Linear measure on plane continua of finite linear measure

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1. Introduction

Let Λ be one-dimensional Hausdorff measure in $\mathbf{C}$. Let B be a continuum in $\mathbf{C}$ of finite linear measure, i.e., $\Lambda(B)<\infty$. Denote the components of the complement of B in the Riemann sphere by $\{V_j\}$; each V_j is simply connected. Let $f_j\colon \mathbf{D}\to V_j$ be a Riemann map, where $\mathbf{D}$ is the open unit disk.

Theorem.

$$2\Lambda(B)=\sum_j \int_0^{2\pi} |f_j'(e^{i\theta})|\,d\theta.$$

The problem of establishing this identity was raised by Ch. Pommerenke in a letter to the author. The proof has two parts. First we show that the j^{th} integral in the theorem is equal to the integral with respect to Λ of the multiplicity function of f_j; this is (I). The second part, (II), is to show that the sum of the multiplicity functions is equal to 2 a.e. on B with respect to Λ.

After the proof of the theorem we shall indicate a generalization. This is a decomposition of the restriction of Λ to B as a sum of measures on the boundaries of the V_j.

Finally I want to thank Professor Ch. Pommerenke for writing to me about this problem. A related inequality had been treated in [1].

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2.

We begin with some general measure theory. For a measurable set $A \subseteq \mathbf{R}$ we denote its Lebesgue measure by $|A|$.

Lemma 1. *Let u be a real-valued absolutely continuous function on an interval (a, b). Then, for every measurable set $A \subseteq (a, b)$, $u(A)$ is measurable and*

$$(*) \qquad |u(A)| \leqq \int_A |u'(x)|\, dx.$$

In particular, if $u'=0$ a.e. on A, then $|u(A)|=0$.

Remark. This is just a case of Sard's lemma. For completeness we shall indicate an outline of a proof.

Proof. Let I be an open subinterval such that $\bar{I} \leqq (a, b)$. Then $u(I)$ is an interval with endpoints $u(c)$ and $u(d)$ for points c and d in $\bar{I}$. Hence $|u(I)|=|u(d)-u(c)|=\left|\int_c^d u'(x)\, dx\right| \leqq \int_I |u'(x)|\, dx$.

Now if W is an open subset of (a, b), $W=\bigcup I_j$ where the I_j are disjoint open intervals. Hence $|u(W)| \leqq \sum |u(I_j)| \leqq \sum \int_{I_j} |u'|\, dx = \int_W |u'|\, dx$. From this it follows that if $N \subseteq (a, b)$ is a null set then $u(N)$ is a null set. Standard arguments then show that $u(A)$ is measurable if A is measurable and then (*) follows easily.

Next, we need the converse to the Sard lemma; cf. [5], p. 322.

Lemma 2. *Let u be a real-valued absolutely continuous function on an interval (a, b). If K is a compact subset of (a, b) such that $|u(K)|=0$, then $u'=0$ a.e. on K.*

Proof. Suppose not. Then, by replacing K by a smaller set, we may assume, without loss of generality, that u' exists and is strictly positive at every point of K. We denote the characteristic function of K by χ_K. Almost every point of K is a Lebesgue point for $|u'|$ and also for $|u'|\chi_K$. Choose such a point $c \in K$. Let $J_\varepsilon=(c-\varepsilon, c+\varepsilon)$. Set $\eta=u'(c)$; $\eta>0$. Then

$$Q(\varepsilon) \equiv \frac{1}{2\varepsilon} \int_{J_\varepsilon \setminus K} |u'|\, dx$$

$$= \frac{1}{2\varepsilon} \int_{J_\varepsilon} |u'|\, dx - \frac{1}{2\varepsilon} \int_{J_\varepsilon} |u'|\chi_K\, dx.$$

Therefore, as $\varepsilon \to 0$, $Q(\varepsilon) \to \eta - \eta = 0$.

We have $u(t)=u(c)+(t-c)\eta\big(1+\sigma(t)\big)$ where $\sigma(t) \to 0$ as $t \to c$. Choose ε so small that $|\sigma(c \pm \varepsilon)| < \frac{1}{3}$. Then $u(J_\varepsilon) \supseteq \big(u(c-\varepsilon), u(c+\varepsilon)\big) \supseteq \big(u(c)-\varepsilon\eta(1-\frac{1}{3}), u(c)+$

$\varepsilon\eta(1-\frac{1}{3}))$. Hence $|u(J_\varepsilon)|\geqq 2\varepsilon\cdot\eta\cdot\frac{2}{3}$. By Lemma 1 we have

$$|u(J_\varepsilon\setminus K)| \leqq \int_{J_\varepsilon\setminus K}|u'|\,dx \leqq 2\varepsilon Q(\varepsilon).$$

Now $u(K\cap J_\varepsilon)\supseteqq u(J_\varepsilon)\setminus u(J_\varepsilon\setminus K)$ and therefore

$$\begin{aligned}|u(K\cap J_\varepsilon)| &\geqq |u(J_\varepsilon)|-|u(J_\varepsilon\setminus K)|\\ &\geqq 2\varepsilon\eta\tfrac{2}{3}-2\varepsilon Q(\varepsilon)\\ &= 2\varepsilon(2\eta/3-Q(\varepsilon)).\end{aligned}$$

If we choose ε so small that $Q(\varepsilon)<\eta/3$ we get $|u(K\cap J_\varepsilon)|>2\varepsilon\frac{\eta}{3}>0$. This contradicts $|u(K)|=0$. Q.E.D.

For $z=x+iy\in\mathbf{C}$, we have the coordinate projections: $\pi_1(z)=x$ and $\pi_2(z)=y$.

Lemma 3. *("Projection lemma.") Let B be a continuum in $\mathbf{C}$ with $\Lambda(B)<\infty$. Let K be a compact subset of B with $\Lambda(K)>0$. Then either $|\pi_1(K)|>0$ or $|\pi_2(K)|>0$.*

Remark. Besicovitch ([2], [3]) has constructed sets of positive Λ measure whose projections on all lines have zero measure. By the lemma, these sets are not contained in continua of finite linear measure. A proof of the lemma can also be obtained from [8], Corollary 3.15 and Theorem 6.10.

Proof. First we consider the special case when B is a rectifiable Jordan arc. Let F: $[0,L]\to B$ parametrize B by arclength. Then $|F(t_2)-F(t_1)|\leqq|t_2-t_1|$ for $0\leqq t_1\leqq t_2\leqq L$; so F is a Lipschitz function, therefore absolutely continuous; $|F'(t)|=1$ a.e. on $[0,L]$. Let $K_0=F^{-1}(K)$, then $|K_0|=\Lambda(K)>0$.

Write $F=u+iv$. Suppose that $|\pi_1(K)|=0$ and $|\pi_2(K)|=0$. Then $u(K_0)=\pi_1(K)$ and so $|u(K_0)|=0$. By Lemma 2, $u'=0$ a.e. on K_0. Likewise $v'=0$ a.e. on K_0. Hence $F'=0$ a.e. on K_0. But $|F'|=1$ a.e. on K_0 and $|K_0|>0$. This is a contradiction and we conclude that $|\pi_1(K)|>0$ or $|\pi_2(K)|>0$.

In the general case of a continuum B of finite linear measure, Besicovitch ([2], [3]) has shown that B is a disjoint union of a countable set of Jordan arcs and a Λ-null set. Therefore $K\subseteqq B$ must meet one of these Jordan arcs in a set K_1 of positive measure. By the first part of the proof, $|\pi_1(K_1)|>0$ or $|\pi_2(K_1)|>0$. Q.E.D.

Suppose that F: $(a,b)\to\mathbf{C}$ is continuous. For $z\in\mathbf{C}$, define $\varphi(z)=\#\{t\in(a,b)\colon F(t)=z\}$, here $\#$ denotes the number of elements in a set. This is the so-called crude multiplicity function. It is known to be measurable under very general conditions; see [6]. For completeness we shall give the proof in our simple setting.

Lemma 4. *φ is a Borel measurable function on $\mathbf{C}$.*

Proof. Let Π be a finite partition of (a, b) by intervals J which are disjoint, say all of the form $(c, d]$ except the right-most interval. As each J is σ-compact, $F(J)$ is σ-compact and so is a Borel set. Hence

$$\psi_\Pi := \sum \{\chi_{F(J)} : J \in \Pi\}$$

is Borel measurable. If Π_n is a nested set of partitions whose maximal interval has length converging to zero, it is straightforward that $\psi_{\Pi_n} \uparrow \varphi$ on $\mathbf{C}$. Q.E.D.

Let V be a simply connected plane domain with $\Lambda(\partial V) < \infty$ and let f: $\mathbf{D} \to V$ be a Riemann map (one-to-one, conformal, onto). It is known that f is continuous on $\bar{\mathbf{D}}$ and that f' is in the Hardy space H^1. Also $f|\partial\mathbf{D}$ is of bounded variation and its variation, denoted by var (f), is equal to $\int_0^{2\pi} |f'(e^{i\theta})|\, d\theta$. Set $\varphi(z) = \#\{e^{i\theta} : f(e^{i\theta}) = z\}$. By Lemma 4, φ is Borel measurable on $\mathbf{C}$. We shall see below (Lemma 8) that $\varphi \leqq 2$, Λ a.e.

Consider a partition Π of $\partial\mathbf{D}$ consisting of a finite number of disjoint half-open subarcs I. Fix an $I \in \Pi$ with endpoints α_I and β_I. Let $J = f(I)$, an arcwise connected subset of ∂V. By an argument of Besicovitch ([2], p. 311) there exists a Jordan arc in J connecting $f(\alpha_I)$ to $f(\beta_I)$. Let g_I: $I \to J$ parametrize such a Jordan arc. Since g_I maps I onto the Jordan arc $g_I(I)$, it follows that the variation of g_I on I, denoted var (g_I), equals the length of $g_I(I) = \Lambda(g_I(I))$.

We claim that var $(g_I) \leqq$ var $(f|I)$. In fact, the construction of g_I proceeds as follows: Choose the largest subarc $\gamma \subseteqq I$ such that the values of f coincide at the endpoints of γ. Set g_I to be the constant value on γ which agrees with f at the endpoints of γ. Continue inductively in this way to modify f on open subarcs until a one-to-one function is obtained. This is g_I. It is then clear that var $(g_I) \leqq$ var $(f|I)$. We have

$$|f(\beta_I) - f(\alpha_I)| \leqq \Lambda(g_I(I)) \leqq \operatorname{var}(f|I). \tag{1}$$

Now define f_Π: $\partial\mathbf{D} \to \partial V$ by $f_\Pi|I = g_I$ for each $I \in \Pi$. Let $\varphi_\Pi(z) = \#\{e^{i\theta} : f_\Pi(e^{i\theta}) = z\}$. Then f_Π is continuous and φ_Π is Borel measurable. Since the g_Π are one-to-one and the $\{I\}$ are disjoint, it follows that $\varphi_\Pi = \sum_{I \in \Pi} \chi_{g_I(I)}$. Hence

$$\sum_I \Lambda(g_I(I)) = \sum_I \int \chi_{g_I(I)}\, d\Lambda = \int \left(\sum_I \chi_{g_I(I)}\right) d\Lambda = \int \varphi_\Pi\, d\Lambda. \tag{2}$$

Summing in (1), using (2), we get

$$\sum_{I \in \Pi} |f(\beta_I) - f(\alpha_I)| \leqq \int \varphi_\Pi\, d\Lambda \leqq \operatorname{var}(f). \tag{3}$$

Denote the sum in (3) by var (f, Π).

Choose a sequence of partitions Π_n so that var $(f, \Pi_n) \to$ var (f). Set

$$\psi_n = \varphi_{\Pi_n}$$

and $F_n = f_{\Pi_n}$. Then $F_n \to f$ uniformly on $\partial\mathbf{D}$. By (3),

$$\operatorname{var}(f, \Pi_n) \leqq \int \psi_n \, d\Lambda \leqq \operatorname{var}(f).$$

It follows that $\int \psi_n \, d\Lambda \to \operatorname{var}(f)$. Also, for any partition Π and $I \in \Pi$, $g_I(I) \subseteqq f(I)$, hence $\chi_{g(I)} \leqq \chi_{f(I)}$. Summing over I we get $\varphi_\Pi \leqq \sum_I \chi_{f(I)} \leqq \varphi$. Taking $\Pi = \Pi_n$ we get $\psi_n \leqq \varphi$ for all n. We have proved the following

Lemma 5. $\int \psi_n \, d\Lambda \to \operatorname{var}(f)$.

We set $\psi = \liminf \psi_n$, then $\psi \leqq \varphi$.

Lemma 6. $\psi = \varphi$ *holds* Λ *a.e.*

Proof. We suppose not, arguing by contradiction. Then there exists a compact subset K of ∂V such that $\Lambda(K) > 0$ and $\psi < \varphi$ on K. Let $E = f^{-1}(K) \subseteqq \partial\mathbf{D}$. Then $|E| > 0$. In fact, by Lemma 3, we may suppose that $|\pi_1(K)| > 0$, then $|u(E)| > 0$ and so, by Lemma 1, $\int_E |u'| \, d\theta \geqq |u(E)| > 0$; i.e. $|E| > 0$. By shrinking K we may assume also that the derivative f' exists at each point of E. Finally set $N(t) = \# \{z \in \partial V\colon \operatorname{Re} z = t\}$ for $t \in \mathbf{R}$. Since $\Lambda(\partial V) < \infty$, it is known that $\int_{-\infty}^{\infty} N(t) \, dt < \infty$. Hence $N(t) < \infty$ a.e. Again by shrinking K, we may suppose that $N(t) < \infty$ for each $t \in \pi_1(K)$ $(|\pi_1(K)| > 0)$.

Since φ takes on only the values 1 and 2 on ∂V, Λ a.e., we may assume that one of the following two cases holds:

(a) $\psi \equiv 0$ on K and $\varphi \equiv 1$ on K or
(b) $\psi \leqq 1$ on K and $\varphi \equiv 2$ on K.

First consider case (a). Take $\zeta \in E$ and let $z = f(\zeta) \in K$. Then $\psi_{n_j}(z) = 0$ for some subsequence. Let δ be the distance from z to the nearest other point of $\partial V \cap l$, where l is the vertical line thru z; we know that $\partial V \cap l$ is finite. Let Δ be the disc centered at z of radius $\delta/2$. Choose a connected neighborhood W of ζ in $\partial\mathbf{D}$ such that $f(W) \subseteqq \Delta$. Since $F_{n_j} \to f$ uniformly on $\partial\mathbf{D}$, $F_{n_j}(W) \subseteqq \Delta$ for j large. But $z \notin F_{n_j}(W) \subseteqq \partial V$ and since $\partial V \cap l \cap \Delta = \{z\}$ it follows that the connected set $F_{n_j}(W)$ lies on one side of l, for large j. Therefore also $f(W)$ lies on one side of l. It follows that $u = \operatorname{Re} f$ has a local maximum or minimum at ζ. Therefore $u'(\zeta) = 0$. Hence $u' \equiv 0$ on E. By Lemma 1, $0 = |u(E)| = |\pi_1(K)|$. This is a contradiction.

Now consider case (b). If $\psi = 0$ on a subset of K of positive measure, then the argument of case (a) carries over to show that $u' = 0$ on a set of positive measure. Just as above this yields a contradiction. Thus we may assume that $\psi \equiv 1$ on K and $\varphi \equiv 2$ on K. Fix $z \in K$. Then $f^{-1}(z)$ consists of two points ζ' and ζ''. Choose a subsequence: $\psi_{n_j}(z) = 1$ for all j. Then $F_{n_j}^{-1}(z) = \zeta_j$, a single point of $\partial\mathbf{D}$. By passing to a subsequence we may assume that $\{\zeta_j\}$ converges in $\partial\mathbf{D}$. The limit must be ζ' or ζ'' because $F_{n_j} \to f$ uniformly and so f maps the limit point to z. Say $\zeta_j \to \zeta'$.

Then F_{n_j} does not take on the value z near ζ''. The connectedness argument of case (a) then shows that $u'(\zeta'')=0$, just as before.

Let $A=E\cap\{\zeta\in\partial\mathbf{D}\colon u'(\zeta)=0\}$. By the last paragraph, $f(A)=K$. Hence $|u(A)|=|\pi_1(K)|>0$. But $u'=0$ on A. This contradicts Lemma 1. Q.E.D.

Lemma 7. *$\int \psi_n\, d\Lambda \to \int \varphi\, d\Lambda$ as $n\to\infty$.*

Proof. By Lemma 6, $\varphi=\liminf \psi_n$, a.e. Also, since $\psi_n\leqq\varphi$ for all n, $\limsup \psi_n\leqq\varphi$. It follows that $\lim \psi=\varphi$, Λ a.e. The lemma then follows from the Lebesgue dominated convergence theorem.

Now by Lemmas 5 and 7 we have

$$\text{(I)}\qquad \int \varphi\, d\Lambda = \int_{\partial\mathbf{D}} |f'|\, d\theta.$$

3.

We now let Λ denote the restriction of one-dimensional Hausdorff measure to B. By assumption $0<\Lambda(B)<\infty$. We have $\overline{\mathbf{C}}\setminus B=\bigcup V_j$ and $f_j\colon \mathbf{D}\to V_j$ a fixed Riemann map. Let $\varphi_j(z)=\#\{\zeta\in\partial\mathbf{D}\colon f_j(\zeta)=z\}$ for $z\in B$. By Lemma 4, φ_j is Borel measurable and therefore

$$\Phi :\equiv \textstyle\sum_j \varphi_j$$

is also Borel measurable. The second half of the proof of the theorem consists of showing that

$$\text{(II)}\qquad \Phi = 2 \quad \text{holds } \Lambda \text{ a.e.}$$

Lemma 8. *$\Phi\leqq 2$ except on a countable set.*

Remark. The case $\varphi_j\leqq 2$ except on a countable set for any j was observed by Rudin [7]. We used this fact in the proof of Lemma 6.

Proof. Let $P=\{p\in B\colon \Phi(p)\geqq 3\}$. We construct a triod $T(p)$ at p as follows. There are three cases:

(1) $\varphi_j(p)\geqq 1$, $\varphi_k(p)\geqq 1$ and $\varphi_l(p)\geqq 1$ for some j, k and l distinct.
(2) $\varphi_j(p)\geqq 2$ and $\varphi_k(p)\geqq 1$ for some $j\neq k$.
(3) $\varphi_j(p)\geqq 3$ for some j.

In Case 1, there are points ζ_j, ζ_k, ζ_l in $\partial\mathbf{D}$ such that $f_i(\zeta_i)=p$, $i=j, k$ or l. Let $T(p)$ be the union of the image by f_i of the half radius $\{r\zeta_i\colon \frac{1}{2}\leqq r\leqq 1\}$ for $i=j, k, l$.

In Case 2, we repeat this for $\zeta_j\neq\zeta_j'$ and ζ_k in $\partial\mathbf{D}$ and in Case 3 for distinct ζ_j, ζ_j', ζ_j'' in $\partial\mathbf{D}$.

We get in each a triod and as p varies over P, these triods $T(p)$ are disjoint. It follows from a theorem of R. L. Moore [4] that P is countable. Q.E.D.

Lemma 9. *$\Phi > 0$ holds Λ a.e.*

Proof. Define $B_0 = \{z \in B\colon \Phi(z) = 0\}$. Since φ_j maps $\partial\mathbf{D}$ onto ∂V_j, $\varphi_j \geqq 1$ on ∂V_j and so $B_0 = B \setminus \bigcup \partial V_j$. Suppose $\Lambda(B_0) > 0$. Then, by the projection lemma, $|\pi_1(B_0)| > 0$ or $\pi_2|(B_0)| > 0$. We may assume that $|\pi_1(B_0)| > 0$. Choose $z \in B_0$ such that the vertical line l through z intersects B only finitely often (cf. the proof of Lemma 6). Then some line segment contained in l lies in some V_j and has z as an endpoint. Then $z \in \partial V_j$, a contradiction. Hence $\Lambda(B_0) = 0$. Q.E.D.

Let $B_1 = \{z\colon \Phi(z) = 1\}$.

Lemma 10. $\Lambda(B_1) = 0$.

Proof. For all j define

$$A_j = \{z \in B\colon \varphi_j(z) = 1 \text{ and } \varphi_k(z) = 0 \text{ if } k \neq j\}.$$

Then $B_1 = \bigcup_j A_j$. It suffices to show $\Lambda(A_j) = 0$ for all j. Without loss of generality, it suffices to show $\Lambda(A_1) = 0$.

Suppose not. Take a compact $K \subseteqq A_1$ such that $\Lambda(K) > 0$. By the projection lemma, we may assume that $|\pi_1(K)| > 0$. We may further assume that every vertical line which meets K intersects B only finitely often and that u_1' exists everywhere on $E = f_1^{-1}(K)$. Here $f_1 = u_1 + iv_1$. We know that $|E| > 0$.

Fix $\zeta \in E$ and let $z = f(\zeta) \in K$. Let l be the vertical line through z. Then l meets B only finitely often. Let l_1 and l_2 be the segments of $l \setminus B$ which have z as an endpoint. Then l_i lies in some V_j for $i = 1, 2$. Since $\varphi_j(z) = 0$ for $j \neq 1$, we conclude that l_1 and l_2 are both contained in V_1. Let p_i be an interior point of l_i for $i = 1, 2$. Let l_3 be the closed line segment joining p_1 to p_2. Then $l_3 \subseteqq V_1 \cup \{z\}$. Let γ_0 be a Jordan arc joining p_1 to p_2 in the simply connected domain $V_1 \setminus l_3$. If γ is the Jordan curve $\gamma_0 \cup l_3$, then $\gamma \subseteqq V_1 \cup \{z\}$.

Consider $\sigma = f_1^{-1}(\gamma)$. Since $\varphi_1(z) = 1$, $f_1^{-1}(z) = \{\zeta\}$ and σ is a Jordan curve in $\mathbf{D} \cup \{\zeta\}$. The inside of σ is mapped by f_1 *onto* the inside of γ. Hence the inside of γ is disjoint from B. It follows that locally, near z, B lies on one side only of l. This means that u_1 has a local maximum or minimum at ζ. Hence $u_1'(\zeta) = 0$. Hence $u_1' \equiv 0$ on E. As $|E| > 0$ and $|u_1(E)| = |\pi_1(K)| > 0$, this contradicts Lemma 1. We conclude that $\Lambda(A_1) = 0$. Q.E.D.

Now (II) follows from Lemmas 8, 9 and 10.

Let χ_j be the characteristic function of ∂V_j.

Proposition. $1 \leqq \sum_j \chi_j \leqq 2$ *Λ a.e., and hence* $\Lambda(B) \leqq \sum \Lambda(\partial V_j) \leqq 2\Lambda(B)$.

Remark. The second inequality on measure was obtained in [1].

Proof. Clearly $\chi_j \leqq \varphi_j$ for all j and therefore $\sum \chi_j \leqq \Phi = 2$ by (II). If $z \in B \setminus B_0$ then $z \in \partial V_j$ for some j and so $1 \leqq \sum \chi_j(z)$. Hence $1 \leqq \sum \chi_j$, Λ a.e. by the proof of Lemma 9. Q.E.D.

4. Proof of the theorem

Integrating (II) w.r.t. Λ we get

$$2\Lambda(B) = \int \Phi \, d\Lambda = \sum_j \int \varphi_j \, d\Lambda,$$

by the monotone convergence theorem. Applying (I) to the last integrals gives the theorem.

5.

We now consider a refinement of the theorem. Let $f\colon \mathbf{D} \to V$ be a Riemann map as in Section 2 with $\Lambda(\partial V) < \infty$ and $\varphi(z) = \#\{\zeta \in \partial \mathbf{D} \colon f(\zeta) = z\}$. Define the "push-forward" measure $\mu = f_*(|f'| \, d\theta)$ on ∂V by $\int g(z) \, d\mu(z) = \int_{\partial \mathbf{D}} g \circ f |f'| \, d\theta$ for every bounded Borel function g on ∂V. By definition $\varphi\Lambda$ is the measure on B given by $(\varphi\Lambda)(E) = \int_E \varphi(z) \, d\Lambda(z)$ for every Borel set $E \subseteqq \partial V$.

Lemma 11. $\varphi\Lambda = \mu$.

Proof. For $S \subseteqq \partial \mathbf{D}$ define, for $z \in \partial V$, $\varphi_S(z) = \#\{\zeta \in S \colon f(\zeta) = z\}$. If J is a subarc of $\partial \mathbf{D}$ then the arguments of Lemmas 4, 5, 6 and 7 show that

$$\int_J |f'| \, d\theta = \int_{\partial V} \varphi_J(z) \, d\Lambda(z).$$

Let W be an open subset of ∂V, write $f^{-1}(W) = \bigcup J_j$ where the J_j are disjoint subarcs of $\partial \mathbf{D}$. Then $\int_{f^{-1}(W)} |f'| \, d\theta = \sum \int_{J_j} |f'| \, d\theta = \sum \int \varphi_{J_j} \, d\Lambda = \int \sum \varphi_{J_j} \, d\Lambda = \int \varphi_{f^{-1}(W)} \, d\Lambda$. It is clear that $\varphi_{f^{-1}(W)} = \varphi \cdot \chi_W$. Hence we get

$$\mu(W) = \int_{f^{-1}(W)} |f'| \, d\theta = \int_W \varphi \, d\Lambda,$$

for every open $W \subseteqq \partial V$. This gives the lemma.

Now for every component V_j of $\overline{\mathbf{C}} \setminus B$ we have $f_j\colon \mathbf{D} \to V_j$ and a φ_j. Define $\mu_j = f_{j*}(|f_j'| \, d\theta)$. By Lemma 11, $\varphi_j \Lambda = \mu_j$. Summing over j and applying (II) gives the following decomposition of Λ on B:

$$2\Lambda = \sum_j \mu_j.$$

In this equality of Borel measures, the sum is taken in the strong norm sense. This decomposition is equivalent to saying that

$$2\int g\,dA=\sum_j\int_{\partial D} g\circ f_j\,|f_j'|\,d\theta$$

for each bounded Borel function on B. Our theorem is just the case $g\equiv 1$.

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