On the compactness of paracommutators

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1. Introduction

In their paper [2], Janson and Peetre consider the paracommutator defined by

$$\widehat{T_b^{st}f}(\xi) = (2\pi)^{-d} \int \hat{b}(\xi-\eta)\, A(\xi, \eta)\, |\xi|^s\, |\eta|^t \hat{f}(\eta)\, d\eta \tag{1}$$

and obtain a series of results about its L^2-boundedness and its S^p-estimates. In § 13, they prove three theorems about the compactness of paracommutators (for notations see below):

Theorem A. *Suppose that A satisfies* A1 *and* A3 (γ) *and that* $s+t<\gamma$ *and* $s, t>0$. *If* $b\in b_\infty^{s+t}$, *then* T_b^{st} *is compact.*

Theorem B. *Suppose that A satisfies* A3 *and*

$$\|A\|_{M(\Delta_j\times\Delta_k)} \leqq a(j-k) \quad \text{with} \quad \sum_{-\infty}^{\infty} a(n) < \infty.$$

If $b\in b_\infty^0$, *then* T_b *is compact.*

Theorem C. *Suppose that A satisfies* A1, A2, A3. *If* $b\in$ CMO, *then* T_b *is compact*.

In this paper, we study the converses of the above theorems. We adopt the notation in [2]. For the sake of completeness, we include some of them, which are used in this paper. Let Δ_k denote the set $\{\xi\in\mathbf{R}^d\colon 2^k\leqq|\xi|\leqq 2^{k+1}\}$. The space of Schur multipliers $M(U\times V)$ is the set of all $\varphi\in L^\infty(U\times V)$ that admit the representation

$$\varphi(\xi, \eta) = \int_\Omega \alpha(\xi, \omega)\, \beta(\eta, \omega)\, d\mu(\omega) \tag{2}$$

for some finite measure space (Ω, μ) and $\|\alpha\|_{L^\infty(U\times\Omega)}, \|\beta\|_{L^\infty(V\times\Omega)} \leqq 1$; the norm $\|\varphi\|_{M(U\times V)}$ is given by the minimum of the $\mu(\Omega)$ over all representations (2).

A0: There exists an $r>1$ such that $A(r\xi, r\eta)=A(\xi, \eta)$.
A1: $\|A\|_{M(\Delta_j\times\Delta_k)}\leqq C$, for all $j, k\in\mathbf{Z}$.
A2: There exist $A_1, A_2\in M(\mathbf{R}^d\times\mathbf{R}^d)$ and $\delta>0$ such that

$$A(\xi, \eta) = A_1(\xi, \eta) \quad \text{for} \quad |\eta| < \delta|\xi|$$

$$A(\xi, \eta) = A_2(\xi, \eta) \quad \text{for} \quad |\xi| < \delta|\eta|.$$

A3: There exist $\gamma>0$ and $\delta>0$ such that if $B=B(\xi, r)$ with $r<\delta|\xi_0|$, then

$$\|A\|_{M(B\times B)} \leqq C\left(\frac{r}{|\xi_0|}\right)^{\gamma}.$$

A4: There exists no $\xi\neq 0$ such that $A(\xi+\eta, \eta)=0$ for a.e. η.
A5: For every $\xi_0\neq 0$ there exist $\delta>0$ and $\eta_0\in\mathbf{R}^d$ such that, with

$$U=\{\xi:\ \left|\xi/|\xi|-\xi_0/|\xi_0|\right|<\delta \text{ and } |\xi|>|\xi_0|\} \quad \text{and} \quad V=B(\eta_0, \delta|\xi_0|),$$
$$A(\xi, \eta)^{-1}\in M(U\times V).$$

We need another non-degeneracy assumption A4$\frac{1}{2}$ on $A(\xi, \eta)$, which is stronger than A4 but weaker than A5.

A4$\frac{1}{2}$: For every $\xi_0\neq 0$ there exist $\eta\in\mathbf{R}^d$ and $\delta>0$ such that, with

$$B_0 = B(\xi_0+\eta_0, \delta|\xi_0|) \quad \text{and} \quad D_0 = B(\eta_0, \delta|\xi_0|), \quad A(\xi, \eta)^{-1}\in M(B_0\times D_0).$$

Remark 1. It is easy to show that the assumption A4$\frac{1}{2}$ is equivalent to the following statement:

For every $\xi_0\neq 0$ there exist $\eta_0\in\mathbf{R}^d$ with $\eta_0\notin\{-\xi_0, 0\}$ and

$$0<\delta<\tfrac{1}{3}\min(|\xi_0+\eta_0|, |\eta_0|, 1)$$

such that, with $B_0=B(\xi_0+\eta_0, \delta|\xi_0|)$ and $D_0=B(\eta_0, \delta|\xi_0|)$, $A(\xi, \eta)^{-1}\in M(B_0\times D_0)$. A4$\frac{1}{2}$ and A5 will be used in the homogeneous case (A0 holds). In that case A5$\Rightarrow$A4$\frac{1}{2}$. In fact, if A5 holds, we choose a finite set of points $\{\xi_0^{(j)}\}_{j=1}^J$ on $\{1\leqq|\xi|\leqq r\}$ with corresponding sets $U^{(j)}$ and $V^{(j)}$ such that $\bigcup_{j=1}^J U^{(j)}\supset\{|\xi|\geqq r\}$ and

$$A(\xi, \eta)^{-1}\in M(U^{(j)}\times V^{(j)}).$$

Consequently, $\bigcup_{j=1}^J r^k U^{(j)}\supset\{|\xi|\geqq r^{k+1}\}$ and $A(\xi, \eta)^{-1}\in M(r^k U^{(j)}\times r^k V^{(j)})$ for every $k\in\mathbf{Z}$. Let $\xi_0\neq 0$, without loss of the generality, we may assume that $1\leqq|\xi_0|\leqq r$. Then there exists $U^{(j)}$ such that $\xi_0\in r^{-2}U^{(j)}$. Choose $\delta'>0$ small enough such that $B(\xi_0, 2\delta'|\xi_0|)\subset r^{-2}U^{(j)}$. If $|\eta_0^{(j)}|<\delta' r^2|\xi_0|$, let $\eta_0=r^{-2}\eta_0^{(j)}$, $\delta=\min(\delta'\ \delta^{(j)}/r^3)$, $B_0=B(\xi_0+\eta_0, \delta|\xi_0|)$ and $D_0=B(\eta_0, \delta|\xi_0|)$, then $B_0\subset r^{-2}U^{(j)}$, $D_0\subset r^{-2}V^{(j)}$ and

$$\|A^{-1}\|_{M(B_0\times D_0)} \leqq \|A^{-1}\|_{M(r^{-2}U^{(j)}\times r^{-2}V^{(j)})} < \infty.$$

If $|\eta_0^j| \geqq \delta' r^2 |\xi_0|$, let $\eta_0 = r^{-k-2}\eta_0^{(j)}$ where $k = [\log_r |\eta_0^{(j)}|/\delta' |\xi_0|] + 1$,

$$\delta = \min\left(\delta',\, \delta' \frac{\delta^{(j)}}{r^3 |\eta^{(j)}|}\right),$$

$B_0 = B(\xi_0 + \eta_0, \delta |\xi_0|)$ and $D_0 = B(\eta_0, \delta |\xi_0|)$, then $r^k B_0 \subset r^{-2} U^{(j)}$, $r^k D_0 \subset r^{-2} V^{(j)}$ and hence

$$\|A^{-1}\|_{M(B_0 \times D_0)} = \|A^{-1}\|_{M(r^k B_0 \times r^k D_0)} \leqq \|A^{-1}\|_{M(r^{-2} U^{(j)} \times r^{-2} V^{(j)})} < \infty,$$

i.e. $A4\frac{1}{2}$ holds.

Remark. 2. The assumptions A4$\frac{1}{2}$ and A5 are asymmetric in ξ and η, consequently the theorems below will be asymmetric too.

As in Triebel [6], let $Z(\mathbf{R}^d)$ denote the set

$$\{f \in S(\mathbf{R}^d) \colon D^\alpha \hat{f}(0) = 0, \text{ for every } \alpha\}.$$

Let b^s_∞ denote the closure of $Z(\mathbf{R}^d)$ in B^s_∞ and CMO denote the closure of $Z(\mathbf{R}^d)$ in BMO.

On examples whose kernels $A(\xi, \eta)$ satisfy A4$\frac{1}{2}$ or A5, see § 1 and § 6 of [2]. In particular, the kernels of Hankel operators, commutators, higher order commutators and paraproducts satisfy A4$\frac{1}{2}$ and A5. As well known, Hartman [1] and Sarason [5] have proved that a Hankel operator Γ_φ is compact if and only if $\varphi \in$ CMO, and Uchiyama [7] has proved that a commutator $[K, b]$ is compact if and only if $b \in$ CMO, so Theorem 2 below is a generalization of their results.

The main results of this paper are the following two theorems.

Theorem 1. *Suppose that A satisfies* A0 *with some* $r > 1$, A1, A3 (γ) *and* A4$\frac{1}{2}$, *then T_b^{st} being compact implies that* $b \in b^{s+t}_\infty$.

Theorem 2. *Suppose that A satisfies* A0 *with some* $r > 1$, A1, A3 (γ) *and* A5, *then T_b being compact implies that* $b \in$ CMO.

We need some lemmas.

Lemma 1. *If T is a compact operator on $L^2(\mathbf{R}^d)$ and $f_j \to 0$ weakly in $L^2(\mathbf{R}^d)$ as $j \to \infty$, then $\|Tf_j\|_2 \to 0$.*

This is well-known.

Lemma 2. *If g is a positive continuous function with compact support, $g_r(x) = r^{d/2} g(rx)$, and if $|f_{r,\omega}(x)| \leqq g_r(x)$ then $f_{r,\omega} \to 0$ weakly in $L^2(\mathbf{R}^d)$ and uniformly in ω as $r \to 0$ or $r \to \infty$.*

This is obvious.

Lemma 3. *Let* $b \in B^s_\infty$. *Then* $b \in b^s_\infty$ *if and only if* b *satisfies the following three conditions*

$$\text{(3)}\quad \begin{aligned} &\text{(i) } 2^{ks}\|b*\psi_k\|_\infty \to 0 \quad \text{as} \quad k \to +\infty,\\ &\text{(ii) } 2^{ks}\|b*\psi_k\|_\infty \to 0 \quad \text{as} \quad k \to -\infty,\\ &\text{(iii) } |b*\psi_k(x)| \to 0 \quad \text{as} \quad |x| \to \infty, \ \textit{for every } k, \end{aligned}$$

where ψ *is an arbitrary test function in* $S(\mathbf{R}^d)$ *such that* $\operatorname{Re}\hat{\psi}(\xi) \geqq c > 0$ *on* Δ_0, $\operatorname{supp}\hat{\psi} \subset \{r \leqq |\xi| \leqq R\}$ *for some* $0<r<1$, $2<R<\infty$, *and* $\hat{\psi}_k(\xi) = \hat{\psi}(2^{-k}\xi)$.

Remark 3. It is easy to see that under the conditions (i) and (ii), the condition (iii) is equivalent to the condition

$$\text{(iii)}' \quad \sup_k 2^{ks}|b*\psi_k(x)| \to 0 \quad \text{as} \quad |x| \to \infty.$$

Lemma 4. *Let* $b \in \mathrm{BMO}$. *Then* $b \in \mathrm{CMO}$ *if and only if* b *satisfies the following three conditions*

$$\text{(4)}\quad \begin{aligned} &\text{(i) } \lim_{a \downarrow 0} \sup_{|Q|=a} M(b, Q) = 0,\\ &\text{(ii) } \lim_{a \uparrow \infty} \sup_{|Q|=a} M(b, Q) = 0,\\ &\text{(iii) } \lim_{|x|\to\infty} M(b, Q+x) = 0 \quad \textit{for each} \quad Q, \end{aligned}$$

where

$$M(b, Q) = \inf_{c \in \mathbf{C}} \left\{ \frac{1}{|Q|} \int_Q |b(y)-c|\, dy \right\}.$$

The proof of Lemma 3 is omitted here. We refer to Peng [4]. Lemma 4 is due to Herz, Strichartz and Sarason, and a proof is given by Uchiyama [7].

We will prove Theorem 1 and 2 in § 2 and § 3, respectively.

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2. Proof of Theorem 1

For the sake of simplicity, we assume that $r=2$ in A0. By the assumption A4$\frac{1}{2}$ and Remark 1, there exist finite sets of points $\{\xi_0^{(j)}\}_{j=1}^J$ in Δ_0 and $\{\eta_0^{(j)}\}_{j=1}^J$ with corresponding open balls $B(\xi_0^{(j)}, \delta^{(j)})$ and $B(\eta_0^{(j)}, \delta^{(j)})$ such that $\eta_0^{(j)} \neq 0$, $\eta_0^{(j)} \neq -\xi_0^{(j)}$, $\bigcup_{j=1}^J B(\xi_0^{(j)}, \delta^{(j)}) \supset \Delta_0$, $\delta^{(j)} < \frac{1}{3}\min(|\xi_0^{(j)}+\eta_0^{(j)}|, |\eta_0^{(j)}|, 1)$, and with $B_j = B(\xi_0^{(j)}+\eta_0^{(j)}, \delta^{(j)})$ and $D_j = B(\eta_0^{(j)}, \delta^{(j)})$, $A^{-1} \in M(B_j \times D_j)$.

We choose the positive functions $h_j'(\xi)$ and $h_j(\eta)$ such that $h_j', h_j \in C_0^\infty(\mathbf{R}^d)$ $\operatorname{supp} h_j' = \bar{B}_j$, $h_j'(\xi) > 0$ on B_j, $\operatorname{supp} h_j = \bar{D}_j$, and $h_j(\eta) > 0$ on D_j. Let

$$\hat{\psi}(\xi) = \sum_{j=1}^J \int |\xi+\eta|^s |\eta|^t h_j'(\xi+\eta) h_j(\eta)\, d\eta.$$

Then $\hat{\psi} \in C_0^\infty(\mathbf{R}^d)$, $\operatorname{supp} \hat{\psi} \subset \{\frac{1}{3} \leqq |\xi| \leqq 2 + \frac{2}{3}\}$, and $\hat{\psi}(\xi) \geqq c > 0$ on Δ_0. Thus ψ can be used to define the norm of B_∞^{s+t}, in particular it can be used to Lemma 3.

Since A4$\frac{1}{2}$⇒A4, by Theorem 9.1 of [2], we know that $b \in B_\infty^{s+t}$. If $b \notin b_\infty^{s+t}$, by Lemma 3, b does not satisfy at least one of (i), (ii) and (iii) in (3).

If b does not satisfy (i) then there exists a subsequence $k_\nu \to +\infty$ as $\nu \to \infty$, a sequence of points $x_\nu \in \mathbf{R}^d$ and $\varepsilon_0 > 0$ such that

$$2^{k_\nu(s+t)} |b * \psi_{k_\nu}(x_\nu)| \geqq \varepsilon_0. \tag{5}$$

We shall show that (5) contradicts the compactness of T_b^{st}. Let

$$\widehat{f_\nu^{(j)}}(\xi) = 2^{-k_\nu d/2} h_j'(2^{-k_\nu}\xi)\, e^{ix_\nu \cdot \xi},$$

$$\widehat{g_\nu^{(j)}}(\eta) = 2^{-k_\nu d/2} h_j(2^{-k_\nu}\eta)\, e^{-ix_\nu \cdot \eta}.$$

Then

$$\|f_\nu^{(j)}\|_2 = C_j', \quad \|g_\nu^{(j)}\| = C_j,$$

thus we have

$$2^{k_\nu(s+t)} |b * \psi_{k_\nu}(x_\nu)| = C \left| \int \hat{b}(\xi) 2^{k_\nu(s+t)} \hat{\psi}_{k_\nu}(\xi) e^{ix_\nu \cdot \xi}\, d\xi \right|$$

$$= C \left| \sum_{j=1}^J \iint \hat{b}(\xi) |\xi+\eta|^s |\eta|^t 2^{-k_\nu d} h_j'(2^{k_\nu}(\xi+\eta)) h_j(2^{-k_\nu}\eta) e^{ix_\nu \cdot \xi}\, d\xi\, d\eta \right|$$

$$= C \left| \sum_{j=1}^J \iint \hat{b}(\xi-\eta) |\xi|^s |\eta|^t \widehat{f_\nu^{(j)}}(\xi) \widehat{g_\nu^{(j)}}(\eta)\, d\xi\, d\eta \right|.$$

Since $A^{-1} \in M(B_j \times D_j)$, it has the representation

$$A(\xi, \eta)^{-1} = \int_\Omega \alpha(\xi, \omega) \beta(\eta, \omega)\, d\mu(\omega)$$

with $\|\alpha\|_{L^\infty(B_j \times \Omega)}, \|\beta\|_{L^\infty(D_j \times \Omega)} \leqq 1$ and $\mu(\Omega) \leqq \|A^{-1}\|_{M(B_j \times D_j)}$. Note that

$$A(2^{-k_\nu}\xi, 2^{-k_\nu}\eta) = A(\xi, \eta),$$

thus we have

$$2^{k_\nu(s+t)} |b * \psi_{k_\nu}(x_\nu)|$$

$$\leqq C \sum_{j=1}^J \left| \iiint_\Omega \hat{b}(\xi-\eta) A(\xi, \eta) |\xi|^s |\eta|^t \alpha(\xi/2^{k_\nu}, \omega) \right.$$

$$\left. \times \widehat{f_\nu^{(j)}}(\xi) \beta(\eta/2^{k_\nu}, \omega) \widehat{g_\nu^{(j)}}\, d\xi\, d\eta\, d\mu(\omega) \right|$$

$$\leqq C \sum_{j=1}^J \int_\Omega \|\widehat{T_b^{st} g_{\nu,\omega}^{(j)}}\|_{L^2(\mathbf{R}^d)} \|\widehat{f_{\nu,\omega}^{(j)}}\|_{L^2(\mathbf{R}^d)}\, d\mu(\omega)$$

(where $\widehat{f_{\nu,\omega}^{(j)}}(\xi)=\alpha(\xi/2^{k_\nu},\omega)\widehat{f_\nu^{(j)}}(\xi)$, $\widehat{g_{\nu,\omega}^{(j)}}(\eta)=\beta(\eta/2^{k_\nu},\omega)\widehat{g_\nu^{(j)}}(\eta)$)

$$\leqq C\sum_{j=1}^J\int_\Omega\|\widehat{T_b^{st}g_{\nu,\omega}^{(j)}}\|_{L^2(\mathbf{R}^d)}\,d\mu(\omega).$$

By Lemma 2, $f_{\nu,\omega}^{(j)}\to 0$ weakly in $L^2(\mathbf{R}^d)$ and uniformly in $\omega\in\Omega$ as $\nu\to\infty$, and by Lemma 1, $\|\widehat{T_b^{st}g_{\nu,\omega}^{(j)}}\|_{L^2(\mathbf{R}^d)}\to 0$ uniformly in $\omega\in\Omega$ as $\nu\to\infty$. This contradicts (5).

Similarly, we can show that b must satisfy (ii).

If b satisfies (i) and (ii), but does not satisfy (iii), then there exist k_0 and a sequence of points $\{x_\nu\}$ and $\varepsilon_0>0$ such that $|x_\nu|\to\infty$ as $\nu\to\infty$ and

$$(6)\qquad |b*\psi_{k_0}(x_\nu)|\geqq\varepsilon_0.$$

We shall now show that (6) contradicts the compactness of T_b^{st}. Without loss of generality, we assume that $k_0=0$. Let

$$\widehat{f_\nu^{(j)}}(\xi)=h_j'(\xi)\,e^{ix_\nu\cdot\xi}$$

$$\widehat{g_\nu^{(i)}}(\eta)=h_j(\eta)\,e^{-ix_\nu\cdot\eta}.$$

Then $\|f_\nu^{(j)}\|_2=C_j'$, $\|g_\nu^{(j)}\|_2=C_j$. Thus we have

$$|b*\psi(x_\nu)|=C\Big|\sum_{j=1}^J\iint\hat b(\xi)\,|\xi+\eta|^s|\eta|^t h_j'(\xi+\eta)\,h_j(\eta)\,e^{ix_\nu\cdot\xi}\,d\xi\,d\eta\Big|$$

$$=C\Big|\sum_{j=1}^J\iint\hat b(\xi)\,|\xi+\eta|^s|\eta|^t\widehat{f_\nu^{(j)}}(\xi)\widehat{g^{(j)}}(\eta)\,d\xi\,d\eta\Big|$$

$$\leqq C\sum_{j=1}^J\Big|\iiint_\Omega\hat b(\xi-\eta)\,A(\xi,\eta)\,|\xi|^s|\eta|^t\alpha(\xi,\omega)\widehat{f_\nu^{(j)}}(\xi)\,\beta(\eta,\omega)\widehat{g_\nu^{(j)}}(\eta)\,d\xi\,d\eta\,d\mu(\omega)\Big|$$

$$\leqq C\sum_{j=1}^J\int_\Omega\|\widehat{T_b^{st}g_{\nu,\omega}^{(j)}}\|_{L^2(\mathbf{R}^d)}\,d\mu(\omega)$$

where $\widehat{g_{\nu,\omega}^{(j)}}(\eta)=\beta(\eta,\omega)\widehat{g_\nu^{(j)}}(\eta)$. By the Riemann—Lebesgue lemma $g_{\nu,\omega}^{(j)}\to 0$ weakly in $L^2(\mathbf{R}^d)$ as $\nu\to\infty_2$ for every $\omega\in\Omega$, and by Lemma 1, $\|\widehat{T_b^{st}g_{\omega\nu,}^{(j)}}\|_{L^2(\mathbf{R}(^d}\to 0$ as $\nu\to\infty$ for every $\omega\in\Omega$, thus $\int_\Omega\|\widehat{T_b^{st}g_{\nu,\omega}^{(j)}}\|_{L^2(\mathbf{R}^d)}\,d\mu(\omega)\to 0$ as $\nu\to\infty$. This contradicts (6).

This completes the proof of Theorem 1.

3. Proof of Theorem 2

For the sake of simplicity, we assume that A is homogeneous of degree 0, i.e. A0 holds for every $r>0$. The proof for the general case is similar. (Cf. Janson and Peetre [2].)

By Theorem 10.1 of [2], we know that $b\in\mathrm{BMO}$, and by Theorem 1, we know that $b\in b^0_\infty$, i.e. (i), (ii) and (iii) in (3) hold for b. By Lemma 4, it suffices to show that (i), (ii) and (iii) in (4) hold for b.

As in the proof of Theorem 10.1 of Janson and Peetre [2], by A5, we may choose a finite set of points $\{\xi_0^j\}_{j=1}^J$ on the unit sphere and $\{\eta_0^{(j)}\}_{j=1}^J$ with corresponding sets $U^{(j)}$ and $V^{(j)}$ such that $\bigcup_{j=1}^J U^{(j)}\supset\{|\xi|\geqq 1\}$ and $A^{-1}\in M(U^{(j)}\times V^{(j)})$. Thus $\bigcup_{j=1}^J (U^{(j)}-\eta_0^{(j)})\supset\{|\xi|\geqq R\}$ for some large R. We fix $g\in L^2$ with $|g(x)|\geqq 1$ when $|x|<1$ and $\operatorname{supp}\hat{g}\subset B(0,\delta)$, where $\delta=\min_{1\leqq j\leqq J}\delta^{(j)}$. We may assume that $\delta<1$.

To show (i) in (4), for every $\varepsilon>0$, by (i) and (ii) in (3), there exists $K_\varepsilon>0$ to be an integer such that

$$\|b*\psi_k\|_\infty<\varepsilon \quad\text{if}\quad |k|>K_\varepsilon.$$

Let $r<\dfrac{\varepsilon}{2^{K_\varepsilon}}$. For $B=B(x_r, r)$, put

$$b(x)=\sum\nolimits_{-\infty}^{K_\varepsilon-1} b_k(x)+\sum\nolimits_{K_\varepsilon}^{m-3} b_k(x)+\sum\nolimits_{m-2}^{\infty} b_k(x) = b^{(1)}(x)+b^{(2)}(x)+b^{(3)}(x),$$

where $b_k(x)=b*\psi_k(x)$, $m=[\log_2 R/r]$. Now we estimate

$$\frac{1}{|B|}\int_{B(x_r,r)}|b(x)-b^{(1)}(x_r)-b^{(2)}(x_r)|\,dx$$

$$\leqq\frac{1}{|B|}\int_{B(x_r,r)}|b^{(1)}(x)-b^{(1)}(x_r)|\,dx+\frac{1}{|B|}\int_{B(x_r,r)}|b^{(2)}(x)-b^{(2)}(x_r)|\,dx$$

$$+\frac{1}{|B|}\int_{B(x_r,r)}|b^{(3)}(x)|\,dx = I_1+I_2+I_3.$$

For I_1, we use the standard estimates

$$\|\nabla b_k\|_\infty\leqq C2^k\|b_k\|_\infty.$$

Hence

$$I_1\leqq C\sum\nolimits_{-\infty}^{K_\varepsilon-1}\frac{1}{|B|}\int_{B(x_r,r)}2^k|x-x_r|\,dx\leqq C\sum\nolimits_{-\infty}^{K_\varepsilon-1}2^k r=Cr2^{K_\varepsilon}<C\varepsilon.$$

For I_2, we have, similarly,

$$I_2\leqq\sum\nolimits_{K_\varepsilon}^{m-3}\frac{1}{|B|}\int_{B(x_r,r)}\|\nabla b_k\|_\infty|x-x_r|\,dx\leqq C\varepsilon r\sum\nolimits_{K_\varepsilon}^{m-3}2^k\leqq C\varepsilon.$$

For I_3, we have

$$I_3 \leqq \left(\frac{1}{|B|}\int_{B(x_r, r)} |b^{(3)}(x)|^2\, dx\right)^{1/2} = |B|^{-1/2} \|b^{(3)}\|_{L^2(B(x_r, r))}$$

$$\leqq |B|^{-1/2} \|b^{(3)} g_{r, x_r}\|_{L^2(\mathbf{R}^d)} \quad \left(\text{where } g_{r, x_r}(x) = g\left(\frac{x-x_r}{r}\right)\right)$$

$$\leqq (2\pi)^{-3d/2} |B|^{-1/2} \|\widehat{b^{(3)}} * \hat{g}_{r, x_r}\|_{L^2(\mathbf{R}^d)}$$

$$\leqq (2\pi)^{-3d/2} |B|^{-1/2} \|\widehat{b^{(3)}} * \hat{g}_{r, x_r}\|_{L^2\left(|\xi| \geqq \frac{R}{r}\right)}$$

$$+(2\pi)^{-3d/2} |B|^{-1/2} \|\widehat{b^{(3)}} * \hat{g}_{r, x_r}\|_{L^2\left(|\xi| \leqq \frac{R}{r}\right)} = I_{31} + I_{32}.$$

Note that when $k > m+2$, $\operatorname{supp} \hat{b}_k * \hat{g}_{r, x_r} \subset \{|\xi| > R/r\}$. Thus we get

$$I_{32} \leqq (2\pi)^{-3d/2} |B|^{-1/2} \sum_{m-2}^{m+2} \|\hat{b}_k * \hat{g}_{r, x_r}\|_{L^2(\mathbf{R}^d)} = |B|^{-1/2} \sum_{m-2}^{m+2} \|b_k\, g_{r, x_r}\|_{L^2(\mathbf{R}^d)}$$

$$\leqq |B|^{-1/2} \sum_{m-2}^{m+2} \|b_k\|_\infty \|g_{r, x_r}\|_{L^2(\mathbf{R}^d)} \leqq C \sum_{m-2}^{m+2} \varepsilon = C\varepsilon.$$

Finally, for I_{31}, note that $\operatorname{supp} \widehat{b^{(i)}} * \hat{g}_{r, x_r} \subset B(0, R/r)$ for $i=1, 2$. Thus we have

$$I_{31} = (2\pi)^{-3d/2} |B|^{-1/2} \|\hat{b} * \hat{g}_{r, x_r}\|_{L^2\left(|\xi| \geqq \frac{R}{r}\right)}$$

$$\leqq (2\pi)^{-3d/2} |B|^{-1/2} \sum_{j=1}^{J} \|\hat{b} * \hat{g}_{r, x_r}\|_{L^2\left(\frac{U^{(j)} - \eta_0^{(j)}}{r}\right)}.$$

Hence, it suffices to show that for every j

$$(7) \qquad (2\pi)^{-3d/2} |B|^{-1/2} \|\hat{b} * \hat{g}_{r, x_r}\|_{L^2\left(\frac{U^{(j)} - \eta_0^{(j)}}{r}\right)} < \varepsilon,$$

when r is small enough.

To show (7), let

$$\hat{f}_r(\eta) = |B_r|^{-1/2} \hat{g}_{r, x_r}(\eta - \eta_0^{(j)}/r) = \omega_d^{-1/2} r^{d/2} \hat{g}(r\eta - \eta_0^{(j)})\, e^{-i(r\eta - \eta_0^{(j)}) \cdot x_r/r}.$$

Then $\|\hat{f}_r\|_2 = C$, $\operatorname{supp} \hat{f}_r \subset B(\eta_0^{(j)}/r, \delta/r) \subset \frac{1}{r} V^{(j)}$. Since $A^{-1} \in M(U^{(j)} \times V^{(j)})$, there is a representation of $A(\xi, \eta)^{-1}$,

$$A(\xi, \eta)^{-1} \chi_{U^{(j)}}(\xi)\, \chi_{V^{(j)}}(\eta) = \int_\Omega \alpha(\xi, \omega)\, \beta(\eta, \omega)\, d\mu(\omega)$$

such that

$$\|\alpha\|_{L^\infty(U^{(j)} \times \Omega)},\ \|\beta\|_{L^\infty(V^{(j)} \times \Omega)} \leqq 1$$

and

$$\int_{\Omega} d\mu(\omega) \leqq \|A^{-1}\|_{M(U^{(j)}\times V^{(j)})}.$$

Thus we have

$$(2\pi)^{-3d/2}|B|^{-1/2}\|\hat{b}*\hat{g}_{r,x_r}\|_{L^2\left(\frac{U^{(j)}-\eta_0^{(j)}}{r}\right)} = C\|\hat{b}*\hat{f}_r\|_{L^2\left(\frac{U^{(j)}}{r}\right)}$$

$$= C\left\|\int \hat{b}(\xi-\eta)\hat{f}_r(\eta)\,d\eta\right\|_{L^2\left(\frac{U^{(j)}}{r}\right)}$$

$$= C\left\|\int\int_{\Omega} \hat{b}(\xi-\eta)A(\xi,\eta)\alpha\left(\frac{\xi}{r},\omega\right)\beta\left(\frac{\eta}{r},\omega\right)\hat{f}_r(\eta)\,d\eta\,d\mu(\omega)\right\|_{L^2\left(\frac{U^{(j)}}{r}\right)}$$

$$\leqq C\int_{\Omega}\|\widehat{T_b f_{r,\omega}}\|_{L^2(\mathbf{R}^d)}\,d\mu(\omega),$$

where $\hat{f}_{r,\omega}(\eta) = \beta\left(\frac{\eta}{r},\omega\right)\hat{f}_r(\eta)$.

By Lemma 2, $\hat{f}_{r,\omega}\to 0$ weakly in $L^2(\mathbf{R}^d)$ and uniformly in $\omega\in\Omega$ as $r\to 0$, and by Lemma 1, $\|\widehat{T_b f_{r,\omega}}\|_{L^2(\mathbf{R}^d)}\to 0$ uniformly in $\omega\in\Omega$ as $r\to 0$, i.e. (7) holds.

A similar, but simpler argument shows that (ii) in (4) holds.

To show (iii) in (4), for a fixed $B=B(0,r)$, we may assume that $r=1$. For every $\varepsilon>0$, by (ii) in (3), there exists $K_\varepsilon>[\log_2 R]$ such that

$$\|b_k\|_\infty<\varepsilon, \quad \text{if} \quad k<-K_\varepsilon.$$

But

$$b(x) = \sum_{-\infty}^{-K_\varepsilon} b_k(x)+\sum_{-K_\varepsilon+1}^{\infty} b_k(x) = b^{(1)}(x)+b^{(2)}(x).$$

For $|x^0|$ large enough, we estimate

$$\int_{B(x^0,1)}|b(x)-b^{(1)}(x^0)|\,dx$$

$$\leqq \int_{B(x^0,1)}|b^{(1)}(x)-b^{(1)}(x^0)|\,dx+\int_{B(x^0,1)}|b^{(2)}(x)|\,dx = I_1+I_2.$$

For I_1, we have

$$I_1 \leqq \sum_{-\infty}^{-K_\varepsilon}\int_{B(x^0,1)}|\nabla b_k(\bar{x})|\,|x-x^0|\,dx \leqq C\sum_{-\infty}^{-K_\varepsilon}2^k\varepsilon < C\varepsilon.$$

For I_2, we have

$$I_2 \leqq \|b^{(2)}\|_{L^2(B(x^0,1))}$$

$$\leqq \|b^{(2)}g_{x^0}\|_{L^2(\mathbf{R}^d)} \text{ (where } g \text{ is as before, and } g_{x^0}(x) = g(x-x^0))$$

$$= (2\pi)^{-3d/2}\|\widehat{b^{(2)}}*\hat{g}_{x^0}\|_{L^2(\mathbf{R}^d)}$$

$$\leqq (2\pi)^{-3d/2}\|\widehat{b^{(2)}}*\hat{g}_{x^0}\|_{L^2(|\xi|\geqq R)}+(2\pi)^{-3d/2}\|\widehat{b^{(2)}}*\hat{g}_{x^0}\|_{L^2(|\xi|\leqq R)} = I_{21}+I_{22}.$$

Note that when $k > m+2 = [\log_2 R]+2$, $\operatorname{supp} \hat{b}_k * \hat{g}_{x^0} \subset \{|\xi| \geqq R\}$, so we have

$$I_{22} \leqq (2\pi)^{-3d/2} \sum_{-K_\varepsilon+1}^{m+2} \|\hat{b}_k * \hat{g}_{x^0}\|_{L^2(\mathbf{R}^d)} = \sum_{-K_\varepsilon+1}^{m+2} \|b_k g_{x^0}\|_{L^2(\mathbf{R}^d)}.$$

The sum has only finitely terms, and each term is bounded by

$$\Big(\int_{|x|>N} |b_k(x)\, g(x-x^0)|^2\, dx\Big)^{1/2} + \Big(\int_{|x|\leqq N} |b_k(x)\, g(x-x^0)|^2\, dx\Big)^{1/2}.$$

Because $|b_k(x)| \to 0$ as $|x| \to \infty$ and $|g(x-x^0)| \leqq C|x-x^0|^{-M}$ for some large M, when $|x-x^0|$ large enough,

$$I_{22} < C\varepsilon, \quad \text{if} \quad |x^0| \quad \text{large enough.}$$

For I_{21}, it suffices to show that

$$(8) \qquad (2\pi)^{-3d/2} \|\hat{b} * \hat{g}_{x^0}\|_{L^2(U^{(j)} - \eta_0^{(j)})} < \varepsilon, \quad \text{when} \quad |x^0| \text{ large enough.}$$

To show (8), let

$$\hat{f}_{x^0}(\eta) = \hat{g}_{x^0}(\eta - \eta_0^{(j)}) = \hat{g}(\eta - \eta_0^{(j)})\, e^{-i(\eta_0^{(j)} - \eta)\cdot x^0}.$$

Then $\|\hat{f}_{x^0}\|_{L^2} = C$ and $\operatorname{supp} \hat{f}_{x^0} \subset B(\eta_0^{(j)}, \delta) \subset V^{(j)}$. Thus we have

$$(2\pi)^{-3d/2} \|\hat{b} * \hat{g}_{x^0}\|_{L^2(U^{(j)} - \eta_0^{(j)})} = (2\pi)^{-3d/2} \|\hat{b} * \hat{f}_{x^0}\|_{L^2(U^{(j)})}$$

$$= C \Big\| \int \hat{b}(\xi - \eta) \hat{f}_{x^0}(\eta)\, d\eta \Big\|_{L^2(U^{(j)})}$$

$$= C \Big\| \int\int_\Omega \hat{b}(\xi - \eta) A(\xi, \eta)\, \alpha(\xi, \omega)\, \beta(\eta, \omega) \hat{f}_{x^0}(\eta)\, d\eta\, d\mu(\omega) \Big\|_{L^2(U^{(j)})}$$

$$\leqq C \Big\| \int_\Omega \widehat{T_b f_{x^0, \omega}} \Big\|_{L^2(\mathbf{R}^d)} d\mu(\omega)$$

where $\widehat{f_{x^0,\omega}}(\eta) = \beta(\eta, \omega) \hat{f}_{x^0}(\eta)$. By the Riemann—Lebesgue lemma, $\hat{f}_{x^0,\omega} \to 0$ weakly in $L^2(\mathbf{R}^d)$ as $|x^0| \to \infty$ for every $\omega \in \Omega$, and by Lemma 1,

$$\|\widehat{T_b f_{x^0,\omega}}\|_{L^2(\mathbf{R}^d)} \to 0 \quad \text{as} \quad |x^0| \to \infty \quad \text{for every } \omega \in \Omega,$$

hence

$$\int_\Omega \|\widehat{T_b f_{x^0,\omega}}\|_{L^2(\mathbf{R}^d)}\, d\mu(\omega) \to 0 \quad \text{as} \quad |x^0| \to \infty,$$

i.e. (8) holds.

This completes the proof of Theorem 2.

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