A-superharmonic functions and supersolutions of degenerate elliptic equations

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1. Introduction

We consider supersolutions of the equation

$$\nabla \cdot A(x, \nabla u) = 0 \tag{1.1}$$

where $A\colon G\times\mathbf{R}^n \to \mathbf{R}^n$ is a strictly monotone (usually non-linear) elliptic differential operator in an open set G in $\mathbf{R}^n$, $n \geqq 2$. The precise assumptions are given in Section 2. In connection with equations of the type (1.1) we refer, for example, to [9], [11], [15], [20], and [21]. Supersolutions of (1.1) are the functions $u \in \operatorname{loc} W_p^1(G)$ satisfying

$$\int_G A(x, \nabla u) \cdot \nabla\varphi \, dx \geqq 0$$

for all non-negative $\varphi \in C_0^\infty(G)$. Supersolutions in general fail to be continuous and, in order to have pointwise estimates, the above definition is not quite adequate. It is our purpose in this paper to show that the classical potential theoretical definition for superharmonic functions is pertinent also in non-linear situations, and that it indeed yields a class of functions which strictly includes the supersolutions of (1.1) and is closed under upper directed monotone convergence. More precisely, we say that a lower semicontinuous function $u\colon G \to \mathbf{R}\cup\{\infty\}$ is *A-superharmonic* if it satisfies the comparison principle: for each domain $D \subset\subset G$ and each function $h \in C(\bar{D})$ which is a solution of (1.1) in D, the condition $h \leqq u$ in ∂D implies $h \leqq u$ in D. The comparison principle is valid for solutions of (1.1), whence potential theoretical aspects can be salvaged.

It is shown that supersolutions of (1.1) can be redefined in a set of measure zero so that they become A-superharmonic and, conversely, that if u is a locally bounded A-superharmonic function, then u belongs locally to the Sobolev space W_p^1 and is a supersolution of (1.1). To sum up, we may say that A-superharmonic functions form a closure of supersolutions with respect to upper directed monotone convergence. It is worth noting that there are singular solutions of (1.1) which are not locally in W_p^1 but are A-superharmonic.

Solving the obstacle problem (to be specified below) is the most effective tool in this connection, and the key point is to have a solution that is continuous up to the boundary. This method is used to show that without any regularity requirement the given definition leads to the existence and the integrability of the gradient of an A-superharmonic function. Interior regularity for solutions to obstacle problems has been studied by several authors in a variety of situations, and the first treatment which covers equations of the type (1.1) is apparently due to J. H. Michael and W. P. Ziemer [18]. We show that the solution to the obstacle problem with a continuous obstacle is continuous not only inside the domain but also at each boundary point where the Wiener criterion is satisfied. (For solutions of (1.1) this is known [15].) As a by-product the lower semicontinuity of supersolutions is attained, cf. [22]. The obstacle problem is discussed in Section 2, and the relation between supersolutions and A-superharmonic functions is examined in Section 3.

The fact that the definition for supersolutions through the comparison principle is useful also in the case of non-linear equations was first observed by S. Granlund, P. Lindqvist and O. Martio [3], [4], [13]. They exploited the obstacle method in studying sub-extremals of convex variational integrals in the borderline case, $p=n$, with applications to function theory. In [12] A-superharmonic functions are introduced for the p-harmonic operator, $A(x, h)=|h|^{p-2}h$, which is indeed a prototype of the operators considered here. We also extend some of their results; for example, the regularity of locally bounded sub-extremals (defined through the comparison principle) for a class of variational integrals is proved in [4], [12].

We close the paper in Section 4, where removable sets for A-superharmonic functions are studied. Some observations seem to be new even for solutions of (1.1); we show, for example, that compact sets with zero $(n-1)$-measure are removable for locally lipschitz solutions.

Notation. We use fairly standard notation. For $G\subset\mathbf{R}^n$ open and $A\subset\mathbf{R}^n$ measurable, $n\geqq 2$, the familiar function spaces are denoted as $C^\infty(G)$, $C_0^\infty(G)$, $C(A)$, $L^p(A)$, $W_p^1(G)$, $W_{p,0}^1(G)$, $\operatorname{loc} L^p(A)$ etc.; the space $L_p^1(G)$ consists of functions u whose distributional gradient ∇u belongs to $L^p(G)$. If $B=B(x_0, r)\subset\mathbf{R}^n$ is an open ball and $\sigma>0$, then $\sigma B=B(x_0, \sigma r)$. Integral averages are marked as usual, $⨍_A u\,dx=\dfrac{1}{|A|}\int_A u\,dx$, $|A|=\int_A dx$. For open sets D and G, $D\subset\subset G$ means that $\bar{D}$, the closure of D, is compact in G.

If $1\leqq q<\infty$, then the q-capacity of the condenser (C, G) is the number

$$\operatorname{cap}_q(C, G)=\inf\int_G |\nabla u|^q\,dx$$

where u runs through all functions in $C_0^\infty(G)$ with $u \geqq 1$ in C; here a condenser means a pair (C, G) where G is open in $\mathbf{R}^n$ and $C \subset G$ is compact. For $G=\mathbf{R}^n$ we let $\operatorname{cap}_q(C, \mathbf{R}^n)=\operatorname{cap}_q C$. The basic properties of variational capacities can be found, for example, in [16], [19].

In general c is a constant which may change from one line to the next.

2. Properties of supersolutions

We investigate supersolutions of the equation (1.1). It is shown that supersolutions are lower semicontinuous after a redefinition in a set of measure zero and that under suitable (but rather weak) conditions the obstacle problem has a continuous solution up to the boundary.

Throughout this section we assume that G is an open set in $\mathbf{R}^n$, $n \geqq 2$, and that the operator $A\colon G \times \mathbf{R}^n \to \mathbf{R}^n$ satisfies the following assumptions for some constants $1<p<\infty$ and $0<\alpha \leqq \beta<\infty$:

(2.1) the function $x \mapsto A(x, h)$ is measurable for all $h \in \mathbf{R}^n$, and the function $h \mapsto A(x, h)$ is continuous for a.e. $x \in G$;

for all $h \in \mathbf{R}^n$ and a.e. $x \in G$

$$A(x, h) \cdot h \geqq \alpha |h|^p, \tag{2.2}$$

$$|A(x, h)| \leqq \beta |h|^{p-1}, \tag{2.3}$$

$$\big(A(x, h_1)-A(x, h_2)\big) \cdot (h_1-h_2) > 0, \tag{2.4}$$

whenever $h_1 \neq h_2$, and

$$A(x, \lambda h) = |\lambda|^{p-2} \lambda A(x, h) \tag{2.5}$$

for all $\lambda \in \mathbf{R}$, $\lambda \neq 0$.

Operators satisfying (2.1)—(2.5) have been studied earlier e.g. by V. G. Maz'ya [15], see also [11], [20], and [21].

A function u in $\operatorname{loc} W_p^1(G)$ is an *A-supersolution* in G if

$$\int_G A(x, \nabla u) \cdot \nabla \varphi \, dx \geqq 0 \tag{2.6}$$

for all non-negative $\varphi \in C_0^\infty(G)$. Observe that (2.5) implies: if u is an A-supersolution, then so is $\lambda u+\mu$ whenever $\lambda \geqq 0$ and $\mu \in \mathbf{R}$. This homogeneity assumption can be used in many places to replace the linearity.

We require the following form of the comparison principle; see also 3.7 below.

2.7. Lemma. *If u, $-v\in W_p^1(G)$ are two A-supersolutions in G and if*

$$w = \min(u-v, 0)\in W_{p,0}^1(G),$$

then $u\geqq v$ a.e. in G.

Proof. Since

$$0 \leqq \int_G A(x, \nabla v)\cdot\nabla w\,dx - \int_G A(x, \nabla u)\cdot\nabla w\,dx$$

$$= -\int_{\{u<v\}} \big(A(x, \nabla v)-A(x, \nabla u)\big)\cdot(\nabla v-\nabla u)\,dx \leqq 0,$$

then $\nabla w=0$ a.e. in G and the lemma follows.

Next suppose that G is bounded, that ψ is a function in G and that $\theta\in W_p^1(G)$ is such that $\theta\geqq\psi$ a.e. in G. A function $u\in W_p^1(G)$ with $u-\theta\in W_{p,0}^1(G)$ and $u\geqq\psi$ a.e. in G is *a solution to the obstacle problem with the obstacle ψ and with boundary values θ* if (2.6) holds for all $\varphi\in W_{p,0}^1(G)$ with $\varphi\geqq\psi-u$ a.e. in G.

The operator A defines a strictly monotone, coercive and continuous mapping from the space $L_p^1(G)$ onto its dual, cf. e.g. [15]. Hence there exists a unique solution u to the above obstacle problem, see [9], p. 87. Clearly u is also an A-supersolution in G.

2.8. Lemma. *Suppose that u is a solution to the obstacle problem in G with the obstacle ψ and with boundary values θ. If $v\in W_p^1(G)$ is an A-supersolution in G with $v\geqq\psi$ a.e. in G and* $\min(v, u)-\theta\in W_{p,0}^1(G)$, *then $v\geqq u$ a.e. in G.*

Proof. The non-negative function

$$\eta = u-\min(v, u)$$

belongs to $W_{p,0}^1(G)$ and $-\eta\geqq\psi-u$ a.e. in G. Thus

$$0 \leqq \int_G \big(A(x, \nabla v)-A(x, \nabla u)\big)\cdot\nabla\eta\,dx$$

$$= \int_G \big(A(x, \nabla\min(v, u))-A(x, \nabla u)\big)\cdot\big(\nabla u-\nabla\min(v, u)\big)\,dx \leqq 0.$$

It follows that $\eta=0$ a.e. in G as required.

2.9. Corollary. *If u and v are two A-supersolutions in G, then* $\min(u, v)$ *is an A-supersolution in G.*

Proof. Choose $D\subset\subset G$. Let $\tilde{w}$ be the solution to the obstacle problem in D with the obstacle and boundary values $w=\min(u, v)$. Then, by Lemma 2.8, $u\geqq\tilde{w}$ and $v\geqq\tilde{w}$ a.e. in G. Thus $w=\tilde{w}$ is an A-supersolution in D as desired.

For each bounded G and $\theta\in W_p^1(G)$ there is a unique solution u of (1.1) with boundary values θ, that is, $u-\theta \in W^1_{p,0}(G)$ and

$$\int_G A(x, \nabla u)\cdot \nabla\varphi\, dx = 0$$

for all $\varphi \in W^1_{p,0}(G)$. As known, weak solutions of (1.1) are actually continuous [20], [21]. Further, if $\theta \in C(\bar{G})$ and the *Wiener criterion*

$$\int_0^1 \frac{\varphi(t)^{1/p-1}}{t}\, dt = \infty, \tag{2.10}$$

$$\varphi(t) = \begin{cases} t^{p-n}\operatorname{cap}_p(\bar{B}(x,t)\cap(\mathbf{R}^n\setminus G)), & p \neq n, \\ \operatorname{cap}_n(\bar{B}(x,t)\cap(\mathbf{R}^n\setminus G), B(x,2t)), & p = n, \end{cases}$$

holds at $x \in \partial G$, then

$$\lim_{y\to x} u(y) = \theta(x). \tag{2.11}$$

Therefore we say that a bounded open set G is *regular* if (2.10) holds at each $x \in \partial G$. Observe that balls and polygons are regular and, in particular, that each open set can be exhausted by regular ones. Finally, if $p > n$, then (2.11) always holds. For this discussion the reader is referred to [15] (see also [13]).

For the next theorem suppose that G is bounded, that $\psi \in C(G)$ and that $\theta \in W^1_p(G)$ is such that $\theta \geqq \psi$ a.e. in G.

2.12. Theorem. *There is a unique function $u \in C(G)\cap W^1_p(G)$ with $u-\theta \in W^1_{p,0}(G)$, $u \geqq \psi$ in G and*

$$\int_G A(x, \nabla u)\cdot \nabla\varphi\, dx \geqq 0 \tag{2.13}$$

for all $\varphi \in W^1_{p,0}(G)$ with $\varphi \geqq \psi - u$ a.e. in G; in the open set $\{x \in G\colon u(x) > \psi(x)\}$ u is a solution of (1.1).

If, moreover, G is regular and $\theta \in C(\bar{G})$, then $u \in C(\bar{G})$ and $u=\theta$ in ∂G.

The existence and the uniqueness of the solution $u \in W^1_p(G)$ were discussed above, and we proceed to prove that u is actually continuous. Our reasoning follows the usual track through Harnack type inequalities and the Moser iteration scheme (cf. [20], [21]), the situation being simpler than that in [18].

The lower semicontinuity of supersolutions is also established, cf. [22].

2.14. Theorem. *Suppose that u is an A-supersolution in G. Then u is locally essentially lower bounded, and there is a lower semicontinuous (lsc) version of u with*

$$u(x) = \operatorname{ess}\varliminf_{y\to x} u(y) \tag{2.15}$$

for each $x \in G$.

We require several estimates in which the regularity of the obstacle plays no role. In what follows we use the notation $u^+ = \max(u, 0)$, $u^- = \min(u, 0)$ and $\varepsilon \in \{+, -\}$.

2.16. Lemma. *Suppose that D is an open subset of G. If either*

(i) $\varepsilon=+$, *and u is a solution to the obstacle problem in G with the obstacle $\psi\leqq 0$ in D,*

or

(ii) $\varepsilon=-$, *and u is an A-supersolution in G,*

then

$$\int_D |u^\varepsilon|^q |\nabla u^\varepsilon|^p \eta^p\,dx \leqq c(p,\beta/\alpha)\int_D |u^\varepsilon|^{p+q}|\nabla\eta|^p\,dx \tag{2.17}$$

whenever $\eta\in C_0^\infty(D)$ is non-negative and $q\geqq 0$.

Proof. Pick $\eta\in C_0^\infty(D)$, $0\leqq\eta\leqq 1$. Since the function $\varphi=-u^\varepsilon\eta^p$ belongs to $W^1_{p,0}(D)$ and since $\varphi\geqq\psi-u$ (case (i)) or $\varphi\geqq 0$ (case (ii)) a.e. in G, then

$$\begin{aligned}0 &\leqq \int_G A(x,\nabla u)\cdot(-\nabla u^\varepsilon\eta^p - pu^\varepsilon\eta^{p-1}\nabla\eta)\,dx\\ &\leqq -\alpha\int_D |\nabla u^\varepsilon|^p\eta^p\,dx + p\beta\int_D |u^\varepsilon||\nabla\eta|\eta^{p-1}|\nabla u^\varepsilon|^{p-1}\,dx.\end{aligned}$$

Now Hölder's inequality yields

$$\int_D |\nabla u^\varepsilon|^p\eta^p\,dx \leqq c(p,\beta/\alpha)\int_D |u^\varepsilon|^p|\nabla\eta|^p\,dx.$$

Thus it follows that

$$\int_D |\nabla(u-k)^\varepsilon|^p\eta^p\,dx \leqq c(p,\beta/\alpha)\int_D |(u-k)^\varepsilon|^p|\nabla\eta|^p\,dx$$

for $k\geqq 0$ (case (i)) or $k\leqq 0$ (case (ii)). Multiplying both sides by $|k|^{q-1}$, $q>0$, and integrating (cf. e.g. [16], Theorem 1.2.3) yields the desired estimate (2.17).

2.18. Lemma. *Suppose that $B\subset G$ is a ball and that either* (i) *or* (ii) *of Lemma* 2.16 *holds for $B=D$. Then*

$$\operatorname*{ess\,sup}_{\sigma B}|u^\varepsilon| \leqq c(1-\sigma)^{-\xi}\Big(\fint_B |u^\varepsilon|^q\,dx\Big)^{1/q} \tag{2.19}$$

whenever $0<\sigma<1$ and $0<q\leqq p$. Here $\xi=\dfrac{n}{q}$ if $p<n$, $\xi=\dfrac{2p}{q}$ if $p\geqq n$, and $c=c(n,p,q,\beta/\alpha)>0$.

Proof. We may assume that $B=B(0,1)$. Write

$$r_l=\sigma+(1-\sigma)2^{-l},$$

$l=0,1,2,\dots$, and let $\eta_l\in C_0^\infty(r_lB)$ be such that $\eta_l=1$ in $r_{l+1}B$ and that $|\nabla\eta_l|\leqq$

$c(1-\sigma)^{-1}2^l$ where $c>0$ does not depend on l. Next, writing

$$w_l = (u^\varepsilon)^{1+s/p}\eta_l$$

for $s\geqq 0$ and using the estimate (2.17) yields

$$\int_{r_l B} |\nabla w_l|^p\, dx \leqq c(p+s)^p \int_{r_l B} |u^\varepsilon|^{s+p} |\nabla \eta_l|^p\, dx$$
$$\leqq c(p+s)^p (1-\sigma)^{-p} 2^{lp} \int_{r_l B} |u^\varepsilon|^{s+p}\, dx.$$

Now we employ the familiar Moser iteration, cf. e.g. [2], [20], [21]. If $\chi=\dfrac{n}{n-p}$ $(p<n)$ or $\chi=2$ $(p\geqq n)$ and $\varkappa=p+s$, then the Sobolev inequality yields

$$\left(\int_{r_{l+1} B} |u^\varepsilon|^{\varkappa\chi}\, dx\right)^{1/\varkappa\chi} \leqq c^{1/\varkappa} 2^{pl/\varkappa} \varkappa^{p/\varkappa} (1-\sigma)^{-p/\varkappa} \left(\int_{r_l B} |u^\varepsilon|^{\varkappa}\, dx\right)^{1/\varkappa}.$$

Choosing $\varkappa_l = p\chi^l$ and iterating we arrive at the desired estimate for $q=p$:

$$\operatorname*{ess\,sup}_{\sigma B} |u^\varepsilon| \leqq c(1-\sigma)^{-\xi} \left(\fint_B |u^\varepsilon|^p\, dx\right)^{1/p}; \tag{2.20}$$

here $c=c(n, p, \beta/\alpha)>0$ and $\xi=\dfrac{n}{p}\,(p<n)$ or $\xi=2\,(p\geqq n)$.

It is by now well known that the exponent p in (2.20) can be arbitrarily decreased (see e.g. [7]); to be precise, we apply an interpolation argument used by E. Di Benedetto and N. S. Trudinger [1], p. 299: for $0<q\leqq p$ let

$$\Phi(q) = \sup_{\frac{1}{2}<\sigma<1} (1-\sigma)^{\tilde q} \left(\fint_{\sigma B} v^p\, dx\right)^{1/p}$$

where $v=|u^\varepsilon|$, $\tilde q=(p-q)\xi/q$ and ξ is as in (2.20). Write $M_\sigma = \operatorname{ess\,sup}_{\sigma B} v$. Then (2.20) implies for every $\sigma\in(0,1)$ and $\sigma'=\frac{1}{2}(1+\sigma)$

$$M_\sigma (1-\sigma)^{\xi p/q} \leqq c(1-\sigma')^{\tilde q} \left(\fint_{\sigma' B} v^p\, dx\right)^{1/p} \leqq c\Phi(q),$$

and using Young's inequality yields

$$M_\sigma (1-\sigma)^{\xi p/q} \leqq c \left(\fint_B v^q\, dx\right)^{1/q}$$

where $c=c(n, p, q, \beta/\alpha)>0$ as desired. The lemma is completely proved.

2.21. *Remark.* It follows from Lemma 2.18 that if u is a solution to the obstacle problem (2.13), then u is locally essentially bounded above. Similarly, each $\mathcal{A}$-supersolution v is locally essentially bounded below.

In view of (2.5) it is not difficult to see that if u is a positive $\mathcal{A}$-supersolution, then the function $-1/u$ is also an $\mathcal{A}$-supersolution; for details see [8], (2.3). Thus Lemma 2.18 produces

2.22. Lemma. *Suppose that u is an A-supersolution, positive in a ball $B\subset G$. Then*

$$\operatorname*{ess\,inf}_{\sigma B} u \geqq c(1-\sigma)^{\xi}\left(\fint_{B} u^{-q}\,dx\right)^{-1/q}, \tag{2.23}$$

whenever $0<\sigma<1$ and $0<q\leqq p$. Here ξ is as in Lemma 2.18 and $c=c(n,p,q,\beta/\alpha)>0$.

2.24. Lemma. *Let u be a positive A-supersolution in G. Then*

$$\int_{C} |\nabla \log u|^{p}\,dx \leqq c(p,\beta/\alpha)\operatorname{cap}_{p}(C,G) \tag{2.25}$$

whenever C is compact in G.

Proof. We may assume that $\operatorname{ess\,inf} u>0$. Pick a non-negative $\varphi\in C_{0}^{\infty}(G)$ with $\varphi=1$ in C. Since the function $\eta=\varphi^{p}u^{1-p}\in W_{p,0}^{1}(G)$ is non-negative, then

$$0\leqq \int_{G} A(x,\nabla u)\cdot\left(p\varphi^{p-1}u^{1-p}\nabla\varphi-(p-1)u^{-p}\varphi^{p}\nabla u\right)dx$$

or

$$\begin{aligned}\int_{\operatorname{spt}\varphi}|\nabla u|^{p}u^{-p}\varphi^{p}\,dx &\leqq c(p,\beta/\alpha)\int_{\operatorname{spt}\varphi}|\nabla u|^{p-1}u^{1-p}|\nabla\varphi|\varphi^{p-1}\,dx\\ &\leqq c(p,\beta/\alpha)\left(\int_{\operatorname{spt}\varphi}|\nabla u|^{p}u^{-p}\varphi^{p}\,dx\right)^{(p-1)/p}\left(\int_{G}|\nabla\varphi|^{p}\,dx\right)^{1/p}.\end{aligned}$$

This establishes (2.25).

As known, the estimate (2.25) implies that the function $v=\log u$ belongs locally to $\operatorname{BMO}(G)$. Indeed, for each ball $B=B(x,R)$ with $2B\subset G$ we have

$$\int_{B}|\nabla v|^{p}dx\leqq\begin{cases}c(n,p,\beta/\alpha)R^{n-p}, & p\neq n,\\ c(n,\beta/\alpha), & p=n,\end{cases}$$

and it follows from the John—Nirenberg lemma [2], Theorem 7.21, that

$$\left(\fint_{B}u^{s}\,dx\right)^{1/s}\leqq c\left(\fint_{B}u^{-s}\,dx\right)^{-1/s}$$

whenever B is a ball with $2B\subset G$; here $s=s(n,p,\beta/\alpha)>0$ and $c=c(n,s)>0$. Thus (2.23) provides

2.26. Lemma. *Suppose that u is a positive A-supersolution in G. Then there is $s=s(n,p,\beta/\alpha)>0$ such that*

$$\operatorname*{ess\,inf}_{\sigma B} u\geqq c(1-\sigma)^{\xi}\left(\fint_{B}u^{s}\right)^{1/s} \tag{2.27}$$

whenever B is a ball with $2B\subset G$ and $0<\sigma<1$. Here ξ is as in Lemma 2.18 and $c=c(n,p,\beta/\alpha)>0$.

Proof of Theorem 2.14. In view of Remark 2.21 it suffices to find a version of u with the property (2.15). For that, suppose first $M=\sup_{G}u<\infty$ and fix $x\in G$.

Write $B_R = B(x, R)$ and

$$m_R = \operatorname*{ess\,inf}_{B_R} u$$

for R positive and small enough. We may assume that $u > 0$ in G and that $M > m_R$ for small R. Then (2.27) implies for some $0 < s \leqq 1$

$$m_{R/2} - m_R \geqq c\left(\fint_{B_R} (u-m_R)^s\, dx\right)^{1/s} \geqq c(M-m_R)^{(s-1)/s}\left(\fint_{B_R} (u-m_R)\, dx\right)^{1/s}.$$

Thus

$$\operatorname*{ess\,\underline{lim}}_{y \to x} u(y) = \lim_{R \to 0} \fint_{B(x,R)} u\, dx \tag{2.28}$$

for every $x \in G$, and, by the Lebesgue theorem, the proof is complete if u is bounded above. The general case follows from this since the functions $u_k = \min(u, k)$ are A-supersolutions and since a locally integrable lsc function can be redefined in a set of measure zero so that (2.15) holds. Theorem 2.14 is then proved.

Proof of Theorem 2.12. Let u be the lsc solution to the obstacle problem (2.13) such that (2.15) holds. Then the set $D = \{x \in G : u(x) > \psi(x)\}$ is open and the standard reasoning shows that u is a solution of (1.1) in D. Indeed, suppose that D is not empty and pick $\eta \in C_0^\infty(D)$ with $|\eta| \leqq 1$; then $-\delta = \sup_{\operatorname{spt}\eta} (\psi - u) < 0$ and for $\varepsilon \in (-\delta, \delta)$, $\varepsilon\eta \geqq \psi - u$ in G, whence

$$0 \leqq \int_D A(x, \nabla u) \cdot \nabla(\varepsilon\eta)\, dx$$

or

$$0 = \int_D A(x, \nabla u) \cdot \nabla\eta\, dx$$

as required. Thus there is a continuous version of u in D, and by (2.15) it equals u at each point $x \in D$.

We show next that

$$\operatorname*{ess\,\overline{lim}}_{y \to x} u(y) = u(x) \tag{2.29}$$

whenever $x \in G$; this, by (2.15) and by Remark 2.21, implies that $u \in C(G)$. So fix $x \in G$. Since, again by (2.15), $u \geqq \psi$ everywhere in G, we may assume that $u(x) = \psi(x)$. Now fix $\varepsilon > 0$ and choose a ball $B = B(x, R) \subset\subset G$ so that $\sup_B \psi < \psi(x) + \varepsilon = \lambda$ and so that $\inf u > u(x) - \varepsilon = \mu$. The estimate (2.19) yields

$$\operatorname*{ess\,sup}_{\frac{1}{2}B} (u-\lambda)^+ \leqq c \fint_B (u-\lambda)^+\, dx,$$

where c does not depend on R. On the other hand,

$$\fint_B (u-\lambda)^+\, dx = \fint_B (u - \min(u, \lambda))\, dx$$

$$\leqq \fint_B (u-\mu)\, dx = \fint_B u\, dx - u(x) + \varepsilon.$$

Since u is locally bounded above, the above inequalities together with (2.28) yield

$$\operatorname{ess}\varlimsup_{y\to x} u(y) \leqq \psi(x)+c\varepsilon = \operatorname{ess}\varliminf_{y\to x} u(y)+c\varepsilon.$$

Letting $\varepsilon\to 0$ establishes (2.29); thus $u\in C(G)$.

To prove the second assertion we first show that

$$\varlimsup_{y\to x} u(y) \leqq \theta(x) \tag{2.30}$$

for each $x\in\partial G$. Suppose that the open set $U=\{x\in G\colon u(x)>\theta(x)\}$ is not empty. Since $u=\theta$ in $\partial U\cap G$ and since $u-\theta\in W^1_{p,0}(G)$, it is not difficult to see that $u-\theta\in W^1_{p,0}(U)$. Thus u is the unique solution of (1.1) in U with boundary values $\theta\in C(\overline{U})\cap W^1_p(U)$. Since G is regular and since $U\subset G$, $\lim_{y\to x, y\in U} u(y)=\theta(x)$ for $x\in \partial U\cap\partial G$, cf. (2.11). Thus (2.30) follows.

To complete the proof, let $h\in C(\overline{G})$ be the unique solution of (1.1) in G with $h=\theta$ in ∂G. Thus it follows from Lemma 2.7 and from (2.15) that $u\geqq h$ in G, whence

$$\varliminf_{y\to x} u(y) \geqq \theta(x)$$

for $x\in\partial G$ as desired.

Theorem 2.12 is thereby completely proved.

2.31. *Remarks.* (a) The above proof yields also that $\lim_{y\to x} u(y)=\theta(x)$ whenever $\theta\in C(\overline{G})$ and (2.10) holds at $x\in\partial G$.

(b) Observe that Theorem 2.12 holds also under weaker requirements on ψ; this is made more explicit in [18].

We conclude this section with a lemma, required in Section 3.

2.32. Lemma. *Suppose that u_i is an increasing and locally bounded sequence of A-supersolutions in G. Then $u=\lim u_i$ is an A-supersolution in G. Moreover, $A(x, \nabla u_i)\to A(x, \nabla u)$ weakly in $L^{p/(p-1)}(D)$ whenever $D\subset\subset G$.*

Proof. Fix open sets $D\subset\subset D_0\subset\subset G$. Since we may clearly assume that $u<0$ in D_0, it follows from (2.17) that $u\in W^1_p(D_0)$ and that $\nabla u_i\to\nabla u$ weakly in $L^p(D_0)$. Now choose $\eta\in C_0^\infty(D_0)$ with $0\leqq\eta\leqq 1$ and $\eta=1$ in D. Write $\psi_i=\eta(u-u_i)$ and

$$I_i = \int_D \big(A(x, \nabla u)-A(x, \nabla u_i)\big)\cdot(\nabla u-\nabla u_i)\,dx.$$

Thus

$$\begin{aligned}
0 \leqq I_i &\leqq \int_{D_0} \eta\big(A(x, \nabla u)-A(x, \nabla u_i)\big)\cdot(\nabla u-\nabla u_i)\,dx\\
&= \int_{D_0}\big(A(x, \nabla u)-A(x, \nabla u_i)\big)\cdot\nabla\psi_i\,dx-\int_{D_0}(u-u_i)\big(A(x, \nabla u)-A(x, \nabla u_i)\big)\cdot\nabla\eta\,dx\\
&\leqq \int_{D_0} A(x, \nabla u)\cdot\nabla\psi_i\,dx-\int_{D_0}(u-u_i)\big(A(x, \nabla u)-A(x, \nabla u_i)\big)\cdot\nabla\eta\,dx\\
&= \int_{D_0}\eta A(x, \nabla u)\cdot(\nabla u-\nabla u_i)\,dx+\int_{D_0}(u-u_i)A(x, \nabla u_i)\cdot\nabla\eta\,dx.
\end{aligned}$$

The last two integrals tend to zero and, accordingly, so does the sequence I_i. This guarantees that $A(x, \nabla u_i) \to A(x, \nabla u)$ weakly in $L^{p/(p-1)}(D)$ (for the details see e.g. [15], Lemma 1). The lemma is thereby proved since the rest is now obvious.

3. A-superharmonic functions

We introduce A-harmonic and A-superharmonic functions and investigate the relation between A-supersolutions and A-superharmonic functions. In particular, it is shown that A-superharmonic functions form the closure of A-supersolutions with respect to upper directed monotone convergence.

Throughout this section let G be an open set in $\mathbf{R}^n$ and suppose that $A: G \times \mathbf{R}^n \to \mathbf{R}^n$ is an operator satisfying (2.1)—(2.5).

A weak solution $u \in \operatorname{loc} W_p^1(G)$ of the equation (1.1) is called *A-harmonic* in G if $u \in C(G)$. Recall that each weak solution of (1.1) is actually Hölder continuous. Also it is worth noting that $\lambda u + \mu$, $\lambda, \mu \in \mathbf{R}$, is A-harmonic whenever u is. Furthermore, Harnack's inequality holds; as in [20] we deduce from Lemmas 2.18 and 2.26 that if u is a non-negative A-harmonic function in a domain G and if C is compact in G, there is a constant $c = c(n, p, \alpha, \beta, C) \geqq 1$ with

$$\sup_C u \leqq c \inf_C u. \tag{3.1}$$

The class of A-harmonic functions is closed under uniform convergence.

3.2. Theorem. *Let u_i, $i = 1, 2, \dots$, be a sequence of A-harmonic functions in G such that $u_i \to u$ uniformly on compact subsets of G. Then u is A-harmonic.*

Proof. (For another proof see [3], Theorem 4.21.) Applying the argument used in the proof of Lemma 2.32 yields that $u \in C(G) \cap \operatorname{loc} W_p^1(G)$ and that $A(x, \nabla u_i) \to A(x, \nabla u)$ weakly in $L^{p/(p-1)}(D)$, $D \subset\subset G$. This shows that u is A-harmonic in G.

Harnack's principle, a celebrated tool in potential theory, also holds.

3.3. Theorem. *Suppose that u_i is an increasing sequence of A-harmonic functions in a domain G. Then the function $u = \lim u_i$ is either A-harmonic or identically $+\infty$ in G.*

Proof. Suppose that $u(x) < \infty$ for some $x \in G$. Then, by Harnack's inequality, u is locally bounded in G, and it follows from the Hölder continuity estimate [20], p. 269, that the sequence u_i is equicontinuous. Thus Ascoli's theorem together with Theorem 3.2 yields the claim.

3.4. *A-superharmonic functions.* A lower semicontinuous (lsc) function $u: G \to \mathbf{R} \cup \{\infty\}$ *is A-superharmonic* in G if for each domain $D \subset\subset G$ and each function

$h\in C(\bar{D})$, A-harmonic in D, $h\leqq u$ in ∂D implies $h\leqq u$ in D. An upper semicontinuous (usc) $u\colon G\to \mathbf{R}\cup\{-\infty\}$ is *A-subharmonic* if $-u$ is A-superharmonic.

The following two lemmas are obvious.

3.5. Lemma. *If u and v are A-superharmonic in G, then* $\min(u, v)$ *and* $\lambda u+\mu$, $\lambda\geqq 0$, $\mu\in\mathbf{R}$ *are also A-superharmonic.*

3.6. Lemma. *Let u_i, $i=1, 2, \ldots$, be A-superharmonic in G. If the sequence u_i is increasing or uniformly converging on compact subsets of G, then $u=\lim u_i$ is A-superharmonic in G.*

We need the following extension of [4], Lemma 2.3, cf. also Lemma 2.7 above.

3.7. Comparison principle. *Suppose that G is bounded and that u is A-subharmonic and v A-superharmonic in G. If*

$$\varlimsup_{y\to x} u(y) \leqq \varliminf_{y\to x} v(y)$$

for all $x\in\partial G$ and if the left and the right hand sides are not simultaneously ∞ or $-\infty$, then $u\leqq v$ in G.

Proof. Fix $x\in G$. Let $\varepsilon>0$ and choose a regular domain $D\subset\subset G$ such that $x\in D$ and that $u<v+\varepsilon$ in ∂D. Then let $\varphi_i\in C^\infty(G)$ be a decreasing and $\psi_i\in C^\infty(G)$ an increasing sequence with $\varphi_i\to u$ and $\psi_i\to v+\varepsilon$ in $\bar{D}$. Since ∂D is compact $\varphi_i\leqq\psi_i$ on ∂D for some i. Choosing A-harmonic functions h and g in D with boundary values φ_i and ψ_i, respectively, yields $u\leqq h\leqq g\leqq v+\varepsilon$ in D. By letting $\varepsilon\to 0$ we obtain $u(x)\leqq v(x)$ as desired.

3.8. *A-superharmonic functions versus A-supersolutions.* We first show that each A-supersolution can be made A-superharmonic after a change in a set of measure zero.

3.9. Theorem. *Suppose that u is an A-supersolution in G with*

$$u(x) = \operatorname{ess}\varliminf_{y\to x} u(y) \tag{3.10}$$

for each $x\in G$. Then u is A-superharmonic.

Proof. Note first that u is lsc in G, cf. Remark 2.21. Let $D\subset\subset G$ be a domain and let $h\in C(\bar{D})$ be A-harmonic in D with $h\leqq u$ in ∂D. Fix $\varepsilon>0$. Let $D_0\subset\subset D$ be an open set with $u+\varepsilon>h$ in $D\setminus D_0$. Write $v(x)=\min(u(x)+\varepsilon, h(x))$ for $x\in D_0$. Then v is an A-supersolution in D_0 and $\min(v-h, 0)\in W^1_{p,0}(D_0)$. Thus Lemma 2.7 yields $v\geqq h$ a.e. in D_0, whence, by (3.10), $u+\varepsilon\geqq h$ everywhere in D. Letting $\varepsilon\to 0$ establishes the desired conclusion.

If u is an A-supersolution in G, then, by Theorem 2.14, u can be redefined in a set of measure zero so that (3.10) holds. Thus we obtain

3.11. Corollary. *Suppose that u is an A-supersolution in G. Then there is an A-superharmonic function v in G such that $u=v$ a.e. in G.*

3.12. Theorem. *Suppose that u is A-superharmonic in G and that $D \subset\subset G$ is a domain. Then there is an increasing sequence of functions $u_i \in C(\bar{D}) \cap W_p^1(D)$ such that u_i are A-supersolutions in D and that $u=\lim u_i$ in $\bar{D}$. Moreover, u_i are A-superharmonic.*

Proof. We may suppose that D is regular. Choose an increasing sequence $\varphi_i \in C^\infty(\mathbf{R}^n)$ with $u=\lim \varphi_i$ in $\bar{D}$ and let $u_i \in C(\bar{D}) \cap W_p^1(D)$ be the solution to the obstacle problem with the obstacle φ_i in D and with boundary values $u_i=\varphi_i$ in ∂D. Then u_i is the desired sequence. In fact, u_i is A-superharmonic by Theorem 3.9; since u_i is A-harmonic in the open set $W=\{x \in D\colon u_i(x) > u_{i+1}(x)\}$, it follows from the comparison principle that $W=\emptyset$. Thus the sequence u_i is increasing. Moreover, $w=\lim u_i \geqq u$ in $\bar{D}$. To complete the proof, fix i. Since u_i is A-harmonic in the open set $U=\{x \in D\colon u_i(x) > \varphi_i(x)\}$ and since $\underline{\lim}_{y \to x} u(y) \geqq \varphi_i(x) = u_i(x)$ for each $x \in \partial U$, the comparison principle yields $u_i \leqq u$ in D, whence $w \leqq u$ in D. This completes the proof.

Now Theorem 3.12 together with Lemma 2.32 yields

3.13. Corollary. *Suppose that u is A-superharmonic in G. If u is locally bounded above, then $u \in \operatorname{loc} W_p^1(G)$ and u is an A-supersolution.*

Applying the proof of Lemma 2.32 to the functions $u_i=\min(u, i)$, $i=1, 2, \ldots$, Corollary 3.13 yields

3.14. Corollary. *Let u be A-superharmonic in G. Then u is an A-supersolution provided that $u \in \operatorname{loc} W_p^1(G)$.*

To obtain a converse to Theorem 3.9 we should establish (3.10) for A-superharmonic functions.

3.15. Theorem. *Suppose that u is A-superharmonic in G. Then*

$$u(x) = \operatorname{ess} \underline{\lim}_{y \to x} u(y)$$

for each $x \in G$.

Theorem 3.15 can be established by arguing as in [12], Theorem 5.4. However, the following lemma calls for a proof.

3.16. Lemma. *Suppose that u is A-superharmonic in G and that $u(x)=0$ for a.e. $x \in G$. Then $u(x)=0$ for each $x \in G$.*

Proof. Since u is lsc, $u \leqq 0$ in G. Then fix $x \in G$. We show that $u(x)=0$. For that, pick domains $D_0 \subset\subset D_1 \subset\subset G$ with $x \in D_0$. Let $u_i \in C(D_1) \cap W_p^1(D_1)$ be an increasing sequence of A-superharmonic functions in D_1 with $u=\lim u_i$. Then $\nabla u_i \to \nabla u = 0$ weakly in $L^p(D_1)$ and, as in the proof for Lemma 2.32, we obtain

$$\lim_{i\to\infty} \int_{D_0} (A(x, \nabla u_i) - A(x, \nabla u)) \cdot (\nabla u_i - \nabla u)\, dx = 0.$$

Then, since

$$\alpha \int_{D_0} |\nabla u_i|^p\, dx \leqq \int_{D_0} A(x, \nabla u_i) \cdot \nabla u_i\, dx$$

$$= \int_{D_0} (A(x, \nabla u_i) - A(x, \nabla u)) \cdot (\nabla u_i - \nabla u)\, dx,$$

$\nabla u_i \to 0$ in $L^p(D_0)$. Here we have made use of the fact that $\nabla u = 0$ a.e. in D_0.

Now choose a ball $B = B(x, r) \subset\subset D_0$, and the unique A-harmonic function $h_i \in C(\bar{B}) \cap W_p^1(B)$ with $h_i = u_i$ in ∂B. Let v_i be the Poisson modification,

$$v_i = \begin{cases} h_i & \text{in } B \\ u_i & \text{in } D_0 \setminus B, \end{cases}$$

which is easily seen to be A-superharmonic in D_0. Moreover, $v_i \leqq v_{i+1} \leqq u_{i+1}$ in D_0 by the comparison principle. Thus Lemma 3.6 and Theorem 3.3 imply that $v = \lim v_i$ is A-superharmonic in D_0 and A-harmonic in B. Since $v \leqq u$, it suffices to show that $v = 0$ in B.

To this end, observe that $\nabla v_i \to 0$ in $L^p(D_0)$ since

$$\alpha \int_B |\nabla h_i|^p\, dx \leqq \int_B A(x, \nabla h_i) \cdot \nabla h_i\, dx = \int_B A(x, \nabla h_i) \cdot \nabla u_i\, dx$$

$$\leqq \beta \left(\int_B |\nabla h_i|^p\, dx\right)^{(p-1)/p} \left(\int_B |\nabla u_i|^p\, dx\right)^{1/p}.$$

Then v is a constant a.e. in D_0. But since $v = u = 0$ a.e. in $D_0 \setminus B$, $v = 0$ a.e. in D_0. Thus, by continuity, $v = 0$ in B as desired.

Now Theorems 3.9 and 3.15 together with Corollaries 3.11 and 3.14 imply our main theorem:

3.17. Theorem. *Let u be a function in* $\operatorname{loc} W_p^1(G)$. *Then u is A-superharmonic in G if and only if u is an A-supersolution in G with*

$$u(x) = \operatorname*{ess\,\underline{lim}}_{y\to x} u(y) \tag{3.18}$$

for each $x \in G$. Moreover, if u is an A-supersolution in G, then (3.18) *holds a.e. in G.*

Using a partition of unity reveals the local nature of A-superharmonic functions; a special case of the following is shown in [3].

3.19. Corollary. *A function $u\colon G \to \mathbf{R} \cup \{\infty\}$ is A-superharmonic in G if and only if each $x \in G$ has a neighborhood D so that $u|_D$ is A-superharmonic in D.*

3.20. *Remarks.* (a) Since an A-supersolution has a lsc version, we may handle it as a pointwise defined function which is, in view of Corollary 3.11, A-superharmonic in G. Moreover, the class of A-superharmonic functions forms the closure of A-supersolutions with respect to upper directed monotone convergence. In fact, this follows from Lemma 3.6, since the functions $\min(u, k)$, $k=1, 2, \dots$, are A-supersolutions whenever u is A-superharmonic.

Furthermore, these two classes of functions are not equal for $p \leqq n$. For example, let A be the p-harmonic operator, $A(x, h)=|h|^{p-2}h$, $1<p\leqq n$. Then the function

$$u(x) = \int_{|x|}^{1} t^{(1-n)/(p-1)}\,dt$$

is A-superharmonic in $G=B(0, 1)$ but $u \notin \mathrm{loc}\, W_p^1(G)$, so u is not an A-supersolution. Similar examples can be constructed also for other operators, see [20]. For $p>n$ the situation differs, since by using the estimate (2.25) and Theorem 3.17 it is not difficult to show that each A-superharmonic function is continuous and therefore an A-supersolution by Corollary 3.13, cf. [12].

(b) Theorem 3.17 also yields: a function h is A-harmonic in G if and only if h and $-h$ are A-superharmonic in G.

(c) Given a regular open set G and a function $f \in C(\partial G)$ there exists a unique $u \in C(\bar{G})$, A-harmonic in G, with $u=f$ in ∂G. This follows easily from Harnack's principle and the comparison principle. It is not known to the authors whether the Wiener criterion (2.10) is also necessary for (2.11) if $p \leqq n-1$. For $p>n-1$ the necessity is proved in [13].

The Perron method to solve the generalized Dirichlet problem works also in this non-linear case [4], [10].

4. Removability theorems

We ask conditions which guarantee that a closed set E in G is removable for a given class of A-superharmonic functions in $G \setminus E$. It is shown that, for lower bounded A-superharmonic functions, E is removable if and only if E is of p-capacity zero, a well known theorem in classical potential theory. Further, larger sets are removable for A-superharmonic functions in $L_s^1(G \setminus E)$ with $s>p$.

Throughout this section let G be an open set in $\mathbf{R}^n$ and E a relatively closed set in G. Suppose that $A\colon G \times \mathbf{R}^n \to \mathbf{R}^n$ is an operator satisfying (2.1)—(2.5).

4.1. *Sets of q-capacity zero.* A compact set C in $\mathbf{R}^n$ is of *q-capacity zero* if $\mathrm{cap}_q(C, D)=0$ for each open neighborhood D of C. It is an easy task to show that C is of q-capacity zero if and only if there is a bounded open neighborhood D of C with $\mathrm{cap}_q(C, D)=0$, cf. [19]. If E is closed in G and if each compact subset of E is of q-capacity zero, we say that E is of *q-capacity zero*.

We let q' denote the conjugate exponent of $q\in[1,\infty]$, i.e. $\frac{1}{q}+\frac{1}{q'}=1$.

4.2. Theorem. *Let $q\in[1,p]$. Suppose that a function $u\in\operatorname{loc} L^1_s(G)$, $s=q'(p-1)$, is A-superharmonic in $G\setminus E$ and that E is of q-capacity zero. Set*

$$u(x)=\varliminf_{y\to x} u(y)$$

for $x\in E$. Then u is A-superharmonic in G.

Proof. Since $s=q'(p-1)\geqq p$, u belongs to $\operatorname{loc} L^1_p(G)=\operatorname{loc} W^1_p(G)$; by Corollary 3.14, u is an A-supersolution in $G\setminus E$. In view of Theorem 3.17 it suffices to show that u is an A-supersolution in all of G. For this, let $\varphi\in C_0^\infty(G)$ be non-negative. Choose an open set $D\subset\subset G$ with $\operatorname{spt}\varphi\subset D$ and a sequence $\varphi_i\in C_0^\infty(D)$ such that $0\leqq\varphi_i\leqq 1$, $\varphi_i=1$ in a neighborhood of $E\cap\operatorname{spt}\varphi$, and $\|\varphi_i\|_{W^1_q(D)}\to 0$ as $i\to\infty$; since $E\cap\operatorname{spt}\varphi$ is of q-capacity zero, such a choice is possible by the Poincaré inequality. Then $(1-\varphi_i)\varphi\in C_0^\infty(G\setminus E)$ is non-negative, whence

$$\begin{aligned}\int_G A(x,\nabla u)\cdot\nabla\varphi\,dx &= \int_D A(x,\nabla u)\cdot\nabla(\varphi\varphi_i)\,dx+\int_{G\setminus E} A(x,\nabla u)\cdot\nabla\big((1-\varphi_i)\,\varphi\big)\,dx\\ &\geqq \int_D A(x,\nabla u)\cdot\nabla(\varphi\varphi_i)\,dx.\end{aligned}$$

Now the last integral tends to zero as $i\to\infty$ since

$$\Big|\int_D A(x,\nabla u)\cdot\nabla(\varphi\varphi_i)\,dx\Big|\leqq\|A(x,\nabla u)\|_{q'}\,(\max|\varphi|\,\|\nabla\varphi_i\|_q+\max|\nabla\varphi|\,\|\varphi_i\|_q)$$

and since

$$\|A(x,\nabla u)\|_{q'}\leqq\beta\|\nabla u\|_s^{p-1}<\infty.$$

This shows that u is an A-supersolution in G as required.

The same reasoning yields a parallel result concerning A-harmonic functions:

4.3. Theorem. *Let $q\in[1,p]$. Suppose that the function $u\in\operatorname{loc} L^1_s(G)$, $s=q'(p-1)$, is A-harmonic in $G\setminus E$ and that E is of q-capacity zero. Then u has an A-harmonic extension to E.*

The following two corollaries have natural counterparts for A-superharmonic functions. We let $\dim_H E$ denote the Hausdorff dimension of E.

4.4. Corollary. *Suppose that u is A-harmonic in $G\setminus E$ and that $\dim_H E<n-1$. If $u\in\operatorname{loc} L^1_s(G)$ for each $s<\infty$, then u extends to an A-harmonic function in G.*

Proof. If $1<q<\min(p,n-\dim_H E)$, then E is of q-capacity zero (see e.g. [19], Theorem 4.2), and the assertion follows from Theorem 4.3.

4.5. Corollary. *Suppose that u is locally K-lipschitz and A-harmonic in $G\setminus E$ and that the $(n-1)$-measure of E is zero. Then u extends to a locally K-lipschitz A-harmonic function in G.*

Proof. Since the $(n-1)$-measure of E is zero, u has a locally K-lipschitz extension in G and E is of 1-capacity zero, cf. [16], Lemma 2.2.5. Then, by Theorem 4.3, this extension of u is A-harmonic in G.

4.6. *Remarks.* (a) Theorem 4.3 and Corollaries 4.4 and 4.5 might be known but we have not found any reference to give. O. Martio [14] makes use of the fact that large sets are removable for lipschitz solutions in constructing counterexamples for unique continuation; he assumes that E is an $(n-2)$-plane and proves a result similar to Corollary 4.5.

Observe that the proofs of Theorem 4.3 and Corollaries 4.4 and 4.5 work also under weaker requirements on the operator A.

(b) L. I. Hedberg [5] obtains Theorem 4.3 for ordinary harmonic functions, and he also shows that it cannot be improved.

Next we extend a well-known result of classical potential theory.

4.7. Theorem. *Suppose that u is an A-superharmonic function in $G\setminus E$ with $\underline{\lim}_{y\to x} u(y) > -\infty$ for each $x\in E$. Set*

$$u(x) = \underline{\lim}_{y\to x} u(y)$$

for $x\in E$. If E is of p-capacity zero, then u is A-superharmonic in G.

Proof. Let $D\subset\subset G$ be a domain. It suffices to show that u is A-superharmonic in D. Replacing u by $\min(u, k)$, $k=1, 2, \ldots$, if necessary, we are free to assume that u is bounded in D. Using the estimate (2.17) we show that $u\in \operatorname{loc} W_p^1(D)$ which, by Theorem 4.2, establishes the desired conclusion. For that, pick a domain $D_0\subset\subset D$ and $\varphi\in C_0^\infty(D)$ with $0\leqq\varphi\leqq 1$ and $\varphi=1$ in D_0. Now, since $E_0=\operatorname{spt}\varphi\cap D_0$ is of p-capacity zero, we may choose a sequence D_i, $i=1, 2, \ldots$, of open sets with $E_0=\bigcap_k D_k\subset D_{i+1}\subset\subset D_i\subset\subset D$, and a sequence $\varphi_i\in C^\infty(D)$ such that $\varphi_i=1$ in $D\setminus D_i$, $\varphi_i=0$ in a neighborhood of E_0, and $\int_D |\nabla\varphi_i|^p\,dx\to 0$ as $i\to\infty$. Since u is bounded and A-superharmonic in $D\setminus E$, (2.17) yields

$$\int_{D_0} |\nabla u|^p \varphi_i^p\,dx \leqq \int_D |\nabla u|^p (\varphi\varphi_i)^p\,dx$$
$$\leqq c\int_D |\nabla(\varphi\varphi_i)|^p\,dx \leqq c\Big(\int_D |\nabla\varphi|^p\,dx + \int_D |\nabla\varphi_i|^p\,dx\Big).$$

Thus letting $i\to\infty$ yields

$$\int_{D_0\setminus E} |\nabla u|^p\,dx < \infty.$$

Since the $(n-1)$-measure of E is zero, it follows that $u\in W_p^1(D_0)$, and the proof is complete.

4.8. *Remarks.* (a) Theorem 4.7 is sharp. Indeed, let C be a compact set in a ball B with $\operatorname{cap}_p(C, B)>0$. Let $\varphi\in C^\infty(\mathbf{R}^n)$ be such that $\varphi=0$ in C, and $\varphi=1$

in ∂B, and choose A-harmonic u in $B\setminus C$ with $u-\varphi\in W^1_{p,0}(B\setminus C)$. Since $u>0$ in $B\setminus C$ and since $\lim_{y\to x} u(y)=0$ for some $x\in C$ (cf. [6], Theorem 2), u has no A-superharmonic extension in B as a consequence of the comparison principle.

(b) Theorem 4.7 is well known for A-harmonic functions [20]; this is also sharp in view of the remark above.

(c) Neither Theorem 4.2 nor 4.3 can be improved by assuming that $u\in \operatorname{loc} L^1_s(G\setminus E)$. For this, note that there is $\varepsilon=\varepsilon(n,p,\alpha,\beta)>0$ such that each A-harmonic function u in G belongs to $\operatorname{loc} W^1_{p+\varepsilon}(G)$, cf. [17]. The function u in (a) above serves as a counterexample by choosing C so that $\operatorname{cap}_q(C, B)=0$ for $q<p$.

Moreover, the same example shows that there does not exist $\varepsilon>0$ not depending on G, such that each A-harmonic u in $W^1_p(G)$ would belong to $W^1_{p+\varepsilon}(G)$.

(d) It is well known that there is no non-constant bounded A-harmonic function in $\mathbf{R}^n$ (see e.g. [1], p. 307). Using the above results yields that if E is a closed set of p-capacity zero, then there is no non-constant bounded A-harmonic function in $\mathbf{R}^n\setminus E$. Conversely, by using [15], the estimate (26), it is not difficult to construct a non-constant A-harmonic function in each domain G provided that $\mathbf{R}^n\setminus G$ is not of p-capacity zero.

In the borderline case, $p=n$, an analogous result holds for A-superharmonic functions; it easily follows from the estimate (2.17) that each lower bounded A-superharmonic function in $\mathbf{R}^n\setminus E$ is a constant whenever E is of n-capacity zero. For $n=2$ this is the well known characterization of Greenian sets.

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