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Quasiconformal Lipschitz maps, Sullivan's convex hull theorem and Brennan's conjecture

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Abstract. We show that proving the conjectured sharp constant in a theorem of Dennis Sullivan concerning convex sets in hyperbolic 3-space would imply the Brennan conjecture. We also prove that any conformal map $f\colon \mathbf{D}\to\Omega$ can be factored as a K-quasiconformal self-map of the disk (with K independent of Ω) and a map $g\colon \mathbf{D}\to\Omega$ with derivative bounded away from zero. In particular, there is always a Lipschitz homeomorphism from any simply connected Ω (with its internal path metric) to the unit disk.

1. Introduction

The purpose of this paper is to point out a connection between three dimensional hyperbolic geometry and the expanding properties of planar conformal maps. In particular, we show that a result of Dennis Sullivan about convex hulls in hyperbolic 3-space implies that there is a $K<\infty$ such that any conformal map $f\colon \mathbf{D}\to\Omega$ can be factored as $f=g\circ h$ where h is a K-quasiconformal self-map of the disk and $|g'|$ is bounded away from zero. One consequence is that if (even a weak version of) Sullivan's theorem could be proved with its conjectured sharp constant $K=2$, then the Brennan conjecture would follow. We begin by recalling Sullivan's result and then explain its connection to conformal mappings.

Let $\Omega\subsetneq\mathbf{R}^2$ be a simply connected domain and let $C(\partial\Omega)\subset\mathbf{H}^3$ be the hyperbolic convex hull of $\partial\Omega$ (this is the hyperbolic convex hull of the set of all hyperbolic geodesics with endpoints in $\partial\Omega$). Let S be the boundary component of $C(\partial\Omega)$ which separates Ω from $C(\partial\Omega)$. Equivalently, let $\widetilde{\Omega}\subset\mathbf{H}^3$ be the union of all hemispheres whose bases are contained in Ω. Then $S=\partial\widetilde{\Omega}\cap\mathbf{H}^3$. Let ϱ_S denote the intrinsic path metric on S (using hyperbolic arclength) and let ϱ denote the usual hyperbolic metric on the unit disk $\mathbf{D}$, the upper half space $\mathbf{H}^3$ or Ω. The following are known results.

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Theorem 1.1. *There is an isometry ι from the metric space (S, ϱ_S) to $(\mathbf{D}, \varrho)$.*

Theorem 1.2. *There is a $K_0<\infty$ so that for any simply connected domain $\Omega \subsetneq \mathbf{R}^2$, there is a K_0-biLipschitz map σ from (Ω, ϱ) to (S, ϱ_S) which extends continuously to the identity on $\partial\Omega$. Consequently, there is a universal $K<\infty$ so that σ is K-quasiconformal. Moreover, the map σ is conformally natural in the sense that if Ω is invariant under a group of Möbius transformations then σ commutes with the group action.*

The first result appears in Thurston's notes [44] (with more detailed proofs in the papers of Epstein–Marden [18] and Rourke [40]). The second was apparently known to Thurston and appeared in Sullivan's paper [42] in the case when Ω is simply connected and invariant under a convex cocompact quasi-Fuchsian group. Epstein and Marden [18] proved the more general statement quoted above. Sullivan's convex hull theorem is part of Thurston's hyperbolization theorem and is used in the case of manifolds that fiber over the circle.

When we refer to the best constant in Sullivan's theorem, there are several things we might mean. Let $\mathcal{S}(\Omega)$ denote the set of quasiconformal maps $\sigma: \Omega \to S$ which extend to the identity on $\partial\Omega$. By "best constant" we might mean the infimum of all K, such that $\mathcal{S}(\Omega)$ always contains a

(1) K-biLipschitz map;
(2) K-quasiconformal map; or
(3) K-quasiconformal map which is also biLipschitz (with some constant).

For our applications to conformal maps, it is the third alternative that is most relevant, and when we say in this paper that "Sullivan's theorem holds with constant K" this is what we mean. However, it turns out that the infimums for (2) and (3) are the same. The following is proven in [6].

Theorem 1.3. *Given $K<\infty$ and $\varepsilon>0$ there is a $C<\infty$ so that the following holds. If $f: \Omega \to \mathbf{D}$ is K-quasiconformal, then there is a $(K+\varepsilon)$-quasiconformal map $g: \Omega \to \mathbf{D}$ which is C-biLipschitz between the hyperbolic metrics and such that $f \circ g^{-1}$ extends continuously to the identity on $\partial\mathbf{D}$.*

Because there are various metrics being considered, we will have to be careful when using the term biLipschitz. For example, on Ω we will consider the hyperbolic metric ϱ, the usual Euclidean metric, the spherical metric and the internal path metric (shortest Euclidean length of a path in Ω connecting two given points). When we say a map is biLipschitz, we will have to specify the metric unless it is obvious from the context. Fortunately, all of these metrics give the same class of quasiconformal maps (with the same constants) so that we do not have to be so careful when discussing quasiconformal maps.

Epstein and Marden's proof of Theorem 1.2 in [18] comes in two steps. First they show that the nearest point retraction from Ω onto S_ε (the ε-distance surface from S) is biLipschitz with constant depending on ε. The nearest point retraction from S_ε to S is not even one-to-one, but by foliating S_ε and averaging the retraction map over intervals in the foliation, they construct a biLipschitz map from S_ε to S. The composition of these two maps is then the desired map σ from Ω to S. Their proof gives biLipschitz constant $K_0\approx 88.2$ and quasiconformal constant $K\approx 82.6$ and they conjecture $K_0=K=2$ is correct. In [5] it is proven that one can take $K=7.82$ and $K_0=13.3$, but the construction there is not group invariant; thus, strictly speaking, those results are not an improvement of the Epstein–Marden estimates. However, it follows that there is also a conformally natural map (with a possibly larger, but uniformly bounded constant). This is because if $\Phi\colon \mathbf{D}\to\Omega$ is conformal, and G is a group of Möbius transformations acting on Ω, then $\iota\circ\sigma\circ\Phi$ is quasiconformal from the disk to itself and hence has quasisymmetric boundary values $h\colon \mathbf{T}\to\mathbf{T}$ (with bounds depending only on the quasiconformal constant of f). Moreover, this boundary map h conjugates the Fuchsian group $G_1=\Phi^{-1}\circ G\circ\Phi$ to the Fuchsian group $G_2=\iota\circ G\circ\iota^{-1}$. Hence by the Douady–Earle extension theorem [17], there is a conformally natural extension $\tilde{h}$ of h to the disk with quasiconformal constant depending only on h (and hence only on the quasiconformal constant of σ). Thus $\Phi^{-1}\circ\tilde{h}\circ\iota^{-1}\colon\Omega\to S$ is the desired conformally natural, quasiconformal map. It is not clear whether the best constant for Sullivan's theorem is the same if we also require the map to be conformally natural.

The map $\iota\circ\sigma\colon\Omega\to S\to\mathbf{D}$ is biLipschitz between the hyperbolic metrics. In general, this does not imply it is biLipschitz in the usual Euclidean sense, but we will show that the map can be taken to be locally Lipschitz between the Euclidean metrics.

Theorem 1.4. *Suppose Sullivan's theorem holds with quasiconformal constant K, i.e., there is a K-quasiconformal map from Ω to S which extends to the identity on $\partial\Omega$ and is C-biLipschitz for some $C<\infty$. Then there is an $M=M(K,C)<\infty$ so that for any simply connected domain Ω which contains the unit disk, there is a K-quasiconformal map $g\colon\Omega\to\mathbf{D}$ (same K as above) such that $|g'|\le M$ and $g\circ\iota^{-1}$ extends to the identity on $\partial\Omega$.*

In this paper, we define

$$|g'(x)|=\limsup_{y\to x}\frac{|g(y)-g(x)|}{|y-x|}.$$

For quasiconformal mappings, this is comparable to taking the lim inf. Recall that a domain Ω is called quasiconvex if there is a $C<\infty$ so that any two points $x,y\in\Omega$

can be joined by a path in Ω of length at most $C|x-y|$ (i.e., the internal path metric is comparable to the Euclidean metric). The following is an immediate corollary of the previous result.

Corollary 1.5. *If Ω is any simply connected domain then there is a Lipschitz homeomorphism from Ω with its internal path metric to the unit disk with its usual Euclidean metric. If Ω is quasiconvex, then there is a Lipschitz homeomorphism from Ω to the disk* (*with respect to the Euclidean metric on both domains*).

This is a new result, so far as I know, although I would not be surprised if it was previously known. The proof will show the stronger statement that there is a locally Lipschitz map with respect to the spherical metric on Ω. If Ω is a quasidisk, then it is quasiconvex. Moreover, in this case there is also a biLipschitz reflection across $\partial\Omega$, so it is easy to prove the following.

Corollary 1.6. *If Γ is a bounded quasicircle then there is a quasiconformal, Euclidean Lipschitz map of the plane which maps Γ to the unit circle.*

The following is another easy consequence.

Corollary 1.7. (The factorization theorem.) *Suppose Sullivan's theorem holds with quasiconformal constant K* (*as in Theorem* 1.4) *and that $f:\mathbf{D}\to\Omega$ is conformal. Then $f=g\circ h$, where $h:\mathbf{D}\to\mathbf{D}$ is a K-quasiconformal self-map of the disk and $g:\mathbf{D}\to\Omega$ is expanding in the sense that $|g'(z)|>C|f'(0)|$ for all $z\in\mathbf{D}$.*

The proof will actually show that g has the property that

$$\min_{z\in Q}|g'(z)|\geq C\max_{z\in T(Q)}|g'(z)|,$$

for any Carleson square Q and its top half, $T(Q)$. Thus $|g'|$ almost behaves as if it were increasing near the boundary.

Corollary 1.7 says that there is a universal $K<\infty$ so that an arbitrary conformal map $f:\mathbf{D}\to\Omega$ can only contract as much as a K-quasiconformal self-map of the disk can contract, and conjecturally we can take $K=2$. This observation can be applied to relate questions about conformal maps to questions about quasiconformal self-maps on the disk (in the hope that these are easier to handle). Some examples include

(1) $\|f'\|_{L^p(\mathbf{D},dx\,dy)}\geq C\|h'\|_{L^p(\mathbf{D},dx\,dy)}$ (Brennan's conjecture),
(2) $\dim(f(E))\geq\dim(h(E))$ (dimension distortion),
(3) $\int_0^{2\pi}|f'(re^{i\theta})|^t\,d\theta\leq C\int_0^{2\pi}|h'(re^{i\theta})|^t\,d\theta$ if $t<0$ (integral means),
(4) $\sum_{n=1}^{\infty}\operatorname{dist}(f(z_n),\partial\Omega)\geq C\sum_{n=1}^{\infty}\operatorname{dist}(h(z_n),\partial\mathbf{D})$ (deformations of Fuchsian groups).

We will discuss each of these briefly below. More details are given in Section 5.

We should also note that there are known "expansion properties" of conformal maps which do not seem to be directly related to Corollary 1.7. For example, Makarov's theorem [29] says that if $E\subset\mathbf{T}$ has dimension one then $\dim(f(E))\geq 1$ for every conformal map f on the disk. However, since a quasiconformal self-map of the disk can lower the dimension of some 1-dimensional sets, it does not seem that Makarov's theorem could be deduced from Corollary 1.7 (but see the remarks in Section 5 about minimal sets). Another example is the Hayman–Wu theorem [23]: for any line L, the inverse image $f^{-1}(L)$ has uniformly bounded length for any conformal f. If $f=g\circ h$ as above then $\gamma=g^{-1}(L)$ will have finite length because $|g'(z)|\geq C|f'(0)|$ for all $z\in\mathbf{D}$, but it is not clear why $h^{-1}(\gamma)$ should. Perhaps a more detailed study of the maps h which actually occur in Corollary 1.7, would give a new proof of the Hayman–Wu theorem.

To explain the connection between the factorization theorem and the Brennan conjecture we first recall what the latter is. Suppose Ω is a simply connected plane domain and $F=f^{-1}:\Omega\to\mathbf{D}$ is a conformal map. It is obvious that $\int_\Omega |F'|^2\,dx\,dy=\operatorname{area}(\mathbf{D})=\pi$ so that $F'\in L^2(\Omega, dx\,dy)$, but it is not clear what other L^p spaces F' must belong to. Gehring and Hayman (unpublished) showed that $F'\in L^p$ for $p\in\left(\frac{4}{3},2\right]$ and showed that the lower bound is sharp. Metzger [34] improved this to $p\in\left(\frac{4}{3},3\right)$. In 1978 James Brennan [13] improved this by showing one can take $p\in\left(\frac{4}{3},p_0\right)$ for some $p_0>3$ and conjectured that $p_0=4$ is possible (this is sharp since the Koebe function mapping $\mathbf{D}\to\mathbf{C}\setminus\left[\frac{1}{4},\infty\right)$ gives an $F'\notin L^4$). If one prefers to consider maps $f:\mathbf{D}\to\Omega$ then it is easy to check by change of variables that $f\in L^p$ it is equivalent to

$$\int_{\mathbf{D}} |f'|^{2-p}\,dx\,dy<\infty.$$

The best estimate (so far as I know) is currently due to Bertilsson [3], [4] who showed $p_0\geq 3.422$. This is a slight improvement of the earlier result of Pommerenke [39], [37], that $p_0\geq 3.399$.

In addition to its intrinsic interest, the Brennan conjecture has interesting consequences (e.g. see Section 5) and is currently under intense investigation. Some recent papers on the Brennan conjecture include the work of Carleson and Makarov [16], Hurri-Syrjänen and Staples [24], Barański, Volberg and Zdunik [2]. Moreover, the Brennan conjecture is now just a special case of the more general "universal spectrum conjecture", see e.g., [26], [33].

Recall Astala's recent (and remarkable) proof of the area distortion conjecture for quasiconformal maps [1]. One consequence of Astala's result is that if h is a K-quasiconformal map of the disk to itself, then $|h'|$ is in weak L^p where $p=$

$2K/(K-1)$. A function F is said to be in weak L^p if

$$\operatorname{area}(\{z : |F(z)| > \lambda\}) \leq \frac{C}{\lambda^p}.$$

In particular h' is in every L^p space with $p<2K/(K-1)$. Let $f:\mathbf{D}\to\Omega$ be conformal and let $f=g\circ h$ be the factorization given by Corollary 1.7. If the theorem holds with constant K then $h^{-1}:\mathbf{D}\to\mathbf{D}$ is also K-quasiconformal and so by Astala's theorem, $(h^{-1})'$ is in $L^p(\mathbf{D})$ for every $2\leq p<2K/(K-1)$. Thus for $p>2$ and $w=u+iv=h(x+iy)$,

$$\begin{aligned}
\int_{\mathbf{D}} |f'(z)|^{2-p}\, dx\, dy &\leq \int_{\mathbf{D}} |h'(z)|^{2-p} |g(h(z))|^{2-p}\, dx\, dy \\
&\leq |Cf'(0)|^{2-p} \int_{\mathbf{D}} |h'(z)|^{2-p}\, dx\, dy \\
&\leq |Cf'(0)|^{2-p} \int_{\mathbf{D}} |(h^{-1})'(w)|^{p}\, du\, dv,
\end{aligned}$$

which is finite if $p<2K/(K-1)$. Thus, if Sullivan's theorem holds for every $K>2$, then the Brennan conjecture is true.

In order to improve Bertilsson's result we would have to prove Sullivan's theorem with

$$K = \frac{3.422}{3.422-2} \approx 2.4065,$$

which is much better than currently known estimates. On the other hand, if suffices to prove something much weaker than Sullivan's theorem. First, we do not need the map to be group invariant (assuming the domain has a group acting on it). Secondly, we do not need a map which agrees with the convex hull map ι on the boundary: any Lipschitz, quasiconformal map to the disk will do. Finally, the map need not be K-quasiconformal on all of Ω, but only in a neighborhood of the boundary. This is because f' is bounded on compact subsets of Ω, so when we apply Astala's result in the argument above we can write $h=h_1\circ h_2$ where the complex dilatation of h_2 is supported in $\{z:1-\varepsilon\leq|z|\leq 1\}$ and h_1 is conformal in a neighborhood of the unit circle. Thus we really only need a map $g:\Omega\to\mathbf{D}$ with

$$K_{\partial\Omega}(g) = \inf_{\varepsilon>0} \sup_{\operatorname{dist}(z,\partial\Omega)<\varepsilon} K_g(z) \leq 2,$$

to deduce that Brennan's conjecture holds for a bounded domain Ω. This version at least has the advantage that it is obvious for polygons and smooth domains. Brennan's conjecture is known in various special cases (e.g., close to convex), but it is not obvious whether Sullivan's theorem holds for $K=2$ in these same cases.

If Sullivan's theorem actually holds with $K=2$ we get something a little stronger than Brennan's conjecture.

Theorem 1.8. *If there is a 2-quasiconformal, C-biLipschitz map from Ω to S, then for every conformal map $f:\Omega\to\mathbf{D}$, we have that f' in in weak $L^4(\Omega)$.*

Our second application of Corollary 1.7 concerns dimension distortion. Given a set $E\subset\partial\mathbf{D}$ we would like to estimate

$$\inf_f \dim(f(E)),$$

where the infimum is over all conformal maps from the disk. For simplicity assume that f is a univalent map onto a quasiconvex domain Ω. In this case, we can write $f=g\circ h$, where g satisfies $|g(x)-g(y)|\geq C|x-y|$ for all points in the disk. Thus for any set F, $\dim(g(F))\geq\dim(F)$. Thus $\dim(f(E))\geq\dim(h(E))$ for any set E. Astala has proven the sharp estimate of K-quasiconformal maps,

$$\dim(h(E))\geq\frac{2\dim(E)}{2K+(1-K)\dim(E)}.$$

Taking $K=2$ would give

$$\dim(f(E))\geq\dim(h(E))\geq\frac{\dim(E)}{2-\frac{1}{2}\dim(E)},$$

which is a conjecture of Carleson and Makarov. To remove the restriction that Ω is quasiconvex requires showing the following result.

Lemma 1.9. *If $g:\mathbf{D}\to\Omega$ is quasiconformal and $|g'(z)|$ is bounded below, then $\dim(g(E))\geq\dim(E)$ for any $E\subset\mathbf{T}$, i.e., g cannot reduce dimensions.*

This can be done by modifying Makarov's crosscut method in Theorem 1.4, [30].

The third application is to integral means. For a conformal map f, we define the integral mean

$$I(t,f)=\limsup_{r\to 1}\frac{\log\int_0^{2\pi}|f'(re^{i\theta})|^t\,d\theta}{-\log(1-r)},$$

and set

$$B(t)=\sup I(t,f),$$

where the supremum is taken over all univalent maps f. For $1\leq K<\infty$, define the integral means for quasiconformal self-maps of the disk by

$$I(t,f)=\limsup_{r\to 1}\frac{\log\int_0^{2\pi}((1-|f(re^{i\theta})|)/(1-r))^t\,d\theta}{-\log(1-r)},$$

and set

$$B(K,t) = \sup I(t,f),$$

where the supremum is over all K-quasiconformal maps of the unit disk to itself. We have used the distance to the boundary instead of h' since we can choose h in Corollary 1.7 so that $|h'(z)| \sim 1-|h(z)|$. Corollary 1.7 implies

$$B(t) \le B(K,t), \quad t<0,$$

for some universal $K<\infty$ and conjecturally $K=2$. It is easy to prove that $B(2,t)= -t-1$ if $t<-2$ (see Theorem 5.5), which would give another conjecture of Carleson and Makarov [16].

Our final application comes from the theory of Kleinian groups. Recall that for a non-elementary Fuchsian or Kleinian group G acting on an invariant domain Ω, the critical exponent of the Poincaré series is given by (see e.g., [11]),

$$\delta(G) = \inf\left\{ s : \sum_{g\in G} \operatorname{dist}(g(z_0), \partial\Omega)^s < \infty \right\},$$

where z_0 is any point of Ω. A Fuchsian group is called divergence type if this series diverges for $s=1$.

Suppose G is a Fuchsian group and $f\colon \mathbf{D} \to \Omega$ is a conformal map such that $G' = f \circ G \circ f^{-1}$ is a Kleinian group acting on Ω. Then we call G' a deformation of G by Φ. The group G' acts by isometries on hyperbolic upper half-space and hence on S by isometries of the path metric. This action gives a Fuchsian group H which is K-quasiconformally equivalent to the original group G (with a uniformly bounded K). By a theorem of Pfluger [35], G is divergence type if and only if H is divergence type, and $|g'|$ bounded below implies G' also diverges for exponent 1 (in fact, its Poincaré series is term-by-term bounded below by a multiple of H's Poincaré series.

Corollary 1.10. *If G' is a deformation of a divergence type Fuchsian group G then the Poincaré series of G' also diverges for $s=1$.*

In a similar vein, Fernández and Rodríguez [20] showed that the property $\delta(G)=1$ is also invariant under quasiconformal conjugation on the disk, so the same proof shows that if $\delta(G)=1$ then $\delta(G')\ge 1$ for any deformation of G (however this is already known and can be proved using the fact that $\delta = \dim(\Lambda_c)=1$ (Theorem 1.1 of [9]) and the theorem of Makarov that this dimension cannot be decreased by a conformal map (Theorem 0.4 of [30])).

In a 1979 paper Rufus Bowen [12] proved that the limit set of a quasiconformal deformation of a cocompact Fuchsian group is either a circle or has dimension strictly larger than 1. Dennis Sullivan [43] soon extended this property to cofinite groups, and in 1990 Kari Astala and Michel Zinsmeister showed that it was false for convergence type groups; every such group has a quasi-Fuchsian deformation whose limit set is a non-circular rectifiable curve. In [8] a version of Corollary 1.10 is used to show that Bowen's property holds if and only if G is divergence type. A similar application is given in [7], to show that Ruelle's property fails for every infinite area divergence group with a positive lower bound on its injectivity radius, i.e., there is an analytic deformation of the group for which the dimension of the limit set does not change analytically.

Now we describe the structure of the rest of this paper. In Section 2 we prove Theorem 1.4 and in Section 3, Theorem 1.8. In Section 4 we give another proof of Theorem 1.4 which does not use Sullivan's theorem (and does not give the "right" boundary values). In Section 5 we list a number of related results and questions.

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Added in proof. A version of the factorization theorem for maps $f:\mathbf{D}\to\Omega$ onto a quasidisk follows from Lemma 3.21 of "Doubling conformal densities" by Mario Bonk, Juha Heinonen and Steffen Rohde, to appear in J. Reine Angew. Math. The factors satisfy estimates depending only on the quasidisk constant of Ω.

2. Proof of Theorem 1.4

In this section we will show that the quasiconformal map $\iota\circ\sigma:\Omega\to S\to\mathbf{D}$ in Sullivan's theorem may be taken to be locally Lipschitz with respect to the Euclidean metrics. Since we assume that this map is biLipschitz between the respective hyperbolic metrics, and it is well known that the hyperbolic metrics are comparable to $ds/\operatorname{dist}(z,\Omega)$, it suffices to show that there is an $M<\infty$ so that

$$1-|\iota(\sigma(z))|\leq M\operatorname{dist}(z,\partial\Omega)$$

for every $z\in\Omega$.

It is convenient to work in the ball model of hyperbolic space. The mapping of Ω with the Euclidean metric to Ω with the spherical metric is Lipschitz, so there is no loss of generality in assuming that Ω has the spherical metric. Let $z\in\Omega$ and let $w\in S$ be its image under the nearest point retraction. Let P be the hyperbolic plane tangent to S at w so that the geodesic connecting w to z is perpendicular to P at w. Let $D\subset\Omega$ be the disk bounded by P. It is easy to see that

$$\operatorname{dist}(z,\partial\Omega)\geq\operatorname{dist}(z,\partial D)\simeq 1-|w|$$

(both distances are in the Euclidean sense). On the other hand, the usual estimates for the hyperbolic metric in a half-space give

$$1-|w| \simeq \exp(-\varrho(w_0,w)) \geq \exp(-\varrho_S(w_0,w)) \simeq 1-|\iota(w)|.$$

Thus $1-|\iota(R(z))| \lesssim \operatorname{dist}(z,\partial\Omega)$.

We also need to know that R is a rough isometry from the hyperbolic metric on Ω to the path metric on S, i.e.,

$$\frac{1}{A}\varrho(z,w)-B \leq \varrho_S(R(z),R(w)) \leq A\varrho(z,w)+B,$$

for some $0<A,B<\infty$ (this is also called a quasi-isometry by some authors). This is proven as part of the proof of Lemma 8 in [7].

Next, note that $T=\iota\circ R\circ\sigma^{-1}\circ\iota^{-1}$ is a rough isometry of the hyperbolic disk to itself which extends to the boundary as the identity. Thus any point and its T-image lie within a uniformly bounded hyperbolic distance of each other (this is left as an exercise for the reader and is also Theorem 7.15 of [45]). Hence

$$1-|\iota(\sigma(z))| \lesssim 1-|\iota(R(z))|,$$

as desired. This completes the proof of Theorem 1.4.

3. Proof of Theorem 1.8

Next we give the proof of Theorem 1.8. If Sullivan's theorem holds for $K=2$, then the conformal map $F=f^{-1}:\Omega\to\mathbf{D}$ can be written as $F=H\circ G$ where $H=h^{-1}$ is a 2-quasiconformal self-map of the disk and $G=g^{-1}:\Omega\to\mathbf{D}$ satisfies $\|G'\|_\infty \leq M$. By Astala's theorem from [1], H' is in weak L^4 on the disk. From this we want to deduce that F' is in weak L^4 on Ω. Let

$$\begin{aligned}
F_\lambda &= \{z\in\Omega : |f'(z)|>\lambda\},\\
E_\lambda &= \{z\in\Omega : 2\lambda \geq |F'(z)|>\lambda\},\\
E_{\lambda,n} &= \{z\in E_\lambda : 2^{-n}M \leq |G'(z)| \leq 2^{-n+1}M\}.
\end{aligned}$$

Since $|F'(z)| \simeq |H'(G(z))|\,|G'(z)|$,

$$G(E_{\lambda,n}) \subset \{z\in\mathbf{D} : |H'(z)| \gtrsim \lambda 2^n\},$$

and hence has area $\lesssim \lambda^{-4}2^{-4n}$, since H' is in weak L^4. Thus $\operatorname{area}(E_{\lambda,n}) \lesssim \lambda^{-4}2^{-2n}$. Summing over $n=0,1,2,\dots$, gives $\operatorname{area}(E_\lambda) \lesssim \lambda^{-4}$. Now summing over λ, 2λ, $4\lambda,\dots$, gives

$$\operatorname{area}(F_\lambda) \lesssim \frac{1}{\lambda^4}\sum_{n\geq 0}\frac{1}{2^{4n}} \simeq \frac{1}{\lambda^4},$$

which is the desired result. Thus Sullivan's theorem with quasiconformal constant $K=2$ implies the weak L^4 version of Brennan's conjecture.

4. A direct proof of Corollary 1.7

In this section we will prove the following result.

Theorem 4.1. *There is a $K<\infty$ so that for any simply connected Ω in the plane, there is a K-quasiconformal map $h:\Omega\to\mathbf{D}$ with bounded gradient, i.e., h is Lipschitz from the internal Euclidean path metric on Ω to the Euclidean metric on $\mathbf{D}$.*

This result is weaker than Corollary 1.7 since this result does not require the map h to agree with ι on the boundary of Ω. However, $K=2$ in this weaker result would still imply Brennan's conjecture, so it seems worthwhile to present this different approach. We will actually prove the following version.

Theorem 4.2. *There is a $K<\infty$ so that the following holds. If $f:\mathbf{D}\to\Omega$ is univalent, then there is a K-quasiconformal map $h:\mathbf{D}\to\mathbf{D}$ which is biLipschitz (from the hyperbolic metric to the hyperbolic metric) and there is a $C<\infty$ such that*

$$1-|h(z)|\le C|f'(z)|(1-|z|), \tag{1}$$

$$|h'(z)|\le C|f'(z)| \tag{2}$$

for every $z\in\mathbf{D}$.

If we can prove this, then $g=h\circ f^{-1}$ will be a K-quasiconformal locally Lipschitz map from Ω to $\mathbf{D}$, as desired in Theorem 4.1. To prove Theorem 4.2, we will use the fact that $u=\log|f'|$ is a harmonic Bloch function. By definition, this means that there is a $B<\infty$ so that

$$|\nabla u(z)|(1-|z|^2)\le B$$

for every $z\in\mathbf{D}$. The infimum of such B's is called the Bloch norm of u. A very useful fact about harmonic Bloch functions is that they can be uniformly approximated by a dyadic Bloch martingale $\{u_I\}$ (see e.g., [32], [10], [21]). A dyadic martingale on the circle is a function on the collection of dyadic intervals so that if I_1 and I_2 are children of I then u_I is the average of u_{I_1} and u_{I_2}. Such a martingale is called Bloch if there is a $B<\infty$ such that $|u_I-u_J|\le B$ whenever I is a child of J. The approximation property says that given a harmonic Bloch function u there is a Bloch martingale $\{u_I\}$ and an $A<\infty$ (depending only on the Bloch norm of u) such that

$$|u(z_I)-u_I|\le A$$

for every dyadic interval I. The point z_I is chosen so that $1-|z|=|I|$ and so that its radial projection is the center of I.

We will also use the fact that every quasisymmetric homeomorphism of the circle to itself has a quasiconformal extension to the disk. A homeomorphism of the circle to itself is called k-quasisymmetric if for any two adjacent intervals I and J of the same length,

$$\frac{1}{k} \leq \frac{|h(I)|}{|h(J)|} \leq k,$$

and is called quasisymmetric if there is a $k<\infty$ for which this is true. It is an easy and well-known fact, that in order to show h is quasisymmetric it suffices to show there is such a k that works for all dyadic intervals (or more generally for all b-adic intervals of lengths b^{-n}, $n=1,2,...$). It is also well known that every k-quasisymmetric homeomorphism of the circle extends to a K-quasiconformal self-map of the disk (see e.g. [17]), and that one may take $K=\min(k^{3/2}, 2k-1)$ (e.g., see p. 33 of [28], [27]). Moreover, we may take this extension to be smooth on $\mathbf{D}$, biLipschitz from the hyperbolic metric on the disk to itself and satisfy

$$|h'(z)| \simeq \frac{|h(I)|}{|I|}$$

for every z within a bounded hyperbolic distance of z_I. Using these facts, it now suffices to show the following theorem.

Theorem 4.3. *Suppose u is a harmonic Bloch function with corresponding Bloch martingale $\{u_I\}$. Then there is a quasisymmetric homeomorphism $h\colon \mathbf{T} \to \mathbf{T}$ and a $C_1<\infty$ such that*

$$|h(I)| \leq C_1 |I| \exp(u_I) \tag{3}$$

for every dyadic interval I. The quasisymmetric constant of h depends only on the Bloch constant B of u.

Proof. Before starting the proof, we recall that to check that h is quasisymmetric, it suffices to consider only adjacent dyadic intervals (this is well known but we include the argument for completeness). Suppose I and J are any two disjoint, adjacent intervals. Then I can be covered by two dyadic intervals I_1 and I_2 with length $\leq |I|$. Similarly for J, J_1 and J_2. Moreover, there is a dyadic $J_3 \subset J$ with length $\geq \frac{1}{2}|J|$. Using the quasisymmetric condition for dyadic intervals and the fact that all these intervals are close to each other, we get

$$|h(I)| \leq |h(I_1)| + h(I_2)| \lesssim |h(J_1)| + |h(J_2)| \lesssim |h(J_3)| \leq |h(J)|,$$

and similarly for the other direction.

Thus it suffices to define the h images of dyadic intervals and check the quasisymmetric condition for them. We define h in two steps: a first approximation based on the size of u, followed by a modification.

Suppose N is a positive integer (to be fixed later in the proof depending only on B). We define h by specifying the length of the image of each dyadic interval of length 2^{-nN}, $n=0,1,2,\dots$, inductively. We start with the trivial step, $h(\mathbf{T})=\mathbf{T}$. Now suppose I is a dyadic interval and N is an integer to be chosen later. Consider the Nth generation dyadic descendants of I, $\{I_k\}_{k=1}^{2^N}$, define

$$\Delta_k = u_{I_k} - u_I,$$

and choose h so that

$$|h(I_k)| = \frac{\varphi(\Delta_k)}{\sum_{j=1}^{2^N}\varphi(\Delta_j)}|h(I)|, \tag{4}$$

where $\varphi(t)=e^t$ if $t\le 0$ and $\varphi(t)=t+1$ if $t\ge 0$. Note that φ is continuous, increasing and convex. We claim that

$$2^N \le \sum_{j=1}^{2^N}\varphi(\Delta_j) \le CBN2^N. \tag{5}$$

The left-hand inequality holds by convexity, and the fact that $\{\Delta_k\}=\{u_{I_k}-u_I\}$ has mean value zero. The right-hand inequality holds since the Bloch property implies that each of the 2^N terms is bounded by CBN.

The left-hand inequality of (5) implies that

$$|h(I_k)| \le \frac{1}{2^N}\exp(u_{I_k}-u_I)|h(I)| = \exp(u_{I_k}-u_I)\frac{|I_k|}{|I|}|h(I)|.$$

Induction and telescoping series then give

$$|h(I_k)| \le |I_k|\exp(u_{I_k}), \tag{6}$$

which is the desired estimate (3) with $C_1=1$.

If we only wished to compare dyadic intervals which have the same parents, then the definition of h given so far would be enough. But to compare intervals with different parents, we need to modify what we have done.

First, consider two adjacent dyadic intervals I and J in generation $(n+1)N$ which have the same ancestor in generation nN. Then by (4) we see that $h(I)$ and

$h(J)$ have comparable lengths, since the denominator in (4) is the same for both intervals and the numerators are comparable by at most a factor of $A_1 = \exp(BN)$.

To handle intervals with different ancestors, we modify h as follows. Using induction, suppose we have shown that all dyadic intervals of generation nN have h-images which are comparable with a factor of A_2 (the value of A_2 will be chosen at the end of the proof depending only on the Bloch norm of u). Suppose I and J are adjacent dyadic intervals in generation $(n+1)N$ with different ancestors I^* and J^* in generation nN. Without loss of generality, assume $|h(I)| \leq |h(J)|$. Since u is Bloch, there is an A_3 (depending only on the Bloch constant of u) such that $|u_I - u_J| \leq A_3$. Thus the numerators in (4) corresponding to I and J differ by a factor of at most A_3 while the denominators differ by a factor of at most BN (using (5)). Thus $|h(J)|/|h(I)| \leq A_2 A_3 BN$.

For each adjacent pair I, J with different (nN)th generation ancestors we modify h as follows to get a new homeomorphism h_0. As above assume that $|h(I)| \leq |h(J)|$. If $|h(J)|/|h(I)| \leq A_2$ we do nothing, i.e., $|h_0(I)| = |h(I)|$. Otherwise, we increase the size of $|h(I)|$ until $|h(J)|/|h(I)| = A_2$. This makes $|h(I)|$ at most $A_3 BN$ times longer than it was before. To make up for this we have to make other intervals shorter. We will take the $2^{N-1}-1$ intervals $\{I_j\}$ descended from I^* in N steps which are closest to I. Together with I these intervals make up half of I^*; we denote this half of I^* by K. Decrease the image length of each I_j by a factor of $\lambda = (|h(K)| - |h_0(I)|)/(|h(K)| - |h(I)|)$. This factor is chosen so that $h_0(K)$ has the same length as $h(K)$ has. Note that

$$|h(I)| \leq \frac{A_4 N}{2^N} |h(K)|$$

for some $A_4 < \infty$ (depending only on the Bloch norm B of u) so that λ is bounded from below by

$$\lambda \geq \frac{1 - A_4 N 2^{-N}}{1 - A_4 N 2^{-N} A_2 A_3 BN} \geq \frac{1}{2},$$

if N is large enough, depending only on the other constants. Fix a value of N so that this holds.

Now suppose we have done the modifications for all such adjacent pairs I, J. To see that h_0 is quasisymmetric we have to see that any two adjacent intervals in generation $(n+1)N$ have comparable images under h_0. There are four cases.

Case 1. Suppose I and J have different ancestors in generation nN. Then by construction, the images differ by a factor of at most A_2.

Case 2. Next suppose I and J have the same ancestor I^* in generation nN and suppose I is adjacent to an endpoint of I^*. Then before the modification, the

h-images of I and J were comparable with a factor of A_1. After the modification the image of J was shrunk by at most a factor of 2 and the image of I was increased by a factor of at most A_3BN. Thus after the modification the images are comparable with a factor of at most $2A_3BN$.

Case 3. Suppose I and J have the same ancestor I^* in generation nN and that they are both adjacent to the midpoint of I^*. Then before the modification their images are comparable with a factor of A_1 and after the modification they have both been decreased by a factor of at most 2. Thus they are now comparable with a factor of $2A_1$.

Case 4. Suppose I and J have the same ancestor I^* in generation nN but neither I nor J is adjacent to the midpoint or either endpoint of I^*. Thus both intervals are in the same half of I^*. Before the modification their images were comparable with a factor of A_1 and when modified both images were decreased by the same amount. Thus they are still comparable with factor A_1.

Thus if we choose $A_2 \geq \max(2A_3BN, 2A_1, \exp(A_3))$, we see that A_2 depends only on B (since this is true of A_1, A_3 and N) and any two adjacent intervals have h_0-images whose lengths are comparable by a factor of at most A_2, as desired.

Finally, we have to check that the modification we made changing h to h_0 did not alter the estimate (3) for nNth generation intervals. For intervals whose images were made smaller, it certainly still holds. Each interval I whose image was enlarged is adjacent to an interval J which was left alone and for which the estimate holds. In this case, $|h_0(I)| = |h_0(J)|/A_2$ and $|u_I - u_J| \leq A_3$,

$$|h_0(I)| \leq \frac{|h_0(J)|}{A_2} \leq \frac{|J| \exp(u_J)}{A_2} \leq \frac{|I| \exp(A_3) \exp(u_I)}{A_2} \leq |I| \exp(u_I),$$

since $A_2 \geq \exp(A_3)$. Thus (3) holds with $C_1 = 1$ for every dyadic interval of generation nN. From this it is easy so see that it holds for every dyadic interval with some C_1 depending only on N (and hence only on B). This completes the proof of the theorem. $\square$

5. Questions and conjectures

In this section we will discuss some related results and questions arising from Sullivan's theorem. There is some overlap with our comments from the introduction.

5.1. The weak type Brennan conjecture

Based on Astala's theorem it seems reasonable to ask if the following slight strengthening of Brennan's conjecture holds.

Question 5.1. *Suppose Ω is a simply connected plane domain and $f:\Omega\to\mathbf{D}$ is conformal, one-to-one and onto. Is f' in weak $L^4(\Omega, dx\,dy)$?*

This version is known to experts on the problem, but I have not seen it in the literature.

5.2. Sharp constants

To apply Sullivan's theorem to the Brennan conjecture, we only need a quasiconformal map from Ω to $\mathbf{D}$ with bounded derivative. We do not need the additional condition given in the theorem that the boundary values agree with the map ι on the boundary. All we really want to know is how to compute

$$K(p,\Omega)=\inf\{K:\ \text{there is } f\in \mathrm{QC}(K,\Omega) \text{ with } f'\in L^p(\Omega)\},$$

(where $\mathrm{QC}(K,\Omega)$ denotes the K-quasiconformal maps from Ω to $\mathbf{D}$) and

$$K(p)=\sup_{\Omega} K(p,\Omega),$$

whereas Sullivan's theorem deals with

$$\begin{aligned} K_S(p,\Omega)&=\inf\{K:\ \text{there is } f\in \mathrm{QC}(K,\Omega) \text{ with } f'\in L^p(\Omega),\ f|_{\partial\Omega}=\iota|_{\partial\Omega}\},\\ K_S(p)&=\sup_{\Omega} K_S(p,\Omega),\end{aligned}$$

where $\iota: S\to\mathbf{D}$ is the isometry of the convex hull boundary to the unit disk.

Question 5.2. *Compute $K(p)$ and $K_S(p)$ for each $p\le\infty$.*

By Pommerenke's result, $K(p)=1$ for $p<3.339$, and the Brennan conjecture claims that $K(p)=1$ for $p<4$. For individual domains Ω it is possible for $K(\infty,\Omega)<K_S(\infty,\Omega)$. For example, let Ω be the convex hull of the unit disk and the point 2. It is easy to check that the conformal map of Ω to $\mathbf{D}$ has bounded derivative and hence $K(\infty,\Omega)=1$. On the other hand, there is no conformal map f from Ω to S which is the identity on $\partial\Omega$ because it is easy to check that then $\iota\circ f$ would be a conformal map of Ω to $\mathbf{D}$ which is Möbius on a circular arc of $\partial\Omega$ which is impossible by the uniqueness of analytic mappings. By compactness of quasiconformal mappings this means that $K_S(\infty,\Omega)>1$.

Question 5.3. *Is $K_S(\infty)>K(\infty)$ or are they equal?*

If they are equal then the convex hull mapping ι somehow picks out the optimal way of mapping $\partial\Omega$ to the circle. Is there an explanation of why this should happen?

5.3. An explicit example

It is not clear what the best constant is even for some simple finitely bent domains (note that Brennan's conjecture always holds for such examples since they are piecewise smooth). An explicit example where I do not know the optimal K is illustrated in Figure 1: it consists of the union of nine unit disks of radius $\frac{1}{2}\sqrt{2}$ centered at 0, ± 1, $\pm i$, ± 2, $\pm 2i$.

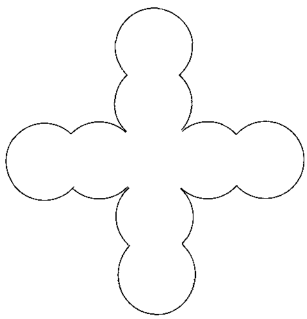

Figure 1. What is the best K for this domain?

5.4. The number of big disks

Another application of Brennan's conjecture concerns harmonic measure. If Ω is simply connected in the plane the harmonic measure on $\partial\Omega$ is the push forward of Lebesgue measure on the circle under a conformal map onto Ω and depends on the choice of base point (the image of zero under the conformal map). If $x\in\partial\Omega$ then Beurling's projection theorem [36] implies that the harmonic measure of $D(x,r)$ is at most $Cr^{-1/2}$ (where C only depends on the choice of base point for harmonic measure). A theorem of Carleson and Makarov [16] says that there are constants $C, M<\infty$ so that for any $\varepsilon>0$, the number of disjoint disks with harmonic measure $\geq r^{\varepsilon+1/2}$ is at most $Mr^{-C\varepsilon}$. Moreover, Brennan's conjecture is true if and only if $C=4$ is sharp, i.e., if and only if for every $C>4$ there is an M which makes this true. If Sullivan's theorem is true for $K=2$, then one can easily show that the Carleson–Makarov result holds for $C=4$ (see the first part of the proof of Theorem 5.5 below).

5.5. The dimension of the convex hull measure

The isometry ι^{-1} from the unit disk to the convex hull surface S defines a measure μ_{CH} (the "convex hull measure") by pushing normalized Lebesgue measure on the circle onto $\partial\Omega$. This is analogous to the usual harmonic measure ω defined by pushing Lebesgue measure forward by the Riemann mapping of $\mathbf{D}$ onto Ω. Makarov's remarkable work shows that ω has dimension 1 for any simply connected domain, i.e., $\omega(E)=0$ for every set with $\dim(E)<1$ but $\omega(E)=1$ for some set with $\dim(E)=1$.

In general, we define the dimension of a measure to be

$$\dim(\mu)=\inf\{\dim(E):\mu(E^c)=0\}.$$

Question 5.4. *What is the dimension of the convex hull measure? More precisely, compute* $\sup_\Omega \dim(\mu_{CH})$, *were the supremum is over all simply connected domains. Is it* $\frac{4}{3}$?

It is not hard to see that this supremum is strictly between 1 and 2. First of all, since the convex hull measure can be written as the image of Lebesgue measure under a quasiconformal map on the disk that has a lower bound on its gradient, the measure of any disk is always bounded above by a uniform multiple of its radius, i.e., $\mu_{CH}(D(x,r))\leq Mr$. This implies that $\dim(\mu_{CH})\geq 1$.

In [5], examples of domains where $\dim(\mu_{CH})>1$ are constructed using special deformations of certain divergence type Fuchsian groups. The construction there is not stated in this language, but it is shown that there are examples such that

$$\limsup_{r\to 1}\frac{\varrho^3_{\mathbf{H}}(\iota^{-1}(re^{i\theta}),\iota^{-1}(0))}{\varrho_{\mathbf{D}}(re^{i\theta},0)}\leq\frac{1}{1+\varepsilon}<1$$

holds for Lebesgue almost every θ. From this condition it is easy to check that

$$\mu_{CH}(D(x,r))\leq Cr^{1+\varepsilon}$$

for μ_{CH} almost every x and for all sufficiently small r (depending on x), and this in turn implies $\dim(\mu_{CH})\geq 1+\varepsilon>1$ by standard estimates.

On the other hand, μ_{CH} can also be written as the image of $f\circ g$ where g is a K-quasiconformal self-map of the disk and f is conformal. The g image of a set of dimension 1 is uniformly bounded below in terms of K (use e.g., Astala's theorem [1]). On the other hand, the push forward under f of α-dimensional Hausdorff measure is singular to $(2-C\alpha)$-dimensional Hausdorff measure by a result of Jones and Makarov (Theorem B-3 of [25]). Combining these results shows that $\dim(\mu_{CH})$ is bounded away from 2 by a constant depending only on the constant K in Sullivan's theorem.

5.6. Integral means

Suppose $f:\mathbf{D}\to\Omega$ is a univalent map and define the integral mean

$$I(t,f)=\limsup_{r\to1}\frac{\log\int_0^{2\pi}|f'(re^{i\theta})|^t\,d\theta}{-\log(1-r)},$$

and set

$$B(t)=\sup I(t,f),$$

where the supremum is taken over all univalent maps f. In other words, $B(t)$ is the smallest number β such that,

$$\int_0^{2\pi}|f'(re^{i\theta})|^t\,d\theta=O\biggl(\Bigl(\frac{1}{1-r}\Bigr)^\beta\biggr)$$

for all univalent f. Clearly $B(t)\geq0$ and Hölder's inequality implies that $B(t)$ is a convex function of t (since L^p norms of a function are log-convex). The Brennan conjecture is equivalent to $B(-2)=1$ (see e.g., [33]). More generally, it is conjectured that $B(t)=\max\bigl(|t|-1,\frac{1}{4}t^2\bigr)$. Brennan's conjecture, easy known results and convexity would imply that this is true for all $t\leq-2$ (it is currently known for all $t\leq t_0<-2$ by a theorem of Carleson and Makarov [16]).

For $1\leq K<\infty$, define the integral means for quasiconformal self-maps of the disk by

$$I(t,f)=\limsup_{r\to1}\frac{\log\int_0^{2\pi}((1-|f(re^{i\theta})|)/(1-r))^t\,d\theta}{-\log(1-r)},$$

and set

$$B(K,t)=\sup I(t,f)$$

where the supremum is over all K-quasiconformal maps of the unit disk to itself. We have used distance to the boundary instead of derivatives so that the integral exists for all r. By Koebe's distortion theorem, this yields the correct values in the conformal ($K=1$) case. Corollary 1.7 shows that Sullivan's theorem with constant K implies $B(t)\leq B(K,t)$ for $t\leq0$, and so for Brennan's conjecture it would be interesting to compute $B(2,t)$. Some values are easy, e.g., Astala's theorem combined with the argument in [16] gives the following result.

Theorem 5.5. *If* $t<-2/(K-1)$ *then* $B(K,t)=-(K-1)t-1$.

Proof. Suppose f is a K-quasiconformal self-map of the disk and let $g=f^{-1}$ be its inverse. By Astala's theorem, g' is in weak L^p with $p=2K/(K-1)$. Fix a

$0<r<1$ and let $N_\varepsilon(\alpha)$ be the maximum number of disjoint intervals I of length r on $\partial\mathbf{D}$ such that $f(I)$ has length

$$r^{1+\alpha+\varepsilon} \leq |f(I)| \leq r^{1+\alpha}.$$

Let $\{I_k\}$ be a maximal collection of such intervals and let $\{J_k\}$ be their images under f. Let $\{Q_k\}$ be the Carleson squares corresponding to the intervals $\{J_k\}$. Then the top half of Q_k has area $\gtrsim r^{2(1+\alpha+\varepsilon)}$ and on this top half, $|g'|\simeq r^{-\alpha}$. Since g' is in weak L^p, this implies that the region where $|g'|\geq r^{-\alpha}$ has total area less than $r^{p\alpha}$. Thus

$$N_\varepsilon(\alpha) \lesssim r^{p\alpha} r^{-2(1+\alpha+\varepsilon)} = r^{(p-2)\alpha-2-2\varepsilon}.$$

Moreover, a K-quasiconformal self-map of the disk is Hölder of order $1/K$, so there can be no such intervals if $\alpha>K-1$, so $N_\varepsilon(\alpha)=0$ if $\alpha>K-1$.

We can now estimate the integral means just as in Section 3.5 of [16]. Let $\alpha_j=\varepsilon j$ for $j=1,\dots,(K-1)/\varepsilon$, then

$$\int_0^{2\pi} |f'((1-r)e^{i\theta})|^t \, d\theta \lesssim \sum_{j=1}^{(K-1)/\varepsilon} r^{1+\alpha_j t} N_\varepsilon(\alpha_j) \lesssim \sum_{j=1}^{(K-1)/\varepsilon} r^{1+\alpha_j t+(p-2)\alpha_j-2-2\varepsilon}.$$

The exponent is negative for $0\leq\alpha_j\leq K-1$ and if $t<-2(K-1)$ it attains its most negative value when $\alpha_j=K-1$ and this value is equal to $(K-1)t+1-2\varepsilon$. Thus

$$\int_0^{2\pi} |f'((1-r)e^{i\theta})|^t \, d\theta \lesssim \sum_{j=1}^{(K-1)/\varepsilon} r^{(K-1)t+1-2\varepsilon} \lesssim \frac{K-1}{\varepsilon}\left(\frac{1}{r}\right)^{-(K-1)t-1+2\varepsilon}.$$

Taking $\varepsilon\to 0$ shows $B(K,t)\leq -(K-1)t-1$ for $t\leq -2/(K-1)$.

To show $B(K,t)\geq -(K-1)t-1$ for $t\leq -1/(K-1)$, simply consider the K-quasiconformal map on the upper halfplane given in polar coordinates by $(r,\theta)\mapsto(r^K,\theta)$. The details are left to the reader. $\square$

Unfortunately, this does not give anything interesting for conformal maps until K is close to 2. For values of t close to 0, we can make the following guess based on the corresponding conjecture for conformal maps.

Question 5.6. *For $-2/(K-1)\leq t\leq 0$, is it true that $B(K,t)=\frac{1}{4}(K-1)^2t^2$?*

Question 5.7. *For $t\leq 0$ is it true that $B(t)=B(2,t)$?*

Even if Sullivan's theorem is true for $K=2$ it is not clear whether we should expect $B(t)=B(2,t)$. First of all, the set of 2-quasiconformal maps which arise from simply connected domains via Sullivan's theorem may only be a "small" subset of all 2-quasiconformal self-maps of the disk, with strictly smaller integral means. Second, saying $B(t)=B(2,t)$ says that the "expanding" factor in Corollary 1.7 can be ignored, which might not be the case.

For $t\geq 0$ we know that $B(t)\geq B(K,t)$ for some K and conjecture that $B(t)\geq B(2,t)$. However, this is probably not useful since for $t>0$ the main contribution to $B(t)$ comes from places where $|f'|$ is large and hence depends on the "expanding" part of the decomposition. In fact, it is not hard to see that $B(K,1)=0$ for all $K\geq 1$, which is known to be strictly less than $B(1)$ (e.g., see [15]).

5.7. Minimal sets

Although conformal maps can reduce the dimension of some subsets of $\mathbf{T}$, they cannot reduce the dimension of every subset. We shall say a set $E\subset\mathbf{T}$ is minimal for conformal maps if $\dim(f(E))\geq\dim(E)$ for every conformal $f:\mathbf{D}\to\Omega$. Obviously intervals are minimal and Makarov proved that every set of dimension 1 is minimal, but it is not obvious whether there are minimal sets with dimension $0<\alpha<1$. Similarly we shall say $E\subset\mathbf{T}$ is conformally thick if $f(E)$ has positive 1-dimensional measure for every conformal map f (otherwise there is a map f so that $f(E)$ has zero linear measure; such an E is called an L-set by Makarov in [31]).

A subset E of the unit circle $\mathbf{T}$ is called minimal for quasisymmetric mappings if $\dim(f(E))\geq\dim(E)$ for every quasisymmetric map $f:\mathbf{T}\to\mathbf{T}$. Jeremy Tyson has conjectured that there are no such minimal sets with dimension $0<\alpha<1$. Using Corollary 1.7 and Lemma 1.9, it is evident that the following result holds.

Corollary 5.8. *If $E\subset\mathbf{T}$ is minimal for quasisymmetric maps it is also minimal for conformal maps.*

Thus Tyson's conjecture can be related to Makarov's theorems. A special type of minimal set for quasisymmetric maps are the quasisymmetrically thick sets, i.e., $f(E)$ has positive length for every quasisymmetric self-map f of the circle. As before it is evident that the following result is true.

Corollary 5.9. *If E is quasisymmetrically thick then it is conformally thick.*

So far as I know, this is a new result, but it is hardly surprising. A result of Staples and Ward [41] says that if $E\subset\mathbf{T}$ is compact with complementary intervals $\{I_j\}$ then E is quasisymmetrically thick if $\sum_j |I_j|^\alpha<0$ for every $\alpha>0$. On the other hand, Makarov [31] has shown E is conformally thick if we have

$$\sum_j |I_j|\sqrt{\log\frac{1}{|I_j|}\log\log\log\frac{1}{|I_j|}}<\infty,$$

which is much, much weaker. Quasisymmetrically thick sets have also been considered by Buckley, Hanson and MacManus in [14].

Question 5.10. *Are there minimal sets for conformal maps with dimension strictly between* 0 *and* 1?

Question 5.11. *Are there minimal sets for quasisymmetric maps with dimension strictly between* 0 *and* 1 (*Tyson's conjecture is that there are not*).

5.8. Conformal welding

Given a closed Jordan curve Γ bounding two simply connected regions we define a corresponding homeomorphism of the circle by taking $\psi = \varphi_1 \circ \varphi_2^{-1}$, where φ_i, $i=1,2$ are conformal maps to the complementary regions to the disk. It is well known that ψ is quasisymmetric if and only if Γ is a quasicircle and that every quasisymmetric ψ occurs in this way. Sullivan's theorem allows us to do a similar thing for every simply connected region by taking $\psi = \varphi \circ \iota$ where $\iota: \mathbf{D} \to S$ is the isometry onto the convex hull boundary and φ is conformal from Ω to the disk. This is formal in the sense that we do not know that ι and φ have continuous boundary values, but the composition $\varphi \circ \sigma \circ \iota$ (where σ is Sullivan's map) defines a quasiconformal map of the disk (with constant independent of Ω) which extends to a quasisymmetric map of the boundary. Obviously, not every quasisymmetric map occurs since those that do have uniformly bounded estimates, but which ones do occur?

Question 5.12. *Does every quasisymmetric map with a* 2*-quasiconformal extension to the disk correspond to a convex hull welding* $\varphi \circ \iota$? *Is this true if we replace* 2 *by* $1+\varepsilon$? *Find some sufficient condition for a homeomorphism to occur as a welding in this way.*

Starting with a simply connected domain Ω we can obtain a quasisymmetric map of the circle by the convex hull welding and then get a quasidisk Ω' by the usual conformal welding. Which quasidisks can occur? What properties of Ω are reflected in the geometry of Ω'?

5.9. The action on quasisymmetric maps

Given any simply connected domain Ω, and a Riemann mapping $f: \mathbf{D} \to \Omega$ the mapping $\iota^{-1} \circ f$ gives a quasisymmetric homeomorphism of the circle to itself (unique up to Möbius transformations). Such a homeomorphism gives rise to another domain Ω' which is a quasidisk by the usual conformal welding procedure. Composing these two procedures gives a self-mapping on the space of quasisymmetric mappings on the circle. The image of this map is actually a bounded set (contained inside maps with a K-quasiconformal extension, where K is from Sullivan's theorem). Is this mapping a contraction? What happens under iteration of the map? Does everything converge to the identity after renormalizing?

5.10. Quasiconformal maps in higher dimensions

Is Theorem 4.1 true in higher dimensions? Of course, there are not enough conformal maps to make it interesting as stated, but we can substitute quasiconformal maps as follows. That the $d=2$ case is true follows immediately from Theorem 1.4.

Question 5.13. *Suppose $\Omega \subset \mathbf{R}^d$ is a C-quasiball with diameter ≥ 1. Then is there a K-quasiconformal, locally Lipschitz map from Ω to the unit ball (with K depending only on C)?*

5.11. Uniformly perfect sets

As noted in [18], Sullivan's theorem does not hold for general multiply connected domains in the plane because the presence of annuli with large moduli causes a problem. Does it hold if we assume there are no such annuli? More precisely, recall that a compact set E is called uniformly perfect if there is an $\varepsilon > 0$ so that for every $x \in E$ and $0 < r < \operatorname{diam}(E)$ there is a $y \in E$ with $\varepsilon r \leq |x-y| \leq r$. There are many equivalent definitions of this condition (e.g., see [36], [19], [22]) and if Ω is an open set such that $E = \partial\Omega$ is uniformly perfect, then the covering map from the disk to Ω has many of the same properties that univalent maps do.

Question 5.14. *If $\partial\Omega$ is uniformly perfect is there a K-quasiconformal map from Ω to S (the boundary component of the convex hull of $\partial\Omega$ facing Ω) which is the identity on $\partial\Omega$?*

5.12. Lipschitz preimages of the disk

We said in the introduction that any quasiconvex simply connected domain can be mapped to the disk by a quasiconformal map which is also Lipschitz with respect to the Euclidean metrics. It is also easy to construct some non-quasiconvex domains with this property. On the other hand, the complement of a ray cannot be so mapped to the disk.

Question 5.15. *Can one give a simple geometric characterization of the domains which can be mapped to the disk by a quasiconformal Lipschitz map?*

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