Green's function for the heat equation as a limit of product integrals

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Let D be a bounded domain with smooth boundary, ∂D, in the complex plane C. Various authors ([1], [2], [3]) have used a representation of the Green's function $g(z, w, t)$ for the heat equation

$$(1) \qquad \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

and D in terms of the so-called elementary solution

$$(2) \qquad p(z, t) = \frac{1}{4\pi t} \exp\left(-\frac{|z|^2}{4t}\right)$$

of (1), amounting to the following:

Take $\Delta t = 2^{-n}$ with n large. Then, if $t > 0$ is an integral multiple of Δt, say $t = (k+1)2^{-n}$, and $z, w \in D$, $g(z, w, t)$ is approximately

$$\iint_D \cdots \iint_D p(w - \zeta_1, \Delta t) p(\zeta_1 - \zeta_2, \Delta t) \cdots p(\zeta_k - z, \Delta t) \, d\xi_1 \, d\eta_1 \cdots d\xi_k \, d\eta_k.$$

(Here, and in all that follows, we write $\zeta_l = \xi_l + i\eta_l$ and similarly $\zeta = \xi + i\eta$, $z = x + iy$, and $w = u + iv$.)

This representation for $g(z, w, t)$ has a simple physical explanation in terms of diffusion ([1], p. 13) and can be established rigorously with the help of Brownian motion theory ([2], p. 238 — although [4] does not give the representation explicitly, it is also worth consulting in this connection). One must know considerable probability theory in order to follow such a rigorous derivation. The formula has, however, such a strong intuitive appeal that one feels it must have a simple direct proof. Such a proof, based essentially on the property of upper semi-continuity, is given here.

I learned about the product integral representation in a lecture given by Serge Dubuc at McGill University in the fall of 1976. He and Gilles Deslauriers used

it in deriving some convexity properties for solutions of (1). Later on C. Borell who had also done similar work, pointed out to me that this application had already been made by Brascamp & Lieb in [3]. I am thankful to Mr Borell, and to H. McKean as well, for having provided me with references and told me something about the history of this material.

After I had submitted the MS of this paper, E. Calabi told me about Yamabe's earlier non-probabilistic proof [5] of a more elaborate version of the product integral representation. Yamabe's argument is different from the one given below, and is more difficult technically.

§ 1. The following treatment applies to $\dot{u} = \nabla^2 u$ in $\mathbf{R}^n$ (after one adjusts, of course, (2) for n spatial dimensions), but we present it for the case $n=2$ in order to preserve the essential difficulties while keeping the notation as simple as possible. We take, then, a bounded domain D in $\mathbf{C}$. *In what follows, we assume that ∂D fulfills at each of its points the Poincaré cone condition, i.e. if $z_0 \in \partial D$ there is a sector S with vertex at z_0 and a disk Δ centered at z_0 such that $\Delta \cap D$ and S do not overlap (save at z_0).*

It is easier to write the product integral approximations to $g(z, w, t)$ in piecewise fashion:

Definition. *If* $n = 1, 2, 3, \ldots, t > 0, w \in D$ *and* $z \in C$(sic!), *put*

$$(3) \qquad g_n(z, w, t) = p(z-w, t) \quad for \quad 0 < t \leq 2^{-n},$$

$$(4) \qquad g_n(z, w, t) = \iint_D p(z-\zeta, t-k \cdot 2^{-n}) g_n(\zeta, w, k \cdot 2^{-n}) d\xi \, d\eta$$

$$for \quad k \cdot 2^{-n} < t \leq (k+1) \cdot 2^{-n}, \quad k = 1, 2, 3, \ldots.$$

We are to prove that, for $z \in \bar{D}$, $w \in D$, and $t > 0$, $g_n(z, w, t)$ tends, as $n \to \infty$, to a function $g(z, w, t)$ having the following properties:

(i) *$g(z, w, t)$ is non-negative and measurable on $\bar{D} \times D \times (0, \infty)$. For each $w \in D$ the function $u(z, t) = g(z, w, t)$ is C^∞ on $D \times (0, \infty)$ and satisfies (1) there.*

(ii) *For each $w \in D$, $g(z, w, t)$ is continuous for $z \in \bar{D}$ and $t > 0$, and $g(z, w, t) = 0$ for $z \in \partial D$, $t > 0$.*

(iii) *For each $z \in D$, $g(z, w, t) dudv$ acts, on D, like $\delta(w-z) dudv$ as $t \to 0$, where δ is the Dirac δ-function on $\mathbf{C}$.*

Taken together, (i), (ii) and (iii) amount to a description of $g(z, w, t)$ as the Green's function of $\dot{u} = \nabla^2 u$ for the domain D.

Lemma 1. *Each function $g_n(z, w, t)$ is ≥ 0 and continuous on $D \times D \times (0, \infty)$. For fixed $t > 0$ and $w \in D$ it is continuous in z on $\mathbf{C}$. For fixed $w \in D$ it is upper semicontinuous in (z, t) on $\bar{D} \times (0, \infty)$, i.e., if $z_0 \in \bar{D}$ and $t_0 > 0$,*

$$(5) \qquad \limsup_{(z, t) \to (z_0, t_0)} g_n(z, w, t) \leq g_n(z_0, w, t_0).$$

Proof. Everything is immediate from (3) and (4) and the well-known approximate identity property of $p(z, t)$, except perhaps the upper semi-continuity. If $k = 1, 2, 3, \ldots, g_n(z, w, t)$ is evidently continuous in (z, t) for $z \in \mathbf{C}$ and $(k-1)2^{-n} < t \leq k \cdot 2^{-n}$, so we need only verify (5) for $z_0 \in \bar{D}$, t_0 of form $k \cdot 2^{-n}$, and t tending to t_0 *from above.* For such t, (4) gives

$$g_n(z, w, t) \leq \iint_C p(z - \zeta, t - k \cdot 2^{-n}) g_n(\zeta, w, k \cdot 2^{-n}) \, d\xi \, d\eta.$$

Now $g_n(\zeta, w, k \cdot 2^{-n})$ is clearly bounded and continuous in ζ over $\mathbf{C}$, so the approximate identity property of $p(z, t)$ implies that the last integral tends to $g_n(z_0, w, k \cdot 2^{-n})$ as $z \to z_0$ and $t \to k \cdot 2^{-n}$. The inequality just written thus yields (5).

Remark. Let $F(w)$ be continuous and of compact support in D, and *nonnegative.* The function

$$v_n(z, t) = \iint_D g_n(z, w, t) F(w) \, du \, dv$$

has the same continuity and semi-continuity in (z, t) as is established for $g_n(z, w, t)$ in Lemma 1. The proof of this is the same as that of the lemma.

We now give two formulas. The first is the well-known reproducing property

$$(6) \quad p(z - w, t + t') = \iint_C p(z - \zeta, t) p(\zeta - w, t') \, d\xi \, d\eta, \quad \text{valid for} \quad t \ \& \ t' > 0.$$

The second is an immediate consequence of (3) and (4), and says that

$$(7) \qquad g_n(z, w, t + r) = \iint_D g_n(z, \zeta, t) g_n(\zeta, w, r) \, d\xi \, d\eta$$

whenever $t > 0$ and r is of the special form $k \cdot 2^{-n}$, $k = 1, 2, 3, \ldots$.

Lemma 2. $g_{n+1}(z, w, t) \leq g_n(z, w, t)$.

Proof. If $0 < t \leq 2^{-n-1}$, $g_{n+1}(z, w, t) = g_n(z, w, t)$ by (3). If $t = t' + 2^{-n-1}$ with $0 < t' \leq 2^{-n-1}$, then by (3), (6) and (4), $g_n(z, w, t) = p(z - w, t' + 2^{-n-1}) \geq \iint_D p(z - \zeta, t') p(\zeta - w, 2^{-n-1}) d\xi d\eta = g_{n+1}(z, w, t)$. The desired inequality is now proved for $0 < t \leq 2^{-n}$. Suppose it has been established for $0 < t \leq k \cdot 2^{-n}$, where $k = 1, 2, 3, \ldots$. To extend its validity to the range $k \cdot 2^{-n} < t \leq (k+1)2^{-n}$, write $t = t' + k \cdot 2^{-n} = t' + (2k) \cdot 2^{-n-1}$ with $0 < t' \leq 2^{-n}$, and apply (7) twice:

$$g_{n+1}(z, w, t) = \iint_D g_{n+1}(z, \zeta, t') g_{n+1}(\zeta, w, k \cdot 2^{-n}) \, d\xi \, d\eta$$

$$\leq \iint_D g_n(z, \zeta, t') g_n(\zeta, w, k \cdot 2^{-n}) \, d\xi \, d\eta$$

$$= g_n(z, w, t).$$

The lemma thus holds by induction on k.

Remark. By the same argument,

(8) $g_n(z, w, t) \leqq p(z-w, t).$

Now $g_n(z, w, t) \geqq 0$, so by Lemma 2, $\lim_{n \to \infty} g_n(z, w, t)$ exists and is $\geqq 0$.

Definition. *For* $t > 0$, $z \in \bar{D}$ *and* $w \in D$,

(9) $g(z, w, t) = \lim_{n \to \infty} g_n(z, w, t).$

Lemma 3. *For each* $w \in D$, $g(z, w, t)$ *is upper semi-continuous in* (z, t).

Proof. By Lemmas 1 & 2, and advanced calculus.

Lemma 4. *If* $t > 0$ *and* $r > 0$ *is a dyadic rational, then*

$$g(z, w, t+r) = \iint_D g(z, \zeta, t) g(\zeta, w, r) \, d\xi \, d\eta.$$

Proof. By (7), (8), (9) and Lebesgue's dominated convergence theorem.

§ 2. We proceed to verify properties (ii), (i) and (iii) (in that order) for the function $g(z, w, t)$ defined by (9).

Theorem 1. $g(z, w, t) = 0$ *for* $t > 0$, $w \in D$ *and* $z \in \partial D$.

Proof. Given $t > 0$, $w \in D$ and $z \in \partial D$ let S be a sector of opening $2\alpha > 0$ with vertex at z which does not contain any $\zeta \in D$ with $0 < |\zeta - z| < c_0$, say, where $c_0 > 0$. (This is the Poincaré cone condition.) Taking any $\varepsilon > 0$ we can, by Lemma 3, find c, $0 < c < c_0$, such that

(10) $g(\zeta, w, t') < g(z, w, t) + \varepsilon$ *for* $|t'-t| < c$ *and* $\zeta \in D$, $|\zeta - z| < c$.

There exist arbitrarily small δ with $0 < \delta < c$ and $t - \delta$ a *dyadic rational*. For such δ, by Lemma 4 and the remark to Lemma 2,

(11)
$$g(z, w, t) = \iint_D g(z, \zeta, \delta) g(\zeta, w, t-\delta) \, d\xi \, d\eta$$
$$\leqq \iint_D p(z-\zeta, \delta) g(\zeta, w, t-\delta) \, d\xi \, d\eta.$$

Break up the right-hand integral in (11) into *two*, the *first* over $D \cap \{\zeta; |\zeta - z| < c\}$ and the *second* over the rest of D. By (10) and our choice of δ, the *first* integral is

$$\leqq \big(g(z, w, t) + \varepsilon \big) \iint_{C \sim S} p(\zeta - z, \delta) \, d\xi \, d\eta$$

which is seen to be $\dfrac{\pi - \alpha}{\pi} [g(z, w, t) + \varepsilon]$ by direct calculation. Using (8) and (2) one shows easily that the *second* integral is

$$\leqq \frac{1}{4\pi(t-\delta)} \iint_{|\zeta - z| \geqq c} p(\zeta - z, \delta) \, d\xi \, d\eta.$$

If we fix c and take $\delta > 0$ small enough (keeping $t-\delta$ a dyadic rational), we can make this last expression $< \varepsilon$. Going back to (11), we get

$$g(z, w, t) \leq \frac{\pi - \alpha}{\pi} g(z, w, t) + \frac{2\pi - \alpha}{\pi} \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary and $\alpha > 0$ depends only on the choice of $z \in \partial D$, we have $g(z, w, t) = 0$. Q.E.D.

Corollary. *For any $w \in D$ and $t > 0$, $g(z, w, t) \to 0$ whenever $z \in D$ approaches ∂D.*

Proof. By the non-negativity of $g(z, w, t)$, the theorem, Lemma 3, and compactness of $\bar{D}$.

Theorem 2. *For each $w \in D$, the function $u(z, t) = g(z, w, t)$ is C^{∞} in $D \times (0, \infty)$ and satisfies $\dot{u} = \nabla^2 u$ there.*

Proof. If the assertion holds for each of the functions $g_n(z, w, t)$ it also holds for $g(z, w, t)$ by Lemma 2, (9), and well known elementary theorems about the heat equation.

So, for any given n and fixed $w \in D$, consider the function $v(z, t) = g_n(z, w, t)$.

From (3) or differentiation under the integral sign in (4) it follows that $v(z, t)$ is C^{∞} on each of the slabs $D \times ((k-1)2^{-n}, k \cdot 2^{-n})$, $k = 1, 2, 3, \ldots$, and satisfies $\dot{u} = \nabla^2 v$ there.

We have to check that the behaviour of $v(z, t)$ is alright when t is near any of the values $k \cdot 2^{-n}$, $k = 1, 2, 3, \ldots$. Take any fixed k. For $t > (k-1)2^{-n}$ and $z \in C$, put

(12) $$V(z, t) = \begin{cases} p(z-w, t) & \text{in case } k = 1 \\ \iint_D p(z-\zeta, t-(k-1)2^{-n}) g_n(\zeta, w, (k-1)2^{-n}) \, d\xi \, d\eta \\ \qquad \text{in case } k > 1. \end{cases}$$

Observe that $V(z, t) = v(z, t)$ for $(k-1)2^{-n} < t \leq k \cdot 2^{-n}$ by (3) or (4). $V(z, t)$ is clearly C^{∞} and satisfies $\dot{V} = \nabla^2 V$ on $D \times ((k-1) \cdot 2^{-n}, \infty)$. If $k \cdot 2^{-n} < t < (k+1) \cdot 2^{-n}$, we have from (12) and (6), followed by (3) or (4) with Fubini's theorem in case $k > 1$,

$$V(z, t) = \iint_C p(z-\zeta, t-k \cdot 2^{-n}) g_n(\zeta, w, k \cdot 2^{-n}) \, d\xi \, d\eta.$$

With (4) this yields finally

(13) $$V(z, t) - v(z, t) = \iint_{C \sim D} p(z-\zeta, t-k \cdot 2^{-n}) g_n(\zeta, w, k \cdot 2^{-n}) \, d\xi \, d\eta.$$

Now if $t - k \cdot 2^{-n} \to 0+$ and $z \to z_0 \in D$ it is easy to see from (13) and (2) that $V(z, t) - v(z, t)$ and all the partial derivatives thereof tend to zero. This is so

because $\exp\left(-\dfrac{R^2}{4\Delta t}\right)$ goes to zero faster than any power of $\dfrac{\Delta t}{R^2}$ when the latter goes to zero — to estimate $g_n(\zeta, w, k \cdot 2^{-n})$ in (13) one may use the remark to Lemma 2.

We see that $v(z, t)$ is C^∞ and satisfies $\dot{v} = \nabla^2 v$ near $t = k \cdot 2^{-n}$ because $V(z, t)$ has that behaviour there. We are done.

Theorem 3. $g(z, w, t)$ *is the Green function for* $\dot{u} = \nabla^2 u$ *and the domain* D.

Proof. Since $g(z, w, t) \geqq 0$ it suffices to establish the following:

Take any continuous and non-negative f with compact support in D, and write for $z \in \bar{D}$ and $t > 0$,

$$u(z, t) = \iint_D g(z, w, t) f(w)\, du\, dv.$$

Then:

    (a) $u(z, t)$ is C^∞ in $D \times (0, \infty)$ and satisfies $\dot{u} = \nabla^2 u$ there.

    (b) If $t > 0$, then $u(z, t) \to 0$ as z approaches ∂D.

    (c) If $t \to 0+$ and $z \to z_0 \in \bar{D}$, then $u(z, t) \to f(z_0)$.

We proceed. Let

(14) $$u_n(z, t) = \iint_D g_n(z, w, t) f(w)\, du\, dv.$$

The argument used in proving Theorem 2 shows that each $u_n(z, t)$ is C^∞ on $D \times (0, \infty)$ and satisfies $\dot{u}_n = \nabla^2 u_n$ there. By (9), the remark to Lemma 2, and Lebesgue's dominated convergence theorem we have $u_n(z, t) \to u(z, t)$ as $n \to \infty$. This convergence is monotone (decreasing) by Lemma 2 and the non-negativity of $f(w)$. So (a) follows from standard theorems about the heat equation.

The corollary to Theorem 1, the remark to Lemma 2, and Lebesgue's dominated convergence theorem now give (b).

We must prove (c). By (14) and (3), for $0 < t \leqq 2^{-n}$,

$$u_n(z, t) = \iint_D p(z - w, t) f(w)\, du\, dv,$$

so, since $f(w)$ is of compact support in D,

(15) $$u_n(z, t) \to f(z_0) \quad \text{whenever} \quad t \to 0 \quad \text{and} \quad z \to z_0 \in \bar{D}.$$

In particular, $u_n(z, t) \to 0$ if $t \to 0$ and $z \to z_0 \in \partial D$, so, if we define $u_n(z, 0)$ to be *zero* for $z \in \partial D$, the remark to Lemma 1 shows that each $u_n(z, t)$ is *upper semi-continuous* on $\partial D \times [0, \infty)$.

By Theorem 1 $u(z, t) = 0$ on $\partial D \times (0, \infty)$, so $u_n(z, t) \to 0$ for $z \in \partial D$ & $t > 0$ as $n \to \infty$. By (14), Lemma 2, and the non-negativity of $f(w)$ we also have $u_n(z, t) \geqq u_{n+1}(z, t)$. Since ∂D is compact, Dini's theorem now implies that $u_n(z, t) \to 0$ *uniformly* on any set of the form $\partial D \times [0, T]$ as $n \to \infty$.

Fix any $T>0$, and let $\varepsilon>0$ be given. As we have just seen, there exists an N such that $0\leqq u_n(z,t)<\varepsilon$ whenever $n>N$, $z\in\partial D$ and $0<t\leqq T$. Let $N<n<m$, $0<t\leqq T$ and z tend to $z_0\in\partial D$ from D. By Lemma 2 and upper semi-continuity of $u_n(z,t)$,

$$(16) \qquad 0 \leqq \limsup_{z\to z_0}\big(u_n(z,t)-u_m(z,t)\big) \leqq \limsup_{z\to z_0} u_n(z,t) \leqq u_n(z_0,t) < \varepsilon.$$

Also by (15), if z tends to $z_0\in\bar{D}$ from D and $t\to 0$,

$$(17) \qquad\qquad u_n(z,t)-u_m(z,t) \to f(z_0)-f(z_0) = 0.$$

However, $\dfrac{\partial}{\partial t}(u_n-u_m)=\nabla^2(u_n-u_m)$ so by (16), (17) and the *principle of maximum for the heat equation,* $0\leqq u_n(z,t)-u_m(z,t)<\varepsilon$ on $\bar{D}\times(0,T]$ whenever $m>n>N$. Now keep $n>N$ fixed and make $m\to\infty$. We get in the limit

$$(18) \qquad\qquad 0 \leqq u_n(z,t)-u(z,t) \leqq \varepsilon \quad \text{on}\quad \bar{D}\times(0,T] \quad\text{if}\quad n > N.$$

Let $t\to 0^+$ and $z\in D$ tend to $z_0\in\bar{D}$. Choosing $n>N$, we have by (15) and (18),

$$\limsup_{\substack{z\to z_0\\ t\to 0}} |u(z,t)-f(z_0)| \leqq \varepsilon.$$

But $\varepsilon>0$ was arbitrary. So $u(z,t)\to f(z_0)$ if $t\to 0^+$ and $z\to z_0\in\bar{D}$. We are done.

Remark. The argument has actually furnished a proof of the *existence* of the Green's function for $\dot{u}=\nabla^2 u$ and the domain D.

Remark. The same method can be applied to some other problems of the form $\dot{u}=Lu$.

References

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