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Recursions for characteristic numbers of genus one plane curves

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Abstract. Characteristic numbers of families of maps of nodal curves to $\mathbf{P}^2$ are defined as intersection of natural divisor classes. (This definition agrees with the usual definition for families of plane curves.) Simple recursions for characteristic numbers of genus one plane curves of all degrees are computed.

1. Introduction

The main results of this paper are recursions calculating characteristic numbers of genus one plane curves of any degree, and of genus one plane curves with fixed complex structure. En route, we derive (known) recursions for characteristic numbers of rational curves.

In Sections 2 and 3, we describe a rigorous framework for discussing characteristic numbers in general (as intersections of natural divisors on Kontsevich's moduli space of stable maps), culminating in Theorem 3.15. This framework will be used in a companion article [V2] verifying Zeuthen's calculation of the characteristic number of smooth plane quartic curves, a project begun by P. Aluffi [A3]. It will also be used in another article [V1] extending formulas of Hurwitz and others on coverings of the sphere. In Section 4, we review facts about maps of low-genus curves to $\mathbf{P}^2$, and in the rest of the article we apply this setup to deduce recursions solving the characteristic number problem for genus one plane curves.

Characteristic number problems motivated a great deal of algebraic geometry in the nineteenth century. For complete historical background and references, see [K1]. After the advent of the intersection theory of Fulton and MacPherson, a modern study of the enumerative geometry of cubics was undertaken successfully in the 1980s (see [A1] for history). The introduction of Kontsevich's moduli space of stable maps in the 1990s has reinvigorated the field by suggesting surprising recursions involving solutions to such enumerative problems, and has led to great advances.

This particular paper was inspired by [KK].

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2. Conventions and background results

2.1. We work over a fixed algebraically closed field k of characteristic 0. By *scheme*, we mean scheme of finite type over k. By *variety*, we mean a separated integral scheme. All morphisms of schemes are assumed to be defined over k, and fibre products are over k unless otherwise specified.

Suppose $f: X \to Y$ is a morphism of varieties. We say that f is *unramified at a point* p if the induced morphism of tangent spaces $T_{p,X} \to f^* T_{f(p),Y}$ is injective, and that f is *unramified* if it is unramified at all points $p \in X$. Let $\operatorname{ram}(f)$ be the set of ramified points of f. Let $\operatorname{Sing}(f) \subset X$ be the set of points that are ramified or are singular points of fibers of f. Let $\operatorname{Sm}(f) := X \setminus \operatorname{Sing}(f)$. Let X^{reg} be the set of regular points of X.

If $f: C \to X$ is a morphism of schemes and Y is a closed subscheme of X, then define $f^{-1}(Y)$ as $C \times_X Y$; $f^{-1}Y$ is a closed subscheme of C.

If $f: X \to Y$ is a finite morphism of varieties of the same dimension, then the ramification (Weil) divisor R_f is the sum over the height 1 associated primes $\mathfrak{p}$ of Ω^1_f of the length (of Ω^1_f) at $\mathfrak{p}$, times the Weil divisor $\mathfrak{p}$. If $\nu: \widetilde{Y} \to Y$ is the normalization, and $p: X \times_Y \widetilde{Y} \to X$ is the projection, then it is simple to check that $g: X \times_Y \widetilde{Y} \to \widetilde{Y}$ is also finite, and $p_* R_g = R_f$.

2.2. A *family of nodal curves over a base scheme* S (or a *nodal curve over* S) is a proper flat morphism $\pi: C \to S$ whose geometric fibers are reduced and of pure dimension 1, with at worst ordinary double points as singularities. (There is no connectedness condition.) If X is a scheme, then a *family of maps of nodal curves to* X *over* S (or a *map of a nodal curve to* X *over* S) is a morphism $\varrho: C \to X \times S$ of schemes over S, where $\pi: C \to S$ is a family of nodal curves over S. A *nodal curve* (with no base scheme specified) is a nodal curve over $\operatorname{Spec} k$, and a *map of a nodal curve to* X is a map over $\operatorname{Spec} k$. Similar definitions hold for families of nodal curves (and maps) over Deligne–Mumford stacks (see [DM] for definitions). We will actually need results in this generality, but for simplicity of exposition we will prove basic results only over schemes. The arguments over Deligne–Mumford stacks are the same.

2.3. Lemma. *Suppose* $\pi: C \to S$ *is a family of nodal curves over a normal variety* S. *Let* $\nu: \widetilde{C} \to C$ *be the normalization map. Then* $\tilde{\pi}: \widetilde{C} \to S$ *is also a family of nodal curves. There is an effective Cartier divisor* $\widetilde{N}$ *on* $\widetilde{C}$ *such that* $\omega_{\widetilde{C}/S} = (\nu^* \omega_{C/S})(-\widetilde{N})$. *The support of* $\widetilde{N}$ *is the locus where* ν *is not an isomorphism (with multiplicity* 1 *along each component), and is contained in* $\mathrm{Sm}(\tilde{\pi})$.

Thus $\nu: \widetilde{C} \to C$ is a clutching morphism ([KN], Section 3). The author was unable to find this precise statement in the literature, but it is surely well known. We will use the notation $\widetilde{N}$ to simultaneously denote the Cartier divisor and the corresponding underlying scheme. We call the Weil divisor $N := \frac{1}{2}\nu(\widetilde{N})$ the *nodes of the family.*

Proof. We first show that $\tilde{\pi}$ is a family of nodal curves, and that $\widetilde{N}$ consists of smooth points of $\tilde{\pi}$. Properness is immediate, and the remaining conditions need only be checked in a formal neighborhood of closed points p of C. If $p \in \mathrm{Sm}(\pi)$, then it is a normal point of C. If p is a node of the fiber, then the complete local ring of C at p is $B \cong A[[u,v]]/(uv-h)$, where A is the complete local ring of S at $\pi(p)$ ([J], 2.23). Here $\mathfrak{m} \subset A$ is the maximal ideal corresponding to $\pi(p)$, $h \in \mathfrak{m}$, and $(x, y, \mathfrak{m})$ is the maximal ideal corresponding to p.

If $h=0$, the normalization clearly has two points, smooth above S. If $h \neq 0$, the local ring is normal. (Sketch of proof: It suffices to show that $A[u,v]/(uv-h)$ is normal, which can be rewritten as $A[x,y]/(x^2-(y^2+h))$. But y^2+h is square-free, and if A' is a normal domain and h' is square-free, then $A'[x]/(x^2-h')$ is normal by same proof as that of [Ha], Exercise II.6.4.)

All that remains is the statement about relative dualizing sheaves. Using the explicit formal-local computations above, there is an exact sequence of sheaves

$$0 \longrightarrow \mathcal{F} \longrightarrow \nu^* \Omega^1_{C/S} \longrightarrow \Omega^1_{\widetilde{C}/S} \longrightarrow 0$$

on $\widetilde{C}$, where $\mathcal{F}$ is an invertible sheaf on $\widetilde{N}$. Hence if det is the determinant functor defined in [KnM], Chapter I,

$$\det \Omega^1_{\widetilde{C}/S} = (\det \nu^* \Omega^1_{C/S}) \otimes (\det \mathcal{F})^{-1} = (\nu^* \det \Omega^1_{C/S}) \otimes (\det \mathcal{F})^{-1}.$$

As $\det \Omega^1_{X/S} = \omega_{X/S}$ for a family of nodal curves ([Kn], Section 1) and $\det \mathcal{F} = \mathcal{O}_{\widetilde{C}}(\widetilde{N})$ (as $\mathcal{F}$ is an invertible sheaf on $\widetilde{N}$ and $\widetilde{N}$ is Cartier), the result follows. $\square$

2.4. We define three different conditions on families of maps of nodal curves $\varrho: C \to \mathbf{P}^2 \times S$.

($*$) Over a dense open subset of S, the curve C is nonsingular, and ϱ factors $C \xrightarrow{\alpha} C' \xrightarrow{\varrho'} \mathbf{P}^2 \times S$ where ϱ' is unramified and gives a birational map from C' to its image; α is a degree d_α map with only simple ramification (i.e. reduced ramification divisor); and the images of the simple ramifications are distinct in $\mathbf{P}^2$. (Whenever this case is discussed, the notation $(\alpha, d_\alpha, C', \varrho')$ will be used.)

($**$) Over the normal locus (a dense open subset) of S, each component of the normalization of C (which is a family of maps of nodal curves by Lemma 2.3) satisfies ($*$).

($***$) No component of the total space C is collapsed by π.

So ($*$) implies ($**$) implies ($***$).

We say that a line L is *tangent* to a map $\varrho: C \to \mathbf{P}^2$ at a point p if $p \in \varrho^{-1} L$ and p is not a reduced point of $\varrho^{-1} L$. We say that L is *simply tangent* if L is tangent to ϱ at p, p is a nonsingular point of C and $\varrho^{-1} L$ has multiplicity exactly 2 at p. Note that the ramified points of ϱ form a closed subset, and a $\mathbf{P}^1$ of lines are tangent to each such point.

2.5. *Remark.* Suppose Y is a scheme of pure dimension d, $\pi: X \to Y$ is a proper flat morphism of relative dimension r, and $\mathcal{L}_1, \dots, \mathcal{L}_s$ are invertible sheaves on X. Then $\pi_*(c_1(\mathcal{L}_1) \cap \dots \cap c_1(\mathcal{L}_s))$ is an element of $A^{s-r}(Y)$ (see [F], Chapter 17): to intersect with a class in $A_* Y$, pull back the class to $A_* X$, intersect with $c_1(\mathcal{L}_1) \cap \dots \cap c_1(\mathcal{L}_s)$, and push forward. In terms of bivariant intersection theory, pulling back and intersecting with a product of Chern classes gives a class in $A^s(X \to Y)$, which we then pushforward to get a class in $A^{s-r}(Y \to Y) = A^{s-r}(Y)$ (see [F], Section 17.2 (P_2)). The same is true if π is a representable morphism of stacks ([Vi], Section 5).

2.6. Stable maps. A *stable map* is a map ϱ from a connected nodal curve C to $\mathbf{P}^2$ (see Section 2.2) such that ϱ has finite automorphism group. The *arithmetic genus* of a stable map is defined to be the arithmetic genus of the nodal curve C. If $[C] \in A_1(C)$ is the fundamental class of C, and $[L] \in A_1(\mathbf{P}^2)$ is the class associated to a line, then $\varrho_*[C] = d[L]$ for some nonnegative integer d. We say that d is the *degree* of the stable map.

A *family of stable maps* is a family of maps of nodal curves to $\mathbf{P}^2$ whose fibers over maximal points are stable maps. Let $\overline{\mathcal{M}}_g(\mathbf{P}^2, d)$ be the stack whose category of sections of a scheme S is the category of families of stable maps to $\mathbf{P}^2$ over S of degree d and arithmetic genus g. For definitions and basic results, see [FP]. It is a fine moduli stack of Deligne–Mumford type. There is a "universal map" over $\overline{\mathcal{M}}_g(\mathbf{P}^2, d)$ that is a family of maps of nodal curves. There is an open substack

$\mathcal{M}_g(\mathbf{P}^2,d)$ that is a fine moduli stack of maps of *nonsingular* curves.

Fix integers d and g. Let $\delta=\binom{d-1}{2}-g$. The locus in $\mathbf{P}H^0(\mathbf{P}^2,\mathcal{O}(d))$ corresponding to irreducible degree d curves with exactly δ simple nodes is an irreducible nonsingular locally closed subvariety of codimension δ, hence dimension $3d+g-1$ ([H], main theorem). There is a unique component of $\overline{\mathcal{M}}_g(\mathbf{P}^2,d)$ that is the closure of such maps (as an easy computation shows that the deformation space to any of these maps has dimension $3d+g-1$); call this component $\overline{\mathcal{M}}_g(\mathbf{P}^2,d)^+$. (If $g>0$ there are other components.) The universal map over $\overline{\mathcal{M}}_g(\mathbf{P}^2,d)^+$ satisfies $(*)$.

In fact something slightly stronger is true (although we will not need it): if $\mathcal{M}$ is an irreducible closed substack of $\overline{\mathcal{M}}_g(\mathbf{P}^2,d)^+$ whose general member corresponds to a map mapping a curve birationally onto its image, and $\dim\mathcal{M}\geq 3d+g-1$, then $\mathcal{M}=\overline{\mathcal{M}}_g(\mathbf{P}^2,d)^+$. This essentially follows from methods of [CH] (see Proposition 2.2); a proof appears (in more generality) in [V3] (Section 3).

We shall see that enumerative questions about plane curves can be usefully interpreted as intersection theory problems on $\overline{\mathcal{M}}_g(\mathbf{P}^2,d)^+$.

In general, if $\pi:\mathcal{U}\to\mathcal{M}$ is a family of maps whose general curve is nonsingular, we will call the locus in $\mathcal{M}$ where the corresponding curve is singular the *boundary* of $\mathcal{M}$, and denote it Δ. By abuse of notation, we sometimes refer to $\pi^*\Delta$ as the boundary as well.

3. Characteristic numbers of families of maps

3.1. Suppose $\varrho: C\to\mathbf{P}^2\times S$ is a family of maps of nodal curves over S, where S is a finite union of dimension d varieties. Let $\check{\mathbf{P}}^2$ be the space of lines in $\mathbf{P}^2$, and let I be the incidence correspondence $I=\{(p,L)\in\mathbf{P}^2\times\check{\mathbf{P}}^2\,|\,p\in L\}$. Let

$$D^{\mathrm{univ}}:=C\times_{\mathbf{P}^2\times S}(I\times S),$$

so we have the following diagram with two fiber squares:

$$\begin{tikzcd}
D^{\mathrm{univ}} \arrow[r] \arrow[d] & C\times\check{\mathbf{P}}^2 \arrow[r] \arrow[d] & C \arrow[d] \\
I\times S \arrow[r] & \mathbf{P}^2\times\check{\mathbf{P}}^2\times S \arrow[r] \arrow[d] & \mathbf{P}^2\times S \\
 & S\times\check{\mathbf{P}}^2. &
\end{tikzcd}$$

As I is a $\mathbf{P}^1$-bundle over $\mathbf{P}^2$, D^{univ} is a $\mathbf{P}^1$ bundle over C. Hence D^{univ} has pure dimension $d+2$ and does not contain any components of $C\times\mathbf{P}^2$. Thus, as I is a Cartier divisor on $\mathbf{P}^2\times\check{\mathbf{P}}^2$, D^{univ} is a Cartier divisor on $C\times\check{\mathbf{P}}^2$.

3.2. *Remark.* For any subvariety Q of C, the divisor D^{univ} intersects $Q\times\check{\mathbf{P}}^2$ transversely: D^{univ} intersects $Q\times\check{\mathbf{P}}^2$ properly, and the components of intersection appear with multiplicity 1. (Reason: $D^{\mathrm{univ}}\times_{C\times\check{\mathbf{P}}^2}(Q\times\check{\mathbf{P}}^2)$ is a $\mathbf{P}^1$-bundle over Q, so it is reduced of dimension $\dim Q+1$.)

3.3. The morphism $\sigma\colon D^{\mathrm{univ}}\to S\times\check{\mathbf{P}}^2$ is proper, as it factors into a sequence of proper morphisms (see the diagram above). Let Σ be the union of the dimension 1 components of fibers of σ, so Σ is a closed subset of D^{univ} ([GD], III.4.4.10 as all fibers have dimension at most 1), and consists of the components of the curves in the family mapped to a line. This includes the subset Σ_p of components of curves mapped to a point (closed by [GD], III.4.4.10 applied to the morphism $D^{\mathrm{univ}}\to\mathbf{P}^2\times S$), and the locally closed subset $\Sigma_L:=\Sigma\setminus\Sigma_p$ of components mapped surjectively to a line. As each component mapped to a line but not a point is mapped to only one line in $\check{\mathbf{P}}^2$, $\dim\Sigma_L\leq d+1$. If the family satisfies $(***)$ then $\dim\Sigma_p\leq d+1$ too.

Hence if the family satisfies $(***)$ then the codimension of $\sigma_*(\Sigma)$ in $S\times\check{\mathbf{P}}^2$ is at least 2, so the morphism σ is quasifinite outside a set of codimension 2. As σ is proper, σ is finite away from this closed subset as well ([GD], III.4.4.2). In this case, define the ramification (Weil) divisor R^{univ} to be the closure in D^{univ} of the ramification divisor of this finite map. Let $\omega^{\mathrm{univ}}:=\omega_{(C\times\check{\mathbf{P}}^2)/(S\times\check{\mathbf{P}}^2)}$.

3.4. Claim. *If the general curve in the family is nonsingular and every component of the generic curve maps with positive degree (so the family satisfies $(***)$), then in $A_{d+1}(D^{\mathrm{univ}})$, $[R^{\mathrm{univ}}]=(D^{\mathrm{univ}}+\omega^{\mathrm{univ}})[D^{\mathrm{univ}}]$. (Here $D^{\mathrm{univ}}+\omega^{\mathrm{univ}}$ is a Cartier divisor on C^{univ} which can be intersected with the class $[D^{\mathrm{univ}}]$.)*

Hence the branch divisor is in the class $\sigma_*((D^{\mathrm{univ}}+\omega^{\mathrm{univ}})|_{D^{\mathrm{univ}}})$.

Proof. It suffices to prove the claim when S is normal; in general, one can pushforward the analogous result on the family over the normalization of S (see Section 2.1). We may discard closed subsets of D^{univ} of codimension at least 2, and we use this to make simplifying assumptions about the family.

(1) The morphism σ is finite away from a codimension 2 subset of $S\times\check{\mathbf{P}}^2$, so we may assume that σ is a finite morphism.

(2) The space $S\times\check{\mathbf{P}}^2$ is regular in codimension 1, so we may assume that $S\times\check{\mathbf{P}}^2$ is regular.

(3) As C is regular in codimension 1, and D^{univ} is a $\mathbf{P}^1$-bundle over C, we may assume that D^{univ} is regular. We may also assume that D^{univ} is disjoint from the

singularities of $C\times\check{\mathbf{P}}^2\to S\times\check{\mathbf{P}}^2$.

The ramification divisor is

$$\sum_P (\mathrm{length}(\Omega^1_{D^{\mathrm{univ}}/(S\times\check{\mathbf{P}}^2)})_P)P,$$

where the summation is over the associated primes P of $\Omega^1_{D^{\mathrm{univ}}/(S\times\check{\mathbf{P}}^2)}$. If $\mathcal{I}$ is the ideal sheaf of D^{univ} on $C\times\check{\mathbf{P}}^2$, then in the exact sequence

$$0\longrightarrow\mathcal{I}/\mathcal{I}^2\longrightarrow\Omega^1_{(C\times\check{\mathbf{P}}^2)/(S\times\check{\mathbf{P}}^2)}\otimes\mathcal{O}_{D^{\mathrm{univ}}}\longrightarrow\Omega^1_{D^{\mathrm{univ}}/(S\times\check{\mathbf{P}}^2)}\longrightarrow 0$$

([Ha], Theorem II.8.17) the first two terms are locally free on D^{univ} of rank 1, and $\mathcal{I}/\mathcal{I}^2\cong\mathcal{O}_{C\times\check{\mathbf{P}}^2}(-D^{\mathrm{univ}})\otimes\mathcal{O}_{D^{\mathrm{univ}}}$. Thus

$$c_1(\Omega^1_{D^{\mathrm{univ}}/(S\times\check{\mathbf{P}}^2)})=c_1(\Omega^1_{(C\times\check{\mathbf{P}}^2)/(S\times\check{\mathbf{P}}^2)}(D^{\mathrm{univ}})\otimes\mathcal{O}_{D^{\mathrm{univ}}}).$$

But $\Omega^1_{(C\times\check{\mathbf{P}}^2)/(S\times\check{\mathbf{P}}^2)}\cong\omega^{\mathrm{univ}}$ on $C^{\mathrm{reg}}\times\check{\mathbf{P}}^2$ by [Kn], Section 1, so the claim follows. □

3.5. *The support of* R^{univ}. Assume the family satisfies $(*)$ (and hence the hypotheses of the previous claim). Recall that $\dim R^{\mathrm{univ}}=d+1$, and R^{univ} is contained in D^{univ}, a $\mathbf{P}^1$-bundle over C. A component R of R^{univ} is of one of two forms. If R maps to $\mathrm{ram}(\varrho)$, then as $\dim\mathrm{ram}(\varrho)\le d$ (and equality holds only for components surjecting onto S), R must be a $\mathbf{P}^1$-bundle over a component of $\mathrm{Sing}(\pi)$ (and R surjects onto S). Loosely speaking, this is a locus where a ramified point of the general curve maps to the universal line (corresponding to the point in $\check{\mathbf{P}}^2$).

Otherwise, there is a morphism from the unramified points of ϱ to $C\times\check{\mathbf{P}}^2$ (sending each point to its tangent line), and the image is an open subvariety of R^{univ} (of dimension $d+1$). Loosely speaking, a component R of R^{univ} mapping to this locus corresponds geometrically to points of tangency from unramified points of the general curve to the universal line. Again, R surjects onto S.

3.6. Claim. (a) *If the family satisfies* $(*)$, *then* R^{univ} *consists of the divisorial components of the closures of the sets*

(i) *ramification points of* α *mapping to the universal line, and*

(ii) *unramified points of maps tangent to the universal line,*

with multiplicity 1.

(b) *In* (i), *"ramification" may be replaced by "simple ramification". In* (ii), *"tangent" may be replaced by "simply tangent".*

(c) *Furthermore,* $\sigma_*R^{\mathrm{univ}}$ *consists of the divisorial components of the closures of the sets*

(i) *maps where a simple ramification of* α *maps to the universal line, and*

(ii) *maps where the image is simply tangent to the universal line.*

Divisors of the first type appear with multiplicity 1, *and divisors of the second type appear with multiplicity* d_α.

Proof. By 3.5, (a) is true set theoretically. By Sard's theorem, we can check the multiplicities by looking over a general point of S (as each component surjects onto S, and the constructions all commute with base change). (See [K2], p. 6, for a discussion of applications of this variant of Sard's theorem to enumerative geometry.) Thus it suffices to prove the result when S is a closed point, so we have reduced to the case of a single map, and the study of the ramification divisor of the morphism from a $\mathbf{P}^1$-bundle over C to $\mathbf{P}^2$.

We first show the result for the unramified map $C' \to \mathbf{P}^2$ (or equivalently, the case $d_\alpha = 1$).

Let $(D^{\mathrm{univ}})' \subset C \times \check{\mathbf{P}}^2$ be the $\mathbf{P}^1$-bundle over C' (defined similarly to D^{univ}). It is simple to show that if $a: X \to Y$ is a finite morphism of nonsingular varieties, $p \in X$ is a general point on a component of the ramification divisor, and $\alpha^{-1}(\alpha(p))$ is a local Artinian scheme of length 2 at p, then the component appears with multiplicity 1. In this case, if $[L] \in \check{\mathbf{P}}^2$, then the pullback of $[L]$ to $(D^{\mathrm{univ}})'$ is isomorphic to the pullback of L to C'. A fundamental fact of duality theory of curves is that the curve $\varrho'(C')$ (or any other reduced curve in $\mathbf{P}^2$) has a finite number of bitangents and flexes, i.e. the map ϱ' has only a finite number of tangencies that are not simple tangencies (see [K3] for a comprehensive survey). Thus if $[L]$ is a general point of the (one-dimensional) branch divisor of $(D^{\mathrm{univ}})' \to \check{\mathbf{P}}^2$, then L is simply tangent to $\varrho(C')$ at exactly one point (and transverse at the rest), so the ramification divisor (and branch divisor) indeed appears with multiplicity 1. This proves the claim for the family $C' \to \mathbf{P}^2$.

Phrased differently, the branch divisor of $(D^{\mathrm{univ}})' \to \check{\mathbf{P}}^2$ is the dual curve to $\varrho(C')$, with multiplicity 1, and the ramification divisor is the set of $(p, L) \in C' \times \check{\mathbf{P}}^2$, where L is the tangent line to $\varrho': C' \to \mathbf{P}^2$ at p. From the fiber square

$$\begin{tikzcd}
D^{\mathrm{univ}} \arrow[r, "\mathbf{P}^1\text{-bundle}"] \arrow[d] & C \arrow[d, "\alpha"] \\
(D^{\mathrm{univ}})' \arrow[r, "\mathbf{P}^1\text{-bundle}"] & C',
\end{tikzcd}$$

the ramification divisor of the left vertical arrow is the pullback of the ramification of α, simply by definition of $(*)$. Thus the ramification of the map $D^{\mathrm{univ}} \to \check{\mathbf{P}}^2$ is as described in (a).

In the course of proving (a), we saw the behavior of the general points of the components of R^{univ} and $\sigma_* R^{\mathrm{univ}}$, so (b) and (c) are also clear. $\square$

3.7. Next assume that S is normal, and that the family satisfies (**). Let $\widetilde{C}$ be the normalization of C (so $\widetilde{C}\to S$ is a family of nodal curves by Lemma 2.3). Let $\widetilde{N}$ (resp. N) be the locus on $\widetilde{C}$ (resp. C) described in Lemma 2.3 (the "branches of the normalization", resp. the "nodes of the family"). Define D^{univ}, ω^{univ}, R^{univ} as above, and $\widetilde{D}^{\mathrm{univ}}$, $\widetilde{\omega}^{\mathrm{univ}}$, $\widetilde{R}^{\mathrm{univ}}$ as the analogous constructions for the family $\widetilde{C}\to S$. Let ν be the normalization $\widetilde{C}\times\check{\mathbf{P}}^2\to C\times\check{\mathbf{P}}^2$.

3.8. Claim. *In* $A_*(C\times\check{\mathbf{P}}^2)$,

$$D^{\mathrm{univ}}\cdot(D^{\mathrm{univ}}+\omega^{\mathrm{univ}})=\nu_*(\widetilde{R}^{\mathrm{univ}}+(\widetilde{N}\times\check{\mathbf{P}}^2)\cdot\widetilde{D}^{\mathrm{univ}})=\nu_*(\widetilde{R}^{\mathrm{univ}})+D^{\mathrm{univ}}\cdot(N\times\check{\mathbf{P}}^2).$$

It should certainly be true that $R^{\mathrm{univ}}=\nu_*(\widetilde{R}^{\mathrm{univ}}+(\widetilde{N}\times\check{\mathbf{P}}^2)\cdot\widetilde{D}^{\mathrm{univ}})$ as cycles, but we will not need that here. It is useful to think of this claim geometrically (but sloppily) as "the divisor where a family is tangent to a line is the divisor where the normalization of the family is tangent to the line, plus twice the divisor where a node of the family is mapped to the line".

Proof. On $\widetilde{C}\times\check{\mathbf{P}}^2$, $\widetilde{D}^{\mathrm{univ}}=\nu^*D^{\mathrm{univ}}$, so

$$\widetilde{R}^{\mathrm{univ}}=\widetilde{D}^{\mathrm{univ}}\cdot(\widetilde{D}^{\mathrm{univ}}+\widetilde{\omega}^{\mathrm{univ}})=\nu^*D^{\mathrm{univ}}\cdot(\nu^*D^{\mathrm{univ}}+\nu^*\omega^{\mathrm{univ}})-(\widetilde{N}\times\check{\mathbf{P}}^2)\cdot\widetilde{D}^{\mathrm{univ}}$$

by Lemma 2.3. Pushing forward by the (finite degree 1) morphism ν gives us the first equality. The second equality follows from the pushpull formula

$$\widetilde{D}^{\mathrm{univ}}\cdot(\widetilde{N}\times\check{\mathbf{P}}^2)=(\nu^*D^{\mathrm{univ}})\cdot(\widetilde{N}\times\check{\mathbf{P}}^2)=D^{\mathrm{univ}}\cdot\nu_*(\widetilde{N}\times\check{\mathbf{P}}^2)=2D^{\mathrm{univ}}\cdot(N\times\check{\mathbf{P}}^2). \quad\square$$

3.9. *Remark.* With the same hypotheses as above, if Q is any subvariety of S, then $\sigma_*(R^{\mathrm{univ}})$ does not contain $Q\times\check{\mathbf{P}}^2$. (As no map is tangent to all lines in $\mathbf{P}^2$, the result is clearly true even if Q is a point.)

3.10. *Incidence and tangency divisors on a family of maps.* For the rest of this section, let $\varrho\colon C\to\mathbf{P}^2\times S$ be a family of maps of nodal curves over some equidimensional reduced base S, satisfying (**). For each line L in $\mathbf{P}^2$ let D_L be the closed subscheme $\varrho^{-1}L$ on C. Let $\omega:=\omega_{C/S}$, and $\mathcal{D}:=\varrho^*\mathcal{O}_{\mathbf{P}^2}(1)$. We will occasionally use ω and $\mathcal{D}$ to also denote their classes in the Chow group.

Let $\alpha:=\pi_*(\mathcal{D}^2)$ and $\beta:=\pi_*(\mathcal{D}\cdot(\mathcal{D}+\omega))$. (In the language of Harris–Morrison's "standard conjecture for the Hilbert scheme" [HM], p. 64, these divisors are A and $A+B$.) The Weil divisor α will be "the divisor of maps through a fixed general point", and the Weil divisor β will be "the divisor of maps simply tangent to a fixed general line." By Remark 2.5, $\alpha,\beta\in A^1S$ (in the operational Chow ring).

3.11. Lemma. *Fix subvarieties Q of C and Q' of S. If $[L]$ is general in $\check{\mathbf{P}}^2$, then*

(i) *D_L is a Cartier divisor on C in class $\mathcal{D}$, and the morphism $D_L \to S$ is quasi-finite in codimension 1;*
thus we can define a ramification divisor R_L as in 3.3, and a branch divisor $B_L := \pi_* R_L$;

(ii) *D_L intersects Q properly, with multiplicity 1 along each component of the intersection;*

(iii) *B_L is the class $\pi_*(\mathcal{D} \cdot (\mathcal{D} + \omega))$; B_L consists of divisorial components of the closure of the locus of maps simply tangent to L, and maps with nodes of the family on L, with multiplicities as described in Claims 3.6 and 3.8; B_L does not contain Q'.*

Proof. Use Kleiman–Bertini ([Ha], III.10.8) on $D^{\text{univ}} \to \check{\mathbf{P}}^2 \times S$ (with group PGL(2)). Use Remark 3.2 and 3.3 for (a), Remark 3.2 for (b), and 3.4–3.9 for (c). $\square$

3.12. Lemma. *If p is general in $\mathbf{P}^2$, then the union of the maximal points of $\varrho^{-1}(p)$ is in class $\mathcal{D}^2$ in $A_{d-1}(C)$.*

Proof. As sets, $\varrho^{-1}(p)$ is the intersection of D_L and D_M where L and M are general lines. By Lemma 3.11, each component of D_L appears with multiplicity 1. If $\{D_L^i\}_i$ are the components of D_L, then by Lemma 3.11 (applied to $Q = D_L^i$), each component of $(\varrho|_{D_L^i})^{-1} M$ appears with multiplicity 1 on D_L^i. Finally, D_M does not contain any component of $D_L^i \cap D_L^j$ (also by Lemma 3.11, taking Q to be any component of $D_L^i \cap D_L^j$). $\square$

3.13. *Remark.* Hence we can interpret α as follows. Fix a general point $p \in \mathbf{P}^2$, and for each component of the generic curve C not mapped to a point, associate the locus where p lies on the image of this component; this is a Weil divisor on S. Associate to this Weil divisor a multiplicity equal to the degree of the map of the component of C onto its image. The formal sum α_p of these divisors (with these multiplicities) is in class α.

We can also interpret β geometrically. Fix a general line $L \subset \mathbf{P}^2$. To each component of the normalization of C (satisfying $(*)$) we associate the locus where the map $C' \to \mathbf{P}^2$ is tangent to L; this is a Weil divisor on S. Assign a multiplicity of d_α to this divisor. To each component of the normalization of C satisfying $(*)$ we also associate the locus where a ramification point of α maps to L, with multiplicity 1. To each node of the family we associate the locus where the node is mapped to L. This is a Weil divisor on S; assign a multiplicity of 2 to it. Then the formal sum β_L of these divisors (with these multiplicities) is in class β.

Moreover, if Q is any subvariety of S (distinct from S), Q is not contained in any component of α_p or β_L on S (for p and L general). Consequently, if the general map in S does not satisfy a closed condition (e.g. cuspidal, tacnodal, or with a node on a fixed line), then neither does the general map in any component of α_p or β_L. In particular, each component in α_p and β_L also satisfies (**).

3.14. We now come to the main result of this section. Suppose $C \to \mathbf{P}^2 \times S$ is a family of maps satisfying (**), and S is a Deligne–Mumford stack. Let the components of C be called $\{C_i\}_{i=1}^{s}$, with α_i and d_{α_i} as defined in 2.4. Let $N_1, \dots, N_t$ be the components of the nodes of the family (see Lemma 2.3)). Fix a general points $p_1, \dots, p_a$ and b general lines $L_1, \dots, L_b$ in $\mathbf{P}^2$.

To each point p_i associate a component $C_{w(i)}$. Partition $\{1, \dots, b\}$ into three sets $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$. To each $j \in \mathcal{L}_1$ associate a component $C_{x(j)}$. To each $j \in \mathcal{L}_2$ associate a component $C_{y(j)}$. To each $j \in \mathcal{L}_3$ associate a component of the nodes of the family $N_{z(j)}$. Consider the following cycle on S, of codimension $a+b$, that is the closure of the subset of $S(k)$ corresponding to maps such that

(i) for each $1 \leq i \leq a$, the image of $C_{w(i)}$ passes through p_i;
(ii) for each $j \in \mathcal{L}_1$, the map restricted to $C_{x(i)}$ is simply tangent to L_j;
(iii) for each $j \in \mathcal{L}_2$, the image of a ramification of $\alpha_{y(j)}$ passes through L_j;
(iv) for each $j \in \mathcal{L}_3$, the image of $N_{z(j)}$ lies on L_j.

(Note that this cycle is empty if more than two lines in $\mathcal{L}_3$ are associated to the same N_k.) To this cycle, associate the multiplicity

$$\left(\prod_{i=1}^{a} d_{\alpha_{w(i)}}\right)\left(\prod_{j \in \mathcal{L}_1} d_{\alpha_{x(j)}}\right) 2^{\#\mathcal{L}(3)}.$$

Let Q be the cycle that is the sum over all choices above of the cycles described above, with multiplicity.

3.15. Theorem. *In $A_{d-a-b}S$, $[Q]=\alpha^a \beta^b [S]$. The components of Q described above are all distinct.*

For a specific example, see Remark 4.6.

Proof. Use induction on $a+b$; the case $a=b=0$ is clear. Assume the result for a fixed $a=a_0$, $b=b_0$, so we have distinct cycles Q_i with multiplicities m_i given by the inductive hypothesis, with $\sum_i m_i [Q_i] = \alpha^{a_0} \beta^{b_0} [S]$. Using Remark 3.13, it is straightforward to verify that the theorem holds for $(a,b)=(a_0+1, b_0)$ (resp. (a_0, b_0+1)), by expressing $\alpha[Q_i]$ (resp. $\beta[Q_i]$) as $\sum_j m_{ij} [Q_{ij}]$. By Remark 3.13, Q_{ij} does not contain $Q_i \cap Q_{i'}$, so $Q_{ij} \neq Q_{i'j'}$ for $(i,j) \neq (i',j')$; this shows inductively that the components of Q described above are distinct. □

Note that if the family satisfies $(*)$, then we may ignore $\mathcal{L}_3$. If furthermore the general map gives a birational isomorphism from the curve to its image, then we may also ignore $\mathcal{L}_2$.

3.16. *Characteristic numbers of families of maps.* If $\varrho: C \to \mathbf{P}^2 \times S$ is a family of maps of nodal curves to $\mathbf{P}^2$ over S, where S is a finite union of varieties of dimension d, then we say that $\alpha^a \beta^b [S]$ $(a+b=d)$ are the *characteristic numbers* of the family of maps. If the family satisfies $(**)$, then these numbers can be interpreted enumeratively using Theorem 3.15, as counting maps (with multiplicity). This construction carries through even if S is a Deligne–Mumford stack (and $\varrho: C \to S$ is a family of nodal curves over a Deligne–Mumford stack, see [DM] for a definition).

The classical *characteristic number problem* for curves in $\mathbf{P}^2$ (studied by, e.g., Chasles, Zeuthen, Schubert) is: *how many irreducible nodal degree d geometric genus g maps are there through a general points, and tangent to b general lines (if $a+b=3d+g-1$)?* (The classical phrasing was somewhat different.) By 2.6, and Theorem 3.15, this number is $\alpha^a \beta^b$ on $\overline{\mathcal{M}}_g(\mathbf{P}^2, d)^+$.

3.17. *Generalizations.* The obvious generalizations to maps to $\mathbf{P}^n$ and with marked points (which will not be needed here) are also true: the arguments are identical. The argument for maps to $\mathbf{P}^1$ (needed in [V1], see also Section 5.11) requires no change: just consider maps to a fixed line in $\mathbf{P}^2$.

4. Genus 0 and 1 facts

In this section, we review known facts about stable maps to $\mathbf{P}^2$.

4.1. *Genus* 0 (*from* [FP]). The Deligne–Mumford stack $\overline{\mathcal{M}}_0(\mathbf{P}^2, d)$ is nonsingular of dimension $3d-1$ ([FP], Section 0.4). The boundary Δ is the union of divisors $\Delta_{0,j}$ where $\Delta_{0,j}$ generically corresponds to a map of two genus 0 curves, joined at a node, one mapped with degree j and the other with degree $d-j$.

4.2. *Genus* 1 (*from* [V4]). Let $\overline{\mathcal{M}}_1(\mathbf{P}^2, d)^*$ be the closure of points in the stack $\overline{\mathcal{M}}_1(\mathbf{P}^2, d)$ corresponding to maps that do not contract a genus 1 union of components. Then $\overline{\mathcal{M}}_1(\mathbf{P}^2, d)^*$ is irreducible of dimension $3d$. The boundary Δ is the union of

(1) divisors $\Delta_{0,j}$ $(j>0)$, where $\Delta_{0,j}$ generically corresponds to a map of a genus 1 curve E and a genus 0 curve R joined at a node, where R is mapped with degree j and E is mapped with degree $d-j$;

(2) the divisor Δ_0 of maps from rational (nodal) curves;

and possibly three more ([V4], Lemma 5.9);

(3) points corresponding to cuspidal rational curves with a contracted elliptic tail;

(4) points corresponding to a contracted elliptic component attached to two rational components, where the images of the rational components meet at a tacnode;

(5) points corresponding to contracted elliptic components attached to three rational components.

Let Δ_{ei} be the union of divisors of type (3)–(5) above.

4.3. Claim. *If Ξ is an irreducible component of Δ_{ei} and $a+b=3d-1$, then in $A_0(\overline{\mathcal{M}}_1(\mathbf{P}^2,d)^*)$, $\alpha^a\beta^b[\Xi]=0$.*

Proof. We prove the result if Ξ is of type (3). Choose a general points and b general lines, and let Q be the 1-dimensional cycle in $\overline{\mathcal{M}}_1(\mathbf{P}^2,d)^*$ described in 3.14, so $[Q]=\alpha^a\beta^b[\overline{\mathcal{M}}_1(\mathbf{P}^2,d)^*]$. Then Q is the set of points corresponding to maps through the a points and tangent to the b lines. We will see that Q is disjoint from Ξ, which will imply the claim.

If $\mathcal{U}\to\Xi$ is the restriction of the universal family to Ξ, let $i:\mathcal{U}'\hookrightarrow\mathcal{U}$ be the component of $\mathcal{U}$ that is the union of the noncontracted genus 0 curves. Then $\varrho\circ i$ is a stable map (of genus 0 curves to $\mathbf{P}^2$) over Ξ, inducing a morphism $f:\Xi\to\overline{\mathcal{M}}_0(\mathbf{P}^2,d)$ whose general fiber (above the image $f(\Xi)$) is 1-dimensional, corresponding to the j-invariant of the elliptic tail. Thus $\dim(f(\Xi))=\dim(\Xi)-1=a+b-1$. There are no maps in $f(\Xi)$ through the a points and tangent to the b lines, for dimensional reasons. But it is easily checked that a map $m\in\Xi$ passes through a point (resp. is tangent to a line) if the map $f(m)$ does, so $Q\cap\Xi=\emptyset$.

The arguments for types (4) and (5) are similar, and will only be sketched. For type (4) divisors, construct the auxiliary family $\mathcal{U}'$ by discarding the contacted genus 1 component and gluing the two genus 0 components together along a node. For type (5), discard the contracted genus 1 component and glue two of the three genus 0 components together along a node; this may require a finite cover. □

For this reason, the components of Δ_{ei} will not contribute enumeratively, so we call them *enumeratively irrelevant* boundary divisors.

Now let $\mathcal{U}$ be the universal curve over $\overline{\mathcal{M}}_1(\mathbf{P}^2,d)^*$, so $\pi:\mathcal{U}\to\overline{\mathcal{M}}_1(\mathbf{P}^2,d)^*$ is a family of nodal genus 1 curves. Then $\overline{\mathcal{M}}_1(\mathbf{P}^2,d)^*\setminus\Delta_{ei}$ is nonsingular away from the enumeratively irrelevant divisors, and the total space $\pi^*(\overline{\mathcal{M}}_1(\mathbf{P}^2,d)^*\setminus\Delta_{ei})$ is nonsingular ([V4], Lemma 4.21 and Proposition 5.5). Away from the codimension 2 subset where boundary divisors intersect (call it S), π is a family of "curves with at most one rational tail". Let R be the closure in $\mathcal{U}$ of the points on rational tails. Then R is a Weil divisor supported over the boundary.

4.4. Claim. *We have, modulo enumeratively irrelevant divisors and torsion, $\omega_{\mathcal{U}/\overline{\mathcal{M}}_1(\mathbf{P}^2,d)^*}=\frac{1}{12}[\Delta_0]+[R]$ as Weil divisor classes.*

Proof. It suffices to prove the result above the open set $\mathcal{M}:=\overline{\mathcal{M}}_1(\mathbf{P}^2,d)^*\setminus(S\cup\Delta_{ei})$. There is a contraction morphism ([Kn])

$$\begin{tikzcd} c\colon \mathcal{U}\times_{\overline{\mathcal{M}}_1(\mathbf{P}^2,d)^*}\mathcal{M} \arrow[rr] \arrow[dr] & & \mathcal{U}' \arrow[dl] \\ & \mathcal{M} & \end{tikzcd}$$

contracting R (take the relatively minimal model of the genus 1 fibration).

As (the total space of) $\mathcal{U}$ is nonsingular above $\mathcal{M}$, a straightforward local calculation shows that the contraction is a blow-up of the image of R along a nonsingular locus, so $c^*\omega_{\mathcal{U}'/\mathcal{M}}(R)=\omega_{\mathcal{U}\times_{\overline{\mathcal{M}}_1(\mathbf{P}^2,d)^*}\mathcal{M}}$. The following lemma shows that $\omega_{\mathcal{U}'/\mathcal{M}}=\frac{1}{12}[\Delta_0]$ modulo torsion, so we are done. $\square$

4.5. Lemma. *Suppose $f\colon\mathcal{U}'\to\mathcal{M}$ is a morphism of nonsingular Deligne–Mumford stacks, and f is a relatively minimal elliptic fibration. Let Δ_0 be the locus of nodal fibers. Then $\omega_{\mathcal{U}'/\mathcal{M}}=\frac{1}{12}[\Delta_0]$ modulo torsion.*

This lemma is implied by the statement $\langle\tau_1\rangle_1=\frac{1}{24}$ on $\overline{\mathcal{M}}_{1,1}$ (see [HM], Exercise 2.58).

4.6. *Remark.* Suppose Ψ is the locus Δ_0 or $\Delta_{0,j}$ in $\overline{\mathcal{M}}_1(\mathbf{P}^2,d)^*$, or the locus $\Delta_{0,j}$ in $\overline{\mathcal{M}}_0(\mathbf{P}^2,d)$. Fix a general points and b general lines, where $a+b=\dim\Psi$. Then by Theorem 3.15, the degree of $\alpha^a\beta^b[\Psi]$ is equal to the number of maps where the map from the normalization passes through the a points and is tangent to the b lines; plus twice the number where the node maps to one of the b lines, and the curve passes through the a points and is tangent to the remaining $b-1$ lines; plus four times the number where the node maps to the intersection of two of the b lines, and the curve passes through the a points and is tangent to the remaining $b-2$ lines.

4.7. *Maps to $\mathbf{P}^n$.* Almost all of the results of Sections 2–4 about maps of curves to $\mathbf{P}^2$ carry over essentially without change to maps to $\mathbf{P}^n$. There are only two additional comments worth making. (1) For $1<j\leq n$, there are classes $\alpha_j\in A^{j-1}(S)$ corresponding to maps intersecting codimension j linear spaces (so $\alpha=\alpha_2$ when $n=2$). All analogous transversality results to α hold. (2) In the genus 1 case, there are (potentially) $n+1$ enumeratively irrelevant boundary divisors.

5. Genus 0 and 1 recursions

5.1. Let $R_d(a,b)$ be the number of irreducible degree d rational curves through a fixed general points and tangent to b fixed general lines if $a+b=3d-1$, and 0 otherwise. By 3.16, this is $\alpha^a\beta^b$ on $\overline{\mathcal{M}}_0(\mathbf{P}^2,d)$. Let $R_d:=R_d(3d-1,0)$ be the number with no tangency conditions. Let $NL_d(a,b)$ be the number of irreducible degree d rational curves through a fixed general points and tangent to b fixed general lines and with a node of the image on a fixed line if $a+b=3d-2$, and 0 otherwise. By [DH], (1.4) and (1.5),

$$NL_d(a,b)=(d-1)R_d(a+1,b)-\tfrac{1}{2}R_d(a,b+1)=\big((d-1)\alpha-\tfrac{1}{2}\beta\big)\alpha^a\beta^b. \tag{1}$$

Let $NP(a,b)$ be the number of irreducible degree d rational curves through a fixed general points and tangent to b fixed general lines and with a node of the image at a fixed point if $a+b=3d-3$, and 0 otherwise. Let $NP_d:=NP_d(3d-3,0)$ be the number with no tangency conditions.

5.2. If $d\geq 2$, let $E_d(a,b)$ be the degree of $\alpha^a\beta^b[\overline{\mathcal{M}}_1(\mathbf{P}^2,d)^*]$ if $a+b=3d$, and 0 otherwise. By 3.16, if $d>2$, $E_d(a,b)$ is the number of irreducible degree d elliptic curves through a fixed general points and tangent to b fixed general lines. Let $E_d:=E_d(3d,0)$.

If $d=2$, $E_d(a,b)$ still has enumerative meaning. Fix a general points and b general lines. Then $E_d(a,b)$ is the number of double covers of a line in the plane by a genus 1 curve with a marked points on the curve mapping to the a fixed points, and with ramifications of the double cover mapping to the b general lines, divided by the order of the automorphism group of such a map. (Recall that the degree of a dimension 0 cycle on a proper Deligne–Mumford stack over k is the degree of the pushforward to $\operatorname{Spec} k$, and may be fractional; see [Vi], Section 1.) Thus $E_2(2,4)=2$, $E_2(1,5)=10$, $E_2(0,6)=\frac{45}{2}$, and $E_2(a,b)=0$ otherwise.

5.3. *Incidences only.* Kontsevich's beautiful recursion ([KM], Claim 5.2.1, or [RT]) computes R_d inductively,

$$R_d=\sum_{i+j=d} i^2 j\left(j\binom{3d-4}{3i-2}-i\binom{3d-4}{3i-1}\right)R_iR_j. \tag{2}$$

One proof involves studying rational curves through $3d-2$ fixed points, two of which are marked p and q, and two marked points r and s on fixed general lines, and pulling back an equivalence on $\operatorname{Pic}\overline{\mathcal{M}}_{0,4}$. The same "cross-ratio" trick gives a recursion for NP_d,

$$\begin{aligned} NP_d = &\sum_{i+j=d} (ij-1)i\left(j\binom{3d-6}{3i-3}-i\binom{3d-6}{3i-2}\right)R_iR_j \\ &+\sum_{i+j=d} ij\left(2ij\binom{3d-6}{3i-4}-i^2\binom{3d-6}{3i-3}-j^2\binom{3d-6}{3i-5}\right)NP_iR_j. \end{aligned} \tag{3}$$

This formula can also be interpreted as a consequence of the Witten–Dijkgraaf–Verlinde–Verlinde equation on $\mathbf{F}_1$; see [KM], Section 5. Pandharipande gives another recursion for NP_d in [P2], Section 3.4. The Eguchi–Hori–Xiong formula (proved by Pandharipande in [P5] and Dubrovin and Zhang in [DZ] using Getzler's relation) gives E_d,

$$E_d = \frac{1}{12}\binom{d}{3}R_d + \sum_{i+j=d}\frac{ij(3i-2)}{9}\binom{3d-1}{3j}R_iE_j. \tag{4}$$

Remarkably, there is still no purely geometric proof known of this result.

5.4. *Swapping incidences for tangencies, genus* 0. From [P4], Lemma 2.3.1, in $\operatorname{Pic}(\overline{\mathcal{M}}_0(\mathbf{P}^2,d))\otimes\mathbf{Q}$,

$$\beta = \frac{d-1}{d}\alpha + \sum_{j=0}^{[d/2]}\frac{j(d-j)}{d}\delta_{0,j}. \tag{5}$$

Intersect this relation with $\alpha^a\beta^b$, where $a+b=3d-2$ (or equivalently, apply this rational equivalence to the one parameter family corresponding to degree d rational curves through a general points and tangent to b general lines) to get

$$\begin{aligned} R_d(a,b+1) = \frac{d-1}{d}R_d(a+1,b) + \sum_{i+j=d}\frac{ij}{2d}\Bigg[&\sum_{\substack{a_i+a_j=a\\ b_i+b_j=b}}\binom{a}{a_i}\binom{b}{b_i}ijR_i(a_i,b_i)R_j(a_j,b_j) \\ &+4b\sum_{\substack{a_i+a_j=a+1\\ b_i+b_j=b-1}}\binom{a}{a_i}\binom{b-1}{b_j}iR_i(a_i,b_i)R_j(a_j,b_j) \\ &+4\binom{b}{2}\sum_{\substack{a_i+a_j=a+2\\ b_i+b_j=b-2}}\binom{a}{a_i-1}\binom{b-2}{b_j}R_i(a_i,b_i)R_j(a_j,b_j)\Bigg]. \end{aligned}$$

In each sum, it is assumed that $i,j>0$; $a_i,a_j,b_i,b_j\geq 0$; $a_i+b_i=3i-1$; $a_j+b_j=3j-1$; and that all of these are integers. The large bracket corresponds to maps from reducible curves. The first sum in the large bracket corresponds to the case where no tangent lines pass through the image of the node; the second sum corresponds to when one tangent line passes through the image of the node; and the third to when two tangent lines pass through the image of the node (see Remark 4.6). Note that in the second sum, $3i-1$ of the $a+b$ conditions fix the component corresponding to R_i (up to a finite number of possibilities). The component corresponding to R_j is then specified by the remaining $3j-2$ conditions, plus the condition that it intersect the other component on a fixed line.

This completes the computation of the characteristic numbers for rational plane curves.

5.5. *Remark.* Pandharipande [P3] earlier obtained (by topological recursion methods and descendants) what can be seen to be the same recursion in the form of a differential equation. If

$$R(x,y,z)=\sum_{a,b,d} R_d(a,b)\frac{x^a}{a!}\frac{y^b}{b!}e^{dz},$$

then

$$R_{yz}=-R_x+R_{xz}-\tfrac{1}{2}R_{zz}^2+(R_{zz}+yR_{xz})^2.$$

(Ernström and Kennedy [EK1], [EK2] showed that the genus 0 characteristic numbers are encoded in a deformed quantum cohomology ring, the *contact cohomology ring.*)

5.6. *Swapping incidences for tangencies, the family NP.* A similar argument applied to the one-parameter family corresponding to degree d rational curves with a node at a fixed point, through a general points and tangent to b general lines (where $a+b=3d-4$) gives the formula shown in Appendix A. The corresponding differential equation is

$$\begin{aligned}NP_{yz}=&-NP_x+NP_{xz}-\tfrac{1}{2}R_{zzx}^2+(R_{zzx}+yR_{zxx})^2\\&+2(R_{zz}+yR_{zx})(NP_{zz}+yNP_{zx})-R_{zz}NP_{zz}.\end{aligned}$$

5.7. *Swapping incidences for tangencies, genus* 1. As $\omega\cong\frac{1}{12}Q+R$ (Claim 4.4), $\beta-\alpha=\pi_*(\mathcal{D}\cdot\omega)=\frac{1}{12}d[\Delta_0]+\sum_i i[\Delta_{0,i}]$, so as Weil divisors,

$$\beta=\alpha+\frac{d}{12}[\Delta_0]+\sum_i i[\Delta_{0,i}]. \tag{6}$$

Restricting this identity to the one parameter family corresponding to degree d elliptic curves through a general points and tangent to b general lines (where $a+b=3d-1$) gives

$$\begin{aligned}
E_d(a,b+1) &= E_d(a+1,b) \\
&\quad + \frac{d}{12}\left(\binom{d-1}{2}R_d(a,b)+2bNL_d(a,b-1)+4\binom{b}{2}NP_d(a,b-2)\right) \\
&\quad + \sum_{i+j=d} i\Bigg[\sum_{\substack{a_i+a_j=a\\ b_i+b_j=b}}\binom{a}{a_i}\binom{b}{b_i}ijR_i(a_i,b_i)E_j(a_j,b_j) \\
&\quad +2b\Bigg(\sum_{\substack{a_i+a_j=a+1\\ b_i+b_j=b-1}}\binom{a}{a_j}\binom{b-1}{b_i}jR_i(a_i,b_i)E_j(a_j,b_j) \\
&\quad + \sum_{\substack{a_i+a_j=a+1\\ b_i+b_j=b-1}}\binom{a}{a_i}\binom{b-1}{b_i}iR_i(a_i,b_i)E_j(a_j,b_j)\Bigg) \\
&\quad +4\binom{b}{2}\sum_{\substack{a_i+a_j=a+2\\ b_i+b_j=b-2}}\binom{a}{a_i-1}\binom{b-2}{b_i}R_i(a_i,b_i)E_j(a_j,b_j)\Bigg].
\end{aligned}$$

Using (1), $NL_d(a,b-1)$ can be found. The large square bracket corresponds to maps of reducible curves. The first sum corresponds to the case when no tangent line passes through the image of the node, the next two sums correspond to when one tangent line passes through the image of the node, and the last sum corresponds to when two tangent lines pass through the image of the node.

The corresponding differential equation is

$$E_y = E_x+\Psi+2(R_{zz}+R_{zx})(E_z+E_x)-R_{zz}E_z,$$

where

$$\Psi = \tfrac{1}{12}\left(\tfrac{1}{2}(R_{zzz}-3R_{zz}+2R_z)+2yNL_z+2y^2NP_z\right).$$

This completes the computation of the characteristic numbers of elliptic plane curves.

5.8. *Characteristic numbers of elliptic curves with fixed j-invariant ($j\neq\infty$).* Let M_j be the Weil divisor on $\overline{\mathcal{M}}_1(\mathbf{P}^2,d)^*$ corresponding to curves whose stable model has fixed j-invariant j. Then $M_j\cong M_\infty$ if $j\neq 0,1728$, $M_0\cong\frac{1}{3}M_\infty$, and $M_{1728}\cong\frac{1}{2}M_\infty$ ([P1], Lemma 4). If $a+b=3d-1$, define $J_d(a,b):=M_\infty\alpha^a\beta^b$. Then if $d\geq 3$, the characteristic numbers of curves with fixed j-invariant, $j \neq 0, 1728, \infty$, are given by $J_d(a,b)$, and if $j=0$ or $j=1728$, then the characteristic numbers are one third and one half $J_d(a,b)$, respectively. But M_∞ parametrizes maps from nodal rational curves, so we can calculate $M_\infty \alpha^a \beta^b$ using Remark 4.6,

$$J_d(a,b) = \binom{d-1}{2} R_d(a,b) + 2bNL_d(a,b-1) + 4\binom{b}{2} NP_d(a,b-2).$$

5.9. *Numbers.* Using the recursions given above, we find the following characteristic numbers for elliptic curves. (The first number in each sequence is the number with only incidence conditions; the last is the number with only tangency conditions.)

Conics: 0, 0, 0, 0, 2, 10, $\frac{45}{2}$.

Cubics: 1, 4, 16, 64, 256, 976, 3424, 9766, 21004, 33616.

Quartics: 225, 1010, 4396, 18432, 73920, 280560, 994320, 3230956, 9409052, 23771160, 50569520, 89120080, 1299962164.

Quintics: 87192, 411376, 1873388, 8197344, 34294992, 136396752, 512271756, 1802742368, 5889847264, 17668868832, 48034104112, 116575540736, 248984451648, 463227482784, 747546215472, 1048687299072.

The cubic numbers agree with those found by Aluffi in [A1]. The quartic numbers agree with the predictions of Zeuthen (see [S], p. 187).

Using the recursion of Subsection 5.8, we find the following characteristic numbers for elliptic curves with fixed j-invariant, $j \neq 0, 1728, \infty$.

Conics: 0, 0, 0, 12, 48, 75.

Cubics: 12, 48, 192, 768, 2784, 8832, 21828, 39072, 50448.

Quartics: 1860, 8088, 33792, 134208, 497952, 1696320, 5193768, 13954512, 31849968, 60019872, 92165280, 115892448.

The cubic numbers agree with those found by Aluffi in [A2], Theorem III(2). The incidence-only numbers necessarily agree with the numbers found by Pandharipande in [P1], as the formula is the same.

5.10. *Characteristic numbers in* $\mathbf{P}^n$. The same method gives a program to recursively compute characteristic numbers of elliptic curves in $\mathbf{P}^n$ that may be simpler than the algorithm of [V5]: Use Kontsevich's cross-ratio method to count irreducible nodal rational curves through various linear spaces and where the node is required to lie on a given linear space (analogous to the derivation of (3)). Use (5) to compute all the characteristic numbers of each of these families of rational curves. Use [V4] to compute the number of elliptic curves through various linear spaces. Finally, use (6) to compute all characteristic numbers of curves in $\mathbf{P}^n$. The same calculations also allow one to compute characteristic numbers of elliptic curves in $\mathbf{P}^n$ with fixed j-invariant.

5.11. *Covers of* $\mathbf{P}^1$. By restricting Pandharipande's relation (5) and relation (6) to degree d covers of a line by a genus 0 and 1 curve, respectively, (so α restricts to 0), where all but one ramification are fixed, we obtain recursions for M_d^g ($g{=}0$, 1), the number of distinct covers of $\mathbf{P}^1$ by irreducible genus g curves with $2d{+}2g{-}2$ fixed ramification points,

$$M_d^0 = \frac{(2d-3)}{d}\sum_{j=1}^{d-1}\binom{2d-4}{2j-2}M_j^0 M_{d-j}^0 j^2(d-j)^2,$$

$$M_d^1 = \frac{d}{6}\binom{d}{2}(2d-1)M_d^0 + \sum_{j=1}^{d-2} 2j(2d-1)\binom{2d-2}{2j-2}M_j^0 M_{d-j}^1 (d-j)j.$$

The first equation was found earlier by Pandharipande and the second by Pandharipande and Graber [GP]. Their proofs used an analysis of the divisors on $\overline{M}_{g,n}(\mathbf{P}^1,d)$. The closed form expression $M_d^0{=}d^{d-3}(2d-2)!/d!$ follows by an easy combinatorial argument from the first equation using Cayley's formula for the number of trees on n vertices. This formula was first proved in [CT]. A more general formula was stated by Hurwitz and was first proved in [GJ1].

By applying the methods of Section 3 to substacks of $\overline{\mathcal{M}}_g(\mathbf{P}^1,d)$, one can recover Hurwitz' general formula, generalize it to genus 1, and interpret it as a graph enumeration problem ([V1]).

Graber and Pandharipande have conjectured a similar formula for $g{=}2$,

$$M_d^2 = d^2\left(\frac{97}{136}d - \frac{20}{17}\right)M_d^1 + \sum_{j=1}^{d-1} M_j^0 M_{d-j}^2\binom{2d}{2j-2}j(d-j)\left(-\frac{115}{17}j + 8d\right)$$
$$+\sum_{j=1}^{d-1} M_j^1 M_{d-j}^1\binom{2d}{2j}j(d-j)\left(\frac{11697}{34}j(d-j) - \frac{3899}{68}d^2\right).$$

This formula was proved in [GJ2], and generalizations are given in [GJV], along with general machinery for dealing with such recursions. It is still geometrically unclear why a genus 2 relation should exist. The relation looks as though it is induced by a relation in the Picard group of the moduli space, but no such relation exists.

5.12. *Divisor theory on* $\overline{\mathcal{M}}_1(\mathbf{P}^2,d)^*$. In [P4], Pandharipande determined the divisor theory on $\overline{\mathcal{M}}_0(\mathbf{P}^n,d)$ (including the top intersection products of divisors). The divisor theory of $\overline{\mathcal{M}}_1(\mathbf{P}^2,d)^*$ is more complicated. In addition to the divisor α and the enumeratively meaningful boundary divisors, there are three enumeratively irrelevant divisors (see 4.2). The Deligne–Mumford stack $\overline{\mathcal{M}}_1(\mathbf{P}^2,d)^*$ is nonsingular away from these divisors. The stack $\overline{\mathcal{M}}_1(\mathbf{P}^2,d)$ is unibranch at the enumeratively irrelevant divisor of type (5); Thaddeus [T] has shown that it is singular there. There are several natural questions to ask about the geometry and topology of $\overline{\mathcal{M}}_1(\mathbf{P}^2,d)^*$. Is it nonsingular at the other two enumeratively irrelevant divisors? Is the normalization of $\overline{\mathcal{M}}_1(\mathbf{P}^2,d)^*$ nonsingular? If $d=3$, how does it compare to Aluffi's space of complete cubics? What are the top intersection products of these divisors? (The arguments here allow us to calculate $\alpha^a\beta^{3d-a}$ and $\alpha^a\beta^{3d-1-a}D$ where D is any boundary divisor.) What about $\overline{\mathcal{M}}_1(\mathbf{P}^n,d)^*$?

Appendix A. A recursive formula for $NP(a,b)$

We have

$$\begin{aligned}
NP(a,b+1) &= \frac{d-1}{d} NP(a+1,b) \\
&+ \sum_{i+j=d} \frac{ij}{2d}\Bigg[\sum_{\substack{a_i+a_j=a+2\\ b_i+b_j=b}} \binom{a}{a_i-1}\binom{b}{b_i}(ij-1)R_i(a_i,b_i)R_j(a_j,b_j) \\
&+2 \sum_{\substack{a_i+a_j=a\\ b_i+b_j=b}} \binom{a}{a_i}\binom{b}{b_i} ijR_i(a_i,b_i)NP_j(a_j,b_j) \\
&+4b \sum_{\substack{a_i+a_j=a+3\\ b_i+b_j=b-1}} \binom{a}{a_i-1}\binom{b-1}{b_i} iR_i(a_i,b_i)R_j(a_j,b_j) \\
&+4b \sum_{\substack{a_i+a_j=a+1\\ b_i+b_j=b-1}} \binom{a}{a_i}\binom{b-1}{b_i} iNP_i(a_i,b_i)R_j(a_j,b_j) \\
&+4b \sum_{\substack{a_i+a_j=a+1\\ b_i+b_j=b-1}} \binom{a}{a_i}\binom{b-1}{b_i} iR_i(a_i,b_i)NP_j(a_j,b_j) \\
&+4\binom{b}{2} \sum_{\substack{a_i+a_j=a+4\\ b_i+b_j=b-2}} \binom{a}{a_i-2}\binom{b-2}{b_i} R_i(a_i,b_i)R_j(a_j,b_j) \\
&+8\binom{b}{2} \sum_{\substack{a_i+a_j=a+2\\ b_i+b_j=b-2}} \binom{a}{a_i-1}\binom{b-2}{b_i} R_i(a_i,b_i)NP_j(a_j,b_j)\Bigg].
\end{aligned}$$

In each sum in the large bracket, it is assumed that $a_i+b_i=3i-1$ if $R_i(a_i,b_i)$

appears in the sum, and $a_i+b_i=3i-3$ if $NP_i(a_i,b_i)$ appears. The same assumption is made when i is replaced by j.

The large square bracket corresponds to maps from reducible curves. (To avoid confusion: the "image of the node" refers to the image of the node of the source curve. The "fixed node" refers to the node of the *image* that is required to be at a fixed point.) Zero, one, or two tangent lines can pass through the image of the node of the source curve. The two branches through the fixed node can belong to the same component, or one can belong to each. Table 1 identifies which possibilities correspond to which sum in the large bracket.

Table 1.

Term in sum	Number of tangent lines through image of node of source	Number of irreducible components through fixed node
First	0	2
Second	0	1
Third	1	2
Fourth and fifth	1	1
Sixth	2	2
Seventh	2	1

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