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Modules of principal parts on the projective line

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Abstract. The modules of principal parts $\mathcal{P}^k(\mathcal{E})$ of a locally free sheaf $\mathcal{E}$ on a smooth scheme X is a sheaf of $\mathcal{O}_X$-bimodules which is locally free as left and right $\mathcal{O}_X$-module. We explicitly split the modules of principal parts $\mathcal{P}^k(\mathcal{O}(n))$ on the projective line in arbitrary characteristic, as left and right $\mathcal{O}_{\mathbf{P}^1}$-modules. We get examples when the splitting-type as left module differs from the splitting-type as right module. We also give examples showing that the splitting-type of the principal parts changes with the characteristic of the base field.

1. Introduction

In this paper we will study the splitting-type of the modules of principal parts of invertible sheaves on the projective line as left and right $\mathcal{O}_{\mathbf{P}^1}$-modules in arbitrary characteristic. The splitting-type of the principal parts $\mathcal{P}^k(\mathcal{O}(n))$ as a left $\mathcal{O}_{\mathbf{P}^1}$-module in characteristic zero has been studied by several authors (see [1], [6] and [7]). The novelty of this work is that we consider the principal parts as left and right $\mathcal{O}_{\mathbf{P}^1}$-modules in arbitrary characteristic. We give examples when the splitting-type as left $\mathcal{O}_{\mathbf{P}^1}$-module differs from the splitting-type as right $\mathcal{O}_{\mathbf{P}^1}$-module. The main theorem of the paper (Theorem 7.1), gives the splitting-type of $\mathcal{P}^1(\mathcal{O}(n))$ as left and right $\mathcal{O}_{\mathbf{P}^1}$-modules for all $n\geq 1$ over any field F. The result is the following: The principal parts $\mathcal{P}^1(\mathcal{O}(n))$ splits as $\mathcal{O}(n)\oplus\mathcal{O}(n-2)$ as right $\mathcal{O}_{\mathbf{P}^1}$-module. f the characteristic of F divides n, then $\mathcal{P}^1(\mathcal{O}(n))$ splits as $\mathcal{O}(n)\oplus\mathcal{O}(n-2)$ as left $\mathcal{O}_{\mathbf{P}^1}$-module. On the other hand, if the characteristic of F does not divide n, then $\mathcal{P}^1(\mathcal{O}(n))$ splits as $\mathcal{O}(n-1)\oplus\mathcal{O}(n-1)$ as left $\mathcal{O}_{\mathbf{P}^1}$-module. Hence the modules of principal parts are the first examples of a sheaf of abelian groups equipped with two non-isomorphic structures as locally free sheaves. In the papers [1], [6] and [7] the

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authors work in characteristic zero, and they only consider the left module structure. In this work we split the principal parts explicitly as left and right modules and the techniques we develop will be used in future papers to get deeper knowledge of the principal parts in positive characteristic. In Sections 3–6 we develop techniques to construct non-trivial maps of $\mathcal{O}$-modules from $\mathcal{O}(n-k)$ to $\mathcal{P}^k(\mathcal{O}(n))$. The main theorem here is Theorem 5.2, where we prove existence of certain systems of linear equations with integer coefficients. Solutions to the systems satisfying extra criteria determines the splitting-type of $\mathcal{P}^k(\mathcal{O}(n))$. Theorem 5.2 is used in Proposition 6.3 to determine the splitting-type of $\mathcal{P}^k(\mathcal{O}(n))$ for all $1 \leq k \leq n$ in characteristic zero, and we recover results obtained in [1], [6] and [7]. We also give examples where the splitting-type can be determined by diagonalizing the structure matrix defining the principal parts (Section 4).

2. Modules of principal parts

We will in the following section define and prove basic properties of the principal parts: existence of fundamental exact sequences, functoriality and existence of bimodule structure. Let X be a scheme defined over a fixed base scheme S. We assume that X is separated and smooth over S. Let Δ in $X \times_S X$ be the diagonal, and let $\mathcal{I}$ in $\mathcal{O}_{X\times_S X}$ be the sheaf of ideals defining Δ. Let X^k be the scheme with topological space Δ and structure sheaf $\mathcal{O}_{\Delta^k} = \mathcal{O}_{X\times_S X}/\mathcal{I}^{k+1}$. By definition, X^k is the *kth order infinitesimal neighborhood of the diagonal.* Throughout the section we omit reference to the base scheme S in products. Let p and q denote the canonical projection maps from $X \times X$ to X.

Definition 2.1. Let $\mathcal{E}$ be a quasi-coherent $\mathcal{O}_X$-module. We define the *kth order modules of principal parts* of $\mathcal{E}$ to be

$$\mathcal{P}^k_X(\mathcal{E}) = p_*(\mathcal{O}_{\Delta^k} \otimes q^*\mathcal{E}).$$

We write $\mathcal{P}^k_X$ for the module $\mathcal{P}^k_X(\mathcal{O}_X)$.

When it is clear from the context which scheme we are working on, we write $\mathcal{P}^k(\mathcal{E})$ instead of $\mathcal{P}^k_X(\mathcal{E})$.

Proposition 2.2. *Let $\mathcal{E}$ be a quasi-coherent $\mathcal{O}_X$-module. There exists an exact sequence*

$$0 \longrightarrow S^k(\Omega^1_X) \otimes \mathcal{E} \longrightarrow \mathcal{P}^k_X(\mathcal{E}) \longrightarrow \mathcal{P}^{k-1}_X(\mathcal{E}) \longrightarrow 0$$

of left $\mathcal{O}_X$-modules, where $k = 1, 2, \ldots$.

See [4], Section 4, for a proof.

From Proposition 2.2 it follows by induction, that for a locally free sheaf $\mathcal{E}$ of rank e, $\mathcal{P}^k_X(\mathcal{E})$ is locally free of rank $e\binom{n+k}{n}$, where n is the relative dimension of X over S.

Proposition 2.3. *Let $f\colon X \to Y$ be a map of smooth schemes over S, and let $\mathcal{E}$ be a locally free $\mathcal{O}_Y$-module. There exists a commutative diagram of exact sequences*

$$\begin{tikzcd}
0 \arrow[r] & S^k(f^*\Omega^1_Y)\otimes f^*\mathcal{E} \arrow[r] \arrow[d] & f^*\mathcal{P}^k_Y(\mathcal{E}) \arrow[r] \arrow[d] & f^*\mathcal{P}^{k-1}_Y(\mathcal{E}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S^k(\Omega^1_X)\otimes f^*\mathcal{E} \arrow[r] & \mathcal{P}^k_X(f^*\mathcal{E}) \arrow[r] & \mathcal{P}^{k-1}_X(f^*\mathcal{E}) \arrow[r] & 0
\end{tikzcd}$$

of left $\mathcal{O}_X$-modules for all $k=1,2,\dots$.

See [5], for a proof.

From Proposition 2.3 it follows that for any open subset U of X, the sheaf $\mathcal{P}^k_X(\mathcal{E})|_U$ is isomorphic to $\mathcal{P}^k_U(\mathcal{E}|_U)$, hence we can do local computations with the principal parts.

Proposition 2.4. *The principal parts $\mathcal{P}^k_X$ define a covariant functor*

$$\mathcal{P}^k_X\colon \mathrm{Mod}(\mathcal{O}_X) \longrightarrow \mathrm{Mod}(\mathcal{P}^k_X),$$

where for all quasi-coherent $\mathcal{O}_X$-modules $\mathcal{E}$, the kth order principal parts $\mathcal{P}^k_X(\mathcal{E})$ is a quasi-coherent $\mathcal{P}^k_X$-module. The functor is right exact and commutes with direct limits. If $\mathcal{P}^k_X$ is flat, the functor is exact.

See [3], Proposition 16.7.3, for a proof.

Note that since we assume X to be smooth over S, it follows that $\mathcal{P}^k_X$ is locally free, hence the functor in Proposition 2.4 is exact.

We next consider the bimodule structure of the principal parts.

Proposition 2.5. *Let $f,g\colon U \to V$ be morphisms of topological spaces, and let s be a section of f and g with $s(V)$ a closed set. Let furthermore $\mathcal{A}$ be a sheaf of abelian groups on U with support in $s(V)$. Then $f_*(\mathcal{A})$ equals $g_*(\mathcal{A})$.*

Proof. We first claim that the natural map from $s_*s^{-1}\mathcal{A}$ to $\mathcal{A}$ is an isomorphism: Since $s(V)$ is closed, and s is a section of f, we see that s is a closed map. Both $s_*s^{-1}\mathcal{A}$ and $\mathcal{A}$ have support contained in $s(V)$, hence we prove that the map is an isomorphism at the stalks for all points p in $s(V)$: The stalk $(s_*s^{-1}\mathcal{A})_{s(p)}$ is isomorphic to $(s^{-1}\mathcal{A})_p$, since s is a closed immersion. Furthermore we have that $(s^{-1}\mathcal{A})_p$ equals $\mathcal{A}_{s(p)}$ since s^{-1} is an exact functor, and the claim follows. We see that $f_*\mathcal{A}$ is isomorphic to $f_*s_*s^{-1}\mathcal{A}$, and since s is a section of f, we get that $f_*\mathcal{A}$ is isomorphic to $s^{-1}\mathcal{A}$. A similar argument proves that $g_*\mathcal{A}$ is isomorphic to $s^{-1}\mathcal{A}$, and the proposition is proved. $\square$

Let Δ be the diagonal in $X\times X$, which is closed since X is separated over S. The sheaf $\mathcal{O}_{\Delta^k}\otimes q^*\mathcal{E}$ has support in Δ, hence by Proposition 2.5 we get an isomorphism between $\mathcal{P}^k_X(\mathcal{E})=p_*(\mathcal{O}_{\Delta^k}\otimes q^*\mathcal{E})$ and $q_*(\mathcal{O}_{\Delta^k}\otimes q^*\mathcal{E})$. By the projection formula, $q_*(\mathcal{O}_{\Delta^k}\otimes q^*\mathcal{E})$ equals $q_*(\mathcal{O}_{\Delta^k})\otimes\mathcal{E}$, hence by Proposition 2.5 the principal parts $\mathcal{P}^k_X(\mathcal{E})$ is isomorphic to $q_*(\mathcal{O}_{\Delta^k})\otimes\mathcal{E}$ as sheaves of abelian groups. Identifying $q_*(\mathcal{O}_{\Delta^k})\otimes\mathcal{E}$ with $\mathcal{P}^k_X(\mathcal{E})$, we have defined two $\mathcal{O}_X$-module structures on $\mathcal{P}^k_X(\mathcal{E})$. It follows that $\mathcal{P}^k_X(\mathcal{E})$ is a sheaf of $\mathcal{O}_X$-bimodules, which means that for any open set U of X, the abelian group $\mathcal{P}^k_U(\mathcal{E}|_U)$ is an $\mathcal{O}_X(U)$-bimodule and all restriction maps are maps of bimodules satisfying obvious compatibility criteria. Let X^k be the kth order infinitesimal neighborhood of the diagonal. Then the two projection maps $p,q\colon X\times X\to X$ induce two maps $l,r\colon\mathcal{O}_X\to\mathcal{P}^k_X$ of $\mathcal{O}_X$-modules. The maps l and r are the maps defining the bimodule structure on $\mathcal{P}^k_X$, and we see that $\mathcal{P}^k_X$ is a sheaf of $\mathcal{O}_X$-bialgebras. The map $d=l-r\colon\mathcal{O}_X\to\mathcal{P}^k_X$ is verified to be a differential operator of order k, called *the universal differential operator.*

3. Transition matrices for principal parts as left modules

In this section we explicitly compute the transition matrices defining the principal parts $\mathcal{P}^k(\mathcal{O}(n))$ on the projective line over the integers. We will use the following notation: Define $\mathbf{P}^1$ as $\operatorname{Proj}\mathbf{Z}[x_0,x_1]$, where $\mathbf{Z}$ are the integers, and put $\mathrm{U}_i=\mathrm{D}(x_i)$ and $\mathrm{U}_{01}=\mathrm{D}(x_0x_1)$ for $i=0,1$, where x_i are homogeneous coordinates on $\mathbf{P}^1$. Consider the modules of principal parts $\mathcal{P}^k$ from Definition 2.1 on $\mathbf{P}^1$ for $k\geq 1$. On the open set U_{01}, the modules of principal parts $\mathcal{P}^k$ equals

$$\mathbf{Z}[t,1/t,u,1/u]/(u-t,1/u-1/t)^{k+1}$$

as an $\mathcal{O}_{\mathrm{U}_{01}}$-module, and $\mathcal{O}_{\mathrm{U}_{01}}$ is isomorphic to $\mathbf{Z}[t,1/t]$.

Lemma 3.1. *On the open set U_{01}, as a left $\mathcal{O}$-module, $\mathcal{P}^k$ is a free $\mathbf{Z}[t,1/t]$-module of rank $k+1$, and there exists two natural bases. The bases are $B=\{1,dt,\dots,dt^k\}$ and $B'=\{1,ds,\dots,ds^k\}$, where $dt^i=(u-t)^i$ and $ds^i=(1/u-1/t)^i$.*

The proof is an easy calculation.

Proposition 3.2. *Consider $\mathcal{P}^k$ as a left $\mathcal{O}$-module on $\mathbf{P}^1$ with $k=1,2,\dots$. On the open set U_{01} the transition matrix $[L]_B^{B'}$ between the two bases B' and B is given by the formula*

$$ds^p = \sum_{i=0}^{k-p} (-1)^{i+p} \frac{1}{t^{i+2p}} \binom{i+p-1}{p-1} dt^{i+p}$$

for all $0 \le p \le k$.

Proof. By definition ds^p equals $(1/u-1/t)^p$ in the module

$$\mathbf{Z}[t, 1/t, u, 1/u](u-t, 1/u-1/t)^{k+1}.$$

It follows that

$$ds^p = \left(\frac{1}{u} - \frac{1}{t}\right)^p = \frac{(-1)^p (u-t)^p}{u^p t^p} = (-1)^p \frac{1}{t^p} dt^p \frac{1}{u^p},$$

and since $u=t+u-t=t+dt$ we get

$$ds^p = (-1)^p \frac{1}{t^p} dt^p \frac{1}{(t+dt)^p}.$$

We have the equality

$$ds^p = (-1)^p \frac{1}{t^{2p}} dt^p \frac{1}{(1+dt/t)^p},$$

and using the identity

$$\frac{1}{(1+\omega)^p} = \sum_{i=0}^{\infty} (-1)^i \binom{i+p-1}{p-1} \omega^i$$

we get

$$ds^p = (-1)^p \frac{1}{t^{2p}} dt^p \sum_{i=0}^{\infty} (-1)^i \binom{i+p-1}{p-1} \left(\frac{dt}{t}\right)^i.$$

We put $dt^{k+1}=dt^{k+2}=\dots=0$ and get

$$ds^p = \sum_{i=0}^{k-p} (-1)^{i+p} \frac{1}{t^{i+2p}} \binom{i+p-1}{p-1} dt^{i+p}$$

and the proposition is proved. $\square$

Consider the invertible sheaf $\mathcal{O}(n)$ on $\mathbf{P}^1$, with $n \ge 1$. We want to study the principal parts $\mathcal{P}^k(\mathcal{O}(n))$ with $1 \le k \le n$ as a left $\mathcal{O}$-module.

Lemma 3.3. *On the open set* U_{01}, *as a left* $\mathcal{O}$*-module,* $\mathcal{P}^k(\mathcal{O}(n))$ *is a free* $\mathbf{Z}[t,1/t]$*-module of rank* $k+1$, *and there exists two natural bases. The bases are* $C=\{1\otimes x_0^n,\ dt\otimes x_0^n, \ldots, dt^k\otimes x_0^n\}$ *and* $C'=\{1\otimes x_1^n,\ ds\otimes x_1^n, \ldots, ds^k\otimes x_1^n\}$, *where* $dt^i=(u-t)^i$ *and* $ds^i=(1/u-1/t)^i$.

The proof is an easy calculation.

Theorem 3.4. *Consider* $\mathcal{P}^k(\mathcal{O}(n))$ *as a left* $\mathcal{O}$*-module on* $\mathbf{P}^1$. *On the open set* U_{01} *the transition-matrix* $[L]_C^{C'}$ *between the bases* C *and* C' *is given by the formula*

$$ds^p\otimes x_1^n=\sum_{i=0}^{k-p}(-1)^p\frac{1}{t^{i+2p-n}}\binom{n-p}{i}dt^{i+p}\otimes x_0^n,$$

where $0\leq p\leq k$.

Proof. By definition

$$ds^p\otimes x_1^n=\left(\frac{1}{u}-\frac{1}{t}\right)^p\otimes t^n x_0^n=(-1)^p\frac{(u-t)^p}{u^pt^p}u^n\otimes x_0^n.$$

Since $n-p\geq 0$ and $u=t+dt$ we get

$$ds^p\otimes x_1^n=(-1)^p\frac{1}{t^p}dt^p(t+dt)^{n-p}\otimes x_0^n.$$

Using the binomial theorem, we get

$$ds^p\otimes x_1^n=(-1)^p\frac{1}{t^p}dt^p\sum_{i=0}^{n-p}\binom{n-p}{i}t^{n-p-i}dt^i\otimes x_0^n.$$

By assumption $dt^{k+1}=dt^{k+2}=\ldots=0$, which gives

$$ds^p\otimes x_1^n=\sum_{i=0}^{k-p}(-1)^p\frac{1}{t^{i+2p-n}}\binom{n-p}{i}dt^{i+p}\otimes x_0^n,$$

and the theorem follows. □

Example 3.5. By Theorem 3.4 the transition matrix $[L]_C^{C'}$ for $\mathcal{P}^1(\mathcal{O}(n))$ is

$$[L]_C^{C'}=\begin{pmatrix} t^n & 0 \\ nt^{n-1} & -t^{n-2}\end{pmatrix}.$$

We compute the determinant $|[L]_C^{C'}|$ and find that it equals $-t^{2n-2}$.

4. Splitting principal parts as left modules by matrix diagonalization

In this section we will explicitly split the modules of principal parts. We will work over $\mathbf{P}^1$ defined over F, where F is a field. By [3], Theorem 2.1, we know that all locally free sheaves of finite rank on $\mathbf{P}^1$ split into a direct sum of invertible sheaves, and we want to explicitly compute the splitting-type for the sheaf $\mathcal{P}^1(\mathcal{O}(n))$ as a left $\mathcal{O}$-module. From Lemma 3.3 it follows that on the basic open set U_0, $\mathcal{P}^1(\mathcal{O}(n))$ is a free $F[t]$-module on the basis $C=\{1\otimes x_0^n, dt\otimes x_0^n\}$. On the open set U_1, $\mathcal{P}^1(\mathcal{O}(n))$ is a free $F[s]$-module on the basis $C'=\{1\otimes x_1^n, ds\otimes x_1^n\}$, where $s=1/t$. When we pass to the open set $\mathrm{U}_{01}=\mathrm{U}_0\cap \mathrm{U}_1$ we see that $\mathcal{P}^1(\mathcal{O}(n))$ has C and C' as bases as $F[t,s]$-module. On U_0 consider the new basis $D=\{1\otimes x_0^n, t\otimes x_0^n+n\,dt\otimes x_0^h\}$. Consider also the new basis $D'=\{1/t\otimes x_1^n+n\,ds\otimes x_1^n, 1\otimes x_1^n\}$ on the open set U_1. Notice that D and D' are bases if and only if the characteristic of F does not divide n, hence let us assume this for the rest of the section. We first compute the base change matrix for $\mathcal{P}^1(\mathcal{O}(n))|_{\mathrm{U}_0}$ from C to D, and we get the matrix

$$[I]_D^C = \begin{pmatrix} 1 & -\dfrac{1}{n}t \\ 0 & \dfrac{1}{n} \end{pmatrix}.$$

We secondly compute the base change matrix for $\mathcal{P}^1(\mathcal{O}(n))|_{\mathrm{U}_1}$ from D' to C', and get the matrix

$$[I]_{C'}^{D'} = \begin{pmatrix} \dfrac{1}{t} & 1 \\ n & 0 \end{pmatrix}.$$

In Example 3.5 we saw that the transition matrix defining $\mathcal{P}^1(\mathcal{O}(n))$ is given by

$$[L]_C^{C'} = \begin{pmatrix} t^n & 0 \\ nt^{n-1} & -t^{n-2} \end{pmatrix}.$$

If we let D be a new basis for $\mathcal{P}^1(\mathcal{O}(n))$ as $F[t]$-module on U_0, and let D' be a new basis for $\mathcal{P}^1(\mathcal{O}(n))$ as $F[s]$-module on U_1, the transition matrix $[L]_D^{D'}$ becomes

$$[L]_D^{D'} = [I]_D^C[L]_C^{C'}[I]_{C'}^{D'}$$

which equals

$$\begin{pmatrix} 1 & -\dfrac{1}{n}t \\ 0 & \dfrac{1}{n} \end{pmatrix} \begin{pmatrix} t^n & \\ nt^{n-1} & -t^{n-2} \end{pmatrix} \begin{pmatrix} \dfrac{1}{t} & 1 \\ n & 0 \end{pmatrix}.$$

We get

$$[L]_D^{D'} = \begin{pmatrix} t^{n-1} & 0 \\ 0 & t^{n-1} \end{pmatrix},$$

hence as a left $\mathcal{O}$-module, the principal parts $\mathcal{P}^1(\mathcal{O}(n))$ splits as $\mathcal{O}(n-1)\oplus\mathcal{O}(n-1)$. By Proposition 2.3, it follows that the splitting $\mathcal{P}^1(\mathcal{O}(n))\cong\mathcal{O}(n-1)\oplus\mathcal{O}(n-1)$ is valid on $\mathbf{P}^1_A$, where A is any F-algebra.

5. Maps of modules and systems of linear equations

We want to study the splitting-type of the the principal parts on the projective line $\mathbf{P}^1$ over any field F as left $\mathcal{O}$-modules. Given $\mathcal{P}^k(\mathcal{O}(n))$ with $1\leq k\leq n$, we will prove existence of systems of linear equations $\{A_r\mathbf{x_r}=b_r\}_{r=0}^k$, where A_r is a rank $r+1$ matrix with integral coefficients. A solution $\mathbf{x_r}$ to the system $A_r\mathbf{x_r}=b_r$ gives rise to a map ϕ_r of left $\mathcal{O}$-modules from $\mathcal{O}(n-k)$ to $\mathcal{P}^k(\mathcal{O}(n))$. The main result is Theorem 5.2 where we prove the following: If there exists, for all $r=0,\dots,k$, solutions $\mathbf{x_r}$ of the systems $A_r\mathbf{x_r}=b_r$ with coefficients in a field F, satisfying certain explicit criteria, then we can completely determine the splitting-type of the principal parts on $\mathbf{P}^1$ defined over the field F.

By Proposition 2.2 we know that $\mathcal{P}^k(\mathcal{O}(n))$ is locally free of rank $k+1$ over $\mathbf{P}^1$ defined over $\mathbf{Z}$, hence by base extension, $\mathcal{P}^k(\mathcal{O}(n))$ is locally free over $\mathbf{P}^1$ defined over any field F. By [2], Theorem 2.1, we know that on $\mathbf{P}^1$ every locally free sheaf of finite rank splits uniquely into a direct sum of invertible $\mathcal{O}$-modules. Recall from Lemma 3.3 that $\mathcal{P}^k(\mathcal{O}(n))$ has two natural bases on the open set U_{01}:

$$C=\{1\otimes x_0^n,\dots,dt^k\otimes x_0^n\}\quad\text{and}\quad C'=\{1\otimes x_1^n,\dots,ds^k\otimes x_1^n\}.$$

By Theorem 3.4, the transition matrix $[L]_C^{C'}$ from the basis C' to C is given by the relation

$$ds^p\otimes x_1^n=\sum_{i=0}^{k-p}(-1)^p\frac{1}{t^{i+2p-n}}\binom{n-p}{i}dt^{i+p}\otimes x_0^n. \tag{5.1}$$

We will use relation (5.1) to construct split injective maps $\mathcal{O}(n-k)\to\mathcal{P}^k(\mathcal{O}(n))$ of left $\mathcal{O}$-modules. On the open set U_0, the sheaf $\mathcal{O}(n-k)$ is isomorphic to $F[t]x_0^{n-k}$ as an $\mathcal{O}$-module. On the open set U_1, $\mathcal{O}(n-k)$ is isomorphic to $F[1/t]x_1^{n-k}$. For $i=0,1$ we want to define maps

$$\phi_r^i\colon\mathcal{O}(n-k)|_{\mathrm{U}_i}\longrightarrow\mathcal{P}^k(\mathcal{O}(n))|_{\mathrm{U}_i},$$

of left $\mathcal{O}_{\mathrm{U}_i}$-modules, agreeing on the open set U_{01}, where $r=0,\dots,k$. The maps $\{\phi_r^i\}_{i=0,1}$ will then glue to give $k+1$ well-defined maps $\phi_r\colon\mathcal{O}(n-k)\to\mathcal{P}^k(\mathcal{O}(n))$ of left $\mathcal{O}$-modules. Let $\phi_0^1(x_1^{n-k})=1\otimes x_1^n$. On the open set U_{01} we have $x_0^{n-k}=$

$t^{k-n}x_1^{n-k}$. We want to define $\phi_0^0(x_0^{n-k})$. Since $x_0^{n-k}=t^{k-n}x_1^{n-k}$ it follows that $\phi_0^0(x_0^{n-k})=t^{k-n}\phi_0^1(x_1^{n-k})=t^{n-k}(1\otimes x_1^{n-k})$. We now use relation (5.1) which proves the equation

$$1\otimes x_1^n=\binom{n}{0}t^n\otimes x_0^n+\binom{n}{1}t^{n-1}dt\otimes x_0^n+\ldots+\binom{n}{k}t^{n-k}dt^{n-k}\otimes x_0^n.$$

We define $\phi_0^0(x_0^{n-k})=t^{k-n}(1\otimes x_0^n)$, and it follows that

$$\phi_0^0(x_0^{n-k})=t^{k-n}\left(\binom{n}{0}t^n\otimes x_0^n+\binom{n}{1}t^{n-1}dt\otimes x_0^n+\ldots+\binom{n}{k}t^{n-k}dt^{n-k}\otimes x_0^n\right)$$

which equals

$$t^{k-n}\sum_{i=0}^{k}\binom{n}{i}t^{n-i}dt^i\otimes x_0^n.$$

Define $c_i^0=\binom{n}{i}$ and $x_{0,0}=1$. We get

$$\phi_0^0(x_0^{n-k})=\sum_{i=0}^{k}c_i^0t^{k-i}dt^i\otimes x_0^n \quad \text{and} \quad \phi_0^1(x_1^{n-k})=x_{0,0}(1\otimes x_1^n).$$

We see that we have defined a map of left $\mathcal{O}$-modules $\phi_0\colon\mathcal{O}(n-k)\to\mathcal{P}^k(\mathcal{O}(n))$, which in fact is defined over the integers $\mathbf{Z}$. We want to generalize the construction made above, and define maps of left $\mathcal{O}$-modules $\phi_r\colon\mathcal{O}(n-k)\to\mathcal{P}^k(\mathcal{O}(n))$ for $r=1,\ldots,k$. Define

$$\phi_r^1(x_1^{n-k})=x_{0,r}\frac{1}{t^r}\otimes x_1^n+x_{1,r}\frac{1}{t^{r-1}}ds\otimes x_1^n+\ldots+x_{r,r}ds^r\otimes x_1^n, \tag{5.2}$$

where the symbols $x_{i,r}$ are independent variables over F for all i and r. Simplifying, we get

$$\phi_r^1(x_1^{n-k})=\sum_{j=0}^{r}x_{j,r}t^{j-r}ds^j\otimes x_1^n.$$

We want to define a map ϕ_r^0 on U_0, such that ϕ_r^0 and ϕ_r^1 glue together to define a map of left $\mathcal{O}$-modules

$$\phi_r\colon\mathcal{O}(n-k)\longrightarrow\mathcal{P}^k(\mathcal{O}(n)).$$

For ϕ_r to be well defined it is necessary that ϕ_r^0 and ϕ_r^1 agree on U_{01}. On U_{01} we see that x_0^{n-k} equals $t^{k-n}x_1^{n-k}$, hence we get

$$\phi_r^0(x_0^{n-k})=\phi_r^0(t^{k-n}x_1^{n-k})=t^{k-n}\phi_r^1(x_1^{n-k}).$$

We get from equation (5.2),

$$\phi_r^0(x_0^{n-k}) = t^{k-n} \sum_{j=0}^{r} x_{j,r} t^{j-r} ds^j \otimes x_1^n$$

which equals

$$\sum_{j=0}^{r} x_{j,r} t^{j+k-n-r} ds^j \otimes x_1^n. \tag{5.3}$$

Using relation (5.1), we substitute $ds^j \otimes x_1^n$ in formula (5.3) and get the expression

$$\phi_r^0(x_0^{n-k}) = \sum_{j=0}^{r} x_{j,r} t^{j+k-n-r} \left(\sum_{i=0}^{k-j} (-1)^j t^{n-i-2j} \binom{n-j}{i} dt^{i+j} \otimes x_0^n \right).$$

Let $l=i+j$ be a change of index. We get the expression

$$\phi_r^0(x_0^{n-k}) = \sum_{j=0}^{r} x_{j,r} t^{j+k-n-r} \left(\sum_{l=j}^{k} (-1)^j t^{n-l-j} \binom{n-j}{l-j} dt^l \otimes x_0^n \right).$$

Since $\binom{a}{-1} = \binom{a}{-2} = \ldots = 0$, we get the expression

$$\phi_r^0(x_0^{n-k}) = \sum_{j=0}^{r} x_{j,r} t^{j+k-n-r} \left(\sum_{l=0}^{k} (-1)^j t^{n-l-j} \binom{n-j}{l-j} dt^l \otimes x_0^n \right).$$

Simplify to obtain

$$\sum_{j=0}^{r} \sum_{l=0}^{k} (-1)^j t^{k-r-l} \binom{n-j}{l-j} x_{j,r} dt^l \otimes x_0^n.$$

Change order of summation and simplify to get

$$\phi_r^0(x_0^{n-k}) = \sum_{l=0}^{k} t^{k-r-l} \left(\sum_{j=0}^{r} (-1)^j \binom{n-j}{l-j} x_{j,r} \right) dt^l \otimes x_0^n.$$

Let

$$c_l^r = \sum_{j=0}^{r} (-1)^j \binom{n-j}{l-j} x_{j,r} \tag{5.4}$$

for $r=1,\dots,k$ and $l=0,\dots,k$. We have defined maps

(5.5) $$\phi_r^0(x_0^{n-k})=\sum_{l=0}^{k} t^{k-r-l}c_l^r\,dt^l\otimes x_0^n$$

and

(5.6) $$\phi_r^1(x_1^{n-k})=\sum_{j=0}^{r} x_{j,r}t^{j-r}ds^j\otimes x_1^n.$$

Note that the definitions from (5.5) and (5.6) are valid for $r=0,\dots,k$ since $c_i^0=\binom{n}{i}$ and $x_{0,0}=1$.

Lemma 5.1. *Let $r=1,\dots,k$. The maps ϕ_r^0 and ϕ_r^1 glue to a well-defined map of left $\mathcal{O}$-modules*

$$\phi_r\colon \mathcal{O}(n-k)\longrightarrow \mathcal{P}^1(\mathcal{O}(n))$$

if and only if $c_k^r=c_{k-1}^r=\dots=c_{k-r+1}^r=0$ and $c_{k-r}^r=1$.

Proof. Consider the expression from (5.5):

$$\begin{aligned}\phi_r^0(x_0^{n-k})=&\sum_{l=0}^{k-r-1} c_l^r t^{k-r-l}dt^l\otimes x_0^n\\ &+c_{k-r}^r dt^{k-r}\otimes x_0^n+c_{k-r+1}^r\frac{1}{t}dt^{k-r+1}\otimes x_0^n+\dots+c_k^r\frac{1}{t^r}dt^k\otimes x_0^n.\end{aligned}$$

We see that the maps ϕ_r^0 and ϕ_r^1 glue if and only if we have

$$c_k^r=c_{k-1}^r=\dots=c_{k-r+1}^r=0 \quad\text{and}\quad c_{k-r}^r=1,$$

and the lemma follows. $\square$

Let $r=1,\dots,k$, and consider the equations from the proof of Lemma 5.1. We have $c_k^r=c_{k-1}^r=\dots=c_{k-r+1}^r=0$ and $c_{k-r}^r=1$. We get from the equation $c_k^r=0$ that

$$\binom{n}{k}x_{0,r}-\binom{n-1}{k-1}x_{1,r}+\binom{n-2}{k-2}x_{2,r}+\dots+(-1)^r\binom{n-r}{k-r}x_{r,r}=0.$$

Writing out $c_{k-1}^r=0$ we get the equation

$$\binom{n}{k-1}x_{0,r}-\binom{n-1}{k-2}x_{1,r}+\binom{n-2}{k-3}x_{2,r}+\dots+(-1)^r\binom{n-r}{k-r-1}x_{r,r}=0.$$

The equation $c^r_{k-r}=1$ gives

$$\binom{n}{k-r}x_{0,r}-\binom{n-1}{k-r-1}x_{1,r}+\binom{n-2}{k-r-2}x_{2,r}+\dots+\binom{n-r}{k-2r}x_{r,r}=1.$$

We get a system of linear equations $A_r\mathbf{x_r}=b_r$, where A_r is the rank $r+1$ matrix

$$\begin{pmatrix} \binom{n}{k} & -\binom{n-1}{k-1} & \binom{n-2}{k-2} & \dots & (-1)^r\binom{n-r}{k-r} \\ \binom{n}{k-1} & -\binom{n-1}{k-2} & \binom{n-2}{k-3} & \dots & (-1)^r\binom{n-r}{k-r-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \binom{n}{k-r} & -\binom{n-1}{k-r-1} & \binom{n-2}{k-r-2} & \dots & (-1)^r\binom{n-r}{k-2r} \end{pmatrix},$$

$\mathbf{x_r}$ is the vector $(x_{0,r},x_{1,r},\dots,x_{r,r})$, and b_r is the vector $(0,0,\dots,0,1)$. Clearly the coefficients of A_r and b_r are in $\mathbf{Z}$. Also, assume that $\mathbf{x_r}$ is a solution to the system $A_r\mathbf{x_r}=b_r$ with coefficients in a field F, then by construction and Lemma 5.1, the map

$$\phi_r\colon \mathcal{O}(n-k)\longrightarrow \mathcal{P}^k(\mathcal{O}(n))$$

defined by

$$\phi^0_r(x_0^{n-k})=\sum_{l=0}^{k-r} c^l_r t^{k-r-l}dt^l\otimes x_0^n$$

and

$$\phi^1_r(x_1^{n-k})=\sum_{j=0}^{r} x_{j,r}t^{j-r}ds^j\otimes x_1^n$$

is a well-defined and nontrivial map of left $\mathcal{O}$-modules. We can prove a theorem.

Theorem 5.2. *Assume that there exists a field F with the property that for all $k=1,\dots,r$ there exists a solution $\mathbf{x_r}$ to $A_r\mathbf{x_r}=b_r$ satisfying $\prod_{i=0}^k x_{i,i}\neq 0$, then $\mathcal{P}^k(\mathcal{O}(n))$ splits as $\bigoplus_{i=0}^k \mathcal{O}(n-k)$ as a left $\mathcal{O}_{\mathbf{P}^1}$-module over F.*

Proof. Assume that there exists a field F and k solutions $\mathbf{x_1},\dots,\mathbf{x_k}$ to the systems $A_r\mathbf{x_r}=b_r$, with coefficients in F satisfying the property that $\prod_{i=0}^k x_{i,i}\neq 0$. On the open set U_0, the module $\bigoplus_{i=0}^k\mathcal{O}(n-k)$ is a free $k[t]$-module on the basis $\{x_0^{n-k}e_0,\dots,x_0^{n-k}e_k\}$. Define the map

$$\phi^0\colon\bigoplus_{i=0}^k\mathcal{O}(n-k)|_{\mathrm{U}_0}\longrightarrow\mathcal{P}^k(\mathcal{O}(n))|_{\mathrm{U}_0},\quad \phi^0(x_0^{n-k}e_r)=\phi^0_r(x_0^{n-k}).$$

On the open set U_1 the module $\bigoplus_{i=0}^{k} \mathcal{O}(n-k)$ is a free $k[1/t]$-module on the basis $\{x_1^{n-k} f_0, \dots, x_1^{n-k} f_k\}$. Define the map

$$\phi^1 \colon \bigoplus_{i=0}^{k} \mathcal{O}(n-k)|_{U_1} \longrightarrow \mathcal{P}^k(\mathcal{O}(n))|_{U_1}, \quad \phi^1(x_1^{n-k} f_r) = \phi_r^1(x_1^{n-k}).$$

Then by construction, the maps ϕ^0 and ϕ^1 glue to a well-defined map ϕ from $\bigoplus_{i=0}^{k} \mathcal{O}(n-k)$ to $\mathcal{P}^k(\mathcal{O}(n))$ of left $\mathcal{O}$-modules. We show explicitly that the map ϕ is an isomorphism: Consider the matrix corresponding to the map $\phi|_{U_0}$,

$$[\phi^0] = \begin{pmatrix} t^k c_0^0 & t^{k-1} c_0^1 & \dots & t c_0^{k-1} & c_0^k \\ t^{k-1} c_1^0 & t^{k-2} c_1^1 & \dots & c_1^{k-1} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ t c_{k-1}^0 & c_{k-1}^1 & \dots & 0 & 0 \\ c_k^0 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

We see the determinant $|[\phi^0]|$ equals $\prod_{i=0}^{k} c_{k-i}^i$ which equals 1 by construction, hence the map ϕ^0 is an isomorphism. Consider the matrix corresponding to $\phi^1|_{U_1}$,

$$[\phi^1] = \begin{pmatrix} x_{0,0} & x_{0,1}\frac{1}{t} & \dots & x_{0,k-1}\frac{1}{t^{k-1}} & x_{0,k}\frac{1}{t^k} \\ 0 & x_{1,1} & \dots & x_{1,k-1}\frac{1}{t^{k-2}} & x_{1,k}\frac{1}{t^{k-1}} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & x_{k,k} \end{pmatrix}.$$

The determinant $|[\phi^1]|$ equals $\prod_{i=0}^{k} x_{i,i}$ which is non-zero by hypothesis. $\square$

6. Application: The left module structure in characteristic zero

In this section we use the results obtained in the previous section to determine the splitting-type of $\mathcal{P}^k(\mathcal{O}(n))$ for all $1 \le k \le n$ on the projective line defined over any field of characteristic zero.

Lemma 6.1. *Let $n, k, a, b \ge 0$, and put $\binom{0}{1} = 0$. Then we have the equality*

$$\binom{n-a+1}{k-a-b+2} - \frac{n-k+b}{k-b+1}\binom{n-a+1}{k-a-b+1} = \frac{\binom{a-1}{1}\binom{n-a+1}{k-a-b+2}}{\binom{k-b+1}{1}}.$$

The proof is an easy calculation.

Proposition 6.2. *Let $n, k \geq 1$, and consider the matrix A_r from Theorem 5.2, with $r=1,2,\dots$. Then the determinant*

$$|A_r| = \begin{vmatrix} \binom{n}{k} & -\binom{n-1}{k-1} & \binom{n-2}{k-2} & \cdots & (-1)^r\binom{n-r}{k-r} \\ \binom{n}{k-1} & -\binom{n-1}{k-2} & \binom{n-2}{k-3} & \cdots & (-1)^r\binom{n-r}{k-r-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \binom{n}{k-r} & -\binom{n-1}{k-r-1} & \binom{n-2}{k-r-2} & \cdots & (-1)^r\binom{n-r}{k-2r} \end{vmatrix} = \pm\prod_{l=0}^{r}\frac{\binom{n-l}{k-r}}{\binom{k-l}{r-l}}.$$

Proof. We prove the formula by induction on the rank of the matrix. Assume first that $r=1$. Adding $-(n-k+1)/k$ times the second row to the first row of A_1 and applying Lemma 6.1 with $a=1$ and $b=1$, we see that the formula is true for $r=1$. Assume the formula is true for rank r matrices A_{r-1}. Consider the matrix

$$M_r = \begin{pmatrix} \binom{n}{k} & \binom{n-1}{k-1} & \binom{n-2}{k-2} & \cdots & \binom{n-r}{k-r} \\ \binom{n}{k-1} & \binom{n-1}{k-2} & \binom{n-2}{k-3} & \cdots & \binom{n-r}{k-r-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \binom{n}{k-r} & \binom{n-1}{k-r-1} & \binom{n-2}{k-r-2} & \cdots & \binom{n-r}{k-2r} \end{pmatrix}$$

which is the matrix A_r with signs removed. Add $-(n-k+1)/k$ times the second row to the first row. Continue and add $-(n-k+1+i)/(k-i)$ times the $(i+1)$th row to the ith row, for $i=2,\dots,r-1$. Apply Lemma 6.1 to get the matrix

$$N_r = \begin{pmatrix} 0 & \frac{\binom{1}{1}\binom{n-1}{k-1}}{\binom{k}{1}} & \frac{\binom{2}{1}\binom{n-2}{k-2}}{\binom{k}{1}} & \cdots & \frac{\binom{r}{1}\binom{n-r}{k-r}}{\binom{k}{1}} \\ 0 & \frac{\binom{1}{1}\binom{n-1}{k-2}}{\binom{k-1}{1}} & \frac{\binom{2}{1}\binom{n-2}{k-3}}{\binom{k-1}{1}} & \cdots & \frac{\binom{r}{1}\binom{n-r}{k-r-1}}{\binom{k-1}{1}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{\binom{1}{1}\binom{n-1}{k-r}}{\binom{k-r+1}{1}} & \frac{\binom{2}{1}\binom{n-2}{k-r-1}}{\binom{k-r+1}{1}} & \cdots & \frac{\binom{r}{1}\binom{n-r}{k-2r+1}}{\binom{k-r+1}{1}} \\ \binom{n}{k-r} & * & * & \cdots & * \end{pmatrix}.$$

The determinant of N_r equals

$$(-1)^r\frac{\binom{n}{k-r}\binom{1}{1}\binom{2}{1}\cdots\binom{r}{1}}{\binom{k-r+1}{1}\binom{k-r+2}{1}\cdots\binom{k-1}{1}\binom{k}{1}}|M'_{r-1}|,$$

where M'_{r-1} is the matrix

$$\begin{pmatrix} \binom{n'}{k'} & \binom{n'-1}{k'-1} & \cdots & \binom{n'-r+1}{k'-r+1} \\ \binom{n'}{k'-1} & \binom{n'-1}{k'-2} & \cdots & \binom{n'-r+1}{k'-r} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{n'}{k'-r+1} & \binom{n'-1}{k'-r} & \cdots & \binom{n'-r+1}{k'-2r+2} \end{pmatrix},$$

and $n'=n-1$ and $k'=k-1$. By the induction hypothesis, we get modulo signs

$$\frac{\binom{n}{k-r}}{\binom{k}{r}} \prod_{i=0}^{r-1} \frac{\binom{n'-i}{k'-r+1}}{\binom{k'-i}{r-1-i}}.$$

Change index by letting $l=i+1$ we get

$$\frac{\binom{n}{k-r}}{\binom{k}{r}} \prod_{l=1}^{r} \frac{\binom{n-l}{k-r}}{\binom{k-l}{r-l}} = \prod_{l=0}^{r} \frac{\binom{n-l}{k-r}}{\binom{k-l}{r-l}}$$

and the proposition follows. □

Proposition 6.3. *Let F be a field of characteristic zero. The maps ϕ_r from Theorem 5.2 exist for $r=1,\dots,k$, and the induced map $\phi=\bigoplus_{i=0}^{r}\phi_r$ defines an isomorphism*

$$\mathcal{P}^k(\mathcal{O}(n)) \cong \bigoplus_{i=0}^{k} \mathcal{O}(n-k)$$

of left $\mathcal{O}$-modules.

Proof. Consider the systems $A_r\mathbf{x_r}=b_r$ for $r=1,\dots,k$, from the proof of Theorem 5.2. By Proposition 6.2 we have that

$$|A_r| = \prod_{l=0}^{r} \frac{\binom{n-l}{k-r}}{\binom{k-l}{r-l}}$$

modulo signs. Since the characteristic of F is zero, the determinant $|A_r|$ is different from 0 for all $r=1,\dots,k$, hence the system $A_r\mathbf{x_r}=b_r$ has a unique solution $\mathbf{x_r}=A_r^{-1}b_r$ for all r. It follows from Theorem 5.2 that the maps

$$\phi_r\colon \mathcal{O}(n-k) \longrightarrow \mathcal{P}^k(\mathcal{O}(n))$$

of left $\mathcal{O}$-modules exist, for $r=1,\dots,k$, and we can consider the map

$$\phi=\bigoplus_{i=0}^{r}\phi_i\colon\bigoplus_{i=0}^{r}\mathcal{O}(n-k)\longrightarrow\mathcal{P}^k(\mathcal{O}(n)).$$

We want to prove that ϕ is an isomorphism. Again by Theorem 5.2, ϕ is an isomorphism if and only if $x_{i,i}\neq 0$ for $i=0,\dots,k$. Assume that $x_{r,r}=0$, and consider the system $A_r\mathbf{x_r}=b_r$. If $x_{r,r}=0$ we get a new system $A_{r-1}\mathbf{y_{r-1}}=0$, where $\mathbf{y_{r-1}}$ is the vector $(x_{0,r},\dots,x_{r-1,r})$. Since the matrix A_{r-1} is invertible, it follows that the system $A_{r-1}\mathbf{y_{r-1}}=0$ only has the trivial solution $\mathbf{y_{r-1}}=0$, hence $x_{0,r}=\dots=x_{r-1,r}=0$, and we have arrived at a contradiction to the assumption that $\mathbf{x_r}$ is a solution to the system $A_r\mathbf{x_r}=b_r$, where b_r is the vector $(0,\dots,0,1)$. It follows that $x_{r,r}\neq 0$ for all $r=0,\dots,k$, and the proposition follows from Theorem 5.2. □

7. Splitting the right module structure

In this section we consider the splitting-type of the principal parts as left and right $\mathcal{O}$-modules on $\mathbf{P}^1$ defined over F, where F is any field. We prove that in most cases the splitting-type as left module differs from the splitting-type as right module. We also show how the splitting-type of the principal parts as a left $\mathcal{O}_{\mathbf{P}^1}$-module changes with the characteristic of the field F. Consider $\mathcal{P}^1(\mathcal{O}(n))$ on $\mathbf{P}^1_F$, where F is any field and $n\geq 1$. It is easy to see that $\mathcal{P}^1(\mathcal{O}(n))$ is locally free as a right $\mathcal{O}$-module.

Theorem 7.1. *If the characteristic of F does not divide n, then $\mathcal{P}^1(\mathcal{O}(n))$ splits as $\mathcal{O}(n-1)\oplus\mathcal{O}(n-1)$ as a left $\mathcal{O}$-module and as $\mathcal{O}(n)\oplus\mathcal{O}(n-2)$ as a right $\mathcal{O}$-module. If the characteristic of F divides n, then $\mathcal{P}^1(\mathcal{O}(n))$ splits as $\mathcal{O}(n)\oplus\mathcal{O}(n-2)$ as left and right $\mathcal{O}$-modules.*

Proof. Recall from Section 4, that the splitting-type of $\mathcal{P}^1(\mathcal{O}(n))$ as a left $\mathcal{O}$-module is $\mathcal{O}(n-1)\oplus\mathcal{O}(n-1)$ if the characteristic of F does not divide n. We next consider the right $\mathcal{O}$-module structure. Let p and q be the projection maps from $\mathbf{P}^1\times\mathbf{P}^1$ to $\mathbf{P}^1$. By definition, $\mathcal{P}^1(\mathcal{O}(n))$ is $p_*(\mathcal{O}_{\Delta^1}\otimes q^*\mathcal{O}(n))$, where $\mathcal{O}_{\Delta^1}$ is the first order infinitesimal neighborhood of the diagonal. By Proposition 2.5 we get the right $\mathcal{O}$-module structure, by considering the module $q_*(\mathcal{O}_{\Delta^1}\otimes q^*\mathcal{O}(n))=q_*(\mathcal{O}_{\Delta^1})\otimes\mathcal{O}(n)$. One checks that $\mathcal{E}|_{\mathrm{U}_0}$ is a free $k[u]$-module on the basis $E=\{1\otimes x_0^n, du\otimes x_0^n\}$, where $du=t-u$. Similarly, $\mathcal{E}|_{\mathrm{U}_1}$ is a free $k[1/u]$-module on the basis $E'=\{1\otimes x_1^n, d(1/u)\otimes x_1^n\}$, where $d(1/u)=1/t-1/u$. On U_{01} the module $\mathcal{E}$ is free on E and E' as an $F[u,1/u]$-module. We compute the transition matrix $[R]_E^{E'}$. We see that $1\otimes x_1^n$ equals $u^n\otimes x_0^n$. By definition, $d(1/u)\otimes x_1^n$ equals $(1/t-1/u)u^n\otimes x_0^n$. We get

$$\left(\frac{1}{t}-\frac{1}{u}\right)u^n\otimes x_0^n=\frac{u-t}{ut}u^n\otimes x_0^n=-u^{n-1}du\left(\frac{1}{t}\right)\otimes x_0^n.$$

By definition, $t=u+du$, and hence we get

$$-u^{n-1}du\frac{1}{u+du}\otimes x_0^n=-u^{n-2}du\frac{1}{1+du/u}\otimes x_0^n.$$

Since $du^2=du^3=...=0$, we get

$$d\left(\frac{1}{u}\right)\otimes x_1^n=-u^{n-2}du\otimes x_0^n,$$

hence the transition matrix looks as

$$[R]_E^{E'}=\begin{pmatrix} u^n & 0 \\ 0 & -u^{n-2} \end{pmatrix},$$

and it follows that $\mathcal{E}$ splits as $\mathcal{O}(n)\oplus\mathcal{O}(n-2)$ as an $\mathcal{O}$-module. Recall the transition matrix for $\mathcal{P}^1(\mathcal{O}(n))$ as a left $\mathcal{O}$-module,

$$[L]_C^{C'}=\begin{pmatrix} t^n & 0 \\ nt^{n-1} & -t^{n-2} \end{pmatrix}.$$

Clearly if the characteristic of F divides n, the splitting-type of $\mathcal{P}^1(\mathcal{O}(n))$ is $\mathcal{O}(n)\oplus\mathcal{O}(n-2)$ as left and right $\mathcal{O}$-modules, and we have proved the theorem. $\square$

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