Some estimates for spectral functions connected with formally hypoelliptic differential operators

NILS NILSSON

University of Lund, Lund, Sweden

1. Introduction

We are going to consider a differential operator $a(x, D) = \sum a_\alpha(x)D^\alpha$ in and open connected subset S of R^n, where D is the differentiation symbol $(2\pi i)^{-1}(\partial/\partial x_1, \ldots, \partial/\partial x_n)$ and the summation is made over a finite number of multi-orders $\alpha = (\alpha_1, \ldots, \alpha_n)$. We assume that the operator $a(x, D)$ is formally hypoelliptic (FHE) of type P in S, i.e. that the (complex-valued) coefficients a_α are in $C^\infty(S)$ and that for every $x \in S$ the polynomial (in $\xi \in R^n)|a(x, \xi) = \sum a_\alpha(x)\xi^\alpha$ is equally strong as the hypoelliptic polynomial P in the sense of Hörmander [4]. We also require that the type polynomial P is not a constant. Moreover, we suppose that $a(x, D)$ is formally self-adjoint in S, i.e. that we have $\sum a_\alpha(x)D^\alpha = \sum D^\alpha \overline{a_\alpha(x)}$ there. Then with no loss of generality we may assume that $\operatorname{Re} a(x, \xi) \to + \infty$ as $|\xi| \to \infty$, $\xi \in R^n$, for every $x \in S$ (Lemma 3).

Suppose now that A is a self-adjoint realization of $a(x, D)$ in the Hilbert space $L^2(S)$ (note that A need not be bounded from below), and let $A = \int_{-\infty}^{\infty} \lambda dE_\lambda$ be its spectral resolution, the E_λ being orthogonal projections in $L^2(S)$, increasing with λ. We shall then prove (Theorem 1) that for every real number λ the projection E_λ is given by a kernel e_λ in $C^\infty(S \times S)$ (e_λ is called the spectral function of A) and that, when $\lambda \to - \infty$, e_λ tends exponentially to zero together with its derivatives (with respect to the variables in $S \times S$), uniformly on compact subsets of $S \times S$.

Further, for an arbitrary n-order α we shall investigate the behaviour as $\lambda \to + \infty$ of the derivative $e_\lambda^{(\alpha, \alpha)}(x, y) = D_x^\alpha(- D_y)^\alpha e_\lambda(x, y)$ when $x = y \in S$. We shall then compare $e_\lambda^{(\alpha, \alpha)}(x, x)$ to the function

$$e_{x,\lambda}^{(\alpha, \alpha)}(x, y) = \int\limits_{\operatorname{Re} a(x, \xi) \leq \lambda} \xi^{2\alpha} \exp\left(2\pi i\langle x - y, \xi\rangle\right)d\xi \ .$$

It will be shown (Theorem 2) that $e_\lambda^{(\alpha,\,\alpha)}(x, x) = (1 + o(1))e_{x,\lambda}^{(\alpha,\,\alpha)}(x, x)$ as $\lambda \to +\infty$, for every $x \in S$.

The proof of these results will rely on the construction and estimation of a local fundamental solution of the operator $a(x, D) - \lambda$ when λ is large and negative. This is made in a series of lemmas, concluded with Lemma 10. Further we shall use a Tauberian theorem for the Stieltjes transformation due to Ganelius.

Results for elliptic operators similar to those of this paper (sometimes sharper) have been given by a great number of authors, and for hypoelliptic operators with constant coefficients by Gorčakov [3] and by the author of the present paper ([8]).

2. Formally hypoelliptic differential operators

In this section we shall, for later reference, state a few properties of FHE differential operators (and, particularly, of hypoelliptic operators with constant coefficients), which are well-known from the works of Hörmander [4], [5], Malgrange [6], and Peetre [10], or follow easily from them.

Assume that the differential operator $P(D)$ with constant coefficients is hypoelliptic, i.e. that $(D^\alpha P(\xi))/P(\xi) \to 0$ as $|\xi| \to \infty$, $\xi \in R^n$, $\alpha \neq 0$. Then one has also the stronger relation

$$|\xi|^c(D^\alpha P(\xi))/P(\xi) = O(1) \quad (|\xi| \to \infty, \ \xi \in R^n, \ \alpha \neq 0) \tag{1}$$

with some positive constant c. In particular, if P is not a constant,

$$|\xi|^c/P(\xi) = o(1) \quad (|\xi| \to \infty, \ \xi \in R^n) . \tag{2}$$

Further there is a largest positive number b such that with some constant C

$$|D^\alpha P(\xi)| \leq C(|P(\xi)| + 1)^{1-b|\alpha|} \quad (\xi \in R^n) \tag{3}$$

for all multi-orders α (writing $|\alpha| = \alpha_1 + \ldots + \alpha_n$). This largest number is not changed if in (3) we restrict ourselves to those α for which $|\alpha| = 1$. If P has degree m we have $b \leq 1/m$, and if r is a positive integer, and if b' is the corresponding number given by (3) for P^r, then $b' = b/r$. If Q is a polynomial weaker than P, i.e. we have $\sum_\alpha |D^\alpha Q(\xi)| \leq C \sum_\alpha |D^\alpha P(\xi)|$ $(\xi \in R^n)$ with some constant C, then for any multi-order α

$$|D^\alpha Q(\xi)| \leq C(|P(\xi)| + 1)^{1-b|\alpha|} \quad (\xi \in R^n) , \tag{4}$$

where C is some constant and b the same as in (3). (This follows easily from Hörmander [5], Remark 2, p. 209.)

If $a(x, D) = \sum a_\alpha(x)D^\alpha$ is a differential operator in the open subset S of R^n and P a hypoelliptic polynomial (in n variables), then $a(x, D)$ is said to be formally hypoelliptic (FHE) of type P in S, if all the coefficients a_α are in $C^\infty(S)$

and if for every $x \in S$ the polynomial (in ξ) $a(x, \xi)$ is equally strong as P. Then $a(x, D)$ can be written in the form

$$a(x, D) = \sum b_j(x)Q_j(D) , \tag{5}$$

where the coefficients b_j are in $C^\infty(S)$ while the polynomials Q_j are all weaker than P. Of course the Q_j can all be chosen real.

If $a_1(x, D)$ and $a_2(x, D)$ are FHE of the types P_1 and P_2, respectively, then the composed operator $a_1(x, D)a_2(x, D)$ is FHE of type P_1P_2.

If $a(x, D)$ is FHE of some type P of degree ≥ 1 and if p is any non-negative integer, then there is another positive integer s such that if u is a distribution in S with $a(x, D)^s u \in L^2(S)$, then $u \in C^p(S)$. We have then for all such u an inequality

$$\sup_{x \in K} |D^\alpha u(x)| \leq C \left(\int\limits_S (|a(x, D)^s u(x)|^2 + |u(x)|^2)dx \right)^{1/2} , \tag{6}$$

for any multi-order α with $|\alpha| \leq p$, and for any compact subset K of S, C being a number which does not depend on u (but e.g. on K).

3. Some lemmas

To prepare our construction and estimation of a fundamental solution of the operator $(a(x, D) - \lambda)$, we shall give some lemmas.

LEMMA 1. *Let P and Q be complex polynomials in n variables such that P is hypoelliptic and Q weaker than P, and let $\xi^0 \in R^n$. Then we have*

$$|D^\alpha Q(\xi + \tau z \xi^0) - D^\alpha Q(\xi)| \leq C|z|\tau^{-c|\alpha|}(|\xi| + 1)^{-c|\alpha|}(|P(\xi)| + \tau^{1/b}) \tag{7}$$

for all $\xi \in R^n$ and real numbers $\tau \geq 1$, and for all complex numbers z with $|z| \leq 1$. Here α is an arbitrary multi-order, b the number given by (3), while c and C are positive and independent of ξ, z, and τ.

Proof. Suppose first $\alpha \neq 0$. By Taylor's formula

$$D^\alpha Q(\xi + \tau z \xi^0) - D^\alpha Q(\xi) = \sum_{j=1}^{m-|\alpha|} (\tau z)^j Q_j(\xi) ,$$

where m is the degree of P (of course $\deg(Q) \leq \deg(P)$, Q being weaker than P). Further Q_j is a linear combination of derivatives of Q of order $|\alpha| + j$. By (4) we have

$$|\tau^j z^j Q_j(\xi)| \leq C\tau^j |z|^j(|P(\xi)| + 1)^{1-b(|\alpha|+j)} \tag{8}$$

with some constant C. From the elementary inequality $x^a y^{1-a} \leq x + y$, valid for $x, y \geq 0$ and $0 \leq a \leq 1$, we get from (8), putting $x = \tau^{1/b}$, $y = |P(\xi)| + 1$ and $jb < a < 1$ (possible, since $b \leq 1/m$ and $j < m$), and inequality

$$|\tau^j z^j Q_j(\xi)| \leq C |z|^j \tau^{(bj-a)/b}(|P(\xi)| + 1 + \tau^{1/b}) \; (|P(\xi)| + 1)^{a-b|\alpha|-bj} \, .$$

So, taking $a - jb$ sufficiently small, the lemma follows from (2) when $\alpha \neq 0$. The case $\alpha = 0$ is clearly simpler.

By use of (4) and the inequality $x^a y^{a-1} \leq x + y$ we readily get also the following lemma.

LEMMA 2. *With P and Q as in Lemma 1 we have*

$$|D^\alpha Q(\xi)| \leq C \tau^{-c|\alpha|}(|\xi| + 1)^{-c|\alpha|}(|P(\xi)| + \tau^{1/b}) \quad (\xi \in R^n)$$

for all multi-orders α, and all $\tau \geq 1$, where C and c are positive constants and b the same as in Lemma 1.

We also have

LEMMA 3. *Let $a(x, D)$ be FHE of type P and formally self-adjoint in an open subset S of R^n. Then $\mathrm{Re}\,(a(x, D))$ (obtained by taking real parts of the coefficients) is also FHE of type P, and $\mathrm{Im}\,(a(x, D))$ is strictly weaker than P, i.e.,*

$$|\xi|^c (D_\xi^\alpha \, \mathrm{Im}\,(a(x, \xi)))/P(\xi) \to 0 \quad (|\xi| \to \infty, \; \xi \in R^n)$$

for any multi-order α, where c is some positive constant and the convergence is uniform for x in compact subsets of S.

Proof. We use the representation (5). Then the formal adjoint $a(x, D)^*$ of $a(x, D)$ is given by $\sum Q_j(D)\overline{b_j(x)}$. By Leibniz's formula

$$a(x, D)^* = \sum \overline{b_j(x)} Q_j(D) + \sum c_k(x) R_k(D) \, ,$$

where every R_k is of the form $D^\alpha(Q_j)$, with $\alpha \neq 0$. Hence, by (4) and (2) it follows that with some positive constant c we have

$$|\xi|^c (D^\beta R_k(\xi))/P(\xi) \to 0 \quad (|\xi| \to \infty, \; \xi \in R^n)$$

for every k and every multi-order β. Since by assumption $a(x, D)^* = a(x, D)$, we get

$$2i \, \mathrm{Im}\, a(x, D) = \sum c_k(x) R_k(D) \, ,$$

from which the lemma immediately follows.

It follows from Lemma 3 that if $a(x, D)$ is FHE and formally self-adjoint, then the type polynomial P can be chosen real. If S is in addition connected and

deg $(P) > 0$, then either $\mathrm{Re}\, a(x, \xi) \to + \infty$ as $|\xi| \to \infty$, $\xi \in R^n$, for all $x \in S$ or this is true with $- \infty$ instead of $+ \infty$. This follows easily from Lemma 3, (4), and a simple continuity argument.

From Nilsson [8], Theorem 1, we take the following result.

LEMMA 4. *Let P be a real polynomial on R^n such that $P(\xi) \to + \infty$ as $|\xi| \to \infty$. Let α be an arbitrary multi-order. When $\lambda \in R$, put*

$$e^{(2\alpha)}(\lambda) = \int_{P(\xi) \leq \lambda} \xi^{2\alpha} d\xi \,.$$

Then there are non-negative real numbers C, a, and t, where $C > 0$ and t is an integer, such that

$$C^{-1}\lambda^a(\log \lambda)^t \leq e^{(2\alpha)}(\lambda) \leq C\lambda^a(\log \lambda)^t \tag{9}$$

when λ is large enough. Further the function $e^{(2\alpha)}$ is differentiable when λ is large enough, and we have

$$de^{(2\alpha)}(\lambda)/d\lambda = O(1)\lambda^{a-1}(\log \lambda)^t \quad (\lambda \to + \infty) \,.$$

Remark. Lemma 4 can be sharpened in various directions. E.g. it is possible to replace (9) by the estimate

$$e^{(2\alpha)}(\lambda) = C'(1 + o(1))\lambda^a(\log \lambda)^t \quad (\lambda \to + \infty) \,,$$

with some constant $C' > 0$. Also, it is not necessary that the exponent of ξ in the integral be even (if not, one must of course allow that $C' = 0$ and that $a < 0$).

We shall also need

LEMMA 5. *Let P be as in Lemma 4 and in addition hypoelliptic, and put*

$$\tilde{e}^{(\alpha)}(\lambda) = \int_{P(\xi) \leq \lambda} |\xi^\alpha| d\xi \,,$$

when α is a multi-order. Then, if $\beta \leq \alpha$ (i.e. if $\beta_j \leq \alpha_j$ for all j), if $\alpha \neq \beta$, and if α has only even coordinates, we have

$$\tilde{e}^{(\beta)}(\lambda) = O(1)\lambda^{-c}\tilde{e}^{(\alpha)}(\lambda) \quad (\lambda \to + \infty) \,,$$

where c is some positive constant.

Proof. When γ is a multi-order, put $\hat{\gamma} = (\gamma_1 + 1,\ \gamma_2, \ldots, \gamma_n)$. It is clearly sufficient to show that for any γ we have

$$\tilde{e}^{(\gamma)}(\lambda) = O(1)\lambda^{-c}\tilde{e}^{(\hat{\gamma})}(4\lambda) \quad (\lambda \to + \infty) \,, \tag{10}$$

with some positive constant c. For in (10) we can, of course, as well add 1 to any other of the component orders than γ_1, and so we get by repeated use of (10)

$$\tilde{e}^{(\beta)}(\lambda) = O(1)\lambda^{-c}\tilde{e}^{(\alpha)}(C\lambda) = O(1)\lambda^{-c}\tilde{e}^{(\alpha)}(\lambda) \quad (\lambda \to + \infty)\,,$$

where c and C are positive constants, and where in the last step we have used Lemma 4 for $\tilde{e}^{(\alpha)} = e^{(\alpha)}$.

When T is a positive real number, put

$$M_T(\lambda) = \{\xi \in R^n | P(\xi) \le \lambda,\ |\xi_1| \le T\} \quad \text{and} \quad N_T(\lambda) = \{\xi \in R^n | P(\xi) \le \lambda,\ |\xi_1| > T\}\,,$$

and let us prove that there is a positive number k such that

$$\int_{M_T(\lambda)} |\xi^\gamma| d\xi \le \int_{N_T(4\lambda)} |\xi^\gamma| d\xi \tag{11}$$

when $T = \lambda^k$ and λ is large enough. For then we should get

$$e^{(\gamma)}(\lambda) = \int_{M_T(\lambda)} |\xi^\gamma| d\xi + \int_{N_T(\lambda)} |\xi^\gamma| d\xi \le 2 \int_{N_T(4\lambda)} |\xi^\gamma| d\xi \le 2T^{-1} \int_{N_T(4\lambda)} |\xi^{\hat{\gamma}}| d\xi \le 2\lambda^{-k}\tilde{e}^{(\hat{\gamma})}(4\lambda)$$

for large λ, proving (10).

Put $\xi = (\xi_1, \xi')$, with $\xi' = (\xi_2, \ldots, \xi_n) \in R^{n-1}$. Then $Q(\xi') = P(0, \xi')$ is a hypoelliptic polynomial on R^{n-1}, and

$$|\xi'|^{-c}Q(\xi') \to + \infty \quad (|\xi'| \to \infty) \tag{12}$$

with some positive constant c. Let $\tilde{e}_1^{(\gamma)}$ correspond to Q (on R^{n-1}) as $\tilde{e}^{(\gamma)}$ to P. Put $\gamma' = (\gamma_2, \ldots, \gamma_n)$ and

$$f(\xi_1, \lambda) = \int_{P(\xi_1, \xi') \le \lambda} |(\xi')^{\gamma'}| d\xi'\,.$$

We contend that there is a positive number c such that when λ is sufficiently large we have

$$e_1^{(\gamma')}(2^{-1}\lambda) \le f(\xi_1, \lambda) \le e_1^{(\gamma')}(2\lambda) \tag{13}$$

for all ξ_1 with $|\xi_1| \le \lambda^{2c}$. Indeed, we have the Taylor expansion

$$P(\xi_1, \xi') = Q(\xi') + \sum_{j=1}^{m} (2\pi i)^j \xi_1^j (j!)^{-1} D^{(j, 0)} P(0, \xi')\,,$$

and since with some positive number c we have

$$|\xi'|^c D^{(j, 0)} P(0, \xi')/Q(\xi') \to 0 \quad (|\xi'| \to \infty)$$

for every $j \ge 1$, it follows that with some positive number c we have

$$2^{-1}Q(\xi') \le P(\xi_1, \xi') \le 2Q(\xi')$$

when $|\xi_1| \le |\xi'|^e$ and $|\xi'|$ is sufficiently large. From this inequality and (12) we conclude (13).

Now we get by (13)

$$\int_{M_T(\lambda)} |\xi^\gamma| d\xi \le T^{\gamma_1} \int_{-T}^{T} f(\xi_1, \lambda) d\xi_1 \le 2T^{\gamma_1+1} e_1^{(\gamma')}(2\lambda) \quad \text{and}$$

$$\int_{N_T(4\lambda)} |\xi^\gamma| d\xi \ge \int_{T}^{\lambda^{2c}} f(\xi_1, 4\lambda) d\xi_1 \ge (\lambda^{2c} - T) \tilde{e}_1^{(\gamma')}(2\lambda)$$

when T is any real number such that $1 \le T \le \lambda^{2c}$ and λ is large enough. Thus we immediately get (11) (with e.g. k equal to $c/(\gamma_1 + 1)$ with the c of the above inequality). The lemma is proved.

Let P be as in Lemma 4 and assume that for the multi-order α we have $\tilde{e}^{(\alpha)}(\lambda) = O(1)\lambda^\varkappa$ as $\lambda \to + \infty$, where $\varkappa$ is a constant < 1. Then the integral

$$\tilde{S}^{(\alpha)}(\lambda) = \int |\xi^\alpha| (P(\xi) - \lambda)^{-1} d\xi$$

is convergent when λ is sufficiently large and negative. For, with some real number c,

$$\tilde{S}^{(\alpha)}(\lambda) = \int_{c}^{\infty} (\mu - \lambda)^{-1} d\tilde{e}^{(\alpha)}(\mu) = \int_{c}^{\infty} (\mu - \lambda)^{-2} \tilde{e}^{(\alpha)}(\mu) d\mu \,, \tag{14}$$

the last step follows by an integration by parts, where the boundary term at ∞ vanishes because of our growth condition on $\tilde{e}^{(\alpha)}$.

From (14) and Lemma 5 we easily get

LEMMA 6. *If $\beta \le \alpha$, $\beta \ne \alpha$, and all the components of α are even, then there is a positive constant c such that*

$$\tilde{S}^{(\beta)}(\lambda) = O(1)|\lambda|^{-c}\tilde{S}^{(\alpha)}(\lambda) \quad (\lambda \to - \infty) \,.$$

When u is a measurable function on R^n and α a multi-order, let us put

$$L^{(\alpha)}(u) = \int |\xi|^\alpha |u(\xi)| d\xi, \quad \text{with} \quad |\xi|^\alpha = \sum_{\beta \le \alpha} |\xi^\beta| \,.$$

We then have

LEMMA 7. *There is a constant C (which may depend on the multi-order α) such that* (with $* =$ convolution)

$$L^{(\alpha)}(u * v) \leq CL^{(\alpha)}(u)L^{(\alpha)}(v)$$

for all measurable functions u *and* v *on* R^n *such that* $L^{(\alpha)}(u)$ *and* $L^{(\alpha)}(v)$ *are finite.*

Proof. By the Fubini theorem we have

$$\int |\xi|^\alpha |(u * v)(\xi)| d\xi \leq \int |\xi|^\alpha |u(\xi - \eta)| |v(\eta)| d\xi d\eta = \int I(\eta)|v(\eta)| d\eta ,$$

where

$$I(\eta) = \int |\xi|^\alpha |u(\xi - \eta)| d\xi = \int |\xi + \eta|^\alpha |u(\xi)| d\xi .$$

Hence the lemma follows from the obvious inequality $|\xi + \eta|^\alpha \leq C|\xi|^\alpha |\eta|^\alpha$, where C is some constant.

4. Construction and estimation of a certain fundamental solution

Throughout this section $a(x, D)$ will be a differential operator on the whole of R^n, and we shall suppose that it is FHE of type P and formally self-adjoint there. The type operator P is chosen real (which is always possible, as remarked after Lemma 3) and such that $P(\xi) \geq 1$ for all $\xi \in R^n$. Moreover, we suppose that

$$P(\xi) \geq C|\xi|^s \quad (\xi \in R^n) \tag{15}$$

where C and s are positive constants and where we shall suppose s to be as large as we need in our estimates. Further it will be assumed that $a(x, D) = P(D)$ for all x outside some compact subset of R^n. In particular, this means that we have assumed that $\mathrm{Re}\, a(x, \xi) \to +\infty$ $(|\xi| \to \infty,\ \xi \in R^n)$ for every $x \in R^n$ which is no essential restriction (see Lemma 3).

We are going to construct and estimate a fundamental solution $g_\lambda(x, y)$ of the operator $a(y, D_y) - \lambda$ when λ is large and negative (saying fundamental solution we mean that

$$(a(y, D_y) - \lambda)g_\lambda(x, y) = \delta_x(y)$$

for every $x \in R^n$, where δ_x is the Dirac measure at the point x). The construction will be carried out by the Levi parametrix method. As the parametric $h_\lambda(x, \cdot)$ we take the tempered fundamental solution with pole x of the operator $a(x, D_y) - \lambda$ (having constant coefficients), i.e.,

$$h_\lambda(x, y) = \int (a(x, \xi) - \lambda)^{-1} \exp (2\pi i \langle y - x, \xi \rangle) d\xi ,$$

where the integrand is in $L^1(R^n)$ for all $x \in R^n$ when λ is large and negative. For from Lemma 3, the representation (5) for $a(x, D)$ and a simple continuity argument it easily follows that for large negative λ

$$|a(x, \xi) - \lambda| \geq C(P(\xi) - \lambda) \quad (x, \xi \in R^n) \tag{16}$$

with a positive constant C (remember also that we may take the number s of (15) as large as we need).

The fundamental solution g_λ will be constructed by the formula

$$g_\lambda(x, y) = h_\lambda(x, y) + \int u_\lambda(x, z) h_\lambda(z, y) dz , \tag{17}$$

where u_λ satisfies the integral equation

$$u_\lambda(x, z) - \int u_\lambda(x, w) A_\lambda(w, z) dw = A_\lambda(x, z) , \tag{18}$$

with $A_\lambda(x, z) = (a(x, D_z) - a(z, D_z)) h_\lambda(x, z)$.

To give the formulas (17) and (18) a sense and to estimate g_λ we shall now introduce convenient norms. For any multi-order α, put

$$M_\alpha(u) = L^{(\alpha)}(\mathscr{F}u) = \int |\xi|^\alpha |\mathscr{F}u(\xi)| d\xi ,$$

where we have again written $|\xi|^\alpha = \sum_{\beta \leq \alpha} |\xi^\beta|$ and where $\mathscr{F}$ denotes the Fourier transformation (taken in the distribution sense of Schwartz [10]), and where further u is a tempered distribution on R^n such that $|\xi|^\alpha \mathscr{F}u(\xi)$ is a function in $L^1(R^n)$. The set of all such distributions obviously form a Banach space B_α (with the norm M_α). It is also clear that B_α is a set of continuous functions and a simple approximation argument shows that $C_0^\infty(R^n)$ is dense in B_α. Further let $N_{\alpha, \beta}$ be the uniform norm of linear mappings L from B_α to B_β:

$$N_{\alpha, \beta}(L) = \sup (M_\beta(Lu)/M_\alpha(u)) ,$$

where the supremum is taken over all non-zero functions $u \in B_\alpha$. If K is a distribution on $R^n \times R^n$, it defines a linear mapping $\tilde{K}$ from $C_0^\infty(R^n)$ to the space of distributions on R^n:

$$\tilde{K}u(x) = \int K(x, y) u(y) dy .$$

We shall then write $N_{\alpha, \beta}(K)$ or even $N_{\alpha, \beta}(K(x, y))$ instead of $N_{\alpha, \beta}(\tilde{K})$. Now let us estimate the distribution

$$A_\lambda(x, z) = (a(x, D_z) - a(z, D_z)) h_\lambda(x, z)$$

appearing in (18).

LEMMA 8. *There are positive numbers* c *and* $\varkappa_0$ *such that*

$$N_{\alpha, \alpha}(A_\lambda(x, z) \exp (\varkappa |\lambda|^b (z_j - x_j))) = O(1) |\lambda|^{-c} \quad (\lambda \to -\infty)$$

for every j, $1 \leq j \leq n$, *every multi-order* α, *and every real* $\varkappa$ *such that* $|\varkappa| \leq \varkappa_0$. *Here* b *is the number corresponding to* P *by* (3).

Proof. First let us prove the lemma in the case $\varkappa = 0$. By (5) we can write

$$A_\lambda(x, z) = \sum (b_j(x) - b_j(z)) Q_j(D_z) h_\lambda(x, z) ,$$

where, for every j, b_j belongs to $C^\infty(R^n)$ and is constant outside some compact subset of R^n, while the polynomial Q_j is weaker than P. By expansion in Taylor series, any term in the above sum may, for any positive integer k, be written on the form

$$F_\lambda(x, z) = \sum_{0 < |\beta| < k} (\beta!)^{-1} (2\pi i)^{|\beta|} (D^\beta b(z)) (x - z)^\beta Q(D_z) h_\lambda(x, z) + R_\lambda(x, z) , \quad (19)$$

where b is one of the functions b_j and Q one of the polynomials Q_j, and where

$$R_\lambda(x, z) = ((k - 1)!)^{-1} \left(\int_0^1 (1 - t)^{k-1} \frac{d^k}{dt^k} b(z + t(x - z)) dt \right) Q(D_z) h_\lambda(x, z) .$$

Now the mapping having as kernel the term with index β in the sum in the right member of (19) is the composition of the two mappings $L_1 : u \mapsto (D^\beta b) u$ and $L_2 : v \mapsto \text{const.} \int (x - z)^\beta Q(D_z) h_\lambda(x, z) v(z) dz$. Here $D^\beta b \in C_0^\infty(R^n)$, and so it follows by Lemma 7 that $N_{\alpha, \alpha}(L_1) < \infty$. Hence, to prove our lemma (in the case $\varkappa = 0$) it suffices to show that with some positive constant c

(i) $N_{\alpha, \alpha}((x - z)^\beta Q(D_z) h_\lambda(x, z)) = O(1) |\lambda|^{-c} \quad (\lambda \to -\infty)$, for any $\beta \neq 0$ and any Q weaker than P, and that

(ii) $N_{\alpha, \alpha}(R_\lambda) = O(1) |\lambda|^{-c} \quad (\lambda \to -\infty)$ if k is large enough.

Let us start by proving (i). Put $h_\lambda = H_\lambda + \tilde{H}_\lambda$, where

$$H_\lambda(x, z) = \int (P(\xi) - \lambda)^{-1} \exp (2\pi i \langle z - x, \xi \rangle) d\xi$$

and thus

$$\tilde{H}_\lambda(x, z) = \int ((a(x, \xi) - \lambda)^{-1} - (P(\xi) - \lambda)^{-1}) \exp (2\pi i \langle z - x, \xi \rangle) d\xi .$$

First let us prove (i) with H_λ instead of h_λ. If $\varphi \in C_0^\infty(R^n)$, we have that the Fourier transform χ of the function $\int (x - z)^\beta (Q(D_z) H_\lambda(x, z)) \varphi(z) dz$ is defined by

$$\chi(-\xi) = (\mathcal{F}\varphi)(-\xi) \cdot D^\beta (Q(\xi)/(P(\xi) - \lambda)) .$$

Hence we get

$$N_{\alpha, \alpha}((x - z)^\beta Q(D_z) H_\lambda(x, z)) \leq \sup_{\xi \in R^n} |D^\beta(Q(\xi)/(P(\xi) - \lambda))| \; .$$

Now $D^\beta(Q(\xi)/(P(\xi) - \lambda))$ is a finite linear combination of terms of the form

$$Q^{(\alpha^1)}(\xi) P^{(\alpha^2)}(\xi) \ldots P^{(\alpha^t)}(\xi)(P(\xi) - \lambda)^{-t} \, ,$$

where we have written e.g. $Q^{(\nu)} = i^{|\nu|} D^\nu Q$ when γ is a multi-order, and where $\sum_{j=1}^t \alpha^j = \beta \neq 0$. Thus, by Lemma 2 (taking $\tau = |\lambda|^b$), we get

$$\sup_{\xi \in R^n} |D^\beta(Q(\xi)/(P(\xi) - \lambda))| = O(1)|\lambda|^{-c} \quad (\lambda \to - \infty) \, ,$$

where c is a positive constant, giving the desired result for H_λ. Now let us prove (i) with $\tilde{H}_\lambda$ instead of h_λ. Again take φ in $C_0^\infty(R^n)$. If $\psi(x) = \int (x - z)^\beta (Q(D_z) \tilde{H}_\lambda(x, z)) \varphi(z) dz$, then

$$(\mathscr{F}\psi)(\eta) = \int (\mathscr{F}^{-1}\varphi)(\xi) J_\lambda(\xi, \xi + \eta) d\xi \, ,$$

where

$$J(\xi, \cdot) = (- 1)^{|\beta|} \mathscr{F}_x(D_\xi^\beta((Q(\xi)/(a(x, \xi) - \lambda) - (Q(\xi)/(P(\xi) - \lambda))) \, .$$

Here the function after the $\mathscr{F}_x$ sign is a finite linear combination of terms of the form

$$K_\lambda(x, \xi) = f(x) T_1^{(\alpha^1)}(\xi) \ldots T_r^{(\alpha^r)}(\xi)(a(x, \xi) - \lambda)^{-\mu}(P(\xi) - \lambda)^{-\nu} \, ,$$

where $f \in C_0^\infty(R^n)$, the T_j are certain of the polynomials Q_j (and hence weaker than P), and where $\mu + \nu = r$ and $\beta = \sum_{j=1}^r \alpha^j$.

Further, if γ is any multi-order, then $D_x^\gamma K_\lambda(x, \xi)$ is a finite sum of terms of the same form as for $K_\lambda(x, \xi)$ itself (though with r etc. different). From the inequality (16) and Lemma 2 we then conclude that for any multi-order γ

$$\sup_{x, \xi \in R^n} |D_x^\gamma K_\lambda(x, \xi)| = O(1)|\lambda|^{-c} \quad (\lambda \to - \infty) \, ,$$

where c is a positive constant. Since for every λ and ξ the support of $K_\lambda(\cdot, \xi)$ is contained in a fixed compact seubset of R^n, we have for large negative λ

$$|J_\lambda(\xi, \zeta)| \leq C|\lambda|^{-c}(|\zeta| + 1)^{-(n+1+|\alpha|)} \quad (\xi, \zeta \in R^n) \, ,$$

where C and c are positive and independent of ξ, ζ, and λ. From Lemma 7 it then follows that when λ is large and negative

$$M_\alpha(\psi) \leq C|\lambda|^{-c} M_\alpha(\varphi) \quad (\varphi \in C_0^\infty(R^n))$$

with positive constants C and c, which concludes the proof of (i).

Now let us turn to (ii). Clearly $R_\lambda(x, z)$ is a finite sum of terms of the form

$$S_\lambda(x, z) = F(x, z)(x - z)^\beta Q(D_z) h_\lambda(x, z)$$

with $|\beta| = k$ and with

$$F(x, z) = C_\beta \int_0^1 (1 - t)^{k-1} D^\beta b(z + t(x - z)) dt , \tag{20}$$

where C_β is a constant. Further we find that if k (and $|\lambda|$) is large enough, then for all x, $z \in R^n$

$$|D_x^\gamma ((x - z)^\beta Q(D_z) h_\lambda(x, z)| \leq C |\lambda|^{-c} (|x - z| + 1)^{-2(n+1)} \tag{21}$$

for any multi-order γ with $|\gamma| \leq n + 1 + |\alpha|$, where C and c are positive constants. For we can estimate the L^1-norm of the Fourier transform with respect to z of

$$(x_k - z_k)^{2(n+1)} D_x^\gamma ((x - z)^\beta Q(D_z) h_\lambda(x, z)$$

with $C |\lambda|^{-c}$, as in the proof of (i) considering the terms that occur in the differentiation, and taking $k > (2n + 2 + |\alpha|)/c$, where c is the constant from Lemma 2 (which is clearly independent of the particular choice of Q there). Now, by (20), $F \in C^\infty(R^n \times R^n)$ and F and all the derivatives of F are bounded on $R^n \times R^n$. Further $R_\lambda(x, z) = 0$ when both $|x|$ and $|z|$ are large (for then $(b(x) - b(z))$ as well as all the Taylor terms with $1 < |\beta| < k$ are equal to zero). Thus we get by (21)

$$|D_x^\gamma R_\lambda(x, z)| \leq C |\lambda|^{-c} (|x| + 1)^{-(n+1)} (|z| + 1)^{-(n+1)} \quad (x, z \in R^n) \tag{22}$$

when $|\gamma| \leq n + 1 + |\alpha|$, and with positive constants C and c. It follows from (22) that

$$|D_x^\gamma (\mathcal{F}_z R_\lambda)(x, \eta)| \leq C |\lambda|^{-c} (|x| + 1)^{-(n+1)} \quad (x, \eta \in R^n) , \tag{23}$$

if $|\gamma| \leq n + 1 + |\alpha|$. Let $\varphi \in C_0^\infty(R^n)$ and put $\psi(x) = \int R_\lambda(x, z) \varphi(z) dz$. Then, by (23)

$$|D^\gamma \psi(x)| \leq C |\lambda|^{-c} (|x| + 1)^{-(n+1)} M_\alpha(\varphi) \quad (x \in R^n)$$

if $|\gamma| \leq n + 1 + |\alpha|$, and so

$$M_\alpha(\psi) \leq C |\lambda|^{-c} M_\alpha(\varphi) \quad (\varphi \in C_0^\infty(R^n))$$

when λ is large and negative, proving (ii).

It remains to treat the case $\varkappa \neq 0$ of the present lemma. Formally, multiplication of a function $f(z - x)$ by $\exp(\varkappa |\lambda|^b (z_j - x_j))$ corresponds to translation of $(\mathcal{F}f)$ by the vector $-(2\pi)^{-1} \varkappa |\lambda|^b i e_j$ in the independent variable, where e_j is the j:th unit vector in R^n. In our case we have to take f such that $(\mathcal{F}f)(\xi) = \text{const.} \cdot D_\xi^\beta (Q(\xi)/(a(x, \xi) - \lambda))$, and we can easily see that if $\varkappa$ is sufficiently small, then the above correspondence is not only formal. For by Lemma 1 and (16) it follows that there are positive constants C and $\varkappa_0$ such that for large negative λ

$$|a(x, \xi - \varkappa(2\pi)^{-1}|\lambda|^b i e_j) - \lambda| \geq C(P(\xi) - \lambda) \quad (x, \xi \in R^n) \tag{24}$$

for all complex numbers $\varkappa$ with $|\varkappa| \leq \varkappa_0$. Hence, as the above-mentioned formal correspondence is trivially also actual when $\varkappa$ is real, it extends by analytic continuation to an actual correspondence for all complex $\varkappa$ with $|\varkappa| \leq \varkappa_0$. Moreover, it follows from Lemma 1 that the estimate of Lemma 2, which we have used to majorize the nominators of our Fourier transforms, is not destroyed if we replace ξ by $\xi + \varkappa(2\pi)^{-1}|\lambda|^b e_j$, with $\varkappa$ sufficiently small. From these observations we easily conclude the lemma also in the general case.

We shall also have to estimate the function h_λ of (17).

LEMMA 9. *There is a positive constant $\varkappa_0$ such that, when $1 \leq j \leq n$, and α is any multi-order, we have for all real numbers $\varkappa$ with $|\varkappa| \leq \varkappa_0$*

$$M_\alpha(\exp(\varkappa|\lambda|^b(y_j - x_j))h_\lambda(x, y)) = O(1)S^{(2\alpha)}(\lambda) \quad (\lambda \to -\infty),$$

where the norm M_α is taken with respect to the variable x, and where the estimate is uniform with respect to $y \in R^n$. Further, as before, b is the positive number corresponding to P by (3). When $\beta \leq \alpha$ we also have, uniformly in $y \in R^n$,

$$M_\alpha(D_y^\beta h_\lambda(\cdot, y)) = O(1)S^{(2\alpha)}(\lambda) \quad (\lambda \to -\infty), \quad \text{where}$$

$$S^{(2\alpha)}(\lambda) = \int \xi^{2\alpha}(P(\xi) - \lambda)^{-1}d\xi.$$

Proof. As with Lemma 8 we first consider the case $\varkappa = 0$. Write again $h_\lambda = H_\lambda + \tilde{H}_\lambda$, with

$$H_\lambda(x, y) = \int (P(\xi) - \lambda)^{-1}\exp(2\pi i\langle y - x, \xi\rangle)d\xi.$$

Then

$$|(\mathscr{F}D_y^\beta H_\lambda(\cdot, y))(\xi)| = |\xi^\beta|(P(-\xi) - \lambda)^{-1} \quad (\xi, y \in R^n),$$

and so (with $\beta \leq \alpha$)

$$M_\alpha(D_y^\beta H_\lambda(\cdot, y)) = O(1)\int |\xi|^{2\alpha}(P|\xi| - \lambda)^{-1}d\xi = O(1)S^{(2\alpha)}(\lambda)$$

as $\lambda \to -\infty$, where the last estimate follows from Lemma 6. Further we have

$$(\mathscr{F}D_y^\beta \tilde{H}_\lambda(\cdot, y))(\eta) = \int \xi^\beta(P(\xi) - \lambda)^{-1}L_\lambda(\xi, \xi + \eta)\exp(2\pi i\langle y, \xi\rangle)d\xi,$$

where

$$L_\lambda(\xi, \cdot) = \mathscr{F}_x((P(\xi) - a(x, \xi))/(a(x, \xi) - \lambda)).$$

Arguing as in the proof of (i) in Lemma 8 it follows that for large negative λ

$$|L_\lambda(\xi, \zeta)| \leq C(|\zeta| + 1)^{-(n+1+|\alpha|)} \quad (\xi, \zeta \in R^n) \,,$$

where C is a constant. Hence, by the Fubini theorem, the inequality $|\eta - \xi|^\alpha \leq \text{const} \cdot |\xi|^\alpha |\eta|^\alpha$, and Lemma 6,

$$\int |\eta|^\alpha |(\mathcal{F} D_y^\beta \tilde{H}_\lambda(\cdot, y))(\eta)| d\eta \leq$$

$$\leq C \int |\xi^\beta| (P(\xi) - \lambda)^{-1} \left(\int |\eta|^\alpha (|\eta + \xi| + 1)^{-(n+1+|\alpha|)} d\eta \right) d\xi \leq$$

$$\leq C' \int |\xi|^\alpha (P(\xi) - \lambda)^{-1} \left(\int |\xi|^\alpha |\eta|^\alpha (|\eta| + 1)^{-(n+1+|\alpha|)} d\eta \right) d\xi \leq$$

$$\leq C'' \int |\xi|^{2\alpha} (P(\xi) - \lambda)^{-1} d\xi = O(1) S^{(2\alpha)}(\lambda) \quad (\lambda \to -\infty) \,,$$

where C' and C'' are also constants. This ends the proof in the case $\varkappa = 0$. The extension to the general case is made exactly as in the proof of Lemma 8.

Remark. From the proof of Lemma 9 it is seen that we even have a majoration

$$|(\mathcal{F} D_y^\beta h_\lambda(\cdot, y))(\eta)| \leq V_\lambda(\eta) \quad (y, \eta \in R^n, \beta \leq \alpha) \,,$$

where $V_\lambda(\eta)$ is independent of y and where

$$\int |\eta|^\alpha V_\lambda(\eta) d\eta = O(1) S^{(2\alpha)}(\lambda) \quad (\lambda \to -\infty) \,.$$

We can now conclude this section by the desired construction and estimation of a fundamental solution of $a(y, D_y) - \lambda$ (where $a(y, D_y)$ still satisfies the conditions at the beginning of this section). (Notation: When $F(x, y)$ is a function of $x, y \in R^n$ and α, β are multi-orders, we shall write $F^{(\alpha, \beta)}(x, y) = (iD_x)^\alpha (iD_y)^\beta F(x, y)$.)

LEMMA 10. *Let N be any positive integer. Then (provided that the number s in (15) is $> n + N$) there is for all sufficiently large negative values of λ a function g_λ on $R^n \times R^n$ with the following properties:*
(i) *for every $x \in R^n$ the function $g_\lambda(x, \cdot)$ is a fundamental solution with pole x of the operator $a(y, D_y) - \lambda$,*
(ii) *g_λ belongs to $C^N(R^n \times R^n)$, and for every multi-order α with $|2\alpha| \leq N$ we have with some positive constant c*

$$g_\lambda^{(\alpha, \alpha)}(x, x) = (1 + O(1)|\lambda|^{-c}) G_{x,\lambda}^{(\alpha, \alpha)}(x, x) \quad (\lambda \to -\infty)$$

for every $x \in R^n$, where

$$G_{x,\lambda}^{(\alpha,\alpha)}(x,y) = \int \xi^{2\alpha}(\text{Re } a(x,\xi) - \lambda)^{-1} \exp{(2\pi i\langle x - y, \xi\rangle)}d\xi \,,$$

(iii) *we have* $g_\lambda(x,y) = O(1)|\lambda|^{-c}$ $(\lambda \to - \infty)$ *uniformly on* $R^n \times R^n$, *with some positive constant* c,

(iv) *when* α, β *are multi-orders such that* $|\alpha| \leq N$ (then $D_x^\alpha g_\lambda(x, \cdot)$ is in $C^\infty(R^n \backslash \{x\})$ for every $x \in R^n$, since by (i) we have $(a(y, D_y) - \lambda)D_x^\alpha g_\lambda(x,y) = 0$ in $R^n \backslash \{x\}$ and since the operator $a(y, D_y) - \lambda$ is FHE) *we have*

$$g_\lambda^{(\alpha,\beta)}(x,y) = O(1) \exp{(- \varkappa |\lambda|^b)} \quad (\lambda \to - \infty) \,,$$

uniformly on compact subsets of the region $x \neq y$ *in* $R^n \times R^n$, *where* $\varkappa$ *is a positive constant that may depend on the compact subset, and where* b *is the positive number corresponding to* P *by* (3).

Proof. As already mentioned, g_λ will be constructed by the formula

$$g_\lambda(x,y) = h_\lambda(x,y) + \int u_\lambda(x,z)h_\lambda(z,y)dz \,, \tag{17}$$

where

$$u_\lambda(x,z) - \int u_\lambda(x,w)A_\lambda(w,z)dw = A_\lambda(x,z) \,, \tag{18}$$

where h_λ and A_λ are the same functions as those estimated in the Lemmas 8 and 9. Let $\mathscr{L}(B_\alpha)$ be the Banach space of all bounded linear mappings from (the whole of) B_α to B_α. Then we shall interpret the integral in (18) as the kernel of the mapping, e.g. in $\mathscr{L}(B_0)$, composed by those defined by A_λ and u_λ (in that order), where u_λ is so far unknown. And in (17) we take, for every $y \in R^n$, the integral as the (continuous representative of the) function in B_0 which is the image of the function $h_\lambda(\cdot, z)$ by the mapping $u_\lambda : B_0 \to B_0$. When α is any multi-order we have by Lemma 8 that $N_\alpha(A_\lambda) < 1$ if λ is negative and large enough. Thus the equation (18) has then a unique solution u_λ in $\mathscr{L}(B_\alpha)$, given by the geometric series

$$A_\lambda(x,z) + \int A_\lambda(x,w)A_\lambda(w,z)dw + \ldots \tag{25}$$

(with integrals interpreted as compositions of mappings in $\mathscr{L}(B_\alpha)$). Clearly the kernel u_λ does not depend on the choice of α, this being a simple consequence of the fact that $M_0(f) \leq M_\alpha(f)$ for any α and any f. Further, there are by Lemma 8 positive constants c and $\varkappa_0$ such that when $|\varkappa| \leq \varkappa_0$ and $1 \leq j \leq n$ we have

$$N_{\alpha,\alpha}(\exp{(\varkappa |\lambda|^b(x_j - z_j))}A_\lambda(x,z)) = O(1)|\lambda|^{-c} \quad (\lambda \to - \infty) \,.$$

If in (25) we replace $A_\lambda(x,z)$ by $\exp{(\varkappa |\lambda|^b(x_j - z_j))}A_\lambda(x,z)$, with $|\varkappa| \leq \varkappa_0$ and λ sufficiently large, then the sum of the series will be changed to

exp $(\varkappa|\lambda|^b(x_j - z_j))u_\lambda(x, z)$, by pairwise cancellation of exponential factors. To see that such cancellation actually takes place, with the sense that we have given to the integrals, we can apply the kernel $A_{1,\lambda} = A_\lambda(x, z) \exp(\varkappa|\lambda|^b(x_j - z_j))$ to a function f such that $f \in B_0$ and $f_1 = \exp(-\varkappa|\lambda|^b z_j)f(z) \in B_0$. Then, if F is the image of f by A_λ, and F_1 of f_1 by $A_{1,\lambda}$, we have $F_1(x) = \exp(\varkappa|\lambda|^b x_j)F(x)$. For this is clearly true when $f \in C_0^\infty(R^n)$, and so it follows for an f as above approximating f with functions in $C_0^\infty(R^n)$ simultaneously in the norms M_0 and $M_{1,0}$, where $M_{1,0}$ is defined by $M_{1,0}(\varphi) = M_0(\exp(-\varkappa|\lambda|^b z_j)\varphi(z))$. (This can be done, first taking the product and then the convolution of f with suitable functions in $C_0^\infty(R^n)$.) It follows that both $M_0(F) < \infty$ and $M_{1,0}(F) < \infty$. Thus, if we apply A_λ resp. $A_{1,\lambda}$ any given finite number of times to a function f as above, letting F resp. F_1 be the images, we get $F_1(x) = \exp(\varkappa|\lambda|^b x_j)F(x)$, implying the desired cancellation.

Thus it follows that

$$N_\alpha(\exp(\varkappa|\lambda|^b(x_j - z_j))u_\lambda(x, z)) = O(1)|\lambda|^{-c} \quad (\lambda \to -\infty), \tag{26}$$

when $|\varkappa| \leq \varkappa_0$ and $1 \leq j \leq n$.

Let us show (ii) by proving that, with a positive constant c,

$$h_\lambda^{(\alpha, \alpha)}(x, x) = (1 + O(1)|\lambda|^{-c})G_{x,\lambda}^{(\alpha, \alpha)}(x, x) \quad (\lambda \to -\infty) \tag{27}$$

and

$$D_x^\alpha D_y^\alpha\left(\int u_\lambda(x, z)h_\lambda(z, x)dz\right) = O(1)|\lambda|^{-c}G_{x,\lambda}^{(\alpha, \alpha)}(x, x) \quad (\lambda \to -\infty) \tag{28}$$

for any $x \in R^n$, together with the order N differentiability of the two terms on the right in (17).

If α, β are multi-orders, then $h_\lambda^{(\alpha, \beta)}(x, \cdot)$ is the inverse Fourier transform (with respect to ξ) of a finite sum of terms of the form

$$f_\lambda(x, \xi) = \varphi(x)\xi^\gamma R_1(\xi) \ldots R_t(\xi)(a(x, \xi) - \lambda)^{-(t+1)}\exp(-2\pi i\langle x, \xi\rangle)$$

and of the term

$$F_\lambda(x, \xi) = i^{|\alpha+\beta|}\xi^{\alpha+\beta}(a(x, \xi) - \lambda)^{-1}\exp(-2\pi i\langle x, \xi\rangle),$$

where $\varphi \in C_0^\infty(R^n)$ and the polynomials R_j are all weaker than P, while $\gamma \leq \alpha + \beta$, $\gamma \neq \alpha + \beta$, and $|\gamma| + t = |\alpha + \beta|$. For in the defining formula

$$h_\lambda(x, y) = \int (a(x, \xi) - \lambda)^{-1}\exp(2\pi i\langle y - x, \xi\rangle)d\xi$$

we get such terms, differentiating under the integral sign, and by Lemma 2 the integrands are in $L^1(R^n)$ if the number s in (15) is $> n + |\alpha + \beta|$, which proves the desired differentiability of the term $h_\lambda(x, y)$. Now let $\alpha = \beta$. By Lemma 3 and (16) applied to the operator Re $a(x, D)$, it follows that for large negative λ

$$|f_\lambda(x, \xi)| \le C|\xi^\gamma|(\mathrm{Re}\, a(x, \xi) - \lambda)^{-1} \quad (\xi \in R^n)\,,$$

where C is a constant, and $\gamma \le 2\alpha$, $\gamma \ne 2\alpha$. So it follows by Lemma 6 that

$$(\mathscr{F}f_\lambda(x, \cdot))(z) = O(1)|\lambda|^{-c}G^{(\alpha,\alpha)}_{x,\lambda}(x, x) \quad (\lambda \to -\infty)\,, \tag{29}$$

uniformly on R^n, with a positive constant c. Further

$$(\mathscr{F}F_\lambda(x, \cdot))(x) =$$
$$= G^{(\alpha,\alpha)}_{x,\lambda}(x, x) - i\int \xi^{2\alpha}(\mathrm{Im}\, a(x, \xi)(a(x, \xi) - \lambda)^{-1}(\mathrm{Re}\, a(x, \xi) - \lambda)^{-1}d\xi\,. \tag{30}$$

Now from Lemma 3, Lemma 2, and (16) it follows that

$$|\mathrm{Im}\, a(x, \xi) \cdot (a(x, \xi) - \lambda)^{-1}| \le C|\lambda|^{-c} \quad (\xi \in R^n)$$

for large negative values of λ, where C and c are positive constants. Hence we get from (30)

$$(\mathscr{F}F_\lambda(x, \cdot))(x) = G^{(\alpha,\alpha)}_{x,\lambda}(x, x) + O(1)|\lambda|^{-c}G^{(\alpha,\alpha)}_{x,\lambda}(x, x) \quad (\lambda \to -\infty)$$

for any $x \in R^n$. Form this and (29) we immediately get (27).

When α, β are multi-orders with $|\alpha + \beta| \le N$, let us see that

$$D^\alpha_x D^\beta_y\left(\int u_\lambda(x, z)h_\lambda(z, y)dz\right) = D^\alpha_x\left(\int u_\lambda(x, z)D^\beta_y h_\lambda(z, y)dz\right) \tag{31}$$

in the distributional sense, where the functions $V_\lambda(x, y) = \int u_\lambda(x, z)D^\beta_y h_\lambda(z, y)dz$ and $D^\alpha_x V_\lambda(x, y)$ are continuous on $R^n \times R^n$.

For this, let us first observe that for any $y^0 \in R^n$ we have

$$M_\alpha(h_\lambda(\cdot, y) - h_\lambda(\cdot, y^0)) \to 0 \quad (y \to y^0) \tag{32}$$

(still assuming that the number s in (15) is large enough). For with the Fourier transform of the function after the M_α sign we have trivially pointwise convergence, and further by the Remark after Lemma 9 we can majorize it with a function in $\mathscr{F}B_\alpha$, and so the Lebesgue theorem on majorized convergence will give (32).

From (26) (with $\varkappa = 0$) and Lemma 9 it follows that, for every $y \in R^n$, $D^\alpha_x V_\lambda(\cdot, y)$ is defined as a distribution in B_0 and hence is a continuous function. By (26) and (32) we get that $M_0(D^\alpha_x V_\lambda(\cdot, y) - D^\alpha_x V_\lambda(\cdot, y^0)) \to 0$ as $y \to y^0$, for any $y^0 \in R^n$, and so the set of all the functions $D^\alpha_x V_\lambda(x, \cdot)$ (with x varying in R^n) is equicontinuous. But continuity in x and equicontinuity in y give continuity in the pair (x, y). Thus $D^\alpha_x V_\lambda$ (and V_λ) is continuous on $R^n \times R^n$.

To see that (31) holds in distribution sense it is sufficient to show that we can change the order of integration:

$$\int\left(\int u_\lambda(x, z)D^\beta_y h_\lambda(z, y)dz\right)\varphi(y)dy = \int u_\lambda(x, z)\left(\int D^\beta_y h_\lambda(z, y)\varphi(y)dy\right)dz\,, \tag{33}$$

when $\varphi \in C_0^\infty(R^n)$. But this is easily done, for trivially

$$(34)$$

$$p^{-1} \sum_{j=-\infty}^{\infty} \left(\int u_\lambda(x, z) D_y^\beta h_\lambda(z, j/p) dz \right) \varphi(j/p) = p^{-1} \int u_\lambda(x, z) \left(\sum_{j=-\infty}^{\infty} D_y^\beta h_\lambda(z, j/p) \varphi(j/p) \right) dz$$

when p is a positive integer, and further $p^{-1} \sum_{j=-\infty}^{\infty} D_y^\beta h_\lambda(x, j/p) \varphi(j/p)$ tends to $\int D_y^\beta h_\lambda(z, y) \varphi(y) dy$ in the norm M_0 when $p \to \infty$, which follows easily by the Remark after Lemma 9 and the Lebesgue theorem on majorized convergence. Hence by Lemma 8 we get (33), letting p tend to ∞ in (34), and also using the continuity of the integral $\int u_\lambda(x, z) D_y^\beta h_\lambda(z, y) dz$, which we have proved above.

Since for any given positive integer N the right hand side of (31) is continuous when $|\alpha + \beta| \leq N$, if the number s in (15) is $> n +.N$, it also follows by well-known theorems that the integral $\int u_\lambda(x, z) h_\lambda(z, y) dz$ is in $C^N(R^n \times R^n)$ and hence that (31) holds also in the classical sense.

Further by Lemma 8 and 9 we immediately get, for any $x \in R^n$,

$$D_x^\alpha \left(\int u_\lambda(x, z) D_y^\beta h_\lambda(z, y) dz \right) = O(1) |\lambda|^{-c} S^{(2\alpha)}(\lambda) = O(1) G_{x, \lambda}^{(\alpha, \alpha)}(x, x) \quad (\lambda \to -\infty)$$

where c is a positive constant and where the last relation follows from the fact that the operator $\text{Re } a(x, D_y)$ is equally strong as $P(D)$ (by Lemma 3). This concludes the proof of (ii).

To show (iii), we observe that from the inequality (16) it easily follows that with some constant $c > 0$

$$h_\lambda(x, y) = O(1) |\lambda|^{-c} \quad (\lambda \to -\infty) \tag{35}$$

uniformly on $R^n \times R^n$. Further Lemma 9 and (26) (with e.g. $\alpha = 0$ and $\varkappa = 0$) give the estimate

$$\int u_\lambda(x, z) h_\lambda(z, y) dz = O(1) |\lambda|^{-c} \quad (\lambda \to -\infty) \tag{36}$$

uniformly on $R^n \times R^n$, with a positive constant c. Now (iii) follows from (35) and (36).

As for (iv), we argue as for (iii), only now considering $\exp(\varkappa |\lambda|^b (z_j - y_j)) h_\lambda(z, y)$ and $\exp(\varkappa |\lambda|^b (x_j - z_j)) u_\lambda(x, z)$ instead of $h_\lambda(z, y)$ and $u_\lambda(x, z)$, respectively, where $\varkappa$ is a sufficiently small real number. Further, we now let α be an arbitrary multi-order with $|\alpha| \leq N$. By Lemma 9 and (26) (and the fact that cancellation of exponential factors takes place, as we have seen above) we get

$$\exp(\varkappa |\lambda|^b (x_j - y_j)) D_x^\alpha g_\lambda(x, y) = O(1) \quad (\lambda \to -\infty), \tag{37}$$

uniformly on $R^n \times R^n$, when $\varkappa$ is a sufficiently small real number. Since j and the sign of $\varkappa$ are arbitrary, it follows from (37) that

$$D_x^\alpha g_\lambda(x, y) = O(1) \exp(-\varkappa |\lambda|^b) \quad (\lambda \to -\infty) \tag{38}$$

uniformly on compact subsets of the region $x \neq y$ in $R^n \times R^n$, with a positive constant $\varkappa$ depending on the compact subset. For the moment accepting (i), which will be proved below, and using the a priori inequality (6) for the function $D_x^\alpha g_\lambda(x, \cdot)$, we at once get (iv) from (38).

It only remains to prove (i), i.e., that $g_\lambda(x, y)$ actually is a fundamental solution of $a(y, D_y) - \lambda$. Let us see that the usual calculation can be carried out, also with the sense of convergence that we have used. We have to show that

$$\int g_\lambda(x, y)\overline{(a(y, D_y) - \lambda)\varphi(y)}dy = \overline{\varphi(x)} \quad (\varphi \in C_0^\infty(R^n)) . \tag{39}$$

On one hand, we have then

$$\int h_\lambda(x, y)\overline{(a(y, D_y) - \lambda)\varphi(y)}dy =$$

$$= \int h_\lambda(x, y)\overline{(a(x, D_y) - \lambda)\varphi(y)}dy + \int h_\lambda(x, y)\overline{(a(y, D_y) - \overline{a(x, D_y)})\varphi(y)}dy = \tag{40}$$

$$= \overline{\varphi(x)} - \int A_\lambda(x, y)\overline{\varphi(y)}dy ,$$

where we have used that $h_\lambda(x, \cdot)$ is, for every $x \in R_n$, a fundamental solution of $a(x, D_y) - \lambda$ with pole x, and, moreover, that $a(y, D_y)$ is formally self-adjoint.

On the other hand, we have, also using (33),

$$\int \left(\int u_\lambda(x, z)h_\lambda(z, y)dz \right) \overline{(a(y, D_y) - \lambda)\varphi(y)}dy =$$

$$= \int u_\lambda(x, z) \left(\int h_\lambda(z, y)\overline{(a(y, D_y) - \lambda)\varphi(y)}dy \right) dz = \tag{41}$$

$$= \int u_\lambda(x, z)\overline{\varphi(z)}dz - \int u_\lambda(x, z) \left(\int A_\lambda(z, y)\overline{\varphi(y)}dy \right) dz .$$

Adding (40) and (41) we get (39), in view of the integral equation (18) for u_λ (and the way of defining the integral there). This concludes the proof of Lemma 10.

5. Estimates for the spectral function

Let S be an open connected subset of R^n, and let $a(x, D)$ be a FHE differential operator in S of type P, with $\deg(P) > 0$. Further suppose that $a(x, D)$ is formally self-adjoint in S. We choose the sign of $a(x, D)$ in such a way (see Section 3) that $\operatorname{Re} a(x, \xi) \to + \infty$ as $|\xi| \to \infty$, $\xi \in R^n$, for every $x \in S$ and the type polynomial P such that $P(\xi) \geq 1$ for all $\xi \in R^n$.

Now assume that A is a self-adjoint realization of $a(x, D)$ in the Hilbert space $L^2(S)$ (in which we have the ordinary inner product $(u, v) = \int_S u(x)\overline{v(x)}dx$ and

the norm $\|u\| = (u, u)^{1/2}$), i.e., A is a self-adjoint operator in $L^2(S)$ such that $A\varphi$ is defined and equal to $a(x, D)\varphi$ for all $\varphi \in C_0^\infty(S)$. When f is in the domain of A, then clearly Af is equal to $a(x, D)f$ with the latter expression taken in the distribution sense. Let $\{E(\lambda)\}_{\lambda \in R}$ be an orthogonal spectral resolution of A. Thus $E(\lambda)$ is for every real λ an orthogonal projection in $L^2(S)$ such that $E(\lambda) \le E(\mu)$ when $\lambda \le \mu$. Further $E(\lambda) \to 0$ as $\lambda \to -\infty$ and $E(\lambda) \to I$ as $\lambda \to +\infty$ (where I is the identical mapping on $L^2(S)$), both in the strong sense, and we have $A = \int_{-\infty}^{\infty} \lambda dE(\lambda)$, with strong convergence. The spectral resolution $\{E(\lambda)\}$ is uniquely determined by A insofar that the functions $E^-(\lambda) = \lim_{\mu \nearrow \lambda} E(\mu)$ and $E^+(\lambda) = \lim_{\mu \searrow \lambda} E(\mu)$ (which are both also spectral resolutions of A) are uniquely determined by A.

We now have the following theorem.

THEOREM 1. *For every real number λ there is a function e_λ in $C^\infty(S \times S)$ such that $e_\lambda^{(\alpha, 0)}(x, \cdot)$ is in $L^2(S)$ for any multiorder α and any $x \in S$ (using again the notation $e_\lambda^{(\alpha, \beta)}(x, y) = (iD_x)^\alpha(iD_y)^\beta e_\lambda(x, y))$, and such that*

$$E(\lambda)u(x) = \int_S e_\lambda(x, y)u(y)dy \quad (u \in L^2(S), \ x \in S),$$

where $E(\lambda)u$ is a function in $C^\infty(S)$. (The function $e_\lambda(x, y)$ is called the spectral function of A; clearly e_λ is Hermitian.) Moreover, for any multiorders α, β and any compact subset K of S there is a real number $\varkappa > 0$ such that

$$e_\lambda^{(\alpha, \beta)}(x, y) = O(1) \exp\left(-\varkappa |\lambda|^b\right) \quad and \quad \|e_\lambda^{(\alpha, 0)}(x, \cdot)\| = O(1) \exp\left(-\varkappa |\lambda|^b\right) \quad (\lambda \to -\infty)$$

uniformly on $K \times K$ and K, resp., where b is the positive number corresponding to the type operator P by (3).

Proof. For the proof we shall in great part refer to the author's paper [7] (the Theorems 3 and 4 there). In that paper the results of our present Theorem 1 are proved for an elliptic operator. However, except the existence of an a priori estimate of the type (6) (which we thus have at our disposal also now), to copy the proofs of [5] we only need a convenient estimate for a local fundamental solution of $a(y, D_y) - \lambda$ when $\lambda \to -\infty$. More precisely, to any point x^0 in S we need a function $g_\lambda(x, y)$ defined when x, y belong to some neighbourhood ω of x^0 (independent of λ) and when $\lambda \le$ some λ_0, such that for any $x \in \omega$ the distribution $(a(y, D_y) - \lambda)g_\lambda(x, \cdot)$ in ω is equal to the Dirac measure at x, and such that for any multi-order α we have an estimate

$$g_\lambda^{(0, \alpha)}(x, y) = O(1) \exp\left(-\varkappa |\lambda|^b\right) \quad (\lambda \to -\infty) \tag{42}$$

uniformly on compact subsets of the region $x \ne y$ of $\omega \times \omega$, where $\varkappa$ is a positive constant that may depend on the compact subset, and where b is the same as in the above formulation of the theorem.

Further we need an estimate for $g_\lambda(x, y)$ over the whole of $\omega \times \omega$, the following being sufficient:

$$g_\lambda(x, y) = O(1) \quad (\lambda \to -\infty) \tag{43}$$

uniformly on $\omega \times \omega$.

If $a(x, D)$ and the type operator P satisfy the conditions at the beginning of Section 4 (in particular, $a(x, D)$ is then defined on the whole of R^n), then Lemma 10 immediately gives a fundamental solution of the desired kind, (42) following from (iv) and (43) from (iii), and we can even take $\omega = R^n$. Let us see that we can remove the assumption that $a(x, D)$ is defined on the whole of R^n an is equal to $P(D)$ when $|x|$ is large. If this is not so, let x^0 be an arbitrary point in S and $\omega \subset S$ a relatively compact open neighbourhood of x^0. Take a function $\psi \in C_0^\infty(R^n)$ with its support contained in S and such that $0 \leq \psi(x) \leq 1$ for all x and that $\psi(x) = 1$ when $x \in \omega$. Putting

$$a'(x, D) = \psi(x)a(x, D)\psi(x) + (1 - \psi(x))P(D)(1 - \psi(x))$$

(interpreting the first term on the right as 0 when x is outside S) we get that $a'(x, D)$ is FHE of type P (as a consequence of the choice of sign of $a(x, D)$ and P) and formally self-adjoint on the whole of R^n. Since $a(x, D)$ and $a'(x, D)$ coincide in ω, it is clear that the fundamental solution g_λ for $a'(x, D)$ obtained from Lemma 10 will do also for $a(x, D)$ in $\omega \times \omega$. Thus, copying the proofs of [5], the theorem follows in the case that the number s in (15) is large enough (i.e. $s > n$ in this case, making $1/P(\xi)$ belong to $L^1(R^n)$). If this condition is not satisfied, we only consider A^r instead of A, where r is an odd integer. Then A^r is a self-adjoint realization of the differential operator $a(x, D)^r$, which is obviously FHE of type P^r. It then follows from the inequality (2) that P^r satisfies (15) with $s > n$ if r is large enough. Thus the theorem is valid for the spectral function $e_{r, \lambda}$ of A^r. But obviously we have (with convenient choice of $e_{r, \lambda}$) the relation $e_\lambda = e_{r, \lambda^r}$ for all real numbers λ, and from this the theorem follows also in the general case (also using the observation after the inequality (3) about the number b corresponding to P^r).

Now we shall investigate the behaviour of $e_\lambda^{(\alpha, \alpha)}(x, x)$ when $\lambda \to +\infty$. We shall compare it to the function

$$e_{x, \lambda}^{(\alpha, \alpha)}(x, x) = \int\limits_{\mathrm{Re}\, a(x, \xi) \leq \lambda} \xi^{2\alpha} d\xi .$$

We have the following theorem.

THEOREM 2. *For any multi-order α and any $x \in S$ we have*

$$e_\lambda^{(\alpha, \alpha)}(x, x) = (1 + O(1)(\log \lambda)^{-1})e_{x, \lambda}^{(\alpha, \alpha)}(x, x) \quad (\lambda \to +\infty) .$$

Proof. First let us suppose that $A \geq I$ and that $\mathrm{Re}\, a(x, \xi) \geq 1$ for all $x \in S$ and all $\xi \in R^n$. (Thus $E(\lambda) = 0$ when $\lambda < 1$.) We shall then use the standard method of first comparing the Stieltjes transforms of $e_\lambda^{(\alpha, \alpha)}(x, x)$ and $e_{x, \lambda}^{(\alpha, \alpha)}(x, x)$. That is, we shall have to study the Green's function of A.

From the a priori estimate (5) we get the following inequality (e.g. as a consequence of the argument in Hörmander [4], Section 3.7) on an arbitrary compact subset K of S and for any multi-orders α, β:

$$\sup_{x, y \in K} |e_\lambda^{(\alpha, \beta)}(x, y)| \leq C(\lambda + 1)^c \quad (\lambda \geq 0) , \tag{44}$$

where C and c are real constants, depending on α and β, while C but not c depends on K.

Copying the proofs of the Theorems 1.2.1 and 1.2.2 in Bergendal [1] we find that $e_\lambda^{(\alpha, \beta)}(x, y)$ is locally of bounded variation as a function of λ, for any $x, y \in S$ and any multi-orders α, β, and also that $e_\lambda^{(\alpha, \alpha)}(x, x)$ is an increasing function of λ for any $x \in S$ and any α. Let us form the function

$$G_\lambda(x, y) = \int_0^\infty (\mu - \lambda)^{-1} de_\mu(x, y) \quad (\lambda < 0; x, y \in S) .$$

It follows by a partial integration that if the constant c in (44) is < 1, then G_λ is continuous in $S \times S$. It is easily verified that G_λ is the kernel of the resolvent of A:

$$(A - \lambda I)^{-1} u(x) = \int_S G_\lambda(x, y) u(y) dy \quad (\text{e.g.} \quad u \in C_0^\infty(S)) .$$

If N is an integer ≥ 0, and if the constant c in (44) is < 1 for all α, β with $|\alpha + \beta| \leq N$, then G_λ is in $C^N(S \times S)$ and

$$G_\lambda^{(\alpha, \beta)}(x, y) = \int_0^\infty (\mu - \lambda)^{-1} de_\mu^{(\alpha, \beta)}(x, y) \quad (|\alpha + \beta| \leq N) , \tag{45}$$

as is seen again using a partial integration.

If further we assume that the number s of (15) is $\leq n + |2\alpha|$, then clearly

$$G_{x, \lambda}^{(\alpha, \alpha)}(x, x) = \int_0^\infty (\mu - \lambda)^{-1} de_{x, \mu}^{(\alpha, \alpha)}(x, x) , \tag{46}$$

where

$$G_{x, \lambda}^{(\alpha, \alpha)}(x, x) = \int \xi^{2\alpha} (\mathrm{Re}\, a(x, \xi) - \lambda)^{-1} d\xi .$$

To estimate the difference $G^{(\alpha,\,\alpha)}_{x,\,\lambda}(x,\,x) - G^{(\alpha,\,\alpha)}_{\lambda}(x,\,x)$ we shall use the fundamental solution $g_\lambda(x,\,y)$ of $a(y,\,D_y) - \lambda$ which we have constructed and estimated in Lemma 10. This we can do when $a(x,\,D)$ satisfies the conditions at the beginning of Section 4. Since we shall only use a local fundamental solution, we can argue as in the proof of Theorem 1 to get rid of the global condition on $a(x,\,D)$. Thus the only remaining extra assumption is that (15) holds for the type operator P with $s \leq n + |2\alpha|$.

Then let x^0 be an arbitrary point in S and ω an open neighbourhood of x^0, which is relatively compact in S. Let g_λ be the fundamental solution of $a(y,\,D_y) - \lambda$ in $\omega \times \omega$, obtained from Lemma 10 when λ is large and negative. Let $\psi \in C^\infty_0(S)$ have its support in ω and be equal to 1 in some neighbourhood ω' of x^0. Then we have for any $x \in \omega'$ the following identity between $L^2(S)$-elements:

$$D^\alpha_x \overline{G_\lambda}(x,\,\cdot) = \psi D^\alpha_x g_\lambda(x,\,\cdot) + (A - \lambda I)^{-1}(D^\alpha_x k_\lambda(x,\,\cdot)), \tag{47}$$

where

$$k_\lambda(x,\,y) = (\psi(y)A_y - A_y\psi(y))g_\lambda(x,\,y)$$

(compare e.g. Odhnoff [7], Prop. 3.5). The identity (47) is proved by differentiation and a simple transcription of the identity

$$u(x) = \int \overline{g_\lambda(x,\,y)}(a(y,\,D_y) - \lambda)(\psi(y)u(y))dy \quad (x \in \omega' \text{ and e.g. } u \in C^\infty(S));$$

we omit the details. We now have to estimate the last term of the right member of (47). From (iv) of Lemma 10 it follows that

$$\|A^r D^\alpha_x k_\lambda(x^0,\,\cdot)\| = O(1) \exp\left(-\varkappa|\lambda|^b\right) \quad (\lambda \to -\infty) \tag{48}$$

for any integer $r \geq 0$, with a positive constant $\varkappa$, while b is the positive number of (3). Now, when $\lambda \leq 0$ we have $\|(A - \lambda I)^{-1}\| \leq 1$, and so it follows from (48) that

$$\|(A - \lambda I)^{-1}A^r D^\alpha_x k_\lambda(x^0,\,\cdot)\| = O(1) \exp\left(-\varkappa|\lambda|^b\right) \quad (\lambda \to -\infty).$$

Using the a priori inequality (5) (and the corresponding regularity statement) it then follows that $(A - \lambda I)^{-1}D^\alpha_x k_\lambda(x^0,\,\cdot)$ is in $C^\infty(S)$ and that (also using the identity (47) and the fact that $G^{(\alpha,\,\alpha)}_\lambda(x^0,\,x^0)$ is real)

$$G^{(\alpha,\,\alpha)}_\lambda(x^0,\,x^0) - g^{(\alpha,\,\alpha)}_\lambda(x^0,\,x^0) = O(1) \exp\left(-\varkappa|\lambda|^b\right) \quad (\lambda \to -\infty). \tag{49}$$

From (49) and (ii) in Lemma 10 we now get, for any $x \in S$,

$$G^{(\alpha,\,\alpha)}_{x,\,\lambda}(x,\,x) - G^{(\alpha,\,\alpha)}_{\lambda}(x,\,x) = O(1)|\lambda|^{-c}G^{(\alpha,\,\alpha)}_{x,\,\lambda}(x,\,x) \quad (\lambda \to -\infty), \tag{50}$$

where c is a positive constant.

To conclude the theorem (in the present special case) from (45), (46), and (50) by the Tauberian theorem (Theorem 2, together with Remark 2) of Ganelius [2]

for the Stieltjes transformation, we must also verify that the following Tauberian condition is fulfilled:

$$\sup_{\lambda \le \mu \le \lambda + \lambda/(c \log \lambda)} \int_{\lambda}^{\mu} d(e_{x,\nu}^{(\alpha,\alpha)}(x,x) - e_{\nu}^{(\alpha,\alpha)}(x,x)) \le O(1)\lambda^a (\log \lambda)^{t-1} \qquad (51)$$

as $\lambda \to +\infty$, where c is the positive number of (50). Further a and t are the numbers corresponding to the polynomial $\mathrm{Re}\, a(x, \cdot)$ and to α by Lemma 4, i.e., we have

$$C^{-1}\lambda^a(\log \lambda)^t \le \int_{\mathrm{Re}\, a(x,\xi) \le \lambda} \xi^{2\alpha}d\xi \le C\lambda^a(\log \lambda)^t$$

when λ is large, with a positive constant C. (Clearly the same numbers a, t correspond to the type polynomial P and α.)

But now we can easily prove (51), by the estimate of Lemma 4 for the derivative with respect to λ of $e_{x,\lambda}^{(\alpha,\alpha)}(x,x)$ and by the fact that $e_{\lambda}^{(\alpha,\alpha)}(x,x)$ increases with λ. So we can apply the Tauberian theorem mentioned and thus prove the theorem under the extra restrictions that we have imposed.

To see that these restrictions can be removed, we consider the operator $B = A^r + kI$, where r is an even integer ≥ 0 and k a real number ≥ 0. B is then a self-adjoint realization in $L^2(S)$ of the differential operator $b(x, D) = a(x, D)^r + k$, and clearly $b(x, D)$ satisfies the said restrictions in any given relatively compact open subset of S, if r and k are large enough. Thus the theorem is then valid for the spectral function $\tilde{e}_\lambda$ of B. But from the relation

$$e_\lambda - e_{-\lambda} = \tilde{e}_{\lambda^r+k} \quad (\lambda > 0)$$

(choosing the pointwise definition of e_λ conveniently when $\lambda < 0$) we get by application of Theorem 1 to the term $e_{-\lambda}$:

$$e_\lambda^{(\alpha,\alpha)}(x,x) = (1 + O(1)/(\log \lambda))\tilde{e}_{x,\lambda^r+k}^{(\alpha,\alpha)}(x,x) \quad (\lambda \to +\infty) \qquad (52)$$

for any $x \in S$. From Lemma 3, (4) and (5) we easily get the estimate

$$(\mathrm{Re}\, a(x,\xi))^r - \mathrm{Re}\, b(x,\xi) + k = O(1)(\mathrm{Re}\, b(x,\xi))^p \quad (|\xi| \to \infty, \ \xi \in R^n)$$

with some real number $p < 1$. This gives

$$\tilde{e}_{x,\lambda^r+k-u(x)\lambda^{pr}}^{(\alpha,\alpha)}(x,x) \le e_{x,\lambda}^{(\alpha,\alpha)}(x,x) \le \tilde{e}_{x,\lambda^r+k+u(x)\lambda^{pr}}^{(\alpha,\alpha)}(x,x)$$

when λ is large, with some positive number $u(x)$. Applying the estimate for the derivative of $\tilde{e}_{x,\lambda}^{(\alpha,\alpha)}(x,x)$, obtained from Lemma 4, we get

$$e_{x,\lambda}^{(\alpha,\alpha)}(x,x) = (1 + O(1)\lambda^{p-1})\tilde{e}_{x,\lambda^r+k}^{(\alpha,\alpha)}(x,x) \quad (\lambda \to +\infty).$$

Inserting this in (52) we get the theorem in the general case.

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Nils Nilsson
Department of Mathematics
University of Lund
Box 725
22007 Lund 7, Sweden