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Some results on embedding Stein spaces with interpolation

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1. Introduction

In this article we give some remarks on embeddings of Stein manifolds and more generally Stein spaces into $\mathbf{C}^n$ with interpolation. This means that given a Stein manifold X together with a countable discrete subset $A=\{a_1,a_2,a_3,...\}$ in X and a countable discrete subset $E=\{e_1,e_2,e_3,...\}$ in $\mathbf{C}^n$ we try to construct a proper holomorphic embedding

$$\varphi\colon X\hookrightarrow \mathbf{C}^N \quad \text{with} \quad \varphi(a_i)=e_i,\ i\in\mathbf{N}. \tag{1.1}$$

According to results of Eliashberg, Gromov [3] and Schürmann [10] every Stein manifold of dimension $n>1$ can be properly holomorphically embedded into $\mathbf{C}^N$ for all $N\geq n+\left[\frac{1}{2}n\right]+1$. Examples of Forster show that this bound is sharp in general [6]. The question of embedding with interpolation was earlier studied by Prezelj in [8]. Her main result is that for a Stein manifold of dimension $n>1$ the embedding with interpolation is always possible into affine space of dimension greater than or equal to $n+\left[\frac{1}{2}(n+1)\right]+1$, thus needing one extra dimension for odd n compared to the general embedding dimension. The proof of Prezelj follows the method of Eliashberg and Gromov, taking care of the interpolation condition at any step. This method of "desingularization" has the disadvantage that even though one has a Stein manifold which can be embedded into affine space of lower dimension, the construction yields an embedding only into dimension $n+\left[\frac{1}{2}(n+1)\right]+1$. For embeddings of the unit disc into $\mathbf{C}^2$ the problem of interpolation was solved by Globevnik in [7], and can be solved for embeddings of $\mathbf{C}$ into $\mathbf{C}^2$ by the methods of that paper and [2]. The question of interpolation on more general subvarieties is considered by Acquistapace, Broglia and Tognoli in [1].

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Our access to the problem is different. We start with the assumption that a given Stein manifold or Stein space can be embedded into affine space of some dimension N. Adding at most two extra dimensions we are able to solve the interpolation question in general (see Proposition 2.7). Combined with known results on embedding (without interpolation) this yields in many cases new results that are really different from Prezelj's. For instance we can prove that $\mathbf{C}^n$ can be embedded into $\mathbf{C}^{n+2}$ with interpolation (2.5). Our version for general Stein manifolds saying that for any Stein manifold of dimension $n>1$ the embedding with interpolation is always possible into affine space of dimension greater than or equal to $n+\left[\frac{1}{2}n\right]+2$ is a little weaker than Prezelj's. The dimension is larger by 1 for even n compared with Prezelj's result and we recover her result for odd n (see Corollary 2.6). Finally we remark that our methods have the advantage to give analogous result for Stein spaces, to generalize Prezelj's proofs from the setting of manifold to complex spaces seems to be a hard technical work.

We thank the referee for proposing a sharper formulation of Lemma 2.3 and pointing out a shortening in its proof.

2. Embedding with interpolation

In this section we give some results on embedding Stein manifolds into $\mathbf{C}^N$ with additional interpolation condition. More precisely we study the following question:

Given a Stein manifold X and a countable discrete subset A of X together with an enumeration $A=\{a_1,a_2,...\}\subset X$. Given furthermore a dimension N and a countable discrete subset E of $\mathbf{C}^N$ also enumerated $E=\{e_1,e_2,...\}\subset\mathbf{C}^N$ we ask the question:

Does there exist a proper holomorphic embedding $\varphi\colon X\hookrightarrow\mathbf{C}^N$ *with*

$$\varphi(a_i)=e_i, \quad i\in\mathbf{N}? \tag{2.1}$$

2.A. Preparations

We first concentrate on the case $X=\mathbf{C}^k$ which we embed into $\mathbf{C}^n$ $(n>k)$. Recall the following definition due to Rosay and Rudin [9]:

Definition 2.1. A subset E of $\mathbf{C}^n$ $(n>1)$ is *tame* if there exists a holomorphic automorphism $\alpha\in\mathrm{Aut}_{\mathrm{hol}}(\mathbf{C}^n)$ which maps the set E onto the set $N:=\{(z_1,0,\dots,0)\colon z_1\in\mathbf{N}\}$ with natural numbers as coordinates contained in the first coordinate line.

Remark 2.2. A tame set is clearly discrete and countable. Furthermore it is easy to prove that any injective map $\psi\colon E\to E$ of a tame subset E of $\mathbf{C}^n$ can be realized by an automorphism of the ambient space $\mathbf{C}^n$, i.e., is the restriction to E of some automorphism, for a proof, modify for instance that of Proposition 3.1 in [9]. Therefore the enumeration of the points in a tame subset E is not of importance if we want to solve the interpolation (2.1). Another observation that follows is that an infinite subset of a tame set is tame too.

It is known that the interpolation is possible in the weaker sense that any discrete subset E can be contained in the image of the embedding $\phi: \mathbf{C}^k \to \mathbf{C}^n$, without specifying the preimage points (see [5]). Even more generally, for any complex subvariety of $\mathbf{C}^n$ the "hitting problem" is solved in [4]. We need a version of this result with some additional property on the preimage points.

Lemma 2.3. *Let $1 \le k < n$. For every countable discrete set $E = \{e_1, e_2, \ldots\}$ there exists a proper holomorphic embedding $\varphi: \mathbf{C}^k \hookrightarrow \mathbf{C}^n$ satisfying $\varphi(j, 0, \ldots, 0) = e_j$ for $j = 1, 2, \ldots$.*

Proof. The case $k=1$ (which is not needed in this paper) is a part of Theorem 1.1 in [2].

In the case $k \ge 2$ the proof is similar to that of Proposition 2 in [5]. The proof given there has to be modified to achieve the tameness of $\varphi^{-1}(E)$ in the following way:

In the very beginning fix a tame subset $\tilde{A}$ of $\mathbf{C}^k$. The embedding $\varphi: \mathbf{C}^k \hookrightarrow \mathbf{C}^n$ is constructed as a limit (uniform on compact subsets) of embeddings $\varphi = \lim_{n\to\infty} \varphi_n$, where φ_0 is the standard embedding and $\varphi_n = \alpha_n \circ \varphi_{n-1}$ with α_n a holomorphic automorphism of $\mathbf{C}^n$. These automorphisms α_n are constructed in an inductive process. In each step (say the nth) of the induction process the image of the nth embedding $\varphi_n(\mathbf{C}^k) = \alpha_n \circ \alpha_{n-1} \circ \ldots \circ \alpha_0 \circ \varphi_0(\mathbf{C}^k)$ contains a new point (e_n) of the set E. We want to modify the sequence of automorphisms $\{\alpha_n\}$ so that the following two additional conditions are satisfied:

(A_n) $e_{n+1} \notin \varphi_n(\mathbf{C}^k)$, meaning that although the first n points of E are contained in the image of the nth embedding, the $(n+1)$th point is not;

(B_n) $\varphi_n^{-1}(e_i) \in \tilde{A}$, $i = 1, 2, \ldots, n$.

Let us show how to modify the nth step of the induction process to satisfy these additional conditions under the assumption that (A_{n-1}) and (B_{n-1}) are satisfied. Since by (A_{n-1}) the point e_n is not contained in $\varphi_{n-1}(\mathbf{C}^k)$, according to the proof in [5] one chooses some point a in $\mathbf{C}^k$ and construct the automorphism α_n so that $\alpha_n \circ \varphi_{n-1}(a) = e_n$. Here the only demand on the choice of the point a is that it is outside some compact set K depending on the previous steps ("far away"). Since the tame set $\tilde{A}$ is discrete we can find a point in $\tilde{A}$ which is outside K. Taking this point as a we satisfy (B_n). Proceed now further like in the proof in [5].

To achieve A_n we modify the constructed automorphism α_n replacing it by $\beta\circ\alpha_n$, where the automorphism β is arbitrarily near to the identity on some compact set K_1 appearing in the proof in [5], fixes all points $e_1, e_2, \dots, e_n$ and moves the embedding away from e_{n+1}.

In case $e_{n+1}\notin\varphi_n(\mathbf{C}^k)$ take simply β to be the identity. If $e_{n+1}\in\varphi_n(\mathbf{C}^k)$ proceed as follows:

Construct β as a shear of the form

$$z\longmapsto z+f(a(z))v$$

(here f is a holomorphic function of one variable, v a non-zero vector in $\mathbf{C}^n$ and a a linear form on $\mathbf{C}^n$ vanishing on the vector v) chosen in such a way that

$$f(a(e_i))=0, \quad i=1,2,\dots,n, \tag{2.2}$$

$$f(a(e_{n+1}))\neq 0 \tag{2.3}$$

and such that the (affine) line $\{e_{n+1}+tv:t\in\mathbf{C}\}$ is not entirely contained in $\varphi_n(\mathbf{C}^k)$.

Furthermore choose a point a on the (affine) line $\{e_{n+1}+tv:t\in\mathbf{C}\}$ very close to e_{n+1} and not on $\varphi_n(\mathbf{C}^k)$. Construct f in such a way that β moves a to e_{n+1} and is small on some compact set (in order to have β small on the compact K_1 as required).

The conditions (B_n) imply that the limit embedding φ not just only contains E but also that $A=\varphi^{-1}(E)$ is a subset of the tame set $\tilde{A}$. By Remark 2.2, A is tame too. The proof of Lemma 2.3 is complete. $\square$

As a last preparation we state a result for the case of embedding $\mathbf{C}^k\hookrightarrow\mathbf{C}^n$, $1<k<n$, with interpolation.

Lemma 2.4. *If the subset $A\subset\mathbf{C}^k$ is tame or the subset $E\subset\mathbf{C}^n$ is tame, then the interpolation problem* (2.1) *is solvable for an embedding $\mathbf{C}^k\hookrightarrow\mathbf{C}^n$, $1<k<n$.*

Proof. Consider first the case that A is tame. By Lemma 2.3 there is an embedding $\varphi:\mathbf{C}^k\hookrightarrow\mathbf{C}^n$ such that $\tilde{A}:=\varphi^{-1}(E)$ is tame too. By the definition of tameness there is an automorphism $\beta\in\mathrm{Aut}_{\mathrm{hol}}(\mathbf{C}^k)$ with $\beta(A)=\tilde{A}$ (in any prescribed order, see Remark 2.2). The embedding $\varphi\circ\beta:\mathbf{C}^k\hookrightarrow\mathbf{C}^n$ solves the problem.

Now to the case that E is tame. Let $i:\mathbf{C}^k\hookrightarrow\mathbf{C}^n$ denote the standard embedding of $\mathbf{C}^k$ into $\mathbf{C}^n$. Every discrete subset of $\mathbf{C}^n$ contained in the image of this embedding is tame (see e.g. [9]). In particular $A_1:=i(A)$ is tame in $\mathbf{C}^n$ and therefore there is an automorphism $\alpha\in\mathrm{Aut}_{\mathrm{hol}}(\mathbf{C}^n)$ with $\alpha(A_1)=E$ (in any prescribed order). The embedding $\alpha\circ i:\mathbf{C}^k\hookrightarrow\mathbf{C}^n$ solves the problem. $\square$

2.B. Results

In this section we use the notation from Definition 2.1.

Proposition 2.5. *Suppose X is a Stein manifold with a discrete subset A such that X admits an embedding φ_0 into $\mathbf{C}^n$ under which the image of A is tame. Then all interpolation problems for embedding X into $\mathbf{C}^{n+l}$ for that subset A (and any $E\subset\mathbf{C}^{n+l}$) are solvable ($l>0$).*

Proof. Since $\varphi_0(A)$ is tame in $\mathbf{C}^n$ we can find, using Lemma 2.4, an embedding $\theta\colon\mathbf{C}^n\hookrightarrow\mathbf{C}^{n+1}$ with $\theta(\varphi_0(A))=E$ (in any prescribed order). The composition $\theta\circ\varphi_0$ solves the problem. □

This result helps to establish a general result near to Prezelj's result mentioned at the end of the introduction.

Corollary 2.6. *Let X be a Stein manifold of dimension $k>1$. Then all interpolation problems for embedding X into $\mathbf{C}^N$ are solvable provided $N\geq k+\left[\frac{1}{2}k\right]+2$.*

Proof. In [10] Schürmann proves that any Stein manifold of dimension $k>1$ can be embedded into C^L for $L\geq k+\left[\frac{1}{2}k\right]+1$ (and this result is optimal by examples of Forster [6]). Moreover his embedding φ is constructed in such a way that $\varphi(A)$ is tame in $\mathbf{C}^L$ for a discrete subset A of X. Thus Proposition 2.5 can be applied. □

This result "loses" one dimension compared to the possibility to embed a general Stein manifold X of dimension k. It coincides with Prezelj's result for even dimensions k and is one dimension weaker for odd k. Our last result shows that adding at most 2 dimensions makes interpolation always possible. This is of course in many cases much stronger than Prezelj's general result (and Corollary 2.6).

Proposition 2.7. *Suppose X is a Stein manifold which admits an embedding $\varphi_0\colon X\hookrightarrow\mathbf{C}^n$. Also suppose that $L\geq n+2$. Then for any two discrete subsets $A\subset X$ and $E\subset\mathbf{C}^L$ the interpolation problem (2.1) is solvable.*

Proof. Compose the embedding φ_0 with the inclusion of $\mathbf{C}^n$ into $\mathbf{C}^{n+1}$ (as the first n coordinates). The image of A under that embedding is tame in $\mathbf{C}^{n+1}$, since it is contained in a hyperplane. The result follows now from Proposition 2.5. □

Remark 2.8. It is clear from the proofs that our Proposition 2.3 is true not only for Stein manifolds but also for Stein spaces. Therefore our main interpolation results, Proposition 2.5 and Proposition 2.7, hold for Stein spaces as well. Also Corollary 2.6 has an analogue for Stein spaces (with finite embedding dimension). The optimal embedding dimensions can be found in the paper of Schürmann [10], and we have to add one extra dimension for proving that the interpolation problem is solvable in general for embeddings into that dimension.

2.C. Open problem

Given a Stein space X which can be embedded into $\mathbf{C}^N$. Is it true that one can solve the general interpolation problem for embeddings of X into $\mathbf{C}^N$?

More precisely this means that given a countable discrete subset A of X together with an enumeration $A=\{a_1, a_2, ...\}\subset X$ and a countable discrete subset E of $\mathbf{C}^N$ also enumerated $E=\{e_1, e_2, ...\}\subset \mathbf{C}^N$ we ask the question:

Does there exist a proper holomorphic embedding $\varphi: X\hookrightarrow \mathbf{C}^N$ *with*

$$\varphi(a_i)=e_i, \quad i\in \mathbf{N}? \tag{2.4}$$

The question has an affirmative answer for X being the unit disc, $X=\mathbf{C}$ by work of Globevnik [7] and for those Stein manifolds of dimension $n=2k$ which have the number $n+\left[\frac{1}{2}n\right]+1$ as their (minimal) embedding dimension by Prezelj's work [8]. Examples of such manifolds can be found in [6].

References

1. Acquistapace, F., Broglia, F. and Tognoli, A., A relative embedding theorem for Stein spaces, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **2** (1975), 507–522.
2. Buzzard, G. T. and Forstnerič, F., A Carleman type theorem for proper holomorphic embeddings, *Ark. Mat.* **35** (1997), 157–169.
3. Eliashberg, Y. and Gromov, M., Embeddings of Stein manifolds of dimension n into the affine space of dimension $3n/2+1$, *Ann. of Math.* **136** (1992), 123–135.
4. Forstnerič, F., Interpolation by holomorphic automorphisms and embeddings in $\mathbf{C}^n$, *J. Geom. Anal.* **9** (1999), 93–117.
5. Forstnerič, F., Globevnik, J. and Rosay, J.-P., Nonstraightenable complex lines in $\mathbf{C}^2$, *Ark. Mat.* **34** (1996), 97–101.
6. Forster, O., Plongements des variétés de Stein, *Comment. Math. Helv.* **45** (1970), 170–184.
7. Globevnik, J., Interpolation by proper holomorphic embeddings of the disc into $\mathbf{C}^2$, *Math. Res. Lett.* **9** (2002), 567–577.
8. Prezelj, J., Interpolation of embeddings of Stein manifolds on discrete sets, *Math. Ann.* **326** (2003), 275–296.
9. Rosay, J.-P. and Rudin, W., Holomorphic maps from C^n to C^n, *Trans. Amer. Math. Soc.* **310** (1988), 47–86.
10. Schürmann, J., Embeddings of Stein spaces into affine spaces of minimal dimension, *Math. Ann.* **307** (1997), 381–399.

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