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Normality and fixed-points of meromorphic functions

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Abstract. Let $\mathcal{F}$ be families of meromorphic functions in a domain D, and let R be a rational function whose degree is at least 3. If, for any $f\in\mathcal{F}$, the composite function $R(f)$ has no fixed-point in D, then $\mathcal{F}$ is normal in D. The number 3 is best possible. A new and much simplified proof of a result of Pang and Zalcman concerning normality and shared values is also given.

1. Introduction

Let D be a domain in $\mathbf{C}$ and $\mathcal{F}$ a family of meromorphic functions defined on D. $\mathcal{F}$ is said to be normal in D, in the sense of Montel, if each sequence $\{f_n\}_{n=1}^{\infty}\subset\mathcal{F}$ has a subsequence $\{f_{n_j}\}_{j=1}^{\infty}$ which converges spherically locally uniformly in D, to a meromorphic function or ∞ (see [6], [10] and [14]).

A fixed-point of a meromorphic function f is a point z at which $f(z)=z$. In 1952, Rosenbloom [9] proved the following results.

Theorem A. *Let f be a transcendental entire function and let $k\in\mathbf{N}$, $k\geq 2$. Then the k^{th} iterate f_k has infinitely many fixed-points.*

Here, $f_2=f(f)$ and f_k is defined inductively via $f_k=f(f_{k-1})$, $k=3,4,\dots$.

Theorem B. *Let P be a polynomial with $\deg P\geq 2$, and let f be a transcendental entire function. Then the composite function $P(f)$ has infinitely many fixed-points.*

Essén and Wu [1] proved a corresponding normality criterion for Theorem A, thereby answering a question of Yang [13, Problem 8].

Theorem C. *Let $\mathcal{F}$ be a family of analytic functions on a domain D. If, for any $f\in\mathcal{F}$, there exists $k=k(f)>1$ such that the k^{th} iterate f_k has no fixed-point in D, then $\mathcal{F}$ is normal in D.*

Fang and Yuan [3] proved a corresponding normality criterion for Theorem B.

Theorem D. *Let $\mathcal{F}$ be a family of analytic functions on a domain D, and let P be a polynomial with* $\deg P \geq 2$. *If, for any $f \in \mathcal{F}$, the composite function $P(f)$ has no fixed-point, then $\mathcal{F}$ is normal in D.*

Let $R(z)=P_1(z)/P_2(z)$, where P_1 and P_2 are relatively prime polynomials. In this paper, $\max\{\deg P_1, \deg P_2\}$ is called the degree of R and denoted by $\deg R$.

Gross and Osgood [5] extended Theorem B to meromorphic functions.

Theorem E. *Let R be a rational function with* $\deg R \geq 3$, *and let f be a transcendental meromorphic function. Then the composite function $R(f)$ has infinitely many fixed-points.*

It is natural to ask whether there exists a corresponding normality criterion for Theorem E. In this paper, using the method of Yang [12], we give an affirmative answer to this question.

Theorem 1. *Let $\mathcal{F}$ be a family of meromorphic functions on a domain D, and let R be a rational function with* $\deg R \geq 3$. *If, for any $f \in \mathcal{F}$, the composite function $R(f)$ has no fixed-point in D, then $\mathcal{F}$ is normal in D.*

Remark 1. If $\mathcal{F}$ is a family of analytic functions, then we need only $\deg R \geq 2$ in Theorem 1. In other words, Theorem D remains valid if the polynomial P is replaced by a rational function R with $\deg R \geq 2$.

Remark 2. The following two examples show that $\deg R \geq 3$ is best possible in Theorem 1.

Example 1. Let

$$f(z)=\frac{\cos\sqrt{z}}{(\sin\sqrt{z})/\sqrt{z}}=\frac{\displaystyle\sum_{j=0}^{\infty}\frac{(-1)^j}{(2j)!}z^j}{\displaystyle\sum_{j=0}^{\infty}\frac{(-1)^j}{(2j+1)!}z^j},$$

and let $\mathcal{F}=\{f_n\}_{n=1}^{\infty}$, where

$$f_n(z)=\frac{i}{\sqrt{n}}f(nz), \quad n=1,2,\dots .$$

Let $D=\{z:|z|<1\}$, and let $R(z)=z^2$. Then

$$R(f_n(z))=-\frac{1}{n}\frac{1-(\sin\sqrt{nz})^2}{\left[(\sin\sqrt{nz})/\sqrt{nz}\right]^2}=-\frac{1}{n\left[(\sin\sqrt{nz})/\sqrt{nz}\right]^2}+z\neq z.$$

On the other hand, the family $\mathcal{F}$ clearly fails to be equicontinuous at 0, as f_n has both zeros and poles in any neighborhood of 0 for large n. Thus $\mathcal{F}$ is not normal at 0.

Example 2. Let $D=\{z:|z-1|<1\}$, and let

$$f(z)=\frac{(\sin\sqrt{z})/\sqrt{z}}{\cos\sqrt{z}}=\frac{\displaystyle\sum_{j=0}^{\infty}\frac{(-1)^j}{(2j+1)!}z^j}{\displaystyle\sum_{j=0}^{\infty}\frac{(-1)^j}{(2j)!}z^j},$$

and $\psi(z)=\sqrt{1+z}$. Let $\mathcal{F}=\{f_n\}_{n=1}^{\infty}$, where

$$f_n(z)=i\sqrt{n}\,\psi(z)f(n(z-1)),\quad n=1,2,\dots,$$

and let $R(z)=(z^2+1)/(z^2-1)$. Then

$$R(f_n(z))=z-\frac{z+1}{1+2n\left(\dfrac{\sin\sqrt{n(z-1)}}{\sqrt{n(z-1)}}\right)^2}\neq z.$$

On the other hand, just as before, $\mathcal{F}$ fails to be normal at $z_0=1$.

In Example 1, $\mathcal{F}$ is not normal at z_0 and $R(z)=z_0$ has a finite solution, while in Example 2, $\mathcal{F}$ is not normal at z_0 and $R(z)=z_0$ has no finite solution.

Let f and g be meromorphic functions on a (fixed) domain D in $\mathbf{C}$, and let a and b be complex numbers. If $g(z)=b$ whenever $f(z)=a$, we write $f(z)=a \Rightarrow g(z)=b$. In a different notation, we have $\overline{E}_f(a)\subset\overline{E}_g(b)$, where

$$\overline{E}_h(c)=h^{-1}(c)\cap D=\{z\in D: h(z)=c\}.$$

If $f(z)=a \Rightarrow g(z)=b$ and $g(z)=b \Rightarrow f(z)=a$, we write $f(z)=a \Leftrightarrow g(z)=b$; in this case $\overline{E}_f(a)=\overline{E}_g(b)$. If $f(z)=a \Leftrightarrow g(z)=a$, we say that f and g *share* the value a in D.

Now let $\mathcal{F}$ be a family of meromorphic functions on D. Schwick [11] was the first to draw a connection between values shared by functions in $\mathcal{F}$ and their derivatives and the normality of the family $\mathcal{F}$. Specifically, he showed that if there exist three distinct complex numbers a_1, a_2 and a_3 such that f and f' share a_j $(j=1,2,3)$ on D for each $f\in\mathcal{F}$, then $\mathcal{F}$ is a normal family on D. Pang and Zalcman [7] extended this result as follows.

Theorem F. *Let $\mathcal{F}$ be a family of meromorphic functions on a domain D, and let a, b, c, and d be complex numbers such that $c\neq a$ and $d\neq b$. If, for each $f\in\mathcal{F}$, $f(z)=a \Leftrightarrow f'(z)=b$ and $f(z)=c \Leftrightarrow f'(z)=d$, then $\mathcal{F}$ is normal in D.*

Choosing $a=b$, $c=d$, we see that Schwick's result actually holds when f and f' share two (rather than three) finite values in D.

In this paper, we improve Theorem F as follows.

Theorem 2. *Let $\mathcal{F}$ be a family of meromorphic functions on a domain D; and let a, b, c, and d be complex numbers such that $b\neq 0$, $c\neq a$, and $d\neq b$. If, for each $f\in\mathcal{F}$, $f(z)=a \Leftrightarrow f'(z)=b$ and $f(z)=c \Rightarrow f'(z)=d$, then $\mathcal{F}$ is normal in D.*

Theorem F is an instant corollary of Theorem 2, since not both b and d can be zero.

Example 3. ([4]) Let

$$f_n(z)=\frac{(nz)^2}{(nz)^2-1}, \quad n=1,2,\dots,$$

and let $\mathcal{F}=\{f_n\}_{n=1}^{\infty}$, $D=\{z:|z|<1\}$. Then

$$f_n'(z)=\frac{-2n^2z}{[(nz)^2-1]^2}.$$

Obviously, if $f\in\mathcal{F}$, f and f' vanish only at 0; also, $f\neq 1$. Thus we have $f(z)=0 \Leftrightarrow f'(z)=0$ and $f(z)=1 \Rightarrow f'(z)=d$ for *any* d (since $f\neq 1$). However, $\mathcal{F}$ is not normal on D. This shows that the condition $b\neq 0$ is necessary in Theorem 2.

For families of analytic functions, b can be allowed to be zero (see [2]).

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2. A useful lemma

The proofs of Theorems 1 and 2 are based on the following result of Pang and Zalcman.

Lemma 1. ([8, Lemma 2]) *Let $\mathcal{F}$ be a family of functions meromorphic on the unit disc, all of whose zeros have multiplicity at least k, and suppose that there exists $A\geq 1$ such that $|f^{(k)}(z)|\leq A$ whenever $f(z)=0$. Then if $\mathcal{F}$ is not normal, there exist, for each $0\leq\alpha\leq k$,*

(a) *a number $0<r<1$;*
(b) *points z_n, $|z_n|<r$;*
(c) *functions $f_n\in\mathcal{F}$;*
(d) *positive numbers $\varrho_n\to 0$,*

such that $\varrho_n^{-\alpha} f_n(z_n+\varrho_n\zeta)=g_n(\zeta)\to g(\zeta)$ *locally uniformly with respect to the spherical metric, where* g *is a nonconstant meromorphic function on* $\mathbf{C}$*, all of whose zeros have multiplicity at least* k*, such that* $g^{\#}(\zeta)\le g^{\#}(0)=kA+1$*. In particular,* g *has order at most two; and, in case* g *is an entire function, it is of exponential type.*

3. Proof of Theorem 1

Let $z_0\in D$. We show that $\mathcal{F}$ is normal at z_0. We consider two cases.

Case 1. $R(z)-z_0$ has at least three finite distinct zeros a, b and c. Assume that $\mathcal{F}$ is not normal at z_0. Then by Lemma 1, there exist points $z_n\to z_0$, positive numbers $\varrho_n\to 0$, and functions $f_n\in\mathcal{F}$ such that

$$g_n(\zeta)=f_n(z_n+\varrho_n\zeta)\to g(\zeta) \tag{3.1}$$

locally uniformly with respect to the spherical metric, where g is a nonconstant meromorphic function on $\mathbf{C}$.

Thus we have

$$R(g_n(\zeta))-(z_n+\varrho_n\zeta)\to R(g(\zeta))-z_0, \tag{3.2}$$

the convergence being uniform on compact subsets of $\mathbf{C}$ disjoint from the poles of g and $R(g)$.

Since $R(g_n(\zeta))-(z_n+\varrho_n\zeta)=R(f_n(z_n+\varrho_n\zeta))-(z_n+\varrho_n\zeta)\neq 0$, by Hurwitz's theorem, either $R(g(\zeta))-z_0\equiv 0$, or $R(g(\zeta))-z_0\neq 0$. If $R(g(\zeta))-z_0\equiv 0$, then g is constant. If $R(g(\zeta))-z_0\neq 0$, then $g(\zeta)\neq a,b,c$; so by Picard's theorem, g is again constant. Thus, whichever alternative holds, we obtain a contradiction. Hence in Case 1, $\mathcal{F}$ is normal at z_0.

Case 2. $R(z)-z_0$ has at most two distinct finite zeros. We claim that there exists a positive number δ_0 such that $\mathcal{F}$ is normal in $D^o_{\delta_0}(z_0)=\{z:0<|z-z_0|<\delta_0\}$. Indeed, by the argument of Case 1, we need only prove that there exists a positive number δ_0 such that for any $z_1\in D^o_{\delta_0}(z_0)$, $R(z)-z_1$ has at least three distinct finite zeros.

Let $S=\{z\in\mathbf{C}:R'(z)=0\}\cup\{\infty\}$ and $E=R(S)=\{R(z):z\in S\}$. Then E is a finite set. Hence there exists a positive number δ_0 such that

$$D^o_{\delta_0}(z_0)\cap E=\emptyset. \tag{3.3}$$

Thus for any $z_1\in D^o_{\delta_0}(z_0)$, $R(z)-z_1$ has no multiple zeros. Hence $R(z)-z_1$ has at least $3(\le\deg R)$ finite distinct zeros. The claim is proved.

Next we consider three subcases.

Case 2.1. $R(z)-z_0$ has at least one multiple finite zero $z=a$. Thus there exists a positive number $\delta_1 \leq \delta_0$ such that

$$R(\{z:|z-a|<\delta_1\})\subset\{z:|z-z_0|<\delta_0\}, \tag{3.4}$$

and

$$R(z)=z_0+\tau\psi^k(z), \tag{3.5}$$

where $k\geq 2$ is an integer, $\tau\neq 0$ is a constant, and $\psi(z)$ is a univalent analytic function in $D_{\delta_1}(a)=\{z:|z-a|<\delta_1\}$ with normalization $\psi(a)=0$, and $\psi'(a)=1$.

Set

$$\mathcal{G}=\{f(R):f\in\mathcal{F}\}. \tag{3.6}$$

Then

(i) $\mathcal{G}$ is normal in $D^o_{\delta_1}(a)=\{z:0<|z-a|<\delta_1\}$;

(ii) for any $z\in D_{\delta_1}(a)$ and $g\in\mathcal{G}$,

$$R(g(z))\neq R(z); \tag{3.7}$$

(iii) $\mathcal{G}$ is normal at a if and only if $\mathcal{F}$ is normal at z_0.

Let η be a positive number such that

$$\psi^{-1}(D_\eta(0))\subset D_{\delta_1}(a).$$

Choose a positive number $\delta_2\leq\delta_1$ such that

$$\psi(D_{\delta_2}(a))\subset D_\eta(0).$$

Thus, for any $z\in D_{\delta_2}(a)$ and any $g\in\mathcal{G}$, we have

$$g(z)\neq\psi^{-1}(\omega_j\psi(z)),\quad j=0,1,\dots,k-1,$$

where $\omega_j=e^{2\pi ij/k}$. Indeed, suppose there exist $z\in D_{\delta_2}(a)$ and $0\leq j\leq k-1$ satisfying

$$g(z)=\psi^{-1}(\omega_j\psi(z)).$$

Since $\psi(D_{\delta_2}(a))\subset D_\eta(0)$, we have $\psi(z)\in D_\eta(0)$ and so also $\omega_j\psi(z)\in D_\eta(0)$. But then $g(z)=\psi^{-1}(\omega_j\psi(z))\in D_{\delta_1}(a)$. Thus $\psi(g(z))=\omega_j\psi(z)$, whence $[\psi(g(z))]^k=[\psi(z)]^k$. But then, by (3.5), $R(g(z))=R(z)$, which contradicts (3.7).

We have shown that

$$g(z) \neq \psi^{-1}(\omega_j \psi(z)), \quad z \in D_{\delta_2}(a),\ j = 0, 1, \dots, k-1.$$

In particular, for any $z \in D_{\delta_2}(a)$, we have

$$g(z) \neq z \quad \text{and} \quad g(z) \neq \psi^{-1}(\omega_1 \psi(z)).$$

Set $\mathcal{H} = \{g - \mathrm{id} : g \in \mathcal{G}\}$, where id denotes the identity mapping. Then
(iv) $\mathcal{H}$ is normal in $D^o_{\delta_2}(a)$;
(v) for any $z \in D_{\delta_2}(a)$ and $h \in \mathcal{H}$,

$$h(z) \neq 0 \quad \text{and} \quad h(z) \neq \psi^{-1}(\omega_1 \psi(z)) - z;$$

(vi) $\mathcal{H}$ is normal at a if and only if $\mathcal{G}$ is normal at a.
Next we prove that $\mathcal{H}$ is normal at $z = a$.

Let $\{h_j\}_{j=1}^{\infty}$ be a sequence in $\mathcal{H}$; then there exists a subsequence of $\{h_j\}_{j=1}^{\infty}$ (which, without loss of generality, we may again denote by $\{h_j\}_{j=1}^{\infty}$) which converges locally spherically uniformly on $D^o_{\delta_2}(a)$ to a function h. We consider two subcases.

Case 2.1.1. $h \not\equiv 0$. Then, by Hurwitz's theorem, $h \neq 0$ in $D^o_{\delta_2}(a)$. Therefore,

$$\min_{0 \leq \theta \leq 2\pi} \left| h\left(a + \tfrac{1}{2}\delta_2 e^{i\theta}\right) \right| > A > 0$$

for some constant A.

Hence for sufficiently large j,

$$\min_{0 \leq \theta \leq 2\pi} \left| h_j\left(a + \tfrac{1}{2}\delta_2 e^{i\theta}\right) \right| > \tfrac{1}{2} A > 0.$$

Since h_j is meromorphic and $h_j \neq 0$ in $D_{\delta_2}(a)$, $1/h_j$ is holomorphic in $D_{\delta_2}(a)$. Thus $1/h_j$ is holomorphic in $\overline{D}_{\delta_2/2}(a) = \{z : |z-a| \leq \frac{1}{2}\delta_2\}$, and

$$\max_{0 \leq \theta \leq 2\pi} \frac{1}{\left| h_j\left(a + \frac{1}{2}\delta_2 e^{i\theta}\right) \right|} < \frac{2}{A}.$$

By the maximum principle, we conclude that

$$\max_{|z-a| \leq \delta_2/2} \frac{1}{|h_j(z)|} < \frac{2}{A},$$

so

$$\min_{|z-a| \leq \delta_2/2} |h_j(z)| > \frac{A}{2} > 0.$$

Hence there exists a subsequence of $\{h_j\}_{j=1}^{\infty}$ which converges locally spherically uniformly in $D_{\delta_2/2}(a)$.

Case 2.1.2. $h \equiv 0$. Then $\{h_j\}_{j=1}^{\infty}$ converges locally uniformly to 0 in $D^o_{\delta_2}(a)$. Thus $\{\psi_j\}_{j=1}^{\infty}$ and $\{\psi'_j\}_{j=1}^{\infty}$ also converge locally uniformly to 0 in $D^o_{\delta_2}(a)$, where

$$\psi_j(z) = \frac{h_j(z)}{\psi^{-1}(\omega_1\psi(z)) - z} \neq 1. \tag{3.8}$$

Hence, denoting by $N(r, a, f)$ the number of poles of f in $D_r(a)$, we have by the argument principle for sufficiently large j,

$$\left|N\left(\frac{\delta_2}{2}, a, \psi_j - 1\right) - N\left(\frac{\delta_2}{2}, a, \frac{1}{\psi_j - 1}\right)\right| = \left|\frac{1}{2\pi i}\int_{|z-a|=\delta_2/2} \frac{\psi'_j(z)}{\psi_j(z)-1}\,dz\right| < 1.$$

Thus

$$N\left(\frac{\delta_2}{2}, a, \psi_j - 1\right) = N\left(\frac{\delta_2}{2}, a, \frac{1}{\psi_j - 1}\right).$$

It follows by (3.8) that for sufficiently large j,

$$N\left(\frac{\delta_2}{2}, a, \psi_j\right) = N\left(\frac{\delta_2}{2}, a, \psi_j - 1\right) = N\left(\frac{\delta_2}{2}, a, \frac{1}{\psi_j - 1}\right) = 0.$$

Thus ψ_j has no pole in $D_{\delta_2/2}(a)$ for sufficiently large j, and so neither does h_j. Hence there exists a subsequence of $\{h_j\}_{j=1}^{\infty}$ which converges locally spherically uniformly in $D_{\delta_2/2}(a)$. Thus $\mathcal{H}$ is normal at a. By (iii)-(vi), $\mathcal{F}$ is normal at z_0.

Case 2.2. $R(z) - z_0$ has only finite simple zeros and has at least one finite zero. Then either

$$R(z) = z_0 + \frac{z-a}{P_1(z)}, \tag{3.9}$$

or

$$R(z) = z_0 + \frac{(z-a)(z-b)}{P_1(z)}, \tag{3.10}$$

where P_1 is a polynomial with $\deg P_1 \geq 3$ and a and b are distinct finite values which are not zeros of P_1.

Since $R(f(z)) \neq z$, $z \in D_{\delta_0}(z_0)$,

$$f(z_0) \neq \infty. \tag{3.11}$$

As in Case 2.1, there exists a positive number δ_3 such that

(vii) R is a univalent analytic function in $D_{\delta_3}(a) = \{z : |z-a| < \delta_3\}$;

(viii) $\mathcal{G}$ is normal in $D^o_{\delta_3}(a) = \{z : 0 < |z-a| < \delta_3\}$;

(ix) $\mathcal{G}$ is normal at a if and only if $\mathcal{F}$ is normal at z_0;

(x) for any $z \in D_{\delta_3}(a)$ and $g \in \mathcal{G}$, $R(g(z)) \neq R(z)$, and $g(a) = f(R(a)) = f(z_0) \neq \infty$.

Now we consider two subcases.

Case 2.2.1. R has the form (3.9). Then by (x), we have

$$(z-a)P_1(g(z))-(g(z)-a)P_1(z)\neq 0, \quad z\in D_{\delta_3}(a). \tag{3.12}$$

Let $P_1(z)=\sum_{j=0}^{p}\lambda_j z^j$ with $p\geq 3$ and $\lambda_p\neq 0$. Then

$$\begin{aligned}(z-a)P_1(\omega)-(\omega-a)P_1(z) &= (z-a)\sum_{j=0}^{p}\lambda_j\omega^j-(\omega-a)P_1(z)\\ &= (z-a)\sum_{j=0}^{p}\lambda_j((\omega-z)+z)^j-(z-a)P_1(z)\\ &\quad -(\omega-z)P_1(z)\\ &= (z-a)\left[\sum_{j=0}^{p}\lambda_j\sum_{t=0}^{j}C_j^t z^{j-t}(\omega-z)^t-P_1(z)\right]\\ &\quad -(\omega-z)P_1(z)\\ &= (\omega-z)\left[(z-a)\sum_{j=1}^{p}\lambda_j\sum_{t=1}^{j}C_j^t z^{j-t}(\omega-z)^{t-1}-P_1(z)\right]\\ &= (\omega-z)\left[\sum_{s=0}^{p-1}Q_s(z)(\omega-z)^s\right], \end{aligned} \tag{3.13}$$

where $C_j^t=j!/t!(j-t)!$ and Q_s $(s=0,1,\dots,p-1)$ are polynomials. In particular,

$$Q_0(z)=(z-a)P_1'(z)-P_1(z), \quad Q_{p-1}(z)=\lambda_p(z-a),$$

and $Q_0(z)\neq 0$, $z\in D_{\delta_4}(a)$, where $\delta_4\leq\delta_3$ is a positive number.

By (3.12) and (3.13), we have

$$g(z)\neq z, \quad \text{and} \quad \sum_{s=0}^{p-1}Q_s(z)(g(z)-z)^s\neq 0. \tag{3.14}$$

Let $\mathcal{H}=\{g-\mathrm{id}:g\in\mathcal{G}\}$. Then

(xi) $\mathcal{H}$ is normal in $D_{\delta_4}^o(a)$;

(xii) $\mathcal{H}$ is normal at a if and only if $\mathcal{G}$ is normal at a;

(xiii) for any $z\in D_{\delta_4}(a)$, and $h\in\mathcal{H}$,

$$(3.15)\qquad h(z)\neq 0,\quad \psi_h(z)=\frac{\sum_{s=1}^{p-1}Q_s(z)h(z)^s}{Q_0(z)}\neq -1\quad\text{and}\quad h(a)=g(a)-a\neq\infty.$$

Using the same argument as in Case 2.1, one can prove that $\mathcal{H}$ is normal at a. We omit the details. It follows that $\mathcal{F}$ is normal at z_0.

Case 2.2.2. R has the form (3.10). Then

$$\frac{(\omega-a)(\omega-b)}{P_1(\omega)}-\frac{(z-a)(z-b)}{P_1(z)}=\frac{(\omega-a)(\omega-b)P_1(z)-(z-a)(z-b)P_1(\omega)}{P_1(\omega)P_1(z)},$$

where $P_1(z)=\lambda z^k+c_1z^{k-1}+\ldots+c_k$ with $k\geq 3$ and $\lambda\neq 0$. We have

$$\begin{aligned}
&(z-a)(z-b)P_1(\omega)-(\omega-a)(\omega-b)P_1(z)\\
&\quad=(z-a)(z-b)P_1(z+\omega-z)-[(\omega-z)+(z-a)][(\omega-z)+(z-b)]P_1(z)\\
&\quad=(z-a)(z-b)\sum_{j=0}^{k}\frac{P_1^{(j)}(z)}{j!}(\omega-z)^j\\
&\qquad-[(\omega-z)^2+(2z-a-b)(\omega-z)+(z-a)(z-b)]P_1(z)\\
&\quad=(z-a)(z-b)\sum_{j=1}^{k}\frac{P_1^{(j)}(z)}{j!}(\omega-z)^j-P_1(z)(\omega-z)^2-P_1(z)(2z-a-b)(\omega-z)\\
&\quad=(\omega-z)\Bigg((z-a)(z-b)P_1'(z)-(2z-a-b)P_1(z)\\
&\qquad+\left[\frac{1}{2}(z-a)(z-b)P_1''(z)-P_1(z)\right](\omega-z)+(z-a)(z-b)\sum_{j=3}^{k}\frac{P_1^{(j)}(z)}{j!}(\omega-z)^{j-1}\Bigg)\\
&\quad=(\omega-z)\sum_{j=1}^{k}Q_j(z)(\omega-z)^{j-1},
\end{aligned}$$

where $Q_1,Q_2,\ldots,Q_k$ are polynomials. In particular,

$$Q_1(z)=(z-a)(z-b)P_1'(z)-(2z-a-b)P_1(z),$$

$Q_1(z)\neq 0$, $z\in D_\delta(a)$ for sufficiently small δ, and $Q_k(z)=\lambda(z-a)(z-b)$. The same argument as in Case 2.2.1 then shows that $\mathcal{F}$ is normal at z_0.

Case 2.3. $R(z)-z_0$ has no finite zero. Thus R has the form

$$(3.16)\qquad R(z)=z_0+\frac{1}{P(z)},$$

where $P(z)$ is a polynomial with $\deg P \geq 3$.

Now for any $f \in \mathcal{F}$ and $z \in D_{\delta_0}(z_0)$, $R(f(z)) \neq z$. It follows that $f(z_0) \neq \infty$ and $(z-z_0)P(f(z))-1 \neq 0$. Hence

$$g_f(z) = \frac{1}{(z-z_0)P(f(z))-1} \tag{3.17}$$

is an analytic function in $D_{\delta_0}(z_0)$. Since $f(z_0) \neq \infty$, we have

$$g_f(z_0) = -1. \tag{3.18}$$

Since $\mathcal{F}$ is normal in $D^o_{\delta_0}(z_0)$, for any $\{f_n\}_{n=1}^{\infty} \subset \mathcal{F}$, there exists a subsequence of $\{f_n\}_{n=1}^{\infty}$ (which we again denote by $\{f_n\}_{n=1}^{\infty}$) which converges locally uniformly with respect to the spherical metric either to ∞ or to a function ψ meromorphic in $D^o_{\delta_0}(z_0)$.

If $f_n \to \infty$ in $D^o_{\delta_0}(z_0)$, then $(z-z_0)P(f_n) \to \infty$ in $D^o_{\delta_0}(z_0)$. Hence by (3.17), $g_{f_n}(z) \to 0$ in $D^o_{\delta_0}(z_0)$. Since g_{f_n} is analytic, the maximum principle shows that $g_{f_n}(z) \to 0$ in $D_{\delta_0}(z_0)$. Hence $g_{f_n}(z_0) \to 0$, which contradicts $g_{f_n}(z_0) = -1$.

Hence $f_n \to \psi$ in $D^o_{\delta_0}(z_0)$. Obviously, we have

$$(z-z_0)P(f_n(z)) \to (z-z_0)P(\psi(z)) \tag{3.19}$$

in $D^o_{\delta_0}(z_0)$. Thus

$$g_{f_n}(z) \to \frac{1}{(z-z_0)P(\psi(z))-1} = G(z) \tag{3.20}$$

in $D^o_{\delta_0}(z_0)$. Since $g_{f_n}(z)$ is analytic, either $G(z) \equiv \infty$ or G is analytic in $D^o_{\delta_0}(z_0)$.

If $G \equiv \infty$, then $(z-z_0)P(\psi(z))-1 \equiv 0$ in $D^o_{\delta_0}(z_0)$. Hence z_0 is a simple pole of $P(\psi)$. But this is impossible, since $\deg P > 1$.

Hence G is analytic in $D^o_{\delta_0}(z_0)$. Thus, by the maximum principle, we have

$$g_{f_n}(z) \to G(z) \tag{3.21}$$

in $D_{\delta_0}(z_0)$. Hence G is analytic in $D_{\delta_0}(z_0)$, and so ψ is meromorphic in $D_{\delta_0}(z_0)$.

By (3.17) and (3.21),

$$(z-z_0)P(f_n(z)) \to (z-z_0)P(\psi(z)) \tag{3.22}$$

in $D_{\delta_0}(z_0)$. Since $f_n(z_0) \neq \infty$, we have $\psi(z_0) \neq \infty$, for otherwise, by (3.22), we should have $0 = \infty$. Thus $\psi(z)$ is analytic on $\bar{D}_{\delta_5}(z_0)$, $(\delta_5 \leq \delta_0)$. Hence by (3.22), for sufficiently large n, f_n is analytic in $D_{\delta_5}(z_0)$. Thus, by the maximum principle, $f_n \to \psi$ in $D_{\delta_5}(z_0)$. Hence $\mathcal{F}$ is normal at z_0.

Thus $\mathcal{F}$ is normal in D. The proof of Theorem 1 is complete.

4. Proof of Theorem 2

We may assume that $D=\Delta$, the unit disc. Suppose that $\mathcal{F}$ is not normal on Δ. Then by Lemma 1, we can find $f_n\in\mathcal{F}$, $z_n\in\Delta$, and $\varrho_n\to0^+$ such that $g_n(\zeta)=\varrho_n^{-1}[f_n(z_n+\varrho_n\zeta)-c]$ converges locally uniformly with respect to the spherical metric to a nonconstant meromorphic function g on $\mathbf{C}$, which satisfies $g^{\#}(\zeta)\leq g^{\#}(0)=|d|+2$.

We claim

(i) $g(\zeta)=0 \Rightarrow g'(\zeta)=d$;

(ii) $g'\neq b$;

(iii) $g\neq\infty$ on $\mathbf{C}$.

Suppose that $g(\zeta_0)=0$. Then by Hurwitz's theorem, there exist $\zeta_n\to\zeta_0$, such that (for n sufficiently large)

$$g_n(\zeta_n)=\varrho_n^{-1}[f_n(z_n+\varrho_n\zeta_n)-c]=0.$$

Thus $f_n(z_n+\varrho_n\zeta_n)=c$. Since $f_n(\zeta)=c \Rightarrow f_n'(\zeta)=d$, we have

$$g_n'(\zeta_n)=f_n'(z_n+\varrho_n\zeta_n)=d.$$

Hence $g'(\zeta_0)=\lim_{n\to\infty}g_n'(\zeta_n)=d$. Thus $g(\zeta)=0 \Rightarrow g'(\zeta)=d$. This proves (i).

Next we prove (ii). Suppose that $g'(\zeta_0)=b$. Then $g(\zeta_0)\neq\infty$. Further, $g'(\zeta)\not\equiv b$; for otherwise, $g(\zeta)=b(\zeta-\zeta_1)$, which is inconsistent with (i). By Hurwitz's theorem, there exist $\zeta_n\to\zeta_0$, such that (for n sufficiently large) $f_n'(z_n+\varrho_n\zeta_n)=g_n'(\zeta_n)=b$. It follows that $f_n(z_n+\varrho_n\zeta_n)=a$, so that $g_n(\zeta_n)=[f_n(z_n+\varrho_n\zeta_n)-c]/\varrho_n=(a-c)/\varrho_n$. Thus $g(\zeta_0)=\lim_{n\to\infty}g_n(\zeta_n)=\infty$, a contradiction. It follows that $g'\neq b$, which is (ii).

Now we prove (iii). Suppose that $g(\zeta_0)=\infty$. Since $g\not\equiv\infty$, there exists a closed disc $K=\{\zeta:|\zeta-\zeta_0|\leq\delta\}$ on which $1/g$ and $1/g_n$ are holomorphic (for n sufficiently large) and $1/g_n\to1/g$ uniformly. Hence, $1/g_n(\zeta)-\varrho_n/(a-c)\to1/g(\zeta)$ uniformly on K. Let the multiplicity of the zero of $1/g$ at ζ_0 be m. Thus $(1/g)^{(m)}(\zeta_0)\neq0$. Since $1/g$ is nonconstant, it follows from Hurwitz's theorem that there exists a positive number $\delta_1(<\delta)$ such that for every sufficiently large n, the equation

$$\frac{1}{g_n(\zeta)}-\frac{\varrho_n}{a-c}=0 \tag{4.1}$$

has exactly m solutions with due count of multiplicity in $D_{\delta_1}(\zeta_0)$. Denote these solutions by $\{\zeta_{jn}\}_{j=1}^m$; then $\lim_{n\to\infty}\zeta_{jn}=\zeta_0$ for $1\leq j\leq m$. Now $f_n(z_n+\varrho_n\zeta_{jn})-c=a-c$, i.e., $f_n(z_n+\varrho_n\zeta_{jn})=a$. Thus $g_n'(\zeta_{jn})=f_n'(z_n+\varrho_n\zeta_{jn})=b$. It follows that

$$\left(\frac{1}{g_n(\zeta)}\right)'\bigg|_{\zeta=\zeta_{jn}}=-\frac{g_n'(\zeta_{jn})}{g_n^2(\zeta_{jn})}=-\frac{b\varrho_n^2}{(a-c)^2}\neq0,\quad j=1,2,\dots,m. \tag{4.2}$$

Thus

$$\zeta_{jn} \neq \zeta_{kn}, \quad 1 \leq j < k \leq m. \tag{4.3}$$

Hence

$$\left(\frac{1}{g_n(\zeta)}\right)' + \frac{b\varrho_n^2}{(a-c)^2} \tag{4.4}$$

has at least m distinct zeros in $D_{\delta_1}(\zeta_0)$ which tend to ζ_0 as $n \to \infty$. By Hurwitz's theorem, ζ_0 is a zero of $(1/g)'$ with multiplicity at least m: and thus $(1/g)^{(m)}(\zeta_0)=0$, a contradiction. This proves (iii).

It follows that g is an entire function and is therefore of exponential type. By (ii), we have

$$g'(\zeta) = b + e^{A\zeta+B}, \tag{4.5}$$

so that

$$g(\zeta) = b\zeta + C + \frac{e^{A\zeta+B}}{A}, \tag{4.6}$$

as long as $A \neq 0$, where A, B and C are constants.

We consider two cases.

Case 1. $A \neq 0$. Let $g(\zeta_0)=0$. Then by (4.6),

$$b\zeta_0 + C + \frac{e^{A\zeta_0+B}}{A} = 0,$$

so by (4.5) and (i), we have

$$b + e^{A\zeta_0+B} = d.$$

Hence

$$\zeta_0 = -\frac{1}{b}\left(C + \frac{d-b}{A}\right).$$

Thus $g(\zeta)=0$ has the unique solution $\zeta=\zeta_0$; but it is evident from (4.6) that $g(\zeta)=0$ has infinitely many solutions.

Case 2. $A=0$. Then by (4.5) and (i), $g'(\zeta) \equiv d$, so $g(\zeta)=d(\zeta-\zeta_1)$. Thus we have

$$g^{\#}(0) = \frac{|g'(0)|}{1+|g(0)|^2} \leq |g'(0)| = |d|,$$

so that $g^{\#}(0) < |d|+2$, a contradiction.

Hence $\mathcal{F}$ is normal in D. The theorem is proved.

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