On Absolutely Convergent Fourier Series

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1. Introduction

Let T denote the circle group and let Z denote the group of integers. We shall consider functions f which are integrable on T and we shall denote their Fourier coefficients by $\hat{f}(n)$, where $n \in Z$. For $\beta > 0$ the set of all $f \in L_1(T)$ such that $\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^\beta < \infty$ will be denoted by $A(\beta)$. Among the classical results in the theory of absolutely convergent Fourier series are the following theorems [6, Vol. 1, Chapter VI, 3].

THEOREM 1 (Bernstein). *If* $f \in \mathrm{Lip}\,\alpha$ *for some* $\alpha > \frac{1}{2}$, *then* $f \in A(1)$.

THEOREM 2 (Zygmund). *If* f *is of bounded variation on* T $(f \in \mathrm{BV})$ *and if* $f \in \mathrm{Lip}\,\alpha$ *for some* $\alpha > 0$, *then* $f \in A(1)$.

Attempts to generalize these theorems have led to the following.

THEOREM 1A (Szász). *If* $f \in \mathrm{Lip}\,\alpha$ *for some* α *with* $0 < \alpha \leq 1$, *then* $f \in A(\beta)$ *for all* β *such that* $\beta > 2/(2\alpha + 1)$.

THEOREM 1B (Hardy). *If* $f \in \mathrm{Lip}\,\alpha$ *for some* α *with* $0 < \alpha \leq 1$, *then* $\sum_{0<|n|<\infty} |n|^{-\beta}|\hat{f}(n)| < \infty$ *for all* β *such that* $\beta > (1 - 2\alpha)/2$.

Definition 1. Let f be a function defined on T and for $r \geq 1$, let

$$V_r[f] = \sup\Big(\sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)|^r\Big)^{1/r},$$

where the supremum is taken over all finite partitions $0 \leq x_0 < x_1 < \ldots < x_n < 2\pi$ of T. The function f is of r-bounded variation $(f \in r\text{-BV})$ if $V_r[f] < \infty$.

THEOREM 2A (Hirschman [2]). *If* $f \in r$-BV *for some* r *with* $1 \leq r < 2$, *and if* $f \in \mathrm{Lip}\,\alpha$ *for some* $\alpha > 0$, *then* $f \in A(1)$.

It is well-known that each of the foregoing theorems is the best possible in a certain sense [6, Vol. 1]. In [4, Exercise I.6.6] Katznelson gave a new and very simple example of a function in Lip $\frac{1}{2}$ that does not belong to $A(1)$. In the remainder of this section we shall give a simple extension of Katznelson's example which can be used to show that all the previous theorems are sharp. We first give the definition of the so-called Rudin-Shapiro polynomials $P_n(x)$ and $Q_n(x)$. Let $P_0(x) = Q_0(x) = 1$, and for $m \geq 0$, let

$$P_{m+1}(x) = P_m(x) + e^{i2^m x}Q_m(x) \quad \text{and} \quad Q_{m+1}(x) = P_m(x) - e^{i2^m x}Q_m(x).$$

Next, let $f_{m+1}(x) = P_{m+1}(x) - P_m(x)$ and for each α with $0 < \alpha < 1$, let

$$g_\alpha(x) = \sum_{k=1}^{\infty} 2^{-k(\alpha+\frac{1}{2})} f_k(x).$$

It follows immediately from the definition of g_α that $\hat{g}_\alpha(n) = 0$ if $n \leq 0$ and that $\hat{g}_\alpha(n) = \varepsilon(n)2^{-k(\alpha+\frac{1}{2})}$ if $2^{k-1} \leq n < 2^k$ for some $k \geq 1$ and with $\varepsilon(n) = \pm 1$.

A proof similar to the one given by Katznelson for the case $\alpha = \frac{1}{2}$ yields the following.

THEOREM 3. *For each* α *with* $0 < \alpha < 1$ *we have*

(i) $g_\alpha \in \mathrm{Lip}\,\alpha$ *and* $g_\alpha \notin \mathrm{Lip}\,\gamma$ *for any* $\gamma > \alpha$,

(ii) $g_\alpha \in \alpha^{-1}$-BV,

(iii) $g_\alpha \notin A(2/(2\alpha + 1))$,

(iv) $\sum_{n=1}^{\infty} n^{(2\alpha-1)/2} |\hat{g}_\alpha(n)| = \infty$.

2. Convolution functions

Throughout this section we shall denote the conjugate of a number $p > 1$ by q, that is, $1/p + 1/q = 1$. For $f, g \in L_1(T)$ the convolution $f * g$ is defined by

$$(f * g)(x) = \int_T f(x - t)g(t)dt.$$

Then $(f * g)\hat{}\,(n) = \hat{f}(n)\hat{g}(n)$ for all $n \in Z$. The following theorem is due to M. Riesz (6, Vol. 1, page 251].

THEOREM 4. *A continuous function* f *has an absolutely convergent Fourier series if and only if there exist functions* $g, h \in L_2(T)$ *such that* $f = g * h$.

The next theorem gives a partial extension of this result.

THEOREM 4A. *If* $g, h \in L_p(T)$ *for some* p *with* $1 < p \leq 2$, *then*

$$g * h \in A(p/(2p-2)).$$

Proof. It follows from Young's inequality and the Hausdorff-Young inequality that

$$\sum_{n=-\infty}^{\infty} |\hat{g}(n)|^{q/2}|\hat{h}(n)|^{q/2} \leq \tfrac{1}{2}\sum_{n=-\infty}^{\infty} |\hat{g}(n)|^{q} + \tfrac{1}{2}\sum_{n=-\infty}^{\infty} |\hat{h}(n)|^{q} \leq \tfrac{1}{2}\|g\|_p^q + \tfrac{1}{2}\|h\|_p^q < \infty.$$

that is, $g * h \in A(q/2) = A(p/(2p-2))$.

We next show that Theorem 4A is sharp.

THEOREM 5. *For every* p *with* $1 < p \leq 2$ *there exist functions* $g, h \in L_p(T)$ *such that* $g * h \notin A(\beta)$ *for any* $\beta < p/(2p-2)$.

Proof. We define the functions g and h by

$$\hat{g}(n) = \hat{h}(n) = \begin{cases} (n^{1/q}\log n)^{-1} & \text{if } n > 1, \\ 0 & \text{if } n \leq 1. \end{cases}$$

Clearly, $\hat{g}(n) \searrow 0$ as $n \to \infty$ and

$$\sum_{n=2}^{\infty} (\hat{g}(n))^p n^{p-2} = \sum_{n=2}^{\infty} n^{p-2-p/q}(\log n)^{-p} = \sum_{n=2}^{\infty} n^{-1}(\log n)^{-p} < \infty,$$

because $p > 1$. A theorem due to Hardy and Littlewood [6, Vol. 2, page 129] implies that g, and hence also h, belongs to $L_p(T)$. Furthermore, if $\beta < p/(2p-2)$ then

$$\sum_{n=-\infty}^{\infty} |(g*h)^{\wedge}(n)|^{\beta} = \sum_{n=2}^{\infty} (n^{1/q}\log n)^{-2\beta} = \infty,$$

because $2\beta/q < 1$. Therefore, $g * h \notin A(\beta)$.

THEOREM 6. *If* $g \in L_p(T)$ *with* $1 < p \leq 2$ *and if* $h \in \operatorname{Lip}\alpha$ *with* $0 < \alpha \leq 1$, *then* $g * h \in A(\beta)$ *for all* β *such that* $2p/(2\alpha p + 3p - 2) < \beta$.

Proof. First choose β such that $2p/(2\alpha p + 3p - 2) < \beta < q$. Then Young's inequality implies that

$$\sum_{n=-\infty}^{\infty} |\hat{g}(n)\hat{h}(n)|^{\beta} \leq \frac{\beta}{q}\sum_{n=-\infty}^{\infty} |\hat{g}(n)|^{q} + \frac{q-\beta}{q}\sum_{n=-\infty}^{\infty} |\hat{h}(n)|^{\beta q/(q-\beta)} = A + B.$$

Since $\beta > 2p/(2\alpha p + 3p - 2)$, we have $\beta q/(q - \beta) > 2/(2\alpha + 1)$. Hence, Theorem 1A implies that B is finite. Also, the Hausdorff-Young inequality implies that A is finite. Therefore, $g * h \in A(\beta)$.

Choosing $\beta = 1$ in Theorem 6 we obtain the following corollary. It shows how we can ameliorate functions in $\operatorname{Lip} \alpha$ with $0 < \alpha \leq \frac{1}{2}$, which are not necessarily in $A(1)$, into functions in $A(1)$ by means of the convolution operator.

COROLLARY 1. *Let* $g \in L_p(T)$, $1 < p \leq 2$, *and* $h \in \operatorname{Lip} \alpha$, $0 < \alpha \leq \frac{1}{2}$. *If* $(2\alpha + 1)p > 2$, *then* $g * h \in A(1)$.

We now show to what extent Theorem 6 and Corollary 1 are the best possible.

THEOREM 7. *Let* p *and* α *satisfy the conditions* $1 < p \leq 2$ *and* $0 < \alpha < 1/p$. *Then,* (i) *for all* α_1 *with* $0 < \alpha_1 < \alpha$ *there exist functions* g *and* h *with* $g \in L_p(T)$ *and* $h \in \operatorname{Lip} \alpha_1$ *and such that* $g * h \notin A(2p/(2\alpha p + 3p - 2))$, (ii) *for all* p_1 *with* $1 < p_1 < p$ *there exist functions* g *and* h *with* $g \in L_{p_1}(T)$ *and* $h \in \operatorname{Lip} \alpha$ *and such that* $g * h \notin A(2p/(2\alpha p + 3p - 2))$.

Proof. (i) If γ is defined by $\gamma = \alpha + 1/q$, then $2/(2\gamma + 1) = 2p/(2\alpha p + 3p - 2)$. Let $h = g_{\alpha_1}$, then, according to Theorem 3(i), $h \in \operatorname{Lip} \alpha_1$. Let g be defined by

$$\hat{g}(n) = \begin{cases} 2^{-k(\gamma - \alpha_1)} & \text{if } 2^{k-1} \leq n < 2^k \text{ for some } k \geq 1, \\ 0 & \text{if } n \leq 0. \end{cases}$$

Then $\hat{g}(n) \searrow 0$ as $n \to \infty$ and

$$\sum_{k=1}^{\infty} \hat{g}(n)^p n^{p-2} \leq \sum_{k=0}^{\infty} 2^{k-1} 2^{-k(\gamma - \alpha_1)p} 2^{(k-1)(p-2)} < \infty,$$

because $1 - (\gamma - \alpha_1)p + p - 2 = (\alpha_1 - \alpha)p < 0$. Thus, $g \in L_p(T)$. Furthermore, if $n \in Z$ and if $2^{k-1} \leq n < 2^k$ for some $k \geq 1$, then

$$\hat{g}(n)\hat{h}(n) = 2^{-k(\gamma - \alpha_1)} \varepsilon(n) 2^{-k(\alpha_1 + \frac{1}{2})} = \varepsilon(n) 2^{-k(\gamma + \frac{1}{2})} = \hat{g}_\gamma(n),$$

that is, $g * h = g_\gamma$. Since, according to Theorem 3(iii), $g_\gamma \notin A(2p/(2\alpha p + 3p - 2))$, we have established (i).

(ii) The proof of (ii) is similar to the proof of (i). In this case the functions g and h are chosen as follows. Let $h = g_\alpha$ and let $\hat{g}(n) = 0$ if $n \leq 0$ and let $\hat{g}(n) = 2^{-k(\gamma - \alpha)}$ if $2^{k-1} \leq n < 2^k$ for some $k \geq 0$ and with $\gamma = \alpha + 1/q$. Then it is clear that the functions g and h satisfy the conditions mentioned in (ii).

Remark 1. The following case of Theorem 7 is of special interest. For each p such that $1 < p < 2$ and each α such that $0 < \alpha < (2 - p)/2p$ there exist functions g and h with $g \in L_p(T)$, $h \in \operatorname{Lip} \alpha$ and $g * h \notin A(1)$. This improves a result of M. and S. Izumi [3, Theorem 3] who proved for each p with $1 < p < 2$ and each s with $s > 2$ the existence of functions g in $L_p(T)$ and h in $L_s(T)$ such that $g * h \notin A(1)$.

3. Multipliers of type $(l_p(Z^n), l_p(Z^n))$

In this section we shall define a collection of functions on the n-dimensional torus T^n. We shall use these functions to show that certain results of Hahn [1] about p-multipliers on T^n are the best possible. Furthermore, for $n = 1$ these new functions will be the same as the functions g_α which were defined in Section 1. Throughout this section we shall use the notation $\boldsymbol{x} = (x_0, x_1, \ldots, x_{n-1})$ for $\boldsymbol{x}$ in T^n and $\boldsymbol{m} = (m_0, m_1, \ldots, m_{n-1})$ for $\boldsymbol{m}$ in Z^n.

Definition 2 [2]. A bounded and measurable function f defined on T^n is a p-multiplier, $1 \leq p \leq \infty$, if for every function F in $l_p(Z^n)$, the function $T(f)F$ is again in $l_p(Z^n)$, where $T(f)F$ is defined by

$$T(f)F(\boldsymbol{m}) = \sum_{\boldsymbol{k} \in Z^n} F(\boldsymbol{m} - \boldsymbol{k})\hat{f}(\boldsymbol{k}).$$

The set of p-multipliers will be denoted by M_p.

Definition 3 [1, page 327]. Let α be a positive real number and let α_* be the largest integer less than α. For $1 \leq p \leq \infty$, Lip (α, p) is the class of all functions f defined on T^n such that for $|k| < \alpha_*$ we have $(\partial/\partial x)^k f \in L_p(T^n)$ and for $|k| = \alpha_*$ we have

$$\left\| \Delta_{\boldsymbol{h}} \left(\frac{\partial}{\partial x}\right)^k f \right\|_p = O(|\boldsymbol{h}|^{\alpha-\alpha_*}) \quad \text{if } \alpha - \alpha_* < 1,$$

$$\left\| \Delta_{\boldsymbol{h}}^2 \left(\frac{\partial}{\partial x}\right)^k f \right\|_p = O(|\boldsymbol{h}|) \qquad \text{if } \alpha - \alpha_* = 1,$$

where for each $\boldsymbol{h}$ and $\boldsymbol{x}$ in T^n we set $\Delta_{\boldsymbol{h}} f(\boldsymbol{x}) = f(\boldsymbol{x} + \boldsymbol{h}) - f(\boldsymbol{x})$.

Obviously, if $p_1 \geq p_2$, then Lip $(\alpha, p_1) \subset$ Lip (α, p_2); so, in particular, Lip $(\alpha, \infty) \subset$ Lip (α, p) for all $p \geq 1$ and for all $\alpha > 0$. Hahn proved the following [1, Theorems 12′ and 20].

THEOREM 8.

(a) *If* $1 < p \leq 2$ *and* $\alpha > n/p$, *then* Lip $(\alpha, p) \subset M_r$ *for* $1 \leq r < \infty$.

(b) *If* $p > 2$ *and* $\alpha > n/p$ *then* Lip $(\alpha, p) \subset M_r$ *for* $2p/(p + 2) \leq r \leq$ $\leq 2p/(p - 2)$.

(c) *If* $n/p < \alpha \leq n/2$, *then* Lip $(\alpha, p) \subset M_r$ *for* $2n/(n + 2\alpha) < r <$ $< 2n/(n - 2\alpha)$.

We shall prove that these results are sharp in the sense that for $p \geq 2$ we cannot replace $\alpha > n/p$ by $\alpha \geq n/p$ in Theorem 8(a) and (b), whereas the conclusion of Theorem 8(c) does not hold for $r = 2n/(n+2\alpha)$ or $r = 2n/(n-2\alpha)$. We do not know whether the conclusion of Theorem 8(a) holds if $1 < p < 2$ and $\alpha = n/p$.

THEOREM 9.
(a) *If* $p \geq 2$ *and if* $\alpha = n/p$, *then* $\mathrm{Lip}(\alpha, \infty) \not\subset M_{2p/(p+2)}$; *in particular*, $\mathrm{Lip}(n/2, \infty) \not\subset M_1$.
(b) If $0 < \alpha \leq n/2$, *then* $\mathrm{Lip}(\alpha, \infty) \not\subset M_{2n/(n+2\alpha)}$.

In order to prove Theorem 9 we first define functions $h_\alpha(\boldsymbol{x})$ for each $\alpha > 0$. For convenience we shall write $\tilde{n}$ for 2^n and ω_n for $\exp(2\pi i/\tilde{n})$. For $i = 0, 1, 2, \ldots$ and $l = 0, 1, \ldots, \tilde{n}-1$ we define the trigonometric polynomials $P_{il}(\boldsymbol{x})$ inductively. Let $P_{00}(\boldsymbol{x}) = \ldots = P_{0\tilde{n}-1}(\boldsymbol{x}) = 1$ for all $\boldsymbol{x} \in T^n$. Next, assume that the polynomials $P_{kl}(\boldsymbol{x})$ have been defined for some $k \geq 0$ and all l with $0 \leq l < \tilde{n}$. Each j with $0 \leq j < \tilde{n}$ has a unique representation of the form

$$j = j_0 + 2j_1 + \ldots + 2^{n-1}j_{n-1},$$

with $j_i \in \{0, 1\}$. Let $\boldsymbol{j} = (j_0, \ldots, j_{n-1}) \in Z^n$ and let $\boldsymbol{j} \cdot \boldsymbol{x} = j_0 x_0 + \ldots + j_{n-1}x_{n-1}$. Next, for l with $0 \leq l < \tilde{n}$ we define $P_{k+1l}(\boldsymbol{x})$ by

$$P_{k+1l}(\boldsymbol{x}) = \sum_{j=0}^{\tilde{n}-1} \omega_n^{lj} e^{i\boldsymbol{j}\cdot\boldsymbol{x}2^k} P_{kj}(\boldsymbol{x}).$$

Since

$$\sum_{j=0}^{\tilde{n}-1} \omega_n^{lj} = \begin{cases} \tilde{n} \text{ if } l = 0, \\ 0 \text{ if } l = 1, 2, \ldots, \tilde{n}-1, \end{cases}$$

we have for arbitrary complex numbers $c_0, \ldots, c_{\tilde{n}-1}$

$$\sum_{l=0}^{\tilde{n}-1} \Big|\sum_{j=0}^{\tilde{n}-1} c_j \omega_n^{lj}\Big|^2 = \tilde{n} \sum_{j=0}^{\tilde{n}-1} |c_j|^2.$$

Therefore,

$$\sum_{l=0}^{\tilde{n}-1} |P_{kl}(\boldsymbol{x})|^2 = \sum_{l=0}^{\tilde{n}-1} \Big|\sum_{j=0}^{\tilde{n}-1} \omega_n^{lj} e^{i\boldsymbol{j}\cdot\boldsymbol{x}2^{k-1}} P_{k-1j}(\boldsymbol{x})\Big|^2 = \tilde{n} \sum_{j=0}^{\tilde{n}-1} |P_{k-1j}(\boldsymbol{x})|^2 = \tilde{n}^{k+1}.$$

Hence, for each $k \geq 0$ we have

$$\|P_{k0}(\boldsymbol{x})\|_\infty \leq \tilde{n}^{(k+1)/2}.$$

Also, $|\hat{P}_{k0}(\boldsymbol{m})| = 1$ if $\boldsymbol{m} = (m_0, \ldots, m_{n-1})$ with $0 \leq m_i < 2^k$ for $i = 0, 1, \ldots, n-1$, and $P_{k0}(\boldsymbol{m}) = 0$ otherwise. For $k \geq 1$ let $f_k(\boldsymbol{x}) = P_{k0}(\boldsymbol{x}) - P_{k-10}(\boldsymbol{x})$, and for $\alpha > 0$ let

$$h_\alpha(\boldsymbol{x}) = \sum_{k=1}^{\infty} \tilde{n}^{-k\left(\frac{\alpha}{n}+\frac{1}{2}\right)} f_k(\boldsymbol{x}).$$

We can show that $h_\alpha \in \mathrm{Lip}\,(\alpha, \infty)$. The proof requires a long and tedious computation which we shall omit. We only observe that we need an n-dimensional version of Bernstein's inequality: if f is a trigonometric polynomial on T^n of degree k, that is,

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{j} \in Z^n} c_{\boldsymbol{j}} e^{i\boldsymbol{j}\cdot\boldsymbol{x}},$$

with $\max_{\boldsymbol{j}} (|j_0| + |j_1| + \ldots + |j_{n-1}|) = k$, then for each of the first order partial derivatives of f we have

$$\left\| \frac{\partial f}{\partial x_i} \right\|_\infty \leq k \, \|f\|_\infty.$$

Proof of Theorem 9. (a) Consider the function F which is defined on Z^n by $F(\boldsymbol{0}) = F(0, \ldots, 0) = 1$ and $F(\boldsymbol{m}) = 0$ for $\boldsymbol{m} \neq \boldsymbol{0}$ and $\boldsymbol{m} \in Z^n$. Clearly, $F \in l_p(Z^n)$. We shall prove that $h_{n/p} \notin M_{2p/(p+2)}$. For each $\boldsymbol{m} \in Z^n$ we have

$$T(h_{n/p})F(\boldsymbol{m}) = \sum_{\boldsymbol{k} \in Z^n} F(\boldsymbol{m} - \boldsymbol{k})\hat{h}_{n/p}(\boldsymbol{k}) = \hat{h}_{n/p}(\boldsymbol{m}).$$

Also,

$$\sum_{\boldsymbol{k} \in Z^n} |\hat{h}_{n/p}(\boldsymbol{k})|^{2p/(p+2)} = \sum_{k=1}^{\infty} (2^{kn} - 2^{(k-1)n})2^{-nk(p+2)/2p \cdot 2p/(p+2)}$$

$$\geq \tfrac{1}{2} \sum_{k=1}^{\infty} 2^{kn}2^{-kn} = \infty,$$

that is, $T(h_{n/p})F \notin l_{2p/(p+2)}$. Therefore, $h_{n/p} \notin M_{2p/(p+2)}$. Since $M_{2p/(p+2)} = M_{2p/(p-2)}$ we also have $h_{n/p} \notin M_{2p/(p-2)}$.

(b) For each α such that $0 < \alpha \leq n/2$ we have

$$\sum_{\boldsymbol{k} \in Z^n} |\hat{h}_\alpha(\boldsymbol{k})|^{2n/(n+2\alpha)} \geq \tfrac{1}{2} \sum_{k=1}^{\infty} 2^{kn} 2^{-nk(2\alpha+n)/2n \cdot 2n/(2\alpha+n)} = \infty.$$

Therefore, an argument as in (a) shows that $h_\alpha \notin M_{2n/(n+2\alpha)}$, and hence also, $h_\alpha \notin M_{2n/(n-2\alpha)}$.

Remark 2. For each n the function $h_{n/2}$ provides an example of a function in $\mathrm{Lip}\,(n/2, \infty)$ which does not have an absolutely convergent Fourier series. Hence the n-dimensional version of Theorem 1 is sharp. This and related results were established by Wainger [5].

Remark 3. The functions g_α as defined in Section 1 also provide new examples that show that several of the results of Hirschman on p-multipliers cannot be improved as was already pointed out by Hirschman in [2].

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