The range of vector-valued analytic functions, II

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The present paper is a continuation of [2]. We keep the notation of [2]. We write $I=\{t\in R: 0\leqq t\leqq 1\}$.

Theorem. *Let F be a pathwise connected set in a separable complex Banach space X and let $O\subset X$ be an open set containing F. There exists a continuous function $f:\bar{\Delta}-\{1\}\to X$, analytic on Δ and such that*

(i) $f(\bar{\Delta}-\{1\})\subset O$

(ii) *the cluster set of f at 1 is $\bar{F}$.*

The proof of this theorem is almost identical with the proof of Theorem in [2] once we have proved the following lemma which enables to pass from balanced sets to arbitrary pathwise connected sets.

Lemma. *Let $f:I\to X$ be a path in a complex Banach space X satisfying $f(0)=0$. Let $\varepsilon>0$ and $0<r<1$. Suppose that E is a closed subset of the boundary of Δ of linear measure 0 which does not contain a point z_0, $|z_0|=1$. Denote $D=\bar{\Delta}\cap K(r, z_0)$. There exists a continuous function $\Phi:\bar{\Delta}\to X$, analytic on Δ and having the following properties*

(a) $\Phi(\bar{\Delta})\subset f(I)+B_\varepsilon(X)$

(b) $\|\Phi(z)\|<\varepsilon \quad (z\in\bar{\Delta}-D)$

(c) $\|f(1)-\Phi(z_0)\|<\varepsilon$

(d) $\Phi(z)=0 \quad (z\in E)$.

Proof. By the Mergelyan theorem for vector-valued functions [1] there exists a polynomial $P: C\to X$ such that $\|f(z)-P(z)\|<\varepsilon/4$ $(z\in I)$. Putting $Q(z)=P(z)-P(0)$ it is easy to see that there exists a neighbourhood U of I such that $Q(U)\subset f(I)+$ $+B_\varepsilon(X)$. Let $V\subset U$ be an open neighbourhood of $I-\{1\}$, containing the point 1 in its boundary and bounded by a Jordan curve contained in U. By the Riemann mapping theorem there exists a one-to-one analytic map φ from Δ onto V which [4, p. 282] has an extension $\bar{\varphi}$ which is a homeomorphism from $\bar{\Delta}$ onto $\bar{V}$. Composing

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it with a Möbius transformation if necessary we may assume that $\bar{\varphi}(0)=0$ and $\bar{\varphi}(1)=1$. By the Rudin—Carleson theorem [3, p. 81] there exists a continuous function $\eta\colon \bar{\Delta}\to\bar{\Delta}$, analytic on Δ and satisfying $\eta(z)=0$ $(z\in E)$ and $\eta(z_0)=1$. Also there exists a peaking function for z_0, i.e. a continuous function $\psi\colon \bar{\Delta}\to\bar{\Delta}$, analytic on Δ and satisfying $\psi(z_0)=1$, $|\psi(z)|<1$ $(z\in\bar{\Delta}-\{z_0\})$ [3, p. 81]. There exists a neighbourhood $W\subset V$ of the point 0 such that $\|Q(z)\|<\varepsilon$ $(z\in W)$. Let $T\subset\Delta$ be a neighbourhood of the point 0 such that $\varphi(T)\subset W$. Multiplying η with a suitable power of ψ if necessary we may assume that $\eta(\bar{\Delta}-D)\subset T$, $\eta(z_0)=1$ and $\eta(z)=0$ $(z\in E)$. Put $\Phi=Q\circ\bar{\varphi}\circ\eta$. It is easy to check that Φ has all the required properties. Q.E.D.

Corollary. *Given any open connected set F in a separable complex Banach space X there exists an analytic function $f\colon \Delta\to X$ whose range is contained and dense in F.*

References

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