n+m}.$$

Then G is a connected, semisimple Lie group with finite center (see TAKAHASHI [9]).

Let $\mathfrak{g} = \mathrm{lie}(G)$ be the Lie algebra of G. Then $\mathfrak{g} = \mathfrak{su}(n, n+k; C)$ and $\mathfrak{g}$ is a real, semisimple Lie algebra.

Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$, with

$$\mathfrak{k} = \left\{ \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} : u^* = -u,\ v^* = -v,\ u \in M_n(\mathbf{C}),\ v \in M_m(\mathbf{C}) \right\}$$

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix} : x \in M_{n,m}(\mathbf{C}) \right\}.$$

Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subalgebra. We may choose for $\mathfrak{a}$ the set of all matrices of the form

$$H_T = \begin{pmatrix} O_{n\times n} & T & O_{n\times k} \\ T & & \\ & O_{m\times m} & \\ O_{k\times n} & & \end{pmatrix}$$

where $O_{p\times q}$ denotes the $(p\times q)$-matrix with only zeros as entries, and

$$T = \operatorname{diag}(t_1, \ldots, t_n)$$

($t_i \in \mathbf{R}$ for all i). Let $\alpha_i \in \mathfrak{a}^*$ $(i=1, \ldots, n)$ be defined by $\alpha_i(H_T)=t_i$. Then the roots of $(\mathfrak{g}, \mathfrak{a})$ are given by $\pm\alpha_i$, $\pm 2\alpha_i$ $(1\leqq i\leqq n)$ and $\pm(\alpha_i\pm\alpha_j)$ $(1\leqq i<j\leqq n)$, with multiplicities $m_{\alpha_i}=2k$, $m_{2\alpha_i}=1$ and $m_{\alpha_i\pm\alpha_j}=2$.

Let $a_T := \exp H_T$, and $A := \{a_T = \exp H_T : H_T \in \mathfrak{a}\}$.

On the root system we choose an ordering such that the positive Weyl chamber C^+ is given by the T with $t_1>t_2>\ldots>t_n>0$. Then the positive roots are α_i, $2\alpha_i$ $(1\leqq i\leqq n)$ and $\alpha_i\pm\alpha_j$ $(1\leqq i<j\leqq n)$. The simple roots are $\alpha_1-\alpha_2, \alpha_2-\alpha_3, \ldots, \alpha_{n-1}-\alpha_n, \alpha_n$.

Let $\sum$ be the set of all roots, and $\sum^+$ the set of all positive roots.

From now on we identify T and H_T.

Let $\varrho := \frac{1}{2}\sum_{\alpha\in\Sigma^+} m_\alpha \alpha$.

Then $\varrho(T)=\sum_{i=1}^n \varrho_i t_i$, with $\varrho_i = k+1+2(n-i)$.

Let $\Delta(a_T) := \prod_{\alpha\in\Sigma^+}(e^{\alpha(T)}-e^{-\alpha(T)})^{m_\alpha}$.

Then we have:

$$\Delta = \sigma\omega^2, \quad \text{with} \quad \sigma(a_T) = 2^{n(2k+1)} \prod_{i=1}^n (\operatorname{sh}^{2k} t_i \operatorname{sh} 2t_i),$$

and

$$\omega(a_T) = 2^{\frac{1}{2}n(n-1)} \prod_{i<j} (\operatorname{ch} 2t_i - \operatorname{ch} 2t_j).$$

Let $\mathbf{D}(G)$ be the algebra of left G-invariant differential operators on G, and let $\mathbf{D}_0(G)$ be the subalgebra of $\mathbf{D}(G)$ of right K-invariant operators. If $D\in\mathbf{D}_0(G)$, let $\delta(D)$ denote the radial part of D.

As usual let $\mathbf{C}$, $\mathbf{R}$, $\mathbf{Z}$, $\mathbf{Z}^+$, $\mathbf{Z}^-$ denote the sets of all complex numbers, real numbers, integers, positive (non zero) integers and negative (non zero) integers, respectively.

2. Radial part of the Laplace—Beltrami operator

Let $\delta(\Omega)$ denote the radial part of the Laplace—Beltrami operator. In [3] HARISH—CHANDRA proved the following lemma:

Lemma 2.1. *Let $H_1, \ldots, H_l$ be a basis of $\mathfrak{a}$, and let $(g^{ij})_{1\leqq i,j\leqq l}$ denote the inverse of the matrix with elements $B(H_i, H_j)$ ($B(.,.)$ Killing form). Then*

$$\delta(\Omega) = \sum_{1\leqq i,j\leqq l} \Delta^{-1} g^{ij} H_i \circ \Delta H_j. \tag{2.1}$$

Take for H_i the matrix H_{T_i}, with $T_i=\operatorname{diag}(0, \ldots, 0, 1, 0, \ldots, 0)$ (with 1 on the i-th place). Then

$$\begin{aligned} B(H_i, H_j) &= \operatorname{tr}(\operatorname{ad} H_i \operatorname{ad} H_j) \\ &= \textstyle\sum_{\beta\in\Sigma} m_\beta \beta(H_i)\beta(H_j) \\ &= 4(k+2n)\delta_{ij}. \end{aligned}$$

So formula (2.1) gives:

$$\delta(\Omega) = \left(4(k+2n)\right)^{-1} \textstyle\sum_{i=1}^{n} \omega^{-2}\sigma^{-1} \frac{\partial}{\partial t_i}\left(\omega^2\sigma\frac{\partial}{\partial t_i}\right).$$

(As a differential operator H_i corresponds with $\partial/\partial t_i$). Hence:

$$\begin{aligned} 4(k+2n)\delta(\Omega) &= \textstyle\sum_i \left(\frac{\partial^2}{\partial t_i^2} + \left(2\omega^{-1}\frac{\partial\omega}{\partial t_i} + \sigma^{-1}\frac{\partial\sigma}{\partial t_i}\right)\frac{\partial}{\partial t_i}\right) \\ &= \textstyle\sum_i \omega^{-1}\left(\frac{\partial^2}{\partial t_i^2} + \sigma^{-1}\frac{\partial\sigma}{\partial t_i}\frac{\partial}{\partial t_i}\right)\circ\omega \\ &\quad - \textstyle\sum_i \omega^{-1}\left(\frac{\partial^2}{\partial t_i^2} + \sigma^{-1}\frac{\partial\sigma}{\partial t_i}\frac{\partial}{\partial t_i}\right)\omega \\ &= \omega^{-1}S_1(L_1, \ldots, L_n)\circ\omega - \omega^{-1}S_1(L_1, \ldots, L_n)\,\omega, \end{aligned}$$

where we have defined

$$L_i := \frac{\partial^2}{\partial t_i^2} + 2\left(k\coth t_i + \coth 2t_i\right)\frac{\partial}{\partial t_i}$$

and

$S_j(L_1, \ldots, L_n) :=$ the j-th elementary symmetric polynomial in $L_1, \ldots, L_n$ (see [9]).

Now define

$$\Delta_j := \omega^{-1}S_j(L_1, \ldots, L_n)\circ\omega,$$

then we have, because of the relation $S_j(L_1, \ldots, L_n)\,\omega = c_j\omega$ (c_j defined by $\sum_{j=0}^{m} c_j\xi^{n-j} = \prod_{i=0}^{n-1}(\xi+4i(i+k+1))$, see [9]):

$$4\left((k+2n)\delta(\Omega)\right) = \Delta_1 - \textstyle\sum_{i=1}^{n-1} 4i(i+k+1). \tag{2.2}$$

3. Eigenfunctions of $\delta(\Omega)$

In this chapter we make use the following lemma (see [4], ch. II, prop. 1.10). Let Λ be the root lattice, that is $\Lambda = \{z_1\beta_1 + \ldots + z_n\beta_n : \beta_i \in \Sigma, \beta_i \text{ is simple}, z_i \in \mathbf{Z}^+ \cup (0)\}$. Let γ denote the natural isomorphism of $\mathbf{D}(X)$ onto $I(A)$ ($X=G/K$, A Lie group corresponding to $\mathfrak{a}$, $I(A)$ set of W-invariant polynomials on A, see [4], ch. II, theorem 1.2).

Lemma 3.1. *The equation*

$$\delta(\Omega)u = -(\langle\lambda, \lambda\rangle + \langle\varrho, \varrho\rangle)u$$

has a unique solution on C^+ *of the form*

$$u(H) = \Phi_\lambda(\exp H) = \sum_{\mu\in\Lambda} \Gamma_\mu \exp\left((\sqrt{-1}\lambda - \varrho - \mu)H\right)$$

with $\Gamma_0 = 1$. $u = \Phi_\lambda \circ \exp$ *is also a solution of the system of differential equations*

$$\delta(D)u = \gamma(D)(\sqrt{-1}\lambda)u, \quad D\in\mathbf{D}_0(G). \tag{3.1}$$

In our case, the function Φ_λ of the lemma takes the form

$$\Phi_\lambda(a_T) = e^{(\sqrt{-1}\lambda-\varrho)(T)} \sum_{\mu\in\Lambda} \Gamma_\mu(\lambda) e^{-\mu(T)} \tag{3.2}$$

where

$$T\in C^+$$

$$\lambda = (\lambda_1, \ldots, \lambda_n)\in\mathfrak{a}_C^*$$

$$\Gamma_0 \equiv 1,$$

in order to be an eigenfunction of all $\delta(D)$, $D\in\mathbf{D}_0(G)$.

So we have to solve

$$(\omega^{-1}S_1(L_1, \ldots, L_n)\circ\omega)u = \mu u,$$

i.e.

$$S_1(L_1, \ldots, L_n)(\omega u) = \mu(\omega u).$$

Let us try a solution $u(T)$ of the form

$$\omega(T)u(T) = v_1(t_1)\cdot\ldots\cdot v_n(t_n),$$

where v_i is a solution of the equation

$$L_i v_i = -(\lambda_i^2 + (k+1)^2)v_i, \quad t_i > 0, \tag{3.3}$$

such that v_i is of the form

$$v_i(t_i) = e^{(\sqrt{-1}\lambda_i-(k+1))t_i} \sum_{n=0}^{\infty} \Gamma_n e^{-nt_i}, \quad \Gamma_0 = 1. \tag{3.4}$$

Definition 3.1. Let $v_i(t_i)$ be a solution of (3.3), which is of the form (3.4). Then we define

$$\Phi_\lambda(a_T) := \frac{v_1(t_1)\cdot\ldots\cdot v_n(t_n)}{\omega(a_T)}.$$

Theorem 1.

a. $\Phi_\lambda(a_T)$ *satisfies* $\delta(\Omega)\Phi_\lambda(a_T) = -(\langle\lambda, \lambda\rangle + \langle\varrho, \varrho\rangle)\Phi_\lambda(a_T)$.

b. $\Phi_\lambda(a_T)$ *has a series expansion* (3.2).

Proof.

a. According to (2.2) we have

$$4(k+2n)\delta(\Omega)\Phi_\lambda(a_T) = \left(\Delta_1 - \sum_{i=0}^{n-1} 4i(i+k+1)\right)\Phi_\lambda(a_T). \tag{3.5}$$

Because of the relation $B(H_i, H_j)=4(k+2n)\delta_{ij}$, the inner product $\langle.,.\rangle$ is given by $\langle\xi, \eta\rangle=(4(k+2n))^{-1}\sum_{i=1}^{n}\xi_i\eta_i$, if $\xi=(\xi_1, \ldots, \xi_n)$, $\eta=(\eta_1, \ldots, \eta_n)$. Hence

$$\begin{aligned}\Delta_1\Phi_\lambda(a_T) &= \omega^{-1}S_1(L_1, \ldots, L_n)\circ\omega\left(\omega^{-1}\prod_{i=1}^{n} v_i(t_i)\right)\\ &= \omega^{-1}\left(-(4(k+2n)\langle\lambda, \lambda\rangle+n(k+1)^2)\right)\prod_{i=1}^{n} v_i(t_i)\\ &= -\left(4(k+2n)\langle\lambda, \lambda\rangle+n(k+1)^2\right)\Phi_\lambda(a_T), \end{aligned}\tag{3.6}$$

because of the relation $L_i v_j(t_j)=-(\lambda_j^2+(k+1)^2)v_j(t_j)\delta_{ij}$. Since $\varrho_i=k+1+2(n-i)$ we have $4(k+2n)\langle\varrho, \varrho\rangle=n(k+1)^2+\sum_{j=0}^{n-1}4j(k+1+j)$, and this together with (3.5) and (3.6) proves *a*.

b. To prove that $\Phi_\lambda(T)$ has a series expansion (3.2) we use the fact that $v_i(t_i)$ is of the form (3.4). We have

$$\Phi_\lambda(a_T) = \frac{v_1(t_1)\cdot\ldots\cdot v_n(t_n)}{\omega(a_T)}.$$

According to (3.4) the numerator is of the form

$$e^{(\sqrt{-1}\lambda_1-(k+1))t_1+\ldots+(\sqrt{-1}\lambda_n-(k+1))t_n}\sum_{l_1=0}^{\infty}\Gamma_{l_1}e^{-l_1t_1}\cdot\ldots\cdot\sum_{l_n=0}^{\infty}\Gamma_{l_n}e^{-l_nt_n}. \tag{3.7}$$

For the denominator we have

$$\begin{aligned}\omega(a_T) &= 2^{\frac{1}{2}n(n-1)}\prod_{i<j}\tfrac{1}{2}\left(e^{2t_i}+e^{-2t_i}-e^{2t_j}-e^{-2t_j}\right)\\ &= 2^{2(n-1)t_1+2(n-2)t_2+\ldots+2t_{n-1}}\prod_{i<j}\left(1-e^{-2(t_i-t_j)}\right)\left(1-e^{-2(t_i+t_j)}\right).\end{aligned}\tag{3.8}$$

In C^+ we have $t_1>t_2>\ldots>t_n>0$, so for all $T\in C^+$ the exponents in the denominator (i.e. $-2(t_i-t_j)$ and $-2(t_i+t_j)$ with $i<j$) are <0, so we have the power series expansions

$$\frac{1}{1-e^{-2(t_i-t_j)}} = \sum_{p=0}^{\infty}e^{-2p(t_i-t_j)},$$

$$\frac{1}{1-e^{-2(t_i+t_j)}} = \sum_{q=0}^{\infty}e^{-2q(t_i+t_j)}.$$

Using these power series expansion and formulas (3.7) and (3.8) we get for Φ_λ:

$$\Phi_\lambda(a_T) = e^{(\sqrt{-1}\lambda_1-(k+1)-2(n-1))t_1+\ldots+(\sqrt{-1}\lambda_{n-1}-(k+1)-2)t_{n-1}+(\sqrt{-1}\lambda_n-(k+1))t_n}\,\cdot$$

$$\cdot\prod_{i=1}^n\left(\sum_{l_i=0}^\infty \Gamma_{l_i} e^{-l_i t_i}\right)\prod_{i<j}\left(\sum_{p=0}^\infty e^{-2p(t_i-t_j)}\sum_{q=0}^\infty e^{-2q(t_i+t_j)}\right),$$

i.e. $e^{(\sqrt{-1}\lambda-\varrho)(T)}$ multiplied with a finite product of convergent series of the form $\sum_{\mu\in\Lambda} b_\mu(\lambda)e^{-\mu(T)}$.

Hence multiplication of the power series gives

$$\Phi_\lambda(a_T) = e^{(\sqrt{-1}\lambda-\varrho)(T)}\sum_{\mu\in\Lambda}\Gamma_\mu(\lambda)\,e^{-\mu(T)}.$$

Clearly we have $\Gamma_0\equiv 1$ which proves b. □

Now we've come to the point where we have to find the function $v_i(t_i)$ which satisfies (3.3) and (3.4). The equation $L_i v_i=\mu_i v_i$ can be seen as a differential equation for Jacobi functions (see [8]). The general equation for Jacobi functions is:

$$(3.9)\qquad (\Delta_{\alpha,\beta}(t))^{-1}\frac{d}{dt}\left\{\Delta_{\alpha,\beta}(t)\frac{du(t)}{dt}\right\} = -(\lambda^2+(\alpha+\beta+1)^2)u(t),$$

where $\Delta_{\alpha,\beta}(t)=(e^t-e^{-t})^{2\alpha+1}(e^t+e^{-t})^{2\beta+1}$.

The left-hand side of (3.9) in the case $\alpha=k$, $\beta=0$, $t=t_i$ is easily seen to be equal to $L_i u$. So let us try to find a solution of

$$(3.10)\qquad (\Delta_{k,0}(t_i))^{-1}\frac{\partial}{\partial t_i}\left\{\Delta_{k,0}(t_i)\frac{\partial u}{\partial t_i}\right\} = -(\lambda_i^2+(k+1)^2)\,u,$$

which is of the form (3.4).

Substitute $t_i := -\mathrm{sh}^2\, t_i$. Then equation (3.10) leads to a hypergeometrical differential equation. If we let $t_i\to\infty$, (3.4) gives the asymptotic behaviour:

$$(3.11)\qquad v_i(t_i) = e^{(\sqrt{-1}\lambda_i-(k+1))t_i}(1+o(1)).$$

According to [2, 2.9(9)] the Jacobi function of the second kind

$$\Phi_{\lambda_i}^{(k,0)}(t_i) = (e^{t_i}-e^{-t_i})^{\sqrt{-1}\lambda_i-(k+1)}\,{}_2F_1\left(\tfrac{1}{2}(-k+1-\sqrt{-1}\lambda_i),\tfrac{1}{2}(k+1-\sqrt{-1}\lambda_i);\right.$$
$$\left.1-\sqrt{-1}\lambda_i;\ -\mathrm{sh}^{-2}t_i\right)$$

is a solution of (3.10) for all λ_i with $\mathrm{Im}\,\lambda_i\notin \mathbf{Z}^-$, having the asymptotic behaviour (3.11).

Lemma 3.2. *$\Phi_{\lambda_i}^{(k,0)}(t_i)$ has a convergent series expansion* (3.4) *for* $t_i>0$.

Proof.

$$\Phi_{\lambda_i}^{(k,0)}(t_i) = (e^{t_i}-e^{-t_i})^{\sqrt{-1}\lambda_i-(k+1)}\,{}_2F_1\Big(\tfrac{1}{2}(-k+1-\sqrt{-1}\lambda_i),\ \tfrac{1}{2}(k+1-\sqrt{-1}\lambda_i);$$
$$1-\sqrt{-1}\lambda_i;\ -\mathrm{sh}^{-2}t_i\Big)$$
$$=(e^{t_i}+e^{-t_i})^{\sqrt{-1}\lambda_i-(k+1)}\,{}_2F_1\Big(\tfrac{1}{2}(k+1-\sqrt{-1}\lambda_i);\ \tfrac{1}{2}(k+1-\sqrt{-1}\lambda_i);$$
$$1-\sqrt{-1}\lambda_i;\ \mathrm{ch}^{-2}t_i\Big)\quad (\text{see } [2,\ 2.10(6)])$$
$$=e^{(\sqrt{-1}\lambda_i-(k+1))t_i}(1+e^{-2t_i})^{\sqrt{-1}\lambda_i-(k+1)}\sum_{n=0}^{\infty}\frac{\Big(\big(\tfrac{1}{2}(k+1-\sqrt{-1}\lambda_i)\big)_n\Big)^2}{(1-\sqrt{-1}\lambda_i)_n\,n!}(\mathrm{ch}^{-2}t_i)^n,$$

absolutely convergent for $t>0$ since $0<ch^{-2}t_i<1$. Hence

$$\Phi_{\lambda_i}^{(k,0)}(t_i)=e^{(\sqrt{-1}\lambda_i-(k+1))t_i}\sum_{n=0}^{\infty}\gamma_n e^{-2nt_i}(1+e^{-2t_i})^{-2n+\sqrt{-1}\lambda_i-k-1}$$

The lemma follows by expansion of $(1+e^{-2t_i})^{-2n+\sqrt{-1}\lambda_i-k-1}$ in powers of e^{-2t_i}. □

Combining theorem 1, lemma 3.1 and lemma 3.2 we get

Theorem 2. *The function*

$$\Phi_\lambda(a_T)=\frac{\Phi_{\lambda_1}^{(k,0)}(t_1)\cdot\ldots\cdot\Phi_{\lambda_n}^{(k,0)}(t_n)}{\omega(a_T)}$$

satisfies

$$\delta(D)\Phi_\lambda(a_T)=\gamma(D)(\sqrt{-1}\lambda)\,\Phi_\lambda(a_T)$$

for all $D\in\mathbf{D}_0(G)$.

4. Spherical functions on $SU(n, n+k;\ \mathbf{C})$

Let φ_λ be a spherical function on G, that is an eigenfunction of all $D\in\mathbf{D}_0(G)$, having value 1 at e. Then we have (see [5]):

(4.1) $$\varphi_\lambda(a_T)=\sum_{s\in W}c(s\lambda)\,\Phi_{s\lambda}(a_T),\quad T\in C^+,$$

where W is the Weyl group of G and $\Phi_\lambda(a_T)$ an eigenfunction of $\delta(\Omega)$ with a series expansion (3.2). Our main goal in this chapter is to find φ_λ, or to find the function c.

Let us first look at the rank 1 case (see [8]). As a solution of the hypergeometrical differential equation (3.10), which is regular for $t=0$, we get:

$$\varphi_{\lambda_i}^{(k,0)}(t_i)={}_2F_1\Big(\tfrac{1}{2}(k+1+\sqrt{-1}\lambda_i),\ \tfrac{1}{2}(k+1-\sqrt{-1}\lambda_i);\ k+1;\ -\mathrm{sh}^2t_i\Big).$$

Now, assume that $\lambda_i \notin \sqrt{-1}\,\mathbf{Z}$. Then we know from [2, 2.10(2)] that

$$
\begin{aligned}
&{}_2F_1\left(\tfrac{1}{2}(k+1+\sqrt{-1}\lambda_i), \tfrac{1}{2}(k+1-\sqrt{-1}\lambda_i);\ k+1;\ \mathrm{sh}^2 t_i\right) \\
&\quad = \sum_{s\in\{1,-1\}} c(s\lambda_i)(e^{t_i}-e^{-t_i})^{\sqrt{-1}s\lambda_i-(k+1)} \\
&\quad \cdot {}_2F_1\left(\tfrac{1}{2}(-k+1-\sqrt{-1}s\lambda_i), \tfrac{1}{2}(k+1-\sqrt{-1}s\lambda_i);\ 1-\sqrt{-1}s\lambda_i;\ -\mathrm{sh}^{-2} t_i\right)
\end{aligned}
$$

with

$$
c(\lambda_i) = \frac{\Gamma(k+1)\Gamma(-\sqrt{-1}\lambda_i)\,2^{\sqrt{-1}\lambda_i+k+1}}{\Gamma\left(\frac{1}{2}(k+1-\sqrt{-1}\lambda_i)\right)\Gamma\left(\frac{1}{2}(k+1+\sqrt{-1}\lambda_i)\right)}. \tag{4.2}
$$

So we have

$$
\varphi_{\lambda_i}(t_j) = c(\lambda_i)\Phi_{\lambda_i}(t_j) + c(-\lambda_i)\Phi_{-\lambda_i}(t_j) \tag{4.3}
$$

(from now on we omit the indices $(k, 0)$, that is we'll write φ_{λ_i} instead of $\varphi_{\lambda_i}^{(k,0)}$ etc.) where c is defined as in (4.2). Because $(-\lambda_i)^2=\lambda_i^2$ the following relation is also valid.

$$
L_i\varphi_{\lambda_i}(t_j) = -(\lambda_i^2+(k+1)^2)\varphi_{\lambda_i}(t_j). \tag{4.4}
$$

Definition 4.1.

$$
\varphi_\lambda(a_T) := \frac{A}{\prod_{i<j}(\lambda_i^2-\lambda_j^2)} \cdot \frac{\det(\varphi_{\lambda_i}(t_j))_{1\leq i,j\leq n}}{\omega(a_T)}.
$$

(A is a normalization constant, independent of T and λ, which has yet to be determined.)

We want to prove that $\varphi_\lambda(a_T)$ is a spherical function on G. Therefore, we'd like to write φ_λ as a combination of Φ_λ's, in a way which is similar to (4.1). According to [9] we have $W=\{s\colon s(t_1, \ldots, t_n)=(\varepsilon_1 t_{\sigma(1)}, \ldots, \varepsilon_n t_{\sigma(n)}),\ \varepsilon_i=\pm 1,\ \sigma\in S_n\}$. We'll denote such an $s\in W$ by $s=(\varepsilon, \sigma)$ with $\varepsilon=(\varepsilon_1, \ldots, \varepsilon_n)$ and $\sigma\in S_n$. Thus

$$
\begin{aligned}
A^{-1}\cdot\omega(a_T)\varphi_\lambda(a_T) &= \frac{\det(\varphi_{\lambda_i}(t_j))}{\prod_{i<j}(\lambda_i^2-\lambda_j^2)} \\
&= \frac{\sum_{\sigma\in S_n}(-1)^{\mathrm{sgn}\,\sigma}\prod_{p=1}^n \varphi_{\lambda_{\sigma(p)}}(t_p)}{(-1)^{\frac{1}{2}n(n-1)}\det((\lambda_i^2)^{j-1})} \\
&= \frac{\sum_{\sigma\in S_n}(-1)^{\mathrm{sgn}\,\sigma}\sum_{\substack{\varepsilon_i=\pm 1\\ i=1,\ldots,n}} c(\varepsilon_1\lambda_{\sigma(1)})\Phi_{\varepsilon_1\lambda_{\sigma(1)}}(t_1)\cdot\ldots\cdot c(\varepsilon_n\lambda_{\sigma(n)})\Phi_{\varepsilon_n\lambda_{\sigma(n)}}(t_n)}{(-1)^{\frac{1}{2}n(n-1)}\det((\lambda_i^2)^{j-1})} \\
&= \sum_{\substack{\sigma\in S_n\\ \varepsilon_i=\pm 1}} \frac{c(\varepsilon_1\lambda_{\sigma(1)})\cdot\ldots\cdot c(\varepsilon_n\lambda_{\sigma(n)})}{(-1)^{\frac{1}{2}n(n-1)}\det((\varepsilon_i\lambda_{\sigma(i)})^{2(j-1)})}\prod_{p=1}^n \Phi_{\varepsilon_p\lambda_{\sigma(p)}}(t_p).
\end{aligned}
$$

Hence

$$\varphi_\lambda(a_T) = \sum_{s\in W} C(s\lambda)\Phi_{s\lambda}(a_T), \tag{4.5}$$

where

$$C(\lambda) = A\cdot\frac{c(\lambda_1)\cdot\ldots\cdot c(\lambda_n)}{(-1)^{\frac{1}{2}n(n-1)}\det(\lambda_i^{2(j-1)})}. \tag{4.6}$$

Since $\langle s\lambda, s\lambda\rangle=\langle\lambda,\lambda\rangle$ for all $s\in W$, it follows from (4.5) and theorem 1 a that

$$\delta(\Omega)\varphi_\lambda(a_T) = -(\langle\lambda,\lambda\rangle+\langle\varrho,\varrho\rangle)\varphi_\lambda(a_T). \tag{4.7}$$

Lemma 4.1. (HUA [6].) *Suppose $f_1(x), \ldots, f_n(x)$ are C^∞-functions on a real interval I. Let*

$$F(x_1, \ldots, x_n) := \frac{\det(f_i(x_j))}{\prod_{i<j}(x_i-x_j)}.$$

Then F is C^∞ and symmetric on I^n and, for $a\in I$,

$$F(a, \ldots, a) = \frac{(-1)^{\frac{1}{2}n(n-1)}}{1!\,2!\ldots(n-1)!}\det(f_i^{(j-1)}(a)).$$

Moreover, if all the f_i are polynomials, then so is F.

Proof. (Sketch) Use complete induction with respect to n, by writing

$$\det(f_i(x_j)) = (x_2-x_1)\ldots(x_n-x_1)\cdot\det\begin{vmatrix} f_1(x_1) & \ldots & f_n(x_1) \\ \frac{f_1(x_2)-f_1(x_1)}{x_2-x_1} & \ldots & \frac{f_n(x_2)-f_n(x_1)}{x_2-x_1} \\ \vdots & & \\ \frac{f_1(x_n)-f_1(x_1)}{x_n-x_1} & \ldots & \frac{f_n(x_n)-f_n(x_1)}{x_n-x_1} \end{vmatrix}$$

and next expanding the determinant with respect to the first row. □

According to [2, 2.8(20)], we have

$$\frac{d^l}{dz^l}\,{}_2F_1(a, b; c; z) = \frac{(a)_l(b)_l}{(c)_l}\,{}_2F_1(a+l, b+l; c+l; z). \tag{4.8}$$

Now

$$\lim_{T\to 0}\frac{\det((\varphi_{\lambda_i})(t_j))}{\omega(T)} =$$

$$= \lim_{T\to 0}\frac{\det\left({}_2F_1\left(\tfrac{1}{2}(k+1+\sqrt{-1}\lambda_i), \tfrac{1}{2}(k+1-\sqrt{-1}\lambda_i); k+1; -\mathrm{sh}^2 t_j\right)\right)}{2^{n(n-1)}\prod_{i<j}(\mathrm{sh}^2 t_i-\mathrm{sh}^2 t_j)}.$$

Using lemma 4.1 and (4.8) we see that this expression is equal to

$$\frac{2^{-n(n-1)}(-1)^{\frac{1}{2}n(n-1)}}{1!\,2!\dots(n-1)!}\cdot$$

$$\cdot\det\begin{vmatrix} 1 & \dots & 1 \\ -\frac{1}{4}\left(\frac{1}{k+1}\right)(\lambda_1^2+(k+1)^2) & \dots & -\frac{1}{4}\left(\frac{1}{k+1}\right)(\lambda_n^2+(k+1)^2) \\ \vdots & & \vdots \\ \left(-\frac{1}{4}\right)^{n-1}\left(\frac{1}{(k+1)\dots(k+(n-1))}\right)(\lambda_1^2+(k+1)^2)\dots(\lambda_1^2+(k+2n-1)^2)\dots & & \end{vmatrix}$$

$$= \frac{1}{2^{2n(n-1)}\prod_{j=1}^{n-1}\{(k+j)^{n-j}j!\}}\det\begin{vmatrix} 1 & \dots & 1 \\ \lambda_1^2+(k+1)^2 & \dots & \lambda_n^2+(k+1)^2 \\ \vdots & & \\ (\lambda_1^2+(k+1)^2)^{n-1} & \dots & (\lambda_n^2+(k+1)^2)^{n-1} \end{vmatrix}$$

$$= \frac{(-1)^{\frac{1}{2}n(n-1)}}{2^{2n(n-1)}\prod_{j=1}^{n-1}\{(k+j)^{n-j}j!\}}\prod_{i<j}(\lambda_i^2-\lambda_j^2).$$

Hence, if we take

$$A = (-1)^{\frac{1}{2}n(n-1)}2^{2n(n-1)}\prod_{j=1}^{n-1}\{(k+j)^{n-j}j!\} \tag{4.9}$$

in definition 3.1 we obtain

$$\varphi_\lambda(a_0) = 1. \tag{4.10}$$

Now, since it is obvious from the definition that φ_λ is W-invariant and C^∞ everywhere on A, it follows from theorem 2 and the relations (4.5) and (4.10) that for all $\lambda\in\mathfrak{a}_{\mathbf{C}}^*$ with $\lambda_p\notin\sqrt{-1}\mathbf{Z}$ for all p, $\varphi_\lambda(a_T)$ is the restriction to A of a spherical function on G. Because the set $\{\lambda\in\mathbf{C}^n\colon \sqrt{-1}\,\lambda_p\notin\mathbf{Z}\ \forall p\}$ is an open, dense subset of $\mathbf{C}^n$, we can catch all λ by analytic continuation (if $\lambda_p=\lambda_q$ for some p, q, $p\neq q$ continuation according to lemma 4.1), so we have proved the first theorem of Berezin and Karpelevič.

Theorem 3. (BEREZIN and KARPELEVIČ [1].) *The zonal spherical functions φ_λ on $G=\mathrm{SU}(n,n+k;\mathbf{C})$ are given by*

$$\varphi_\lambda(a_T) =$$

$$= \frac{A}{\prod_{i<j}(\lambda_i^2-\lambda_j^2)}\cdot\frac{\det\left({}_2F_1\left(\frac{1}{2}(k+1+\sqrt{-1}\lambda_i),\frac{1}{2}(k+1-\sqrt{-1}\lambda_i);\ k+1;\ -\mathrm{sh}^2 t_j\right)\right)}{2^{\frac{1}{2}n(n-1)}\prod_{i<j}(\mathrm{ch}\,2t_i-\mathrm{ch}\,2t_j)}$$

where A is as in (4.9).

5. The algebra $\delta(\mathbf{D}_0(G))$

Now we come to the point where we can prove the second theorem of Berezin and Karpelevič. We proceed as follows. First, we show that the functions φ_λ satisfy $\Delta_j\varphi_\lambda = a_j(\lambda)\varphi_\lambda$ for all j, and next, by using a method of KOORNWINDER (see [7], § 6), we show that every differential operator, which has all the φ_λ as eigenfunctions, is a polynomial in the Δ_j $(j=1, \dots, n)$, and this polynomial is uniquely determined. Thus, because of the fact that $\delta(D)\varphi_\lambda = \gamma(D)(\sqrt{-1}\,\lambda)\varphi_\lambda$ $(D\in\mathbf{D}_0(G))$ (this folows from theorem 2 and (4.5)) it follows that the algebra $\delta(\mathbf{D}_0(G))$ is generated by the Δ_j $(j=1, \dots, n)$.

For reasons of convenience we'll work with a slightly larger set than $\delta(\mathbf{D}_0(G))$.

Lemma 5.1. *$\Delta_j\varphi_\lambda(a_T) = a_j(\lambda)\varphi_\lambda(a_T)$ for all j.*

Proof. In 1 variable t we have

$$L_i\Phi_{\lambda_j}(t) = -(\lambda_j^2+(k+1)^2)\Phi_{\lambda_j}(t)\delta_{ij}.$$

Hence

$$\prod_{i=1}^n(\xi+L_i)\prod_{j=1}^n\Phi_{\lambda_j}(t_j) = \prod_{i=1}^n(\xi-(\lambda_i^2+(k+1)^2))\prod_{j=1}^n\Phi_{\lambda_j}(t_j).$$

Define on $\mathfrak{a}_\mathbf{C}^*$ the functions $a_j(\lambda)$ by

$$\prod_{i=1}^n(\xi-(\lambda_i^2+(k+1)^2)) = \sum_{j=0}^n a_j(\lambda)\,\xi^{n-j}.$$

Then

$$S_j(L_1, \dots, L_n)\prod_{i=1}^n\Phi_{\lambda_i}(t_i) = a_j(\lambda)\prod_{i=1}^n\Phi_{\lambda_i}(t_i) \quad \text{for all } j.$$
$$\Rightarrow (\omega^{-1}S_j(L_1, \dots, L_n)\circ\omega)\,\Phi_\lambda(a_T) = a_j(\lambda)\,\Phi_\lambda(a_T) \quad \text{for all } j.$$
$$\Rightarrow (\omega^{-1}S_j(L_1, \dots, L_n)\circ\omega)\,\varphi_\lambda(a_T) = a_j(\lambda)\,\varphi_\lambda(a_T) \quad \text{for all } j.$$
$$\Rightarrow \Delta_j\varphi_\lambda(a_T) = a_j(\lambda)\varphi_\lambda(a_T) \quad \text{for all } j. \quad \square$$

For the second part: remark first that every differential operator which is a polynomial in the Δ_j, has to have all φ_λ as eigenfunctions, because of lemma 5.1. So we have to prove that every D which has all φ_λ as eigenfunctions must be a polynomial in the Δ_j. We'll restrict ourself to those φ_λ which are polynomials, that is $\frac{1}{2}(k+1\pm\sqrt{-1}\,\lambda_i)\in\mathbf{Z}^-$. If we can prove that this, i.e. every D which has all polynomial φ_λ as eigenfunctions, is a polynomial in the Δ_j, we are done because of the remark above.

Let $\mathcal{N}$ be the ordered set of all n-tuples $\mu=(\mu_1, \dots, \mu_n)$ with $\mu_i\in\mathbf{Z}$ for all i, and $\mu_1\geqq\mu_2\geqq\dots\geqq\mu_n\geqq 0$, and let $\prec$ denote the lexicographical ordering on $\mathcal{N}$.

Let $t=(t_1, \dots, t_n)$ with $t_i\in\mathbf{Z}$ for all i.

Now, let $\varphi_{\lambda_i}(t)$ be a polynomial. Say

$$\tfrac{1}{2}(k+1-\sqrt{-1}\lambda_i) = -m_i-n+i \quad \text{for} \quad i=1, \dots, n \text{ and } m\in\mathcal{N}.$$

Then $\varphi_{\lambda_i}(t)$ becomes

$$\varphi_{\lambda_i}(t) = {}_2F_1(-(m_i+n-i),\ m_i+n-i+k+1;\ k+1;\ -\mathrm{sh}^2 t).$$

We'll denote such a $\varphi_{\lambda_i}(t)$ with $\frac{1}{2}(k+1-\sqrt{-1}\,\lambda_i)=-m_i-n+i$ by $p_{m_i}(t)$. Thus $p_{m_i}(t)$ is a polynomial of degree m_i+n-i in $-\mathrm{sh}^2 t$. Then it follows from lemma 4.1 that $\varphi_\lambda(a_T)$ is a polynomial of the form $\varphi_\lambda(a_T)=c(-\mathrm{sh}^2 t_1)^{m_1}\dots(-\mathrm{sh}^2 t_n)^{m_n}+$terms of lower order (according to the lexicographical ordering of the n-tuples $(m_1, \dots, m_n)$). This polynomial function we'll denote by $P_m(a_T)$ $(m\in\mathcal{N})$.

Definition 5.1. Let $\mathbf{D}^W(G)$ be the set of all W-invariant differential operators on $\mathbf{R}^n$, regular in the interior of all Weyl chambers, and having all the P_m as eigenfunctions, that is $D\in\mathbf{D}^W(G)$ implies $DP_m=b(m)P_m$.

Clearly $\mathbf{D}^W(G)$ includes both $\delta(\mathbf{D}_0(G))$ and all polynomials in the Δ_j.

Lemma 5.2. *Let $D\in\mathbf{D}^W(G)$. Let $m=(m_1, \dots, m_n)\in\mathcal{N}$ be the order of D. Then D is completely determined by its eigenvalues of P_μ, $b(\mu)$, with $\mu\underline{\prec}m$.*

Proof. By the W-invariance of D, D can be written as a symmetric operator in $-\mathrm{sh}^2 t_1, \dots, -\mathrm{sh}^2 t_n$. Let $-\mathrm{sh}^2 \mathrm{t}_\sigma$ denote the vector $(-\mathrm{sh}^2 t_{\sigma(1)}, \dots, -\mathrm{sh}^2 t_{\sigma(n)})$ $(\sigma\in S_n)$. Then

$$D = \sum\nolimits_\mu^{\prime(m)} \sum\nolimits_{\sigma\in S_n} c_\mu(-\mathrm{sh}^2 t_\sigma)\left(\frac{\partial}{\partial(-\mathrm{sh}^2 t_1)}\right)^{\mu_{\sigma(1)}} \dots \left(\frac{\partial}{\partial(-\mathrm{sh}^2 t_n)}\right)^{\mu_{\sigma(n)}},$$

where the sum $\sum_\mu^{\prime(m)}$ is extended to those μ for which $\mu\leqq m$.

We'll prove by complete induction with respect to μ that c_μ is completely determined by $b(\mu)$ $(\mu\prec m)$. We have $c_0=b(0)$. It follows from $DP_\mu=b(\mu)P_\mu$ that

$$b(\mu)P_\mu = \sum\nolimits_{\sigma\in S_n} c_\mu(-\mathrm{sh}^2 t_\sigma)\left(\frac{\partial}{\partial(-\mathrm{sh}^2 t_1)}\right)^{\mu_{\sigma(1)}} \dots \left(\frac{\partial}{\partial(-\mathrm{sh}^2 t_n)}\right)^{\mu_{\sigma(n)}} \cdot P_\mu$$

$$+\sum\nolimits_\nu^{\prime\prime(\mu)} \sum\nolimits_{\tau\in S_n} c_\nu(-\mathrm{sh}^2 t_\tau)\left(\frac{\partial}{\partial(-\mathrm{sh}^2 t_1)}\right)^{\nu_{\tau(1)}} \dots \left(\frac{\partial}{\partial(-\mathrm{sh}^2 t_n)}\right)^{\nu_{\tau(n)}} \cdot P_\mu,$$

where the sum $\sum_\nu^{\prime\prime(\mu)}$ is extended to those ν for which $\nu \underset{\neq}{\prec} \mu$, because the terms of D with $\nu \underset{\neq}{\succ} \mu$ annihilate P_μ. Hence

$$n!\,\beta_\mu c_\mu(-\mathrm{sh}^2 t) = b(\mu)P_\mu - \sum\nolimits_\nu^{\prime\prime(\mu)} \sum\nolimits_{\tau\in S_n} c_\nu(-\mathrm{sh}^2 t_\tau)\left(\frac{\partial}{\partial(-\mathrm{sh}^2 t_1)}\right)^{\nu_{\tau(1)}} \dots$$

$$\dots \left(\frac{\partial}{\partial(-\mathrm{sh}^2 t_n)}\right)^{\nu_{\tau(n)}} \cdot P_\mu$$

where $\beta_\mu=\mu_1!\dots\mu_n!$ times the coefficient of the term of order $(\mu_1, \dots, \mu_n)$ in P_μ.

The lemma now follows by the induction hypothesis. □

Lemma 5.2 immediately implies:

Lemma 5.3. *Let $D_1, D_2\in\mathbf{D}^W(G)$. Then $D_1D_2=D_2D_1$.*

We have by definition $D \in \mathbf{D}^W(G) \Rightarrow D$ is W-invariant. W is the set of all maps s such that

$$s\colon (t_1, \ldots, t_n) \to (\varepsilon_1 t_{\sigma(1)}, \ldots, \varepsilon_n t_{\sigma(n)}) \quad \varepsilon_i = \pm 1 \quad \forall i, \quad \sigma \in S_n.$$

This implies:

Lemma 5.4. *Let $D \in \mathbf{D}^W(G)$. Suppose D is written in the form*

$$D = \sum_\mu c_\mu(t) \left(\frac{\partial}{\partial t_1}\right)^{\mu_1} \cdots \left(\frac{\partial}{\partial t_n}\right)^{\mu_n}.$$

Then D is invariant under the operations

$$t_i \to -t_i \qquad \forall i,$$
$$(t_1, \ldots, t_n) \to (t_{\sigma(1)}, \ldots, t_{\sigma(n)}) \quad \forall \sigma \in S_n.$$

Lemma 5.5. *Let $D \in \mathbf{D}^W(G)$, and let $d =$ degree D. Then D can be written in the form*

(5.1) $$D = \sum_{\substack{\mu \\ \Sigma \mu_i = d}} c_\mu \left(\frac{\partial}{\partial t_1}\right)^{\mu_1} \cdots \left(\frac{\partial}{\partial t_n}\right)^{\mu_n} + \text{l.o.}$$

*(*l.o. *means lower order terms), where the c_μ are constants.*

Proof. Lemma 5.3 implies that D commutes with all the Δ_j, hence

(5.2) $$D\Delta_j - \Delta_j D = 0 \quad \textit{for all } j.$$

We have

$$\Delta_j = S_j \left(\frac{\partial^2}{\partial t_1^2}, \ldots, \frac{\partial^2}{\partial t_n^2}\right) + l.o.$$

Let D be written in the form given by (5.1), only with $c_\mu = c_\mu(t)$. Now we use (5.2), in particular we use the fact that the terms of order $d+2j-1$ disappear. This yields:

$$(d+2j-1)^{th} \text{ order part of } \left[S_j \left(\frac{\partial^2}{\partial t_1^2}, \ldots, \frac{\partial^2}{\partial t_n^2}\right) \left\{\sum_{\substack{\mu \\ \Sigma \mu_i = d}} c_\mu(t) \left(\frac{\partial}{\partial t_1}\right)^{\mu_1} \cdots \left(\frac{\partial}{\partial t_n}\right)^{\mu_n}\right\}\right] = 0.$$

Hence

$$\sum_{\substack{\nu \\ \Sigma \nu_i = d+2j-1}} \left(\sum_{p=1}^n \sum_{\pi \in V_p^{j-1}} \frac{\partial}{\partial t_p} (c_{\nu_1 - i_1(p,\pi), \ldots, \nu_n - i_n(p,\pi)}(t)) \cdot \left(\frac{\partial}{\partial t_1}\right)^{\nu_1} \cdots \left(\frac{\partial}{\partial t_n}\right)^{\nu_n}\right) = 0,$$

where we have defined:

$-V_p^{j-1} :=$ the set of all $(j-1)$-subsets of $\{1, \ldots, p-1, p+1, \ldots, n\}$,

$$-i_q(p, \pi) := \begin{cases} 1 & \text{if } q = p, \\ 2 & \text{if } p \in \pi, \\ 0 & \text{else;} \end{cases}$$

$-c_{j_1, \ldots, j_n} = 0$ if one or more $j_i < 0$.

Hence we have to solve the system of equations

$$(5.3)\qquad \sum_{p=1}^{n} \sum_{\pi\in V_p^{j-1}} \frac{\partial}{\partial t_p}\left(c_{\nu_1-i_1(p,\pi),\dots,\nu_n-i_n(p,\pi)}(t)\right) = 0$$

$$\text{for all } 1 \leqq j \leqq n,\ \nu \text{ with } \sum \nu_i = d+2j-1.$$

We'll prove by complete induction with respect to the lexicographical ordering that (5.3) implies

$$(5.4)\qquad \frac{\partial}{\partial t_q} c_{\nu_1,\dots,\nu_n}(t) = 0 \quad \forall q\colon 1 \leqq q \leqq n,\ \forall \nu\colon \sum \nu_i = d.$$

(Remember that $(\mu_1, \dots, \mu_n) \prec (m_1, \dots, m_n)$ iff $\exists l$ such that $\mu_i = m_i$ if $1 \leqq i \leqq l-1$ and $\mu_l < m_l$.)

i. By taking $j=1$ and $\nu_q=1$, $\nu_i=0$ for $i\neq q$ it is clear from (5.3) that $\frac{\partial}{\partial t_q} c_{0,\dots,0}(t)=0$ $\forall q$.

ii. Let $(l_1, \dots, l_n)=(0, \dots, 0, l_{p+1}, \dots, l_n)$ with $l_{p+1}\neq 0$, and assume that for all q

$$\frac{\partial}{\partial t_q} c_{l'_1,\dots,l'_n}(t) = 0 \quad \text{if} \quad (l'_1, \dots, l'_n) \prec (l_1, \dots, l_n)$$

(induction hypothesis).

a. Assume $1 \leqq q \leqq p$.
By taking $j=n-i+1$, $\nu_q=1$, $\nu_i=0$ if $1\leqq i\leqq p$, $i\neq q$ and $\nu_i=l_i+2$ if $i\geqq p+1$ (5.3) becomes

$$\frac{\partial}{\partial t_q} c_{0,\dots,0,l_{p+1},\dots,l_n}(t) = 0.$$

b. Assume $q \geqq p+1$.
By taking $j=n-q$, $\nu_i=0$ if $1\leqq i\leqq p$, $\nu_i=l_i$ if $p+1\leqq i\leqq q-1$, $\nu_q=l_q+1$ and $\nu_i=l_i+2$ if $i\geqq q+1$ (5.3) becomes

$$\frac{\partial}{\partial t_q} c_{0,\dots,0,l_{p+1},\dots,l_n}(t) = 0,$$

where we have used the induction hypothesis.

So it is proved that (5.3) implies (5.4). Hence $c_{\nu_1,\dots,\nu_n}(t)$=constant for all ν, so the lemma is proved. □

Theorem 4. *Let* $D\in \mathbf{D}^W(G)$. *Then*

a. *D can be written as a polynomial in the* Δ_j*;*

b. *this expression is unique, that is if* $P_1(\Delta_1, \dots, \Delta_n)=P_2(\Delta_1, \dots, \Delta_n)$, *then* $P_1\equiv P_2$.

Proof. a. Let $D \in \mathbf{D}^W(G)$, and suppose D cannot be written as a polynomial in the Δ_j. Let d:=degree D, and assume that d is minimal. According to lemma 5.5 we can write

$$D = \sum_{\substack{\mu \\ \Sigma \mu_i = d}} c_\mu \left(\frac{\partial}{\partial t_1}\right)^{\mu_1} \dots \left(\frac{\partial}{\partial t_n}\right)^{\mu_n} + l.o.$$

Since D satisfies the symmetry relations of lemma 5.4, the d-th order part of D has to be a symmetric polynomial in $\frac{\partial^2}{\partial t_1^2}, \dots, \frac{\partial^2}{\partial t_n^2}$, and hence a polynomial in $S_1, \dots, S_n$, where S_j is the j-th elementary symmetric polynomial in $\frac{\partial^2}{\partial t_1^2}, \dots, \frac{\partial^2}{\partial t_n^2}$. Thus we have

$$D = P(S_1, \dots, S_n) + D',$$

where D' is an operator of degree $<d$. We also have $\Delta_j = S_j + \text{l.o.}$, so $S_j = \Delta_j + \text{l.o.}$ Hence

$$D = P(\Delta_1, \dots, \Delta_n) + D'', \tag{5.5}$$

where D'' is an operator of degree $d'' < d$.

Since $D \in \mathbf{D}^W(G)$ and $P \in \mathbf{D}^W(G)$ (because all $\Delta_j \in \mathbf{D}^W(G)$) we have $D'' \in \mathbf{D}^W(G)$. Because $d'' < d$, D'' can be written as a polynomial in $\Delta_1, \dots, \Delta_n$, and because of (5.5) this implies that D can be written as a polynomial in $\Delta_1, \dots, \Delta_n$. This contradiction proves *a*.

b. It is sufficient to show: $Q(\Delta_1, \dots, \Delta_n) = 0 \Rightarrow Q \equiv 0$, if Q is a polynomial. So, suppose $Q(\Delta_1, \dots, \Delta_n) = 0$, and $Q \not\equiv 0$. So for some $e \in \mathbf{Z}^+$

$$Q(u) = \sum_{\substack{\mu \\ 2\mu_1 + 4\mu_2 + \dots + 2n\mu_n \leqq e}} k_\mu u_1^{\mu_1} u_2^{\mu_2} \dots u_n^{\mu_n},$$

where not for all μ with $2\mu_1 + 4\mu_2 + \dots + 2n\mu_n = e$ we have $k_\mu = 0$. Taking $u_i = \Delta_i$, and using the fact that $\Delta_j = S_j \left(\frac{\partial^2}{\partial t_1^2}, \dots, \frac{\partial^2}{\partial t_n^2}\right) + \text{l.o.}$ we obtain

$$0 = Q(\Delta_1, \dots, \Delta_n) = \sum_{\substack{\mu \\ 2\mu_1 + \dots + 2n\mu_n \leqq e}} k_\mu (S_1 + l.o.)^{\mu_1} (S_2 + l.o.)^{\mu_2} \dots (S_n + l.o.)^{\mu_n}$$

$$= \sum_{\substack{\mu \\ 2\mu_1 + \dots + 2n\mu_n = e}} k_\mu \left(S_1 \left(\frac{\partial^2}{\partial t_1^2}, \dots, \frac{\partial^2}{\partial t_n^2}\right)^{\mu_1}\right) \dots \left(S_n \left(\frac{\partial^2}{\partial t_1^2}, \dots, \frac{\partial^2}{\partial t_n^2}\right)\right)^{\mu_n} + l.o.$$

Hence, the e-th order term of the above expression must be 0. But this is a combination of elementary symmetric polynomials, and this combination can only be 0 if all coefficients are 0, hence

$$k_\mu = 0 \quad \forall \mu \colon 2\mu_1 + 4\mu_2 + \dots + 2n\mu_n = e,$$

which is a contradiction, so $Q \equiv 0$. □

Because of theorem 4 we have proved the second theorem of BEREZIN and KARPELEVIČ [1].

Theorem 5. *Let* $G=SU(n, n+k; \mathbf{C})$. *The operators* $\Delta_j=\omega^{-1} S_j(L_1, \ldots, L_n)\circ\omega$ $(1\leqq j\leqq n)$, *where* S_j *=j-th elementary symmetric polynomial and* $L_i=\frac{\partial^2}{\partial t_i^2}+2(k \coth t_i+\coth 2t_i)\frac{\partial}{\partial t_i}$, *form a system of generators for* $\delta(\mathbf{D}_0(G))$.

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References

1. BEREZIN, F. A. & F. I. KARPELEVIČ, Zonal spherical functions and Laplace operators on some symmetric spaces, *Dokl. Akad. Nauk. SSSR (N. S.)* **118** (1958), 9—12. (In Russian.)
2. ERDÉLYI, A., W. MAGNUS, F. OBERHETTINGER & F. TRICOMBI, *Higher Transcendental Functions,* vol. 1, McGraw-Hill, New York, 1953.
3. HARISH—CHANDRA, Spherical Functions on a Semisimple Lie Group, I. *Am. J. Math.* **80** (1958), 241—310.
4. HELGASON, S., Analysis on Lie Groups and Homogeneous Spaces, *Conf. Board of the Math. Sci. Regional Conf. Ser. Math. no.* 14. Am. Math. Soc., Providence, R.I. 1972.
5. HELGASON, S., Functions on Symmetric Spaces, in: *Proceedings of Symposia in Pure Mathematics, vol. XXVI; Harmonic Analysis on Homogeneous Spaces,* Am. Math. Soc. Providence, R.I. 1973.
6. HUA, L. K., *Harmonic Analysis of Functions of Several Complex Variables in the Classical Domains,* Am. Math. Soc. Providence, R.I. 1963.
7. KOORNWINDER, T. H., *Orthogonal Polynomials in Two Variables which are eigenfunctions of two Algebraically Independent Partial Differential Operators,* I, II. Nederl. Akad. Wetensch. Proc. Ser. A 77=Indag. Math. **36** (1974), pp. 48—58, 59—66.
8. KOORNWINDER, T. H., *A new proof of a Paley—Wiener Type Theorem for the Jacobi Transform.* Ark. Mat. **13** (1975), 145—159.
9. TAKAHASHI, R., *Fonctions Sphériques zonales sur* $U(n, n+k; \mathbf{F})$, in "*Séminaire d'Analyse Harmonique*" (1976—77) Faculté des Sciences de Tunis, Departement de Mathematique, 1977.

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