Ark. Mat., 40 (2002), 363–382

Sobolev spaces in several variables in L^1-type norms are not isomorphic to Banach lattices

Aleksander Pełczyński(1) and Michał Wojciechowski(1)

Abstract. A Sobolev space in several variables in an L^1-type norm is not complemented in its second dual. Hence it is not isomorphic as a Banach space to any complemented subspace of a Banach lattice.

1. Introduction and results

K. Borsuk essentially observed that Sobolev spaces in one variable are isomorphic as Banach spaces to the corresponding classical Banach spaces $L^p=L^p(0,1)$ and $C=C(0,1)$ (cf. [B], [PS] and [PW2]). The situation is more involved for Sobolev spaces in several variables. For the definition of Sobolev spaces see Section 2. If $1<p<\infty$ then, under mild conditions imposed on the domain $\Omega\subset\mathbf{R}^n$, the Sobolev space $L^p_{(k)}(\Omega)$ is still isomorphic to $L^p(0,1)$ for $k=1,2,\dots;\ n=2,3,\dots$ (cf. [PS] and [PW1]). However this is not true in the limit cases $p=1$, $p=\infty$ and $C^{(k)}(\Omega)$. Assume that

$$(*)\qquad \Omega\subset\mathbf{R}^n \text{ is a non-empty open set, } k=1,2,\dots;\ n=2,3,\dots\ .$$

Then the spaces $C^{(k)}(\Omega)$ and $L^\infty_{(k)}(\Omega)$ have the following properties: (a) they are not isomorphic to quotients of $\mathcal{L}_\infty$-spaces, in particular they are not isomorphic to C and L^∞, respectively; (b) they fail to have lust (cf. [DJP], p. 345 for the definition), in particular they are not isomorphic to any complemented subspaces of Banach lattices (cf. [Gr], [H], [K2], [KwP] and [PW2]). S. V. Kislyakov [K1] (cf. also [PW2]) has discovered that if Ω, k and n satisfy $(*)$ then there exist bounded non-two-summing operators from the Sobolev spaces $L^1_{(k)}(\Omega)$ and $\mathrm{BV}_{(k)}(\Omega)$ into a Hilbert space. Hence these Sobolev spaces as Banach spaces are not isomorphic to $\mathcal{L}_1$-spaces, in particular not to the Lebesgue spaces $L^1(\mu)$.

---

(1) Both authors were supported in part by the Polish KBN grant 2P03A 036 14.

In the present paper we establish another invariant which distinguish the spaces $L^1_{(k)}(\Omega)$ from the Lebesgue spaces $L^1(\mu)$. Our main result is the following theorem.

Theorem 1.1. *If Ω, k and n satisfy $(*)$ then the space $L^1_{(k)}(\Omega)$ is uncomplemented in its second dual.*

We identify here and in the sequel a Banach space X with its canonical image in its second dual X^{**}.

Note that $L^1_{(k)}(\Omega)$, being isomorphic to a subspace of $L^1(\mu)$ for appropriate μ (cf., e.g. [PW2]), contains no subspace isomorphic to c_0. Thus combining several facts on Banach lattices ([LT], vol. II, Proposition 1.c.6; Proposition 1.a.11 and Theorem 1.b.16) with Theorem 1.1 we get the following corollary.

Corollary 1.2. *If Ω, k and n satisfy $(*)$ then $L^1_{(k)}(\Omega)$ is not isomorphic to any complemented subspace of any Banach lattice.*

The space $\mathrm{BV}_{(k)}(\Omega)$ is a dual Banach space (cf. Section 6). Thus $\mathrm{BV}_{(k)}(\Omega)$ is complemented in its second dual (cf. [D]). On the other hand, the canonical image of a Banach space X is complemented in X^{**} if and only if X is isomorphic to a complemented subspace of a dual Banach space (cf. [L], p. 540). Thus Theorem 1.1 yields the following corollary.

Corollary 1.3. *If Ω, k and n satisfy $(*)$ then $L^1_{(k)}(\Omega)$ is uncomplemented in $\mathrm{BV}_{(k)}(\Omega)$.*

The Lebesgue decomposition provides the natural projection from the space $M(\Omega)$ of all scalar-valued Borel measures on Ω with finite variation onto $L^1(\Omega)$—the Lebesgue space on Ω with respect to the n-dimensional Lebesgue measure. Thus Corollary 1.3 roughly speaking says that there is no counterpart of the Lebesgue decomposition theorem for Sobolev measures.

Our proof of Theorem 1.1—presented in Section 4—uses the technique developed in [KaP] for translation invariant subspaces of $L^1(G)$ on a compact abelian group G spanned by the complement of a Sidon set. The method goes back to Lindenstrauss [L]. On the "abstract side", our proof uses the *Lindenstrauss lifting principle* (see Section 3, and [KaP]). The main analytic tool is Peetre's theorem (cf. [P]) on the non-existence of the right inverse for the Gagliardo trace (cf. [G]). In Section 5 we present an alternative proof of Peetre's theorem. Section 6 contains some facts about the space of Sobolev measures $\mathrm{BV}_{(k)}(\Omega)$. Using the fact that the latter space is a dual space, we outline a variant of the proof of Theorem 1.1 which does not use the Lindenstrauss lifting principle.

2. Preliminaries and notation

By $\partial^\alpha f$ and $D^\alpha f$ we denote the αth partial derivative and the αth distributional partial derivative of a scalar-valued function f in n variables corresponding to the multiindex $\alpha\in\mathbf{Z}_+^n:=\prod_{j=1}^n\{\{0\}\cup\mathbf{N}\}$, where $\mathbf{N}:=\{1,2,\ldots\}$. For open $\Omega\subset\mathbf{R}^n$ we denote by $\mathcal{D}(\Omega)$ the space of infinitely differentiable scalar-valued functions ϕ on Ω with compact support,

$$\operatorname{supp}\phi=\overline{\{x\in\Omega:\phi(x)\neq 0\}}\subset\Omega.$$

Here and in the sequel $\bar{A}$ stands for the closure of a subset A of $\mathbf{R}^n$ and $\operatorname{bd}A:=\bar{A}\setminus A$ stands for the boundary of an open set A. Recall that given a scalar-valued function f defined on an open set $\Omega\subset\mathbf{R}^n$ (respectively a Borel measure μ on Ω) a function g on Ω (respectively a measure ν on Ω) is called the αth *distributional derivative* of f (respectively of μ), in symbols $g:=D^\alpha f$ (respectively $\nu=D^\alpha\mu$) provided

$$\int_\Omega f\partial^\alpha\phi\,dx=(-1)^{|\alpha|}\int_\Omega g\phi\,dx\quad\left(\text{respectively }\int_\Omega\partial^\alpha\phi\,d\mu=(-1)^{|\alpha|}\int_\Omega\phi\,d\nu\right)$$

for $\phi\in\mathcal{D}(\Omega)$. For the multiindex $\alpha=(\alpha_j)_{j=1}^n$ the quantity $|\alpha|:=\sum_{j=1}^n\alpha_j$ is called the order of the derivative D^α. (We also denote by $|x|$ the absolute value of a scalar x; it will be clear from the context each time which of these we have in mind). For $\alpha=0:=(0,\ldots,0)$ we admit for convenience $D^0f=f$ and $D^0\mu=\mu$. The symbol $\int\ldots dx$ denotes the integral against λ_n—the n-dimensional Lebesgue measure on $\mathbf{R}^n$. By $L^p(\Omega)$ we denote the Lebesgue space L^p on $\Omega\subset\mathbf{R}^n$ with respect to λ_n. By $M(\Omega)$ we denote the space of all scalar-valued Borel measures on Ω with bounded total variation, with the total variation (of the measure) of Ω as the norm. The field of scalars is either the real numbers—$\mathbf{R}$—, or the complex numbers—$\mathbf{C}$. Our definition of spaces of continuous functions is "unorthodox". For open $\Omega\subset\mathbf{R}^n$, by $C(\Omega)$ we denote the space of uniformly continuous scalar-valued functions on Ω which vanishes at infinity. (The latter condition is meaningful only for unbounded Ω.) Each $f\in C(\Omega)$ uniquely extends to $\bar{\Omega}$; we shall identify f with its extension to $\bar{\Omega}$. We equip $C(\Omega)$ with the usual sup norm, $\|f\|_{C(\Omega)}=\sup_{x\in\Omega}|f(x)|$.

Let $1\le p\le\infty$, $k=0,1,\ldots$, and $n=1,2,\ldots$. The Sobolev space $L^p_{(k)}(\Omega)$ is the Banach space of scalar-valued functions f on open $\Omega\subset\mathbf{R}^n$ such that $D^\alpha f$ exists and belongs to $L^p(\Omega)$ for $|\alpha|\le k$, equipped with the norm

$$\|f\|_{L^p_{(k)}(\Omega)}=\begin{cases}\left(\sum_{|\alpha|\le k}\|D^\alpha f\|^p_{L^p(\Omega)}\right)^{1/p} & \text{for } 1\le p<\infty,\\ \max_{|\alpha|\le k}\|D^\alpha f\|_{L^\infty(\Omega)} & \text{for } p=\infty.\end{cases}$$

By $\mathrm{BV}_{(k)}(\Omega)$ we denote the space of measures $\mu \in M(\Omega)$ such that $D^\alpha \mu$ exists and belongs to $M(\Omega)$ for $|\alpha| \leq k$, equipped with the norm

$$\|\mu\|_{\mathrm{BV}_{(k)}(\Omega)} = \sum_{|\alpha| \leq k} \|D^\alpha \mu\|_{M(\Omega)}.$$

Note that $L^1_{(k)}(\Omega)$ can be identified with a subspace of $\mathrm{BV}_{(k)}(\Omega)$ consisting of μ with all distributional derivatives of order $\leq k$ absolutely continuous with respect to λ_n. If $\mu \in \mathrm{BV}_{(k)}(\Omega)$ then $D^\alpha \mu$ is absolutely continuous with respect to λ_n for $|\alpha| < k$. For $k=1$ this follows by convolving μ with a C^∞ approximate identity, and applying the Sobolev embedding theorem (cf. [M], Theorem 1.4.3); the case $k>1$ follows by induction.

By $C^{(k)}(\Omega)$ we denote the space of all scalar-valued functions f on Ω with derivatives $\partial^\alpha f \in C(\Omega)$ for $|\alpha| \leq k$, equipped with the norm

$$\|f\|_{C^{(k)}(\Omega)} = \max_{|\alpha| \leq k} \|\partial^\alpha f\|_{C(\Omega)}.$$

By $C_0^{(k)}(\Omega)$ we denote the subspace of $C^{(k)}(\Omega)$ being the closure of $\mathcal{D}(\Omega)$ in the norm $\|\cdot\|_{C^{(k)}(\Omega)}$. Clearly $C^{(k)}(\Omega)$ can be regarded as a subspace of $L^\infty_{(k)}(\Omega)$; we have $L^\infty_{(k)}(\Omega) \supset C^{(k)}(\Omega) \supset C_0^{(k)}(\Omega)$.

Warning: According to our definition of $C(\Omega)$ we have $C^{(k)}(\mathbf{R}^n) = C_0^{(k)}(\mathbf{R}^n)$.

By $\mathcal{D}(\bar{\Omega})$ we denote the space of functions being the restrictions to Ω of functions from $\mathcal{D}(\mathbf{R}^n)$. Recall (cf. [A], Chapter 3, Theorem 3.18) the following lemma.

Density lemma 2.1. *The space $\mathcal{D}(\bar{\Omega})$ is dense in the space $L^p_{(k)}(\Omega)$ in the norm $\|\cdot\|_{L^p_{(k)}(\Omega)}$ for $1 \leq p < \infty$, and in the space $C^{(k)}(\Omega)$ in the norm $\|\cdot\|_{C^{(k)}(\Omega)}$, provided Ω has the* segment property, *i.e. every $x \in \operatorname{bd} \Omega$ has an open neighborhood U_x in $\mathbf{R}^n$ and there exists a non-zero vector y_x such that for every $z \in \bar{\Omega} \cap U_x$ one has $z + t y_x \in \Omega$ for $0 < t < 1$.*

3. Preparation for the proof of Theorem 1.1

3.1. Lindenstrauss lifting principle

Recall (cf., e.g. [KaP] for the proof) the following principle.

Lindenstrauss lifting principle. *Let $Q: X \to Y$ be a bounded linear surjection (X, Y Banach spaces). Assume that* $\ker Q$ *is complemented in its second dual. Then for every $\mathcal{L}^1$-space L, every operator $T: L \to Y$ admits a lifting, i.e. there is an operator $\widetilde{T}: L \to X$ such that $T = Q\widetilde{T}$.*

Recall that an operator $S: Y \to X$ is a *right inverse* of an operator $Q: X \to Y$ provided $QS = \mathrm{Id}_Y$, where Id_Y denotes the identity on Y. Clearly an operator which has a right inverse is a surjection. Specifying in the Lindenstrauss lifting principle $L = Y = L^1$ and $T = \mathrm{Id}_{L^1}$ we get the following corollary.

Corollary 3.1. *If $Q: X \to L^1$ admits no right inverse then* $\ker Q$ *is uncomplemented in* $(\ker Q)^{**}$.

3.2. A priori inequalities related to traces

We need some properties of traces of functions in $L^1_{(k)}(\Omega)$ in the particular case where Ω is a halfspace. Representing $\mathbf{R}^n = \mathbf{R}^{n-1} \times \mathbf{R}$ for $x \in \mathbf{R}^n$ we write $x = (y, x_n)$ with $y \in \mathbf{R}^{n-1}$ and $x_n \in \mathbf{R}$. We identify $\mathbf{R}^{n-1}$ with the hyperplane $\{(y, x_n) \in \mathbf{R}^n: x_n = 0\}$. We put

$$\mathbf{R}^n_- := \{(y, x_n) \in \mathbf{R}^n : x_n < 0\} \quad \text{and} \quad \mathbf{R}^n_+ := \{(y, x_n) \in \mathbf{R}^n : x_n > 0\}.$$

Each $\phi \in \mathcal{D}(\overline{\mathbf{R}}^n_-)$ uniquely extends onto $\overline{\mathbf{R}}^n_-$, hence it can be regarded as infinitely differentiable function on $\overline{\mathbf{R}}^n_-$. In particular the restriction $\partial^\alpha \phi|_{\mathbf{R}^{n-1}}$ is well defined for each multiindex $\alpha \in \mathbf{Z}^n_+$. For brevity we put $\phi^{(0,s)} := \partial^{(0,\dots,0,s)} \phi$ and $\partial^{(\beta,s)} := \partial^{(\beta_1,\dots,\beta_{n-1},s)}$ for $\beta = (\beta_j)_{j=1}^{n-1} \in \mathbf{Z}^{n-1}_+$ and for $s \in \mathbf{Z}_+$. To define a surjective trace onto $L^1(\mathbf{R}^{n-1})$ we need two a priori inequalities. The first is well known (cf. e.g., [Ko], proof of Proposition 3.2).

Lemma 3.2. *Let $k = 1, 2, \dots$; $n = 2, 3, \dots$. Let $\phi \in \mathcal{D}(\overline{\mathbf{R}}^n_-)$. Then*

$$\|\phi^{(0,k-1)}|_{\mathbf{R}^{n-1}}\|_{L^1(\mathbf{R}^{n-1})} \le \|\phi\|_{L^1_{(k)}(\mathbf{R}^n_-)}. \tag{3.1}$$

Proof. We have $\phi^{(0,k-1)}(y,0) = \int_{-\infty}^0 \phi^{(0,k)}(y,s)\,ds$. Thus

$$\begin{aligned} \|\phi^{(0,k-1)}\|_{L^1(\mathbf{R}^{n-1})} &= \int_{\mathbf{R}^{n-1}} \left| \int_{-\infty}^0 \phi^{(0,k)}(y,s)\,ds \right| dy \\ &\le \iint_{\mathbf{R}^n_-} |\phi^{(0,k)}(y,s)|\,dy\,ds \le \|\phi\|_{L^1_{(k)}(\mathbf{R}^n_-)}. \quad \square \end{aligned}$$

The second lemma insures the surjectivity of the trace.

Lemma 3.3. *Let $k=1,2,\dots$; $n=2,3,\dots$. Then there exists $C=C(k)$ such that for every $g\in\mathcal{D}(\mathbf{R}^{n-1})$ there exists $\phi\in\mathcal{D}(\overline{\mathbf{R}}^n_-)$ such that*

$$\phi^{(0,k-1)}|_{\mathbf{R}^{n-1}}=g; \tag{3.2}$$

$$\text{if } k>1 \text{ then } \phi^{(0,s)}(\,\cdot\,,0)\equiv 0 \quad \text{for } s=0,\dots,k-2; \tag{3.3}$$

$$\|\phi\|_{L^1_{(k)}(\mathbf{R}^n_-)}\le C\|g\|_{L^1(\mathbf{R}^{n-1})}. \tag{3.4}$$

Proof. If $g=0$ we put $\phi=0$. If $g\neq 0$ then $0<c=\max\{\|\partial^\beta g\|_{L^1(\mathbf{R}^{n-1})}:|\beta|\le k\}$.

Pick $h\in\mathcal{D}(\mathbf{R})$ such that $h^{(k-1)}(0)=1$ and if $k>1$ then $h^{(s)}(0)=0$ for $s=0,\dots,k-2$. For sufficiently small $t>0$, which will be chosen later, we put

$$\phi(y,x_n):=t^{k-1}h\Big(\frac{x_n}{t}\Big)g(y),\quad y\in\mathbf{R}^{n-1},\ x_n\le 0.$$

Then for $s=0,1,\dots$, and for $\beta\in\mathbf{Z}^{n-1}_+$ one has

$$\phi^{(\beta,s)}(y,x_n)=t^{k-1-s}\partial^\beta g(y)h^{(s)}\Big(\frac{x_n}{t}\Big),\quad y\in\mathbf{R}^{n-1},\ x_n\le 0. \tag{3.5}$$

In particular for $s=k-1$, $\beta=0:=(0,\dots,0)$, $x_n=0$ we get (3.2). Similarly (3.5) yields (3.3). Specifying in (3.5), $s=k$ and $\beta=0$ we get

$$\begin{aligned}\int_{-\infty}^0\int_{\mathbf{R}^{n-1}}|\phi^{(0,k)}(y,x_n)|\,dx_n\,dy&=\int_{-\infty}^0\Big|h^{(k)}\Big(\frac{x_n}{t}\Big)\Big|\frac{dx_n}{t}\int_{\mathbf{R}^{n-1}}|g(y)|\,dy\\&=\int_{-\infty}^0|h^{(k)}(x_n)|\,dx_n\|g\|_{L^1(\mathbf{R}^{n-1})}.\end{aligned}$$

If $0\le s<k$ and $|\beta|\le k-s$ then, for $0<t<1$, (3.5) implies

$$\begin{aligned}\int_{-\infty}^0\int_{\mathbf{R}^{n-1}}|\phi^{(\beta,s)}(y,x_n)|\,dx_n\,dy&=t^{k-s}\int_{-\infty}^0\Big|h^{(s)}\Big(\frac{x_n}{t}\Big)\Big|\frac{dx_n}{t}\int_{\mathbf{R}^{n-1}}|\partial^\beta g(y)|\,dy\\&=t^{k-s}\int_{-\infty}^0|h^{(s)}(x_n)|\,dx_n\|\partial^\beta g\|_{L^1(\mathbf{R}^{n-1})}\\&\le t\int_{-\infty}^0|h^{(s)}(x_n)|\,dx_n\|\partial^\beta g\|_{L^1(\mathbf{R}^{n-1})}.\end{aligned}$$

Thus setting

$$t=\Big(cN\max_{0\le s\le k}\int_{-\infty}^0|h^{(s)}(x_n)|\,dx_n\Big)^{-1}\|g\|_{L^1(\mathbf{R}^{n-1})},$$

where N is the number of elements of the set of multiindices, $\{(\beta,s):|\beta|+s\le k\}$, we get (3.4) with $C=\int_{-\infty}^0|h^{(k)}(x_n)|\,dx_n+1$. $\square$

By the density lemma 2.1, $\mathcal{D}(\overline{\mathbf{R}}^n_-)$ is a dense linear manifold in $L^1_{(k)}(\mathbf{R}^n_-)$ and $\mathcal{D}(\mathbf{R}^{n-1})$ is dense in $L^1(\mathbf{R}^{n-1})$.

Thus Lemmas 3.2 and 3.3 imply the following result (cf. Gagliardo [G]).

Corollary 3.4. *Let $k=1,2,\dots$, and $n=2,3,\dots$. Then there is a surjection* $\mathrm{Tr}^{(k)}: L^1_{(k)}(\mathbf{R}^n_-)\to L^1(\mathbf{R}^{n-1})$ *which is the unique continuous extension of the map* $\phi\mapsto\phi^{(0,k-1)}|_{\mathbf{R}^{n-1}}$.

3.3. Peetre's theorem on non-existence of right inverses of some traces

The next result discovered by Jaak Peetre [P] provides the crucial ingredient of the proof of Theorem 1.1.

Peetre's theorem 3.5. *There is no right inverse of* $\mathrm{Tr}^{(1)}: L^1_{(1)}(\mathbf{R}^n_-)\to L^1(\mathbf{R}^{n-1})$ *for* $n=2,3,\dots$.

In Section 5 we present a proof of Peetre's theorem and some of its counterparts for the n-dimensional torus. Here we state only a simple consequence of it which we need for the proof of Theorem 1.1.

Corollary 3.6. *Let $k=1,2,\dots$; $n=2,3,\dots$. Let X be a subspace of $L^1_{(k)}(\mathbf{R}^n_-)$ such that $\mathrm{Tr}^{(k)}$ maps X onto $L^1(\mathbf{R}^{n-1})$. Then $\mathrm{Tr}^{(k)}|_X$ admits no right inverse.*

Proof. If $S: L^1(\mathbf{R}^{n-1})\to X$ were a right inverse of $\mathrm{Tr}^{(k)}|_X$ then $I_{(k)}J_XS$ would be a right inverse of $\mathrm{Tr}^{(1)}$. Here $J_X: X\to L^1_{(k)}(\mathbf{R}^n_-)$ denotes the inclusion map and $I_{(k)}: L^1_{(k)}(\mathbf{R}^n_-)\to L^1_{(1)}(\mathbf{R}^n_-)$ is defined by $I_{(k)}(f)=D^{(0,\dots,0,k-1)}f$ for $f\in L^1_{(k)}(\mathbf{R}^n_-)$. □

4. Proof of Theorem 1.1

We begin with the special case $\Omega=\mathbf{R}^n$.

Proposition 4.1. *The space $L^1_{(k)}(\mathbf{R}^n)$ is uncomplemented in its second dual.*

Proof. It suffices to exhibit a complemented subspace of $L^1_{(k)}(\mathbf{R}^n)$ which is uncomplemented in its second dual. Let ${}^eL^1_{(k)}(\mathbf{R}^n)$ (respectively ${}^oL^1_{(k)}(\mathbf{R}^n)$) denote the subspaces of $L^1_{(k)}(\mathbf{R}^n)$ consisting of the functions which are even (respectively odd) with respect to the variable x_n. The subspace is complemented in $L^1_{(k)}(\mathbf{R}^n)$ via the projection $f\mapsto{}^ef$ (respectively $f\mapsto{}^of$), where ${}^ef(y,x_n)=\frac12[f(y,x_n)+f(y,-x_n)]$ (respectively ${}^of(y,x_n)=\frac12[f(y,x_n)-f(y,-x_n])$ for a.e. $(y,x_n)\in\mathbf{R}^{n-1}\times\mathbf{R}$.

Next we define a subspace X_k of $L^1_{(k)}(\mathbf{R}^n_-)$ by

$$X_k=\begin{cases} L^1_{(k)}(\mathbf{R}^n_-) & \text{for } k=1,2,\\ \mathrm{cl}\{\phi\in\mathcal{D}(\overline{\mathbf{R}}^n_-):\phi^{(0,2s)}(\,\cdot\,,0)\equiv0 \text{ for } 0\le s<\frac12(k-1)\} & \text{for } k=3,5,\dots,\\ \mathrm{cl}\{\phi\in\mathcal{D}(\overline{\mathbf{R}}^n_-):\phi^{(0,2s-1)}(\,\cdot\,,0)\equiv0 \text{ for } 1\le s<\frac12 k\} & \text{for } k=4,6,\dots.\end{cases}$$

Here *cl* stands for the closure in the norm $\|\cdot\|_{L^1_{(k)}(\mathbf{R}^n_-)}$.

It follows from Lemmas 3.2 and 3.3 that $\mathrm{Tr}^{(k)}|_{X_k}: X_k \to L^1(\mathbf{R}^{n-1})$ is a surjection which, by Corollary 3.6, has no right inverse. Hence, by Corollary 3.1, $\ker(\mathrm{Tr}^{(k)}|_{X_k})$ is uncomplemented in its second dual. To complete the proof we shall show that (here $\sim$ stands for "is isomorphic to")

$$\ker(\mathrm{Tr}^{(k)}|_{X_k}) \sim \begin{cases} {}^oL^1_{(k)}(\mathbf{R}^n) & \text{for } k \text{ odd}, \\ {}^eL^1_{(k)}(\mathbf{R}^n) & \text{for } k \text{ even}. \end{cases} \tag{4.1}$$

We verify (4.1). First consider the case k odd. Pick $f \in \ker(\mathrm{Tr}^{(k)}|_{X_k})$. Then there exists a sequence $(\phi_l)_{l\in\mathbf{N}} \subset X_k \cap \mathcal{D}(\overline{\mathbf{R}}^n_-)$ such that $\lim_{l\to\infty} \|f-\phi_l\|_{L^1_{(k)}(\mathbf{R}^n_-)}=0$. The existence of such a sequence for $k=1$ is a consequence of the density lemma 2.1, and for $k=3,5,\dots$, it follows from the definition of X_k. Thus remembering that $f \in \ker(\mathrm{Tr}^{(k)}|_{X_k})$, we have

$$\lim_{l\to\infty} \|\phi_l^{(0,k-1)}(\,\cdot\,,0)\|_{L^1(\mathbf{R}^{n-1})} = \lim_{l\to\infty} \|\,\mathrm{Tr}^{(k)}(f-\phi_l)\|_{L^1(\mathbf{R}^{n-1})} = 0.$$

By Lemma 3.3, there exists a sequence $(\psi_l)_{l\in\mathbf{N}} \in \mathcal{D}(\overline{\mathbf{R}}^n_-)$ such that, for $l=1,2,\dots$,

$$\|\psi_l\|_{L^1_{(k)}(\mathbf{R}^n_-)} \le C\|\,\mathrm{Tr}^{(k)}(\phi_l)\|_{L^1(\mathbf{R}^{n-1})}; \quad \mathrm{Tr}^{(k)}(\psi_l) = \mathrm{Tr}^{(k)}(\phi_l);$$
$$\text{if } k>1 \text{ then } \psi_l^{(0,s)}(\,\cdot\,,x_n) \equiv 0 \text{ for } s=0,\dots,k-2.$$

In particular

$$\psi_l \in X_k \text{ and } \phi_l - \psi_l \in \ker(\mathrm{Tr}^{(k)}|_{X_k}) \text{ for } l=1,2,\dots, \quad \lim_{l\to\infty} \|\psi_l\|_{L^1_{(k)}(\mathbf{R}^n_-)} = 0.$$

Put $f_l = \phi_l - \psi_l$ and define $\tilde{f}_l: \mathbf{R}^n \to \mathbf{C}$ by

$$\tilde{f}_l(y,x_n) = \begin{cases} f_l(y,x_n) & \text{for } x_n \le 0, \\ -f_l(y,-x_n) & \text{for } x_n > 0. \end{cases} \tag{4.2}$$

Since f_l belong to $\mathcal{D}(\overline{\mathbf{R}}^n_-)$ and have pure right derivatives with respect to x_n of even order $\le k-1$ identically equal 0 on the hyperplane $x_n=0$, we easily verify that $\tilde{f}_l$ is odd with respect to x_n, belongs to $C^{(k)}(\mathbf{R}^n)$ and has compact support. Thus $\tilde{f}_l \in {}^oL^1_{(k)}(\mathbf{R}^n)$. Clearly, by (4.2),

$$\|\tilde{f}_l - \tilde{f}_{l'}\|_{L^1_{(k)}(\mathbf{R}^n)} \le 2\|f_l - f_{l'}\|_{L^1_{(k)}(\mathbf{R}^n_-)} \quad \text{for } l,l' \in \mathbf{N}.$$

Since $\lim_{l\to\infty} \|f_l - f\|_{L^1_{(k)}(\mathbf{R}^n_-)}=0$, we infer that the sequence $(\tilde{f}_l)_{l\in\mathbf{N}}$ is a Cauchy sequence in $L^1_{(k)}(\mathbf{R}^n)$. Thus there is $\tilde{f} \in {}^oL^1_{(k)}(\mathbf{R}^n)$ so that $\lim_{l\to\infty} \|\tilde{f}_l - \tilde{f}\|_{L^1_{(k)}(\mathbf{R}^n)}=0$.

Clearly $\|f\|_{L^1_{(k)}(\mathbf{R}^n_-)} \le \|\tilde{f}\|_{L^1_{(k)}(\mathbf{R}^n)} \le 2\|f\|_{L^1_{(k)}(\mathbf{R}^n_-)}$. It is easy to verify that the map $f \mapsto \tilde{f}$ is linear. Hence it is an isomorphism from $\ker(\mathrm{Tr}^{(k)}|_{X_k})$ into ${}^oL^1_{(k)}(\mathbf{R}^n)$. To complete the proof in the case k is odd, we show that the map $f \mapsto \tilde{f}$ is onto ${}^oL^1_{(k)}(\mathbf{R}^n)$.

Let $(\Phi_l)_{l\in\mathbf{N}} \subset C_0^{(k)}(\mathbf{R}^n) \cap {}^oL^1_{(k)}(\mathbf{R}^n)$ be an approximate identity for $L^1(\mathbf{R}^n)$ consisting of functions symmetric with respect to the coordinate x_n. Then, for each $g \in {}^oL^1_{(k)}(\mathbf{R}^n)$, the convolution $g*\Phi_l \in C_0^{(k)}(\mathbf{R}^n) \cap {}^oL^1_{(k)}(\mathbf{R}^n)$. Thus $(g*\Phi_l)^{(0,2s)}(\cdot,0) \equiv 0$ for $s=0,\dots,\frac{1}{2}(k-1)$. Moreover $\lim_{l\to\infty} \|g-g*\Phi_l\|_{L^1_{(k)}(\mathbf{R}^n)}=0$. Pick $\eta \in \mathcal{D}(\mathbf{R}^n)$ such that $\eta(y,x_n)$ depends only on y for $|x_n|<1$ and $\eta(0,0)=1$. Let $\eta_n(x)=\eta(x/n)$ for $x \in \mathbf{R}^n$; $n \in \mathbf{N}$. Put $g_l=(g*\Phi_l)\eta_{n_l}$. If the sequence $(n_l)_{l\in\mathbf{N}} \subset \mathbf{N}$ sufficiently rapidly tends to infinity then $\lim_{l\to\infty} \|g-g_l\|_{L^1_{(k)}(\mathbf{R}^n)}=0$ (cf. [S], Chapter V, §2, proof of Proposition 1 for details). Obviously $g_l \in \mathcal{D}(\mathbf{R}^n)$, and $g_l^{(0,2s)}(\cdot,0) \equiv 0$ because in the neighborhood of $\mathbf{R}^{n-1}$ the function η_{n_l} depends on the variable y only. Let $\phi_l := g_l|_{\mathbf{R}^n_-}$. Then $\phi_l \in \mathcal{D}(\overline{\mathbf{R}}^n_-) \cap X_k$ and $\lim_{l\to\infty} \|f-\phi_l\|_{L^1_{(k)}(\mathbf{R}^n_-)}=0$ for some $f \in \ker(\mathrm{Tr}^{(k)}|_{X_k})$. Clearly $\tilde{f}=g$.

The case k even is proved similarly. Instead of (4.2) we define $\tilde{f}_l$ by

$$\tilde{f}_l(y,x_n) = \begin{cases} f_l(y,x_n) & \text{for } x_n \le 0, \\ f_l(y,-x_n) & \text{for } x_n > 0. \end{cases} \quad \square$$

Proof of Theorem 1.1. Every non-empty open $\Omega \subset \mathbf{R}^n$ contains an open cube U with edges parallel to the coordinate axes. There is a linear extension operator from $L^1_{(k)}(U)$ to $L^1_{(k)}(\mathbf{R}^n)$. (This is a very special case of Stein's extension theorem (cf. [S], Chapter VI); this case can be proved in an elementary manner (cf., e.g. [PW2], proof of Theorem 57.) Thus it follows from [PW1], Theorem 1, that $L^1_{(k)}(U)$ is isomorphic to $L^1_{(k)}(\mathbf{R}^n)$. On the other hand, the existence of a linear extension operator in question implies the existence of a linear extension operator, say $\mathcal{E}: L^1_{(k)}(U) \to L^1_{(k)}(\Omega)$ (because $U \subset \Omega \subset \mathbf{R}^n$). Thus $L^1_{(k)}(U)$ is isomorphic to a complemented subspace of $L^1_{(k)}(\Omega)$; the subspace is $\mathcal{E}(L^1_{(k)}(U))$; the projection is given by $f \mapsto \mathcal{E}(f|_U)$ for $f \in L^1_{(k)}(\Omega)$. Thus, $L^1_{(k)}(\Omega)$ is uncomplemented in its second dual because, by Proposition 4.1, it has a complemented subspace with this property (cf. [L], p. 540). $\square$

5. Proof of Peetre's theorem and its relatives

To make the paper self-contained we present a proof of Peetre's theorem. Our argument uses some ideas of the elegant proof of Brudnyi and Shvartsman (cf.

[BS], Theorem 5.5). However we derive the theorem from its counterpart for the two-dimensional torus.

Let $I^n:=\{x=(x_j)_{j=1}^n\in\mathbf{R}^n:|x_j|<\frac{1}{2}$ for $j=1,\dots,n\}$ and $I^n_\pm:=\mathbf{R}^n_\pm\cap I^n$. We identify I^{n-1} with the set $\{x=(x_j)_{j=1}^n\in\mathbf{R}^n:x_n=0\}$. Clearly $L^p_{(k)}(I^n)$ can be identified with $\{f|_{I^n}:f\in L^p_{(k)}(\mathbf{R}^n)\}$. Similar identifications hold for $L^p_{(k)}(\mathbf{R}^n_\pm)$. The n-dimensional torus $\mathbf{T}^n$ is identified with the cube $\overline{I^n}$ with the boundary points identified coordinatewise modulo 1. We let $\mathbf{T}^n_\pm=\{t=(t_j)_{j=1}^n\in\mathbf{T}^n:\pm t_n>0\}$ and we identify $\mathbf{T}^{n-1}$ with the set $\{t=(t_j)_{j=1}^n\in\mathbf{T}^n:t_n=0\}$. For $k=0,1,\dots,$ the Sobolev space $L^p_{(k)}(\mathbf{T}^n)$ can be regarded as the subspace of $L^p_{(k)}(I^n)$ generated by the characters $\{\exp 2\pi i\langle\,\cdot\,,m\rangle:m\in\mathbf{Z}^n\}$. For $k=0$ we have $L^p(\mathbf{T}^n)=L^p(I^n)$. The space $L^p_{(k)}(\mathbf{T}^n_\pm)$ is defined to be the subspace of $L^p_{(k)}(I^n_\pm)$ consisting of the restriction to $I^n_\pm$ of functions from $L^p_{(k)}(\mathbf{T}^n)$. For $k=1,2,\dots,$ we define the trace $\mathrm{tr}^{(k)}:L^1_{(k)}(I^n_-)\to L^1(I^{n-1})$ by $\mathrm{tr}^{(k)}g=(\mathrm{Tr}^{(k)}f)|_{I^{n-1}}$, where $f\in L^1_{(k)}(\mathbf{R}^n_-)$ satisfies $f|_{I^n_-}=g$; we further define $\widetilde{\mathrm{Tr}}^{(k)}:L^1_{(k)}(\mathbf{T}^n_-)\to L^1(\mathbf{T}^{n-1})$ as the restriction of $\mathrm{tr}^{(k)}$ to $L^1_{(k)}(\mathbf{T}^n_-)$. To prove the surjectivity of the trace $\widetilde{\mathrm{Tr}}^{(k)}$ we need to change the proof of Lemma 3.3 slightly. We have to assume additionally that we pick $h\in\mathcal{D}(\mathbf{R}^n)$ to be symmetric with respect to 0; we fix a one-periodic function ϱ (in the variable x_n) such that $\varrho|_I\in\mathcal{D}(I)$ and $\varrho(0)=1$. After constructing $\phi\in\mathcal{D}(\mathbf{R}^n)$ as in the proof of Lemma 3.3 we replace it by $\varrho\phi$. Thus if we started from a one-periodic $g\in\mathcal{D}(\mathbf{R}^{n-1})$ then we can regard $\varrho\phi$ as a function in $L^1_{(k)}(\mathbf{T}^n)$ which satisfies $\|\varrho\phi\|_{L^1_{(k)}(\mathbf{T}^n)}\le C\|\phi\|_{L^1_{(k)}(\mathbf{R}^n)}$, where the positive constant C depends only on the sup norms of ϱ and its partial derivative of order $\le k$. Thus we have the following result.

Proposition 5.1. *The traces* $\mathrm{tr}^{(k)}:L^1_{(k)}(I^n_-)\to L^1(I^{n-1})$ *and* $\widetilde{\mathrm{Tr}}^{(k)}:L^1_{(k)}(\mathbf{T}^n_-)\to L^1(\mathbf{T}^{n-1})$ *are surjections* $(k=1,2,\dots;\ n=2,3,\dots)$.

Now we show that Peetre's theorem reduces to the case of the two-dimensional torus.

Lemma 5.2. (i) *If for some* $n=2,3,\dots,$ *there exists a right inverse of* $\mathrm{tr}^{(1)}:L^1_{(1)}(I^n_-)\to L^1(I^{n-1})$ *then there exists a right inverse of* $\mathrm{Tr}^{(1)}:L^1_{(1)}(\mathbf{R}^n_-)\to L^1(\mathbf{R}^{n-1})$.

(ii) *If for some* $n=2,3,\dots,$ *there exists a right inverse of* $\mathrm{Tr}^{(1)}:L^1_{(1)}(\mathbf{R}^n_-)\to L^1(\mathbf{R}^{n-1})$ *then there exists a right inverse of* $\widetilde{\mathrm{Tr}}^{(1)}:L^1(\mathbf{T}^n_-)\to L^1(\mathbf{T}^{n-1})$.

(iii) *If for some* $n=3,4,\dots,$ *there exists a right inverse of* $\widetilde{\mathrm{Tr}}^{(1)}:L^1_{(1)}(\mathbf{T}^n_-)\to L^1(\mathbf{T}^{n-1})$ *then there exists a right inverse of* $\widetilde{\mathrm{Tr}}^{(1)}:L^1_{(1)}(\mathbf{T}^2_-)\to L^1(\mathbf{T})$.

Proof. (i) For $m\in\frac{1}{2}\mathbf{Z}^{n-1}$ we put $I_m^{n-1}=I^{n-1}+m\subset\mathbf{R}^{n-1}$; the shift operators $\sigma_m\colon L^1(\mathbf{R}^{n-1})\to L^1(\mathbf{R}^{n-1})$ and $\tau_{(m,0)}\colon L^1_{(1)}(\mathbf{R}^n_-)\to L^1_{(1)}(\mathbf{R}^n_-)$ induced by translation by m, and $(m,0)$, respectively, are defined by $\sigma_m(f)(y)=f(y-m)$ for λ_{n-1}-a.e. $y\in\mathbf{R}^{n-1}$, and $\tau_{(m,0)}(g)(x)=g(y-m,x_n)$ for λ_n-a.e. $x=(y,x_n)\in\mathbf{R}^n_-$. Let $\psi\colon\mathbf{R}^{n-1}\to[0,1]$ and $\phi\colon\mathbf{R}\to[0,1]$ be infinitely differentiable functions such that

$$\operatorname{supp}\psi\subset I^{n-1};\qquad \sum_{m\in\frac{1}{2}\mathbf{Z}^{n-1}}\sigma_m(\psi)\equiv 1\quad\text{(the sum is locally finite);}$$
$$\phi|_{[-1/3,0]}\equiv 1;\qquad\qquad \phi|_{(-\infty,-2/3]}\equiv 0.$$

Let $M_{\psi\phi}\colon L^1_{(1)}(I^n_-)\to L^1_{(1)}(\mathbf{R}^n_-)$ be the operator of multiplication defined by

$$M_{\psi\phi}(g)(y,x_n)=\begin{cases}\psi(y)\phi(x_n)g(y,x_n) & \text{for } \lambda_n\text{-a.e. } (y,x_n)\in I^n_-,\\ 0, & \text{otherwise.}\end{cases}$$

We assign to a bounded linear operator $S^{\#}\colon L^1(I^{n-1})\to L^1_{(1)}(I^n_-)$ the bounded linear operator $S\colon L^1(\mathbf{R}^{n-1})\to L^1_{(1)}(\mathbf{R}^n_-)$ defined by

$$S(f)=\sum_{m\in\frac{1}{2}\mathbf{Z}^{n-1}}(\tau_{(0,-m)}M_{\psi\phi}S^{\#}\sigma_m)(f|_{I^{n-1}_{-m}}).$$

It is not hard to verify that if $S^{\#}$ is a right inverse of $\mathrm{tr}^{(1)}$ then S is a right inverse of $\mathrm{Tr}^{(1)}$.

(ii) Let $S\colon L^1(\mathbf{R}^{n-1})\to L^1_{(1)}(\mathbf{R}^n_-)$ be a right inverse of $\mathrm{Tr}^{(1)}$. Let $\phi\in\mathcal{D}(\overline{\mathbf{R}}_-)$ be a fixed function in the variable x_n such that

$$(5.1)\qquad \phi(x_n)=0 \text{ for } x_n<-\tfrac{1}{4}\quad\text{and}\quad \phi(x_n)=1 \text{ for } -\tfrac{1}{8}<x_n<0.$$

Then there is a constant $C>0$ depending only on the sup norms of ϕ and its partial derivatives of order $\le k$ such that $\|\phi F\|_{L^1_{(1)}(\mathbf{R}^n_-)}\le C\|F\|_{L^1_{(1)}(\mathbf{R}^n_-)}$ for $F\in L^1_{(1)}(\mathbf{R}^n_-)$.

Let $\varepsilon>0$ and $\eta>0$. Let $(h_k)_{k\in\mathbf{N}}$ be a Schauder basis for $L^1(I^{n-1})=L^1(\mathbf{T}^{n-1})$. We regard functions in $L'(I^{n-1})$ as functions on $\mathbf{R}^{n-1}$ extended by 0 on $\mathbf{R}^{n-1}\setminus I^{n-1}$. Remembering that $\mathcal{D}(\overline{\mathbf{R}}^n_-)$ is dense in $L^1_{(1)}(\mathbf{R}^n_-)$, for $k=1,2,\dots$, we choose $F_k\in\mathcal{D}(\overline{\mathbf{R}}^n_-)$ so that $\|F_k-Sh_k\|_{L^1_{(k)}(\mathbf{R}^n_-)}<2^{-k}\eta/C$. Put $\Psi_k=\phi F_k$ and $g_k=\Psi_k|_{\overline{I^{n-1}}}$. Then, remembering that $\operatorname{supp}h_k\subset\overline{I^{n-1}}$ and applying Lemma 3.2, we get

$$(5.2)\qquad \begin{aligned}\|h_k-g_k\|_{L^1(\mathbf{R}^{n-1})}&\le\|h_k-\mathrm{Tr}^{(1)}\,\Psi_k\|_{L^1(\mathbf{R}^{n-1})}=\|\,\mathrm{Tr}^{(1)}(\phi Sh_k-\Psi_k)\|_{L^1(\mathbf{R}^{n-1})}\\ &\le\|\phi Sh_k-\Psi_k\|_{L^1_{(1)}(\mathbf{R}^n_-)}\le\eta 2^{-k}.\end{aligned}$$

Pick $\psi\in\mathcal{D}(\mathbf{R}^{n-1})$ such that $\psi(y)=1$ for $y\in I^{n-1}$ and $\operatorname{supp}\psi\subset\frac{3}{2}I^{n-1}$. Let $A=\{m\in\mathbf{Z}^{n-1}:\overline{I^{n-1}+m}\cap\overline{I^{n-1}}\neq\emptyset\}$. Clearly A is a finite set, more precisely it has 3^{n-1} elements. Let us observe that the properties of ψ imply that for every $F\in L^1_{(1)}(\mathbf{R}^n_-)$ the function $\Phi(F):=\sum_{m\in\mathbf{Z}^{n-1}}\tau_{(m,0)}(\psi F)$ is one-periodic in each of the variables $x_1,\dots,x_{n-1}$. It follows from the previous observation and (5.1) that $\Phi(\Psi_k)=\Phi(\phi F_k)$ extends to a function in $L^1_{(1)}(\mathbf{R}^n)$ which is one-periodic with respect to each of the variables $x_1,\dots,x_n$; equivalently $\Phi(\Psi_k)\in L^1_{(1)}(\mathbf{T}^n_-)$. Since $(\psi_{(m,0)}(\psi\Psi_k))|_{I^{n-1}}=0$ for $m\notin A$, we have

$$\widetilde{\operatorname{Tr}}^{(1)}(\Phi(\Psi_k))=\operatorname{tr}^{(1)}(\Phi(\Psi_k)|_{I^n_-})=\operatorname{tr}^{(1)}\Big(\sum_{m\in A}\tau_{(m,0)}(\psi\Psi_k)|_{I^n_-}\Big)$$
$$=\sum_{m\in A}\operatorname{tr}^{(1)}(\tau_{(m,0)}(\psi\Psi_k)|_{I^n_-})=\sum_{m\in A}\tau_{(m,0)}(\psi\Psi_k)|_{I^{n-1}},\qquad k=1,2,\dots\,.$$

Therefore, remembering that $0\le\psi\le1$ and $\Psi_k|_{I^{n-1}}=g_k$, we get

$$\begin{aligned}\|g_k-\widetilde{\operatorname{Tr}}^{(1)}\Phi(\Psi_k)\|_{L^1(\mathbf{T}^{n-1})}&=\int_{I^{n-1}}\Big|\sum_{0\neq m\in A}\tau_{(m,0)}(\psi\Psi_k)\Big|\,d\lambda_{n-1}\\&\le\int_{I^{n-1}}\sum_{0\neq m\in A}|\tau_{(m,0)}(\psi\Psi_k)|\,d\lambda_{n-1}\\&=\int_{\mathbf{R}^{n-1}\setminus I^{n-1}}|\operatorname{Tr}^{(1)}(\psi\Psi_k)|\,d\lambda_{n-1}\\&\le\int_{\mathbf{R}^{n-1}\setminus I^{n-1}}|\operatorname{Tr}^{(1)}\Psi_k|\,d\lambda_{n-1}\\&=\|g_k-\operatorname{Tr}^{(1)}\Psi_k\|_{L^1(\mathbf{R}^{n-1})}.\end{aligned}$$

Combining the latter inequality with (5.2) we get

$$\|g_k-\widetilde{\operatorname{Tr}}^{(1)}(\Phi(\Psi_k))\|_{L^1(\mathbf{T}^{n-1})}\le\eta2^{-k+1}\quad\text{for } k=1,2,\dots\,.$$

It follows from the Krein–Milman–Rutman standard perturbation argument (cf. [LT], vol. I, Proposition 1.a.9) that if we choose $\eta>0$ sufficiently small for given ε, $C>0$, and $\|S\|$ then $(g_k)_{k\in\mathbf{N}}$ is a basis for $L^1(\mathbf{T}^{n-1})$ and there is a unique bounded linear operator $\widetilde{S}_\varepsilon\colon L^1(\mathbf{T}^{n-1})\to L^1_{(1)}(\mathbf{T}^n_-)$ such that $\widetilde{S}_\varepsilon g_k=\Psi_k$ and $\|\mathrm{Id}_{L^1(\mathbf{T}^{n-1})}-\widetilde{\operatorname{Tr}}^{(1)}\circ\widetilde{S}_\varepsilon\|<\varepsilon$ for $k=1,2,\dots$. Thus the operator $\widetilde{\operatorname{Tr}}^{(1)}\circ\widetilde{S}_\varepsilon\colon L^1(\mathbf{T}^{n-1})\to L^1(\mathbf{T}^{n-1})$ is invertible for $\varepsilon<1$. Hence $\widetilde{S}_\varepsilon(\widetilde{\operatorname{Tr}}^{(1)}\circ\widetilde{S}_\varepsilon)^{-1}$ is a right inverse for $\widetilde{\operatorname{Tr}}^{(1)}$.

(iii) If $\widetilde{S}\colon L^1(\mathbf{T}^{n-1})\to L^1_{(1)}(\mathbf{T}^n_-)$ is a right inverse for $\widetilde{\mathrm{Tr}}^{(1)}\colon L^1_{(1)}(\mathbf{T}^n_-)\to L^1(\mathbf{T}^{n-1})$ for some $n>2$ then $P\widetilde{S}J$ is a right inverse for $\widetilde{\mathrm{Tr}}^{(1)}\colon L^1_{(1)}(\mathbf{T}^2_-)\to L^1(\mathbf{T})$, where J and P are defined by

$$Jf(t_1,\dots,t_{n-1})=f(t_{n-1}) \qquad \text{for } f\in L^1(\mathbf{T});$$

$$PF(t_{n-1},t_n)=\int_{\mathbf{T}}\dots\int_{\mathbf{T}}F(t_1,\dots,t_n)\,dt_1\,\dots\,dt_{n-2} \quad \text{for } F\in L^1_{(1)}(\mathbf{T}^n_-). \qquad \square$$

Next we need the following variant of the Gagliardo trace theorem. For $s'\in\mathbf{T}_-$ define $R_{s'}\colon L^1_{(1)}(\mathbf{T}^2_-)\to L^1(\mathbf{T})$ by $R_{s'}F=\widetilde{\mathrm{Tr}}^{(1)}(F^*_{s'}|_{\mathbf{T}^2_-})$ for $F\in L^1_{(1)}(\mathbf{T}^2_-)$, where F^* denotes an arbitrary extension of F to a function in $L^1_{(1)}(\mathbf{T}^2)$ and $F^*_{s'}(t,s)=F^*(t,s-s')$ for λ_2-a.e. (s,t). It is easy to see that the function $R_{s'}F$ is independent of the particular choice of the extension (cf. [S], Chapter VI for the existence of an extension).

Lemma 5.3. *For every $F\in L^1_{(1)}(\mathbf{T}^2_-)$ one has*

$$(5.3)\qquad (R_sF)(t)=F(t,s) \qquad \textit{for } \lambda_2\textit{-a.e. } (t,s);$$

$$(5.4)\qquad \|R_{s'}F-R_{s''}F\|_{L^1(\mathbf{T})}\le\left|\int_{s'}^{s''}\int_{\mathbf{T}}|\nabla F(s,t)|\,dt\,ds\right|,\quad F\in L^1_{(1)}(\mathbf{T}^2_-),\ s',s''\in\mathbf{T}_-.$$

Proof. If $F\in\mathcal{D}(\overline{\mathbf{T}^2_-})$ then (5.3) holds for every $(t,s)\in\mathbf{T}^2_-$. For $F\in L^1_{(1)}(\mathbf{T}^2_-)$ there exists a sequence $(F_n)_{n\in\mathbf{N}}\subset\mathcal{D}(\overline{\mathbf{T}^2_-})$ such that $\lim_{n\to\infty}\|F_n-F\|_{L^1_{(1)}(\mathbf{T}^2_-)}=0$. In particular $\lim_{n\to\infty}\|F_n-F\|_{L^1(\mathbf{T}^2)}=0$. Thus, passing to a subsequence if necessary, we may assume that $\lim_{n\to\infty}F_n(t,s)=F(t,s)$ for λ_2-a.e. (t,s). Thus, by the Fubini theorem, there is a set A with $\lambda_1(\mathbf{T}_-\setminus A)=0$ such that, if $s\in A$ then $\lim_{n\to\infty}F_n(t,s)=F(t,s)$ for λ_1-a.e. t. It follows from the definition of the trace that for every $s\in\mathbf{T}_-$ there is a subsequence, say $(n_k)_{k\in\mathbf{N}}$ of the indices such that $\lim_{k\to\infty}F_{n_k}(t,s)=R_sF(t)$ for λ_1-a.e. t. Thus if $s\in A$ then $F(t,s)=R_sF(t)$ for λ_1-a.e. t. Applying the Fubini theorem again we get (5.3).

It suffices to verify (5.4) for $F\in\mathcal{D}(\overline{\mathbf{T}^2_-})$. For such an F we have the identity

$$(R_{s'}F)(t)-(R_{s''}F)(t)=F(t,s')-F(t,s'')=\int_{s''}^{s'}\frac{\partial}{\partial s}F(t,s)\,ds.$$

Integrating the latter identity against dt, putting properly absolute values, and remembering that the gradient satisfies the inequality

$$\|\nabla F\|_{L^1(\mathbf{T}\times[s'',s'])}\ge\left\|\frac{\partial}{\partial s}F\right\|_{L^1(\mathbf{T}\times[s'',s'])},$$

we get (5.4). $\square$

We also need the following embedding lemma (cf. [BIN], Theorem 10.1; Theorem 10.2 in the English translation of the first edition).

Lemma 5.4. *One has*

$$\int_{\mathbf{T}_-} \operatorname*{ess\,sup}_{t\in\mathbf{T}} |F(t,s)|\,ds \le \|F\|_{L^1_{(1)}(\mathbf{T}^2_-)}, \quad F\in L^1_{(1)}(\mathbf{T}^2_-). \tag{5.5}$$

Proof. Since the norm $\int_{\mathbf{T}_-} \operatorname{ess\,sup}_{t\in\mathbf{T}} |F(t,s)|\,ds$ is complete in the appropriate function space on $\mathbf{T}^2_-$ and $\mathcal{D}(\overline{\mathbf{T}^2_-})$ is dense in $L^1_{(1)}(\mathbf{T}^2_-)$, it is enough to verify (5.5) for $F\in\mathcal{D}(\overline{\mathbf{T}^2_-})$. For fixed $s\in\mathbf{T}_-$ let $\sup_{t\in\mathbf{T}}|F(t,s)|$ be attained at the point $t=t(s)$. Thus for arbitrary $\xi\in\mathbf{T}$ we have

$$\sup_{t\in\mathbf{T}}|F(t,s)| \le |F(t(s),s)-F(\xi,s)|+|F(\xi,s)| = \left|\int_\xi^{t(s)} \frac{\partial}{\partial t}F(t,s)\,dt\right| + |F(\xi,s)|.$$

Integrating against ds and $d\xi$ we get

$$\begin{aligned}\int_{s\in\mathbf{T}_-}\sup_{t\in\mathbf{T}}|F(t,s)|\,ds &\le \int_{s\in\mathbf{T}_-}\int_{\xi\in\mathbf{T}}\int_\xi^{t(s)}\left|\frac{\partial}{\partial t}F(t,s)\right|dt\,d\xi\,ds + \|F\|_{L^1(\mathbf{T}^2_-)}\\ &\le \int_{s\in\mathbf{T}_-}\int_{t\in\mathbf{T}}\int_{\xi\in\mathbf{T}}\left|\frac{\partial}{\partial t}F(t,s)\right|d\xi\,dt\,ds + \|F\|_{L^1(\mathbf{T}^2_-)}\\ &= \left\|\frac{\partial}{\partial t}F\right\|_{L^1(\mathbf{T}^2_-)} + \|F\|_{L^1(\mathbf{T}^2_-)} \le \|F\|_{L^1_{(1)}(\mathbf{T}^2_-)}. \quad \square\end{aligned}$$

Now we are ready for the next proposition which combined with Lemma 5.2 completes the proof of Peetre's theorem.

Proposition 5.5. *There is no right inverse for* $\widetilde{\mathrm{Tr}}^{(1)}: L^1_{(1)}(\mathbf{T}^2_-)\to L^1(\mathbf{T})$.

Proof. Let $\{U_h\}_{h\in\mathbf{T}}$ and $\{V_h\}_{h\in\mathbf{T}}$ be the representations of the circle group $\mathbf{T}$ in the spaces of bounded linear operators on $L^1(\mathbf{T})$ and $L^1_{(1)}(\mathbf{T}^2_-)$, respectively, defined for $h\in\mathbf{T}$ by

$$\begin{aligned} U_h(f)(t) &= f(t+h) &&\text{for } \lambda_1\text{-a.e. } t\in\mathbf{T};\\ V_h(F)(t,s) &= F(t+h,s) &&\text{for } \lambda_2\text{-a.e. } (t,s)\in\mathbf{T}^2_-.\end{aligned}$$

Obviously the family $\{R_s\}_{s\in\mathbf{T}_-}$ (defined before Lemma 5.3) consists of operators which intertwine with the representations, i.e.

$$R_sV_h = U_hR_s, \quad s\in\mathbf{T}_-;\ h\in\mathbf{T}.$$

Assume on the contrary that there is a right inverse for $R_0=\widetilde{\mathrm{Tr}}^{(1)}$, say $S\colon L^1(\mathbf{T})\to L^1_{(1)}(\mathbf{T}^2_-)$. Then there exists another right inverse, say $\widetilde{S}$, which intertwines with the representations $\{U_h\}_{h\in\mathbf{T}}$ and $\{V_h\}_{h\in\mathbf{T}}$, i.e. $\widetilde{S}U_h=V_h\widetilde{S}$ for $h\in\mathbf{T}$. We put

$$\widetilde{S}:=\int_{\mathbf{T}} V_h S U_h^{-1}\,dh.$$

The representations are continuous in the norms of the corresponding Banach spaces of operators. Thus the integrand is a continuous function on $\mathbf{T}$. The integral exists in the Riemann sense. Hence $\widetilde{S}$ is a right inverse for R_0 being a norm limit of convex combinations of right inverses $V_h S U_h^{-1}$.

Clearly $R_s\widetilde{S}$ is, for every $s\in\mathbf{T}_-$, an invariant operator on $L^1(\mathbf{T})$, i.e. it commutes with every U_t for $t\in\mathbf{T}$. Thus (cf. e.g. [SW], Chapter VII, §3, Theorem 3.4) there is a complex valued Borel measure μ_s on $\mathbf{T}$ with finite total variation $\|\mu_s\|$ such that

$$R_s\widetilde{S}f=f*\mu_s;\ \|R_s\widetilde{S}\|=\|\mu_s\|,\quad s\in\mathbf{T}_-.$$

Here $*$ stands for convolution. Let $(\phi_n)_{n\in\mathbf{N}}$ be an approximate unity for $L^1(\mathbf{T})$ consisting of bounded functions with $\|\phi_n\|_{L^1(\mathbf{T})}=1$ for $n=1,2,\dots$. For every $s\in\mathbf{T}_-$ put

$$a(s):=\lim_{n\to\infty}\|\phi_n*\mu_s\|_\infty=\lim_{n\to\infty}\|R_s\widetilde{S}\phi_n\|_\infty,$$

where (for f measurable) $\|f\|_\infty=\operatorname{ess\,sup}_{t\in\mathbf{T}}|f(t)|$; the latter for some f may be $+\infty$. Clearly $a(\cdot)$ is a positive measurable function which may take the value $+\infty$. Since $(\phi_n)_{n\in\mathbf{N}}$ is an approximate unity for $L^1(\mathbf{T})$, we have

$$a(s)=\begin{cases}\|\psi_s\|_\infty, & \text{if } \mu_s=\psi_s\,d\lambda_1 \text{ for some } \psi_s\in L^1(\mathbf{T}),\\ \infty, & \text{otherwise.}\end{cases}$$

Therefor, taking into account the Fatou theorem, (5.3) and (5.5), we get

$$\begin{aligned}\int_{s\in\mathbf{T}_-} a(s)\,ds &\le \liminf_{n\to\infty}\int_{s\in\mathbf{T}_-}\|\phi_n*\mu_s\|_\infty\,ds=\liminf_{n\to\infty}\int_{s\in\mathbf{T}_-}\|R_s(\widetilde{S}\phi_n)\|_\infty\,ds\\ &=\liminf_{n\to\infty}\int_{s\in\mathbf{T}_-}\operatorname*{ess\,sup}_{t\in\mathbf{T}}|\widetilde{S}\phi_n(t,s)|\,ds\le\|\widetilde{S}\phi_n\|_{L^1_{(1)}(\mathbf{T}^2_-)}\le\|\widetilde{S}\|.\end{aligned}$$

Thus $\mu_s=\psi_s\,d\lambda_1$ with $\psi_s\in L^1(\mathbf{T})$ for λ_1-a.e. $s\in\mathbf{T}_-$.

Let $(s_k)_{k\in\mathbf{N}}\subset\mathbf{T}_-$ be an increasing sequence tending to 0, such that μ_{s_k} is absolutely continuous with respect to λ_1 for $k=1,2,\dots$. By (5.4), for $f\in L^1(\mathbf{T})$,

$$\sum_{k=1}^{\infty}\|\psi_{s_k}*f-\psi_{s_{k+1}}*f\|_{L^1(\mathbf{T})}\le\|\widetilde{S}f\|_{L^1_{(1)}(\mathbf{T}^2_-)}\le\|\widetilde{S}\|\,\|f\|_{L^1(\mathbf{T})}.$$

Putting the approximate unity $(\phi_n)_{n\in\mathbf{N}}$ in place of f and passing with n to infinity, we get

$$\sum_{k=1}^{\infty} \|\psi_{s_k}-\psi_{s_{k+1}}\|_{L^1(\mathbf{T})}\le \|\widetilde{S}\|.$$

Hence there exists $\psi\in L^1(\mathbf{T})$ such that $\lim_{k\to\infty}\|\psi_{s_k}-\psi\|_{L^1(\mathbf{T})}=0$. On the other hand $R_0\widetilde{S}f=f$ and, by (5.4), for every $f\in L^1(\mathbf{T})$,

$$\lim_{k\to\infty}\|(\psi_{s_k}*f)-f\|_{L^1(\mathbf{T})}=\lim_{k\to\infty}\|R_{s_k}\widetilde{S}f-R_0\widetilde{S}f\|_{L^1(\mathbf{T})}=0.$$

Thus $\psi*f=f$ for every $f\in L^1(\mathbf{T})$, a contradiction because the algebra $L^1(\mathbf{T})$ with convolution as multiplication has no unit. □

Proposition 5.5 combined with Lemma 5.2 in fact gives the following corollary.

Corollary 5.6. *The traces* $\mathrm{Tr}^{(1)}$, $\mathrm{tr}^{(1)}$ *and* $\widetilde{\mathrm{Tr}}^{(1)}$ *admit no right inverses.*

6. Some remarks on $\mathbf{BV}_{(k)}(\Omega)$

6.1. Proof of Corollary 1.3

As indicated in Section 1 to derive Corollary 1.3 from Theorem 1.1 it is enough to prove the following result.

Proposition 6.1. *For every open non-empty* $\Omega\subset\mathbf{R}^n$ *the space* $\mathrm{BV}_{(k)}(\Omega)$ *is a dual Banach space,* $k,n=1,2,\dots$.

Proof. For a positive integer N let l^p_N denote the space of scalar-valued sequences $x=(x_j)_{j=1}^N$ equipped with the norm

$$|x|_p=\begin{cases}\left(\sum_{j=1}^N|x_j|^p\right)^{1/p} & \text{for } 1\le p<\infty,\\ \max\limits_{1\le j\le N}|x_j| & \text{for } p=\infty.\end{cases}$$

Since the spaces l^1_N and l^∞_N are in duality, it follows from the Riesz representation theorem that the vector-valued space $M(\Omega;l^1_N)$ can be identified with the dual of the space $C_0(\Omega;l^\infty_N)$. The latter is the closure in the sup-norm of l^∞_N-valued continuous functions with compact support in Ω. Now let $N=N(k,n)$ be the number of all multiindices corresponding to all partial derivatives in n variables of order $\le k$. Then the map $J\colon \mathrm{BV}_{(k)}(\Omega)\to M(\Omega;l^1_N)$, defined by

$$J(\mu)=(D^\alpha\mu)_{0\le|\alpha|\le k}\quad\text{for } \mu\in\mathrm{BV}_{(k)}(\Omega), \tag{6.1}$$

is an isometrically isomorphic embedding of $\mathrm{BV}_{(k)}(\Omega)$ into $M(\Omega;l_N^1)$. Clearly $J(\mathrm{BV}_{(k)}(\Omega))$ is a subspace of $M(\Omega;l_N^1)$ which is closed in the weak* topology ($=C_0(\Omega;l_N^\infty)$ topology of $M(\Omega;l_N^1)$). The verification of this fact is routine and follows from the definition of distributional partial derivatives of measures. Thus $J(\mathrm{BV}_{(k)}(\Omega))$ can be identified with the dual of the quotient $C_0(\Omega;l_N^\infty)/\mathrm{BV}_{(k)}(\Omega)_\perp$, where

$$\mathrm{BV}_{(k)}(\Omega)_\perp=\Big\{(f_\alpha)_{0\le|\alpha|\le k}\in C_0(\Omega,l_N^\infty):\sum_{0\le|\alpha|\le k}\int_\Omega f_\alpha\,d(D^\alpha\mu)=0 \text{ for } \mu\in\mathrm{BV}_{(k)}(\Omega)\Big\}.$$

This completes the proof of Proposition 6.1. □

6.2. An alternative proof of Proposition 4.1

We outline how one can use Proposition 6.1 to obtain Proposition 4.1 and therefore Theorem 1.1 without making use of the Lindenstrauss lifting principle.

Denote by $\varkappa_X\colon X\to X^{**}$ the canonical map. For $k=0,1,\dots$, let us define $\iota_{(k)}\colon L^1_{(k)}(\mathbf{R}^n)\to\mathrm{BV}_{(k)}(\mathbf{R}^n)$ by $\iota_{(k)}(f)=f\lambda_n$ for $f\in L^1_{(k)}(\mathbf{R}^n)$. Put $\iota=\iota_{(0)}$. Note that $\iota(D^\alpha f)=D^\alpha(\iota(f))$ and $\|\iota(D^\alpha f)\|_{M(\mathbf{R}^n)}=\|D^\alpha f\|_{L^1(\mathbf{R}^n)}$ whenever $D^\alpha f$ exists and belongs to $L^1(\mathbf{R}^n)$. Thus $\iota_{(k)}\colon L^1_{(1)}(\mathbf{R}^n)\to\mathrm{BV}_{(k)}(\mathbf{R}^n)$ is an isometrically isomorphic embedding, $k=0,1,\dots$.

Proposition 6.2. *Let $k=0,1,\dots$. There exists an isometrically isomorphic embedding $\mathfrak{d}_{(k)}\colon\mathrm{BV}_{(k)}(\mathbf{R}^n)\to L^1_{(k)}(\mathbf{R}^n)^{**}$ such that $\mathfrak{d}_{(k)}(\mathrm{BV}_{(k)}(\mathbf{R}^n))$ is a complemented subspace of $L^1_{(k)}(\mathbf{R}^n)^{**}$ and $\mathfrak{d}_{(k)}\circ\iota_{(k)}=\varkappa_{L^1_{(k)}(\mathbf{R}^n)}$.*

Proof. For simplicity we identify $\mathrm{BV}_{(k)}(\mathbf{R}^n)$ with $J(\mathrm{BV}_{(k)}(\mathbf{R}^n))$, where J is defined by (6.1), and we identify $L^1_{(k)}(\mathbf{R}^n)$ with its image via $\iota_{(k)}$. Thus $L^1_{(k)}(\mathbf{R}^n)$ can be regarded as the subspace of $L^1(\mathbf{R}^n;l_N^1)$ defined by

$$\{(f_\alpha)_{0\le|\alpha|\le k}\in L^1(\mathbf{R}^n;l^1):f_\alpha=D^\alpha f \text{ for } 0\le|\alpha|\le k \text{ and for } f\in L^1_{(k)}(\mathbf{R}^n)\},$$

where $N=N(k,n)$ is defined in the proof of Proposition 6.1. The dual of $L^1(\mathbf{R}^n;l_N^1)$ can be identified with $L^\infty(\mathbf{R}^n;l_N^\infty)$. Thus, by the Hahn–Banach extension principle every $z^*\in L^1_{(k)}(\mathbf{R}^n)^*$ has a norm-preserving extension to some $(\phi_\alpha^{[z^*]})_{0\le|\alpha|\le k}\in L^\infty(\mathbf{R}^n;l_N^\infty)$. Now let $(G_\varepsilon)_{\varepsilon>0}$ be a C^∞-approximate identity of $L^1(\mathbf{R}^n)$, for instance $G_\varepsilon(x)=\varepsilon^{-n}G(x/\varepsilon)$ for $x\in\mathbf{R}^n$, where $G(x)=(2\pi)^{-n/2}\exp(-\frac12|x|_2^2)$.

Let us denote the operator of convolution with G_ε by Φ_ε, i.e. $\Phi_\varepsilon(\nu)(x)=\int_{\mathbf{R}^n}G_\varepsilon(x-y)\nu(dy)$ for λ_n-a.e. x. Then $\Phi_\varepsilon(\mathrm{BV}_{(k)}(\mathbf{R}^n))\subset L^1_{(k)}(\mathbf{R}^n)$ for $k=0,1,\dots$, and

$$(6.2)\qquad \lim_{\varepsilon\to0}\int_{\mathbf{R}^n}\Phi_\varepsilon(\nu)(x)f(x)\,dx=\int_{\mathbf{R}^n}f(x)\,d\nu \quad\text{for } f\in\mathcal{D}(\mathbf{R}^n) \text{ and } \nu\in M(\mathbf{R}^n).$$

Given $\nu\in\mathrm{BV}_{(k)}(\mathbf{R}^n)$ we define $\eth_{(k)}(\nu)$ by

$$\eth_{(k)}(\nu)(z^*)=\operatorname*{LIM}_{\varepsilon\to0}\sum_{0\le|\alpha|\le k}\int_{\mathbf{R}^n}\Phi_\varepsilon(D^\alpha\nu)\phi_\alpha^{[z^*]}\,dx \quad \text{for } z^*\in(L^1_{(k)}(\mathbf{R}^n))^*,$$

where $\mathrm{LIM}_{\varepsilon\to0}$ denotes a generalized (Banach) limit (cf. [DS], Chapter II.3, (23)).

Hence

$$\begin{aligned}|\eth_{(k)}(\nu)(z^*)| &\le \|\phi^{[z^*]}\|_{L^\infty(\mathbf{R}^n;l^\infty_N)}\sup_{\varepsilon>0}\|\Phi_\varepsilon(D^\alpha\nu)\|_{L^1(\mathbf{R}^n;l^1_N)}\\ &=\|\nu\|_{\mathrm{BV}_{(k)}(\mathbf{R}^n)}\|z^*\|_{L^1_{(k)}(\mathbf{R}^n)^*}.\end{aligned}$$

Thus $\|\eth_{(k)}(\nu)\|_{L^1_{(k)}(\mathbf{R}^n)^{**}}\le\|\nu\|_{\mathrm{BV}_{(k)}(\mathbf{R}^n)}$. It follows from (6.2) and the density of $\mathcal{D}(\mathbf{R}^n;l^\infty_N)$ in $L^\infty(\mathbf{R}^n;l^\infty_N)$ in the weak* topology induced by $L^1(\mathbf{R}^n;l^1_N)$ that the latter inequality becomes equality. Hence $\eth_{(k)}:\mathrm{BV}_{(k)}(\mathbf{R}^n)\to L^1_{(k)}(\mathbf{R}^n)^{**}$ is an isometric embedding.

If $\nu=\iota_{(k)}(f)$ for some $f\in L^1_{(k)}(\mathbf{R}^n)$ then $D^\alpha\nu=(D^\alpha f)\lambda_n$ for $0\le|\alpha|\le k$, and for every $\phi\in L^\infty(\mathbf{R}^n)$ one has $\lim_{\varepsilon\to0}\int_{\mathbf{R}^n}\Phi_\varepsilon(D^\alpha f)\lambda_n\phi\,dx=\int_{\mathbf{R}^n}(D^\alpha f)\phi\,dx$. Thus, for every $z^*\in L^1_{(k)}(\mathbf{R}^n)^*$,

$$\eth_{(k)}(f\lambda_n)(z^*)=\lim_{\varepsilon\to0}\sum_{0\le|\alpha|\le k}\int_{\mathbf{R}^n}\Phi_\varepsilon((D^\alpha f)\lambda_n)\phi^{[z^*]}\,dx=z^*(f).$$

Therefore $\eth_{(k)}\circ\iota_{(k)}=\varkappa_{L^1_{(k)}(\mathbf{R}^n)}$.

The desired projection from $L^1_{(k)}(\mathbf{R}^n)^{**}$ onto $\eth_{(k)}(\mathrm{BV}_{(k)}(\mathbf{R}^n))$ is the operator $\eth_{(k)}\circ U^*$ where $U^*:L^1_{(k)}(\mathbf{R}^n)^{**}\to\mathrm{BV}_{(k)}(\mathbf{R}^n)$ is the adjoint operator to the isometric embedding $U:C_0(\mathbf{R}^n;l^\infty_N)/\,\mathrm{BV}_{(k)}(\mathbf{R}^n)_\perp\to L^1_{(k)}(\mathbf{R}^n)^*$ defined as follows. Let $g=(g_\alpha)_{0\le|\alpha|\le k}$ be a representative of a coset $[g]$. Then $U([g])\in L^1_{(k)}(\mathbf{R}^n)^*$ is defined by

$$U([g])(\nu)=\sum_{0\le|\alpha|\le k}\int_{\mathbf{R}^n}g_\alpha\,d(D^\alpha\nu),\quad \nu\in\mathrm{BV}_{(k)}(\mathbf{R}^n).\quad\square$$

Now we are ready for the alternative proof of Proposition 4.1.

Outline of an alternative proof of Proposition 4.1. For simplicity we consider the case $k=1$. By Proposition 6.2 it is enough to exhibit a subspace, say X of $\mathrm{BV}_{(1)}(\mathbf{R}^n)$, such that $X\supset L^1_{(1)}(\mathbf{R}^n)$ but $L^1_{(1)}(\mathbf{R}^n)$ is uncomplemented in X. We put

$$f_-:=f|_{\mathbf{R}^n_-};\quad f_+:=f|_{\mathbf{R}^n_+}\quad\text{and}\quad X:=\{f:\mathbf{R}^n\to\mathbf{C}:f_\pm\in L^1_{(1)}(\mathbf{R}^n_\pm)\}.$$

To see that $X \subset \mathrm{BV}_{(1)}(\mathbf{R}^n)$ fix $f \in X$. Note that if the multiindex α corresponds to the partial derivative $\partial/\partial x_j$ for $j=1,\dots,n-1$, then $(D^\alpha f)|_{\mathbf{R}^n_\pm} = (D^\alpha f_\pm)$ and $(D^\alpha f)|_{\mathbf{R}^{n-1}} = 0$, while for $\alpha=(0,\dots,0,1)$ corresponding to the partial derivative $\partial/\partial x_n$ we add the one-dimensional measure concentrated on $\mathbf{R}^{n-1}$, $D^\alpha f|_{\mathbf{R}^{n-1}} = (\mathrm{Tr}^{(1)} f_+ - \mathrm{Tr}^{(1)} f_-)\lambda_{n-1}$; we define $\mathrm{Tr}^{(1)} f_+ = \mathrm{Tr}^{(1)} f_-^*$, where $f^*(x)=f(-x)$ for $x \in \mathbf{R}^n$. With the help of Lemmas 3.2 and 3.3 we infer that $f \in L^1_{(1)}(\mathbf{R}^n)$ if and only if $\mathrm{Tr}^{(1)} f_+ - \mathrm{Tr}^{(1)} f_- \equiv 0$.

Assume on the contrary that there is a projection, say P, from X onto $L^1_{(1)}(\mathbf{R}^n)$. Then for each $h \in L^1(\mathbf{R}^{n-1})$ there would exist a unique $f[h] \in (\mathrm{Id} - P)(X)$ such that $\mathrm{Tr}^{(1)}(f[h])_+ - \mathrm{Tr}^{(1)}(f[h])_- = h$. Now for $h \in L^1(\mathbf{R}^{n-1})$ put $S(h) = (f[h]^* - f[h])|_{\mathbf{R}^n_-}$. Then $S\colon L^1(\mathbf{R}^{n-1}) \to L^1_{(1)}(\mathbf{R}^n_-)$ would be the right inverse of $\mathrm{Tr}^{(1)}$, a contradiction to Peetre's theorem. $\square$

References

[A] Adams, R., *Sobolev Spaces*, Academic Press, New York, 1975.

[BIN] Besov, O. V., Il'in, V. P. and Nikolskiĭ, S. M., *Integral Representations of Functions and Embedding Theorems*, 2nd ed., Nauka, Moscow, 1996 (Russian) English transl. of 1st ed.: vol. **1**, Winston, Washington, D. C., 1978.

[B] Borsuk, K., Über Isomorphie der Funktionalräume, *Bull. Int. Acad. Polon. Sci. Cl. Sci. Math. Nat. Sér. A* **1933** (1933), 1–10.

[BS] Brudnyi, Y. and Shvartsman, P., *The Whitney Extension Problem*, Manuscript, Haifa, 1999.

[DJP] Diestel, J., Jarchow, H. and Tonge, A., *Absolutely Summing Operators*, Cambridge Univ. Press, Cambridge, 1995.

[D] Dixmier, J., Sur une théorème de Banach, *Duke Math. J.* **15** (1948), 1057–1071.

[DS] Dunford, N. and Schwartz, J. T., *Linear Operators*, Part **I**, Interscience, New York, 1958.

[G] Gagliardo, E., Caratterizazioni delle trace sulla frontiera relative ad alcune classi funzioni in piu variabili, *Rend. Sem. Mat. Univ. Padova* **27** (1957), 284–305.

[Gr] Grothendieck, A., Erratum au mémoire: Produits tensoriels topologiques et espaces nucléaires, *Ann. Inst. Fourier (Grenoble)* **6** (1956), 117–120.

[H] Henkin, G. M., The nonisomorphy of certain spaces of functions of different number of variables, *Funktsional. Anal. i Prilozhen.* **1**:4 (1967), 57–78 (Russian). English transl.: *Funct. Anal. Appl.* **1** (1967), 306–315.

[KaP] Kalton, N. and Pełczyński, A., Kernels of surjections from $\mathcal{L}_1$-spaces with an applications to the Sidon sets, *Math. Ann.* **309** (1997), 135–158.

[K1] Kislyakov, S. V., Sobolev embedding operators and the nonisomorphism of certain Banach spaces, *Funktsional. Anal. i Prilozhen.* **9**:4 (1975), 22–27 (Russian). English transl.: *Funct. Anal. Appl.* **9** (1975), 290–294.

[K2] Kislyakov, S. V., In the space of continuously differentiable function on the torus there is no local unconditional structure, *Preprint* LOMI P1-77, 1977.

[Ko] KOLYADA, V., Rearrangements of functions and embedding of anisotropic spaces of Sobolev type, *East J. Approx.* **4** (1998), 111–199, 431.

[KwP] KWAPIEŃ, S. and PEŁCZYŃSKI, A., Absolutely summing operators and translation-invariant spaces of functions on compact abelian groups, *Math. Nachr.* **94** (1980), 303–340.

[L] LINDENSTRAUSS, J., On a certain subspace of l_1, *Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys.* **12** (1964), 539–542.

[LT] LINDENSTRAUSS, J. and TZAFRIRI, J., *Classical Banach Spaces I, II*, Springer-Verlag, Berlin–Heidelberg, 1977, 1979.

[M] MAZ'YA, V. G., *Sobolev Spaces*, Springer-Verlag, Berlin, 1985.

[P] PEETRE, J., A counterexample connected with Gagliardo's trace theorem, *Comment. Math., Special Issue* **2** (1979), 277–282.

[PS] PEŁCZYŃSKI, A. and SENATOR, K., On isomorphisms of anisotropic Sobolev spaces with "classical Banach spaces" and a Sobolev type embedding theorem, *Studia Math.* **84** (1986), 169–215.

[PW1] PEŁCZYŃSKI, A. and WOJCIECHOWSKI, M., Contribution to the isomorphic classification of Sobolev spaces $L^p_{(k)}(\Omega)$ $(1\le p<\infty)$, in *Recent Progress in Functional Analysis, Proceedings Valdivia Conference (Valencia, 2000)* (Bierstedt, K. D., Bonet, J., Maestre, J. and Schmets, J., eds.), North-Holland Math. Studies **189**, North-Holland, Amsterdam, 2001.

[PW2] PEŁCZYŃSKI, A. and WOJCIECHOWSKI, M., Sobolev spaces, in *Handbook of the Geometry of Banach Spaces, Vol. II* (Johnson, W. B. and Lindenstrauss, J., eds.), North Holland, Amsterdam, to appear.

[S] STEIN, E. M., *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, Princeton, 1970.

[SW] STEIN, E. M. and WEISS, G., *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton Univ. Press, Princeton, 1971.

Received May 16, 2001

Aleksander Pełczyński
Institute of Mathematics
Polish Academy of Sciences
Sniadeckich 8, Ip.
PL-00-950 Warszawa
Poland
email: olek@impan.gov.pl

Michał Wojciechowski
Institute of Mathematics
Polish Academy of Sciences
Sniadeckich 8, Ip.
PL-00-950 Warszawa
Poland
email: miwoj@impan.gov.pl