Monotonicity properties of interpolation spaces II

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1. Introduction

In this paper we continue our investigation begun in [3] of problems of characterizing all the interpolation spaces with respect to a given Banach couple. More specifically we show that a rather large class of Banach couples $\bar{A}=(A_0, A_1)$ are Calderón pairs, that is, A is an interpolation space with respect to $\bar{A}$ if and only it has the property that $a\in A$ and $K(t, b; \bar{A})\leqq K(t, a; \bar{A})$ for all $t\geqq 0$ implies that $b\in A$. We refer to [3] and also [17] and [1] pp. 83, 128, for detailed definitions and for a discussion and bibliography of earlier results of this type, and take this opportunity to correct our inadvertent omission in [3] of the contributions of Mitjagin [10] and Cotlar (unpublished).

We shall continue to use the notation and terminology of [3] together with that of [1]. In some cases we have made minor and unambiguous modifications of terminology from [3] in favour of the usage in [1].

We shall also use the following notions (cf. [4]).

Definition. Let $\bar{A}=(A_0, A_1)$ and $\bar{B}=(B_0, B_1)$ be two Banach couples and let A and B be intermediate spaces with respect to $\bar{A}$ and $\bar{B}$ respectively. Then A and B are *relative interpolation spaces* with respect to $\bar{A}$ and $\bar{B}$ if every linear operator $T\in\mathscr{L}(A_0, B_0)\cap\mathscr{L}(A_1, B_1)$ also maps A into B. A and B are *relative K spaces* if $a\in A$, $b\in\Sigma(\bar{B})=B_0+B_1$ and $K(t, b; \bar{B})\leqq K(t, a; \bar{A})$ for all $t>0$ implies that $b\in B$. $\bar{A}$ and $\bar{B}$ are *relative Calderón pairs* if all relative interpolation spaces A and B are also relative K spaces, that is, all possible interpolation results can be described in terms of K functional inequalities.

Note that in all the above definitions one must take care to write the two couples or two spaces in the correct order. For example (see [4]) (L^1, L^1_w), (L^∞, L^∞_w) are relative Calderón pairs but (L^∞, L^∞_w) and (L^1, L^1_w) are not.

Analogously to the case when $\bar{A}=\bar{B}$, to show that $\bar{A}$ and $\bar{B}$ are relative Calderón pairs it clearly suffices to show that for any $a\in\Sigma(\bar{A})$, $b\in\Sigma(\bar{B})$ with $K(t, b; \bar{B})\leqq$

$K(t, a, \bar{A})$ for all $t>0$ there exists an operator $T\in\mathcal{L}(A_0, B_0)\cap\mathcal{L}(A_1, B_1)$ with $Ta=b$.

The plan of the paper is as follows:

In Section 2 we show that for any Banach couple $\bar{A}=(A_0, A_1)$ and any numbers θ_0, θ_1 in $(0, 1)$ and p_0, p_1 in $[1, \infty]$ the couple of real interpolation spaces $\bar{A}_{\bar{\theta},\bar{p}}=(\bar{A}_{\theta_0,p_0}, \bar{A}_{\theta_1,p_1})$ is a Calderón pair. In fact by a result which is proved in Section 4 we have the same result for the couple $(\bar{A}_{\theta,p}, \bar{A}_{\theta,\infty})$ for all $\theta\in(0, 1)$ and all $p\in(0, \infty]$. Furthermore the same methods enable us to show that $(\bar{A}_{\theta_0,p_0}, \bar{A}_{\theta_1,p_1})$ and $(\bar{B}_{\alpha_0,p_0}, \bar{B}_{\alpha_1,p_1})$ are relative Calderón pairs, where $\bar{B}=(B_0, B_1)$ is another arbitrary Banach couple, for any choice of $\theta_0, \theta_1, \alpha_0, \alpha_1\in(0, 1)$ and $p_0, p_1\in[1, \infty]$. The basic result of Section 2 together with a very brief indication of its proof was announced in [1] and also in [17]. In a more recent note [6] V. I. Dmitriev and V. I. Ovčinnikov gave a more elaborate outline of the proof and some more abstract generalisations of the result.

In Section 3 we answer a question posed by A. A. Sedaev by showing that the couple $(\Lambda^p(\varphi), L^\infty)$ is Calderón. Here $\Lambda^p(\varphi)$ is the space normed by $\|f\|_{\Lambda^p(\varphi)}=\left(\int_0^\infty f^*(t)^p\varphi(t)\,dt\right)^{1/p}$ where $1\leqq p<\infty$ and $\varphi(t)$ is any nonincreasing locally integrable function. In fact we show that $(\Lambda^p(\varphi_1), L^\infty)$ and $(\Lambda^p(\varphi_2), L^\infty)$ are relative Calderón pairs for any functions $\varphi_1(t)$ and $\varphi_2(t)$ each having the same properties as $\varphi(t)$.

In Section 4 we consider spaces which may fail to be Banach and show that (l^p, l^∞) is Calderón for $0<p\leqq\infty$. As already mentioned this enables an extension of the results of Section 2.

In Section 5 we ask some questions related to the preceding results and also pose a problem for the couple $(L^p, W^{1,p})$ consisting of an L^p space and Sobolev space on $\mathbf{R}^n$ or $\mathbf{T}^n$. For $p\neq 2$ this is not a Calderón pair but we are able to obtain necessary conditions on all its interpolation spaces along the lines of results given in [3] Section 3.

We mention that a forthcoming paper [4] will give a detailed discussion of K-monotone spaces or K-spaces and also some results concerning relative Calderón pairs.

It is a pleasure to acknowledge some helpful discussions with Gunnar Sparr.

2. $(\bar{A}_{\theta_0,p_0}, \bar{A}_{\theta_1,p_1})$ is a Calderón paii

For $0<p<\infty$ and $0\leqq\theta\leqq 1$, let l_θ^p denote the complete (quasi) normed space of scalar valued sequences $(c_n)_{n=-\infty}^\infty$ with $\|(c_n)\|_{l_\theta^p}=\left(\sum_{n=-\infty}^\infty |2^{-n\theta}c_n|^p\right)^{1/p}<\infty$. Analogously l_θ^∞ is defined by $\|(c_n)\|_{l_\theta^\infty}=\sup_{n=-\infty}^\infty |2^{-n\theta}c_n|$. Let $\bar{A}=(A_0, A_1)$ be an arbitrary Banach couple. The space $\bar{A}_{\theta,p}$ may be defined to consist of those elements $a\in\sum(\bar{A})$ for which $\|a\|_{\bar{A}_{\theta,p}}=\left\|\left(K(2^n, a; \bar{A})\right)\right\|_{l_\theta^p}$ is finite, for $0<\theta<1$, $0<p\leqq\infty$.

For $1 \leqq p_0, p_1 \leqq \infty$ and $0<\theta_0, \theta_1<1$ Sparr's theorem [16, 17, 3] implies that $(l^{p_0}_{\theta_0}, l^{p_1}_{\theta_1}) = l^{\bar p}_{\bar\theta}$ is a Calderón pair. The proof which we shall give that $(\bar A_{\theta_0,p_0}, \bar A_{\theta_1,p_1}) = \bar A_{\bar p,\bar\theta}$ is a Calderón pair amounts to showing that the above two couples are almost "bl-pseudoretracts" of each other (as defined in [11] p. 22).

Lemma 1. *Given any $a \in \Sigma(\bar A)$ there exists a linear map S from $\Sigma(\bar A)$ into $l_0^\infty + l_1^\infty$ such that $Sa = (K(2^n, a; \bar A))_{n=-\infty}^{\infty}$ and for all $b \in \bar A_{\theta,p}$ $0<\theta<1$, $0<p\leqq\infty$, $Sb \in l^p_\theta$ with $\|Sb\|_{l^p_\theta} \leqq \|b\|_{\bar A_{\theta,p}}$.*

Proof. For each $n=0, \pm 1, \pm 2, \ldots$ $K(2^n, \cdot\,; \bar A)$ is a norm on $\Sigma(\bar A)$. There exist linear functionals l_n, $n=0, \pm 1, \pm 2, \ldots$ on $\Sigma(\bar A)$ such that $|l_n(b)| \leqq K(2^n, b; \bar A)$ for all $b \in \Sigma(\bar A)$ and in particular $l_n(a) = K(2^n, a; \bar A)$. Let $Sb = (l_n(b))_{n=-\infty}^{\infty}$, then S clearly has all the required properties.

Lemma 2. *Let $\theta_0, \theta_1 \in (0,1)$, $p_0, p_1 \in (0, \infty]$. Given any $a \in \Sigma(\bar A_{\bar\theta,\bar p}) = \bar A_{\theta_0,p_0} + \bar A_{\theta_1,p_1}$, there exists a linear operator T mapping $\Sigma(l^{\bar p}_{\bar\theta})$ into $\Sigma(\bar A_{\bar\theta,\bar p})$ such that T maps $l^{p_j}_{\theta_j}$ boundedly into $\bar A_{\theta_j,p_j}$, $j=0,1$ and $T((K(2^n, a; \bar A)) = a$.*

Proof (cf. the "fundamental lemma" [1] p. 45). For each $n=0, \pm 1, \pm 2, \ldots$ choose $g_n \in A_0$, $h_n \in A_1$ such that $g_n + h_n = a$ and $\|g_n\|_{A_0} + 2^n \|h_n\|_{A_1} \leqq (1+\varepsilon) K(2^n, a; \bar A)$ for some $\varepsilon > 0$. Since $a \in \Sigma(\bar A_{\bar\theta,\bar p})$ one can readily deduce that for $a_n = g_n - g_{n+1} = h_{n+1} - h_n$, $\sum_{n=-\infty}^{\infty} \|a_n\|_{\Sigma(\bar A)} < \infty$ and $\sum_{n=-\infty}^{\infty} a_n = a$. For each sequence $(b_n)_{n=-\infty}^{\infty} \in \Sigma(l^{\bar p}_{\bar\theta})$ let $T((b_n)) = \sum_{n=-\infty}^{\infty} \dfrac{b_n a_n}{K(2^n, a; \bar A)}$.

From slight variants of estimates which we shall obtain below it will be clear that the above sum converges absolutely with respect to the norm of $\Sigma(\bar A)$, and so $T((b_n))$ is a well defined element of $\Sigma(\bar A)$. Clearly $T((K(2^n, a; \bar A))) = a$. It remains to show that T maps $l^{p_j}_{\theta_j}$ boundedly into $\bar A_{\theta_j,p_j}$ (cf. [13] Thm. 5.7 p. 243 and [14] Thm. 10, p. 49). Suppose $(b_n) \in l^{p_j}_{\theta_j}$. Then

$$\begin{aligned} K(2^m, T((b_n)); \bar A) &\leqq \textstyle\sum_{n=-\infty}^{\infty} |b_n| K(2^m, a_n; \bar A)/K(2^n, a; \bar A) \\ &\leqq \textstyle\sum_{n=-\infty}^{\infty} |b_n| \min(\|a_n\|_{A_0}, 2^m \|a_n\|_{A_1})/K(2^n, a; \bar A) \\ &< 3(1+\varepsilon) \textstyle\sum_{n=-\infty}^{\infty} |b_n| \min(K(2^n, a; \bar A), 2^{m-n} K(2^n, a; \bar A))/K(2^n, a; \bar A) \\ &= 3(1+\varepsilon) \textstyle\sum_{n=-\infty}^{\infty} |b_n| \min(1, 2^{m-n}). \end{aligned}$$

If $p_j \geqq 1$

$$\begin{aligned} \|T((b_n))\|_{\bar A_{\theta_j,p_j}} &= \|2^{-n\theta_j} K(2^n, T((b_n)); \bar A)\|_{l^{p_j}} \\ &\leqq 3(1+\varepsilon) \|(b_n)\|_{l^{p_j}_{\theta_j}} \|\min(2^{-n\theta_j}, 2^{n(1-\theta_j)})\|_{l^1}. \end{aligned}$$

For $p_j < 1$

$$\begin{aligned} &[2^{-m\theta_j} K(2^m, T((b_n)); \bar A)]^{p_j} \\ &\leqq 3^{p_j}(1+\varepsilon)^{p_j} \textstyle\sum_{n=-\infty}^{\infty} (2^{-n\theta_j}|b_n|)^{p_j} \min(2^{-p_j\theta_j(m-n)}, 2^{p_j(1-\theta_j)(m-n)}). \end{aligned}$$

So

$$\|T((b_n))\|_{\bar{A}_{\theta_j,p_j}} \leqq 3(1+\varepsilon)\|(b_n)\|_{l^{p_j}_{\theta_j}}\|\min(2^{-\theta_j n}, 2^{(1-\theta_j)n})\|_{l^{p_j}}.$$

Thus in all cases T maps $l^{p_j}_{\theta_j}$ into $\bar{A}_{\theta_j,p_j}$ with norm bounded by a number which can be chosen as close as we please to

$$C_{\theta_j,p_j} = 3\left(\frac{1}{(1-2^{-\theta_j p_j^*})}+\frac{1}{(1-2^{-(1-\theta_j)p_j^*}}-1\right)^{1/p_j^*}$$

where $p_j^*=\min(1, p_j)$.

Remark 1. There are obvious analogues of the above two lemmata with l^p_θ replaced by $L^p((0,\infty), t^{-\theta p}dt/t)$, but the analogue of the second lemma only holds for $p_j\geqq 1$.

Corollary 1. *From Lemmata* 1 *and* 2 *it follows immediately that for each* $a\in\Sigma(\bar{A}_{\bar\theta,\bar p})$ *and each* $t>0$,

$$K(t, (K(2^n, a; \bar{A}))_{n=-\infty}^{\infty}; l^{\bar p}_{\bar\theta}) \leqq K(t, a; \bar{A}_{\bar\theta,\bar p})$$

$$\leqq \max_{j=0,1}(C_{\theta_j,p_j})K(t, (K(2^n, a; \bar{A}))_{n=-\infty}^{\infty}; l^{\bar p}_{\bar\theta}).$$

Remark 2. The above corollary provides a formula for $K(t, a; \bar{A}_{\bar\theta,\bar p})$ to within equivalence, cf. Holmstedt [7]. Note that the formula here also applies when $\theta_0=\theta_1$. One could seek a more explicit expression using the K functional for a pair of weighted L^p spaces. (See [1] exercise 2, p. 124, and [3] p. 234; the "transformation" used in [3] was in fact introduced long ago by Stein and Weiss.) By such a procedure, for $p_0=p_1$, $\theta_0\neq\theta_1$ one readily recovers Holmstedt's formula. However, for $p_0\neq p_1$ we obtain an expression, which appears to be rather more unwieldy than Holmstedt's, in terms of the non increasing rearrangement of the sequence $2^{\alpha n}K(2^n, a; \bar{A})$ or the function $t^\alpha K(t, a; \bar{A})$ with respect to a suitably weighted measure n^β on the integers or $t^{\beta-1}dt$ on $(0,\infty)$. Here $\alpha=(p_0\theta_0-p_1\theta_1)/(p_1-p_0)$ and

$$\beta = p_0p_1(\theta_1-\theta_0)/(p_1-p_0).$$

Theorem 1. *Let* $\theta_0, \theta_1\in(0, 1)$ *and* $p_0, p_1\in(0,\infty]$ *be chosen such that* $l^{\bar p}_{\bar\theta}=(l^{p_0}_{\theta_0}, l^{p_1}_{\theta_1})$ *is a Calderón pair, then for any Banach couple* $\bar{A}=(A_0, A_1)$, *the couple* $\bar{A}_{\bar\theta,\bar p}=(\bar{A}_{\theta_0,p_0}, \bar{A}_{\theta_1,p_1})$ *is a Calderón pair.*

Proof. It suffices to show that if $g, f\in\Sigma(\bar{A}_{\bar\theta,\bar p})$ with $K(t, g; \bar{A}_{\bar\theta,\bar p})\leqq K(t, f; \bar{A}_{\bar\theta,\bar p})$ for all $t>0$ then there exists an operator $U\in\mathscr{L}(\bar{A}_{\theta_0,p_0})\cap\mathscr{L}(\bar{A}_{\theta_1,p_1})$ satisfying $Uf=g$. Now by Corollary 1 and the hypothesis on f and g there exists an operator $V\in\mathscr{L}(l^{p_0}_{\theta_0})\cap\mathscr{L}(l^{p_1}_{\theta_1})$ which maps the sequence $K(2^n, f; \bar{A})$ to the sequence $K(2^n, g; \bar{A})$.

By Lemma 1 there exists an operator $S\in\mathscr{L}(\bar{A}_{\theta_0,p_0}, l^{p_0}_{\theta_0})\cap\mathscr{L}(\bar{A}_{\theta_1,p_1}, l^{p_1}_{\theta_1})$ such that $Sf=\big(K(2^n,f;\bar{A})\big)_{n=-\infty}^{\infty}$ and by Lemma 2 there exists an operator $T\in\mathscr{L}(l^{p_0}_{\theta_0}, \bar{A}_{\theta_0,p_0})\cap\mathscr{L}(l^{p_1}_{\theta_1}, A_{\theta_1,p_1})$ such that $T\big(K(2^n,g;\bar{A})\big)=g$. The required operator U is given by $U=TVS$.

Remark 3. By Sparr's theorem [16, 17, 3] the hypotheses of the above theorem hold for all $p_0, p_1\in[1,\infty]$ $\theta_0,\theta_1\in(0,1)$. We shall show in Section 4 that $(l^p_\theta, l^\infty_\theta)$ is a Calderón pair for all $p\in(0,\infty]$ $\theta\in(0,1)$ thus enlarging the range of parameters for which the theorem is valid. We note that Sparr [16, 17] also obtained results for L^p spaces with exponents $p<1$, but in the case of nonatomic measure spaces for which an analogue of Lemma 2 does not hold.

If $\bar{A}=(A_0,A_1)$ and $\bar{B}=(B_0,B_1)$ are two different Banach couples then an obvious adaptation of Theorem 1 shows that $\bar{A}_{\bar{\theta},\bar{p}}$ and $\bar{B}_{\bar{\theta},\bar{p}}$ are relative Calderón pairs. In fact by almost identical reasoning the corresponding result holds for the two couples $\bar{A}_{\bar{\theta},\bar{p}}=(\bar{A}_{\theta_0,p_0},\bar{A}_{\theta_1,p_1})$ and $\bar{B}_{\bar{\alpha},\bar{p}}=(\bar{B}_{\alpha_0,p_0},\bar{B}_{\alpha_1,p_1})$ where the parameters θ_0, θ_1 α_0, α_1 may take any values in $(0,1)$ and p_0, p_1 are in $[1,\infty]$.

3. The couple $\big(\Lambda^p(\varphi), L^\infty\big)$

In this section we give another example of the use of "pseudoretract" techniques which arose in the proof of Theorem 1. Let (X,Σ,μ) be a measure space. For each μ-measurable function $f(x)$ on X we denote the non increasing rearrangement of $f(x)$ by $f^*(t)$, $0<t<\infty$. Let $\varphi(t)$ be a locally Lebesgue integrable decreasing function on $(0,\infty)$. For $1\leqq p<\infty$ we define the Lorentz space $\Lambda^p(\varphi)$ on (X,Σ,μ) to consist of all (equivalence classes of) measurable functions $f(x)$ for which the norm $\|f\|_{\Lambda^p(\varphi)}=\left(\int_0^\infty f^*(t)^p\varphi(t)\,dt\right)^{1/p}$ is finite.

An example of a Banach couple which is not an "exact" Calderón pair was constructed by Sedaev and Semenov using a suitable three dimensional version of the couple $\big(\Lambda^1(\varphi), L^\infty\big)$, (see [15] or [1], p. 127). We shall show here however that in general $\big(\Lambda^1(\varphi), L^\infty\big)$ is a Calderón pair. Specifically if f and g are in $\Lambda^1(\varphi)+L^\infty$ with $K\big(t,g;\Lambda^1(\varphi),L^\infty\big)\leqq K\big(t,f;\Lambda^1(\varphi),L^\infty\big)$ for all $t\geqq 0$ then there exists an operator $T\in\mathscr{L}_4\big(\Lambda^1(\varphi)\big)\cap\mathscr{L}_1(L^\infty)$ such that $Tf=g$. This answers a question posed by A. A. Sedaev. In fact we obtain the following more general result which immediately implies that for any functions φ_1 and φ_2 satisfying the above conditions, $\big(\Lambda^p(\varphi_1), L^\infty\big)$ and $\big(\Lambda^p(\varphi_2), L^\infty\big)$ are relative Calderón pairs (cf. also [5]).

Theorem 2. *Let* (X_1,Σ_1,μ_1) *and* (X_2,Σ_2,μ_2) *be measure spaces and let* $\varphi_1(t)$, $\varphi_2(t)$ *be non increasing locally integrable functions on* $(0,\infty)$. *Let* $\Lambda^p(\varphi_k)$ *denote the Lorentz space of functions on* (X_k,Σ_k,μ_k) *corresponding to* $\varphi_k(t)$, $k=1,2$ *where* $p\in[1,\infty)$. *Let* $\bar{A}=\big(\Lambda^p(\varphi_1), L^\infty(d\mu_1)\big)$ *and* $\bar{B}=\big(\Lambda^p(\varphi_2), L^\infty(d\mu_2)\big)$. *If* $f\in\Sigma(\bar{A})$ *and*

$g \in \Sigma(\bar{B})$ and $K(t, g; \bar{B}) \leqq K(t, f; \bar{A})$ for all $t > 0$, then there exists an operator

$$T \in \mathscr{L}_{4^{1/p}\alpha_p}\big(\Lambda^p(\varphi_1), \Lambda^p(\varphi_2)\big) \cap \mathscr{L}_{\alpha_p}\big(L^\infty(d\mu_1), L^\infty(d\mu_2)\big)$$

such that $Tf = g$. The constant $\alpha_p \leqq 2$ depends only on p and $\alpha_1 = 1$.

Proof. In the light of results of [2] (for the non σ-finite case see also [3]) we can assume without loss of generality that $(X_1, \Sigma_1, \mu_1) = (X_2, \Sigma_2, \mu_2) = (0, \infty)$ equipped with Lebesgue measure, and also that f and g are non negative non increasing and right continuous, so that $f(t) = f^*(t)$, $g(t) = g^*(t)$. As in the proof of Theorem 1 we shall construct T as the composition of three operators $T = UVW$ as indicated in the following diagram:

$$\big(\Lambda^p(\varphi_1), L^\infty\big) \xrightarrow{W} \big(L^p(\varphi_1\, dx), L^\infty\big) \xrightarrow{V} \big(L^p(\varphi_2\, dx), L^\infty\big) \xrightarrow{U} \big(\Lambda^p(\varphi_2), L^\infty\big).$$

Here $L^p(\varphi_k\, dx)$ denotes the L^p space on $(0, \infty)$ normed by

$$\|h\|_{L^p(\varphi_k dx)} = \Big(\int_0^\infty |h(x)|^p \varphi_k(x)\, dx\Big)^{1/p} \quad \text{for} \quad k = 1, 2.$$

(i) Construction and properties of the operator W: We simply take the identity operator $Wh(x) = h(x)$. In view of the inequality $\int_0^\infty h(x)^p \varphi_1(x)\, dx \leqq \int_0^\infty h^*(x)^p \varphi_1(x)\, dx$ (cf. e.g. [8], p. 257) $W \in \mathscr{L}_1\big(\Lambda^p(\varphi_1), L^p(\varphi_1 dx)\big) \cap \mathscr{L}_1(L^\infty, L^\infty)$.

(ii) Construction and properties of U: If $\int_0^\infty \varphi_2(t)\, dt = \infty$ we define positive numbers b_n, $n = 0, \pm 1, \pm 2, \ldots$ for which $\int_0^{b_n} \varphi_2(t)\, dt = 2^n$. If $\int_0^\infty \varphi_2(t)\, dt = I < \infty$ define b_n only for negative n, such that $\int_0^{b_n} \varphi_2(t)\, dt = 2^n I$; thus we may consistently take $b_0 = \infty$ in this case. We now define an auxiliary operator U_1 by

$$U_1 h = \sum_n \Big(\int_{b_{n-1}}^{b_n} h(t)\varphi_2(t)\, dt\Big)\Big(\int_{b_{n-1}}^{b_n} \varphi_2(t)\, dt\Big)^{-1} \chi_{[b_n, b_{n+1})}$$

for each $h \in L^p(\varphi_2 dt) + L^\infty$. The summation over n is from $-\infty$ to ∞ if $I = \infty$ and from $-\infty$ to -1 if $I < \infty$.

$$|U_1 h(x)|^p \leqq \sum_n \Big(\int_{b_{n-1}}^{b_n} |h(t)|\varphi_2(t)\, dt\Big)^p \Big(\int_{b_{n-1}}^{b_n} \varphi_2(t)\, dt\Big)^{-p} \chi_{[0, b_{n+1})}(x)$$

$$\leqq \sum_n \Big(\int_{b_{n-1}}^{b_n} |h(t)|^p \varphi_2(t)\, dt\Big)\Big(\int_{b_{n-1}}^{b_n} \varphi_2(t)\, dt\Big)^{-1} \chi_{[0, b_{n+1})}(x).$$

Since this last sum is a non increasing function of x we see that

$$\|U_1 h\|_{\Lambda^p(\varphi_2)}^p \leqq \sum_n \int_{b_{n-1}}^{b_n} |h(t)|^p \varphi_2(t)\, dt \int_0^{b_{n+1}} \varphi_2(x)\, dx \Big(\int_{b_{n-1}}^{b_n} \varphi_2(t)\, dt\Big)^{-1}$$

$$\leqq 4 \sum_n \int_{b_{n-1}}^{b_n} |h(t)|^p \varphi_2(t)\, dt = 4\|h\|_{L^p(\varphi_2 dx)}^p.$$

Since g is non increasing we obviously have $U_1 g(x) \geqq g(x)$ for all x. Thus if we define U by $Uh(x)=\dfrac{g(x)U_1h(x)}{U_1g(x)}$ then $Ug=g$ and

$$U\in\mathscr{L}_{4^{1/p}}\big(L^p(\varphi_2\,dx),\ \Lambda^p(\varphi_2)\big)\cap\mathscr{L}_1(L^\infty, L^\infty).$$

(iii) Construction and properties of V: Using arguments almost identical to those in the proof of Theorem 5.2.1 [1] p. 109, we can readily show that for each $t>0$

$$\Big(\int_0^{u_2(t)} g^*(x)^p\,\varphi_2(x)\,dx\Big)^{1/p}\leqq K\big(t, g;\ \Lambda^p(\varphi_2), L^\infty\big)$$

and

$$K\big(t, f;\ \Lambda^p(\varphi_1), L^\infty\big)\leqq\alpha_p\Big(\int_0^{u_1(t)} f^*(x)^p\,\varphi_1(x)\,dx\Big)^{1/p}$$

where the functions $u_k(t)$ $k=1, 2$ are defined by the conditions

$$\int_0^{u_k(t)}\varphi_k(x)\,dx=t^p\quad\text{for all}\quad t<\Big(\int_0^\infty\varphi_k(x)\,dx\Big)^{1/p}\quad\text{and}\quad u_k(t)=\infty$$

otherwise, and the constant α_p satisfies $1\leqq\alpha_p\leqq 2$ with $\alpha_1=1$.

We may consider f and g as functions on a measure space (Y, S, ν) where Y consists of two disjoint copies of $(0, \infty)$. $Y=\mathbf{R}_{+,1}\cup\mathbf{R}_{+,2}$ with $d\nu=\varphi_k(x)dx$ on $\mathbf{R}_{+,k}$, and f and g are supported on $\mathbf{R}_{+,1}$ and $\mathbf{R}_{+,2}$ respectively. Letting $F(t)$ and $G(t)$ denote the non increasing rearrangements of f and g respectively with respect to the measure ν we see that for each $t>0$:

$$\int_0^{t^p} G(s)^p\,ds=\int_0^{u_2(t)} g(x)^p\varphi_2(x)\,dx\leqq\alpha_p^p\int_0^{u_1(t)} f(x)^p\varphi_1(x)\,dx=\alpha_p^p\int_0^{t^p} F(s)^p\,ds$$

and thus by the results of [9] there exists an operator $V\in\mathscr{L}_{\alpha_p}\big(L^p(d\nu)\big)\cap\mathscr{L}_{\alpha_p}\big(L^\infty(d\nu)\big)$ with $Vf=g$. In fact we can have

$$V\in\mathscr{L}_{\alpha_p}\big(L^p(\varphi_1\,dx), L^p(\varphi_2\,dx)\big)\cap\mathscr{L}_{\alpha_p}\big(L^\infty(\varphi_1\,dx), L^\infty(\varphi_2\,dx)\big).$$

From the properties of U, V and W it is now evident that

$$T=UVW\in\mathscr{L}_{4^{1/p}\alpha_p}\big(\Lambda^p(\varphi_1), \Lambda^p(\varphi_2)\big)\cap\mathscr{L}_{\alpha_p}(L^\infty, L^\infty)\quad\text{and}\quad Tf=g,$$

completing the proof of the theorem.

4. $(l_\theta^p, l_\theta^\infty)$ for $0<p<\infty$

As already mentioned in Section 2, $(l_\theta^p, l_\theta^\infty)$ is a Calderón pair for all p, $0<p<\infty$ and all $\theta\in(0, 1)$, in fact for all real θ. To establish this it suffices of course to consider the case $\theta=0$ and to prove the following theorem which is of course well known for $p\geqq 1$ ([2, 9, 10]).

Theorem 3. *Let* $0<p<\infty$ *and let* $f, g \in l^p + l^\infty$ *with*

$$\int_0^t g^*(s)^p\,ds \leqq \int_0^t f^*(s)^p\,ds \tag{1}$$

for all positive t. *Then there exists an operator* $Q \in \mathscr{L}_{8^{1/p}}(l^p) \cap \mathscr{L}_{2^{1/p}}(l^\infty)$ *such that* $Qf=g$.

Proof. We can of course suppose that both $f=(f_n)_{n=1}^\infty$ and $g=(g_n)_1^\infty$ are non negative sequences. Let $f^*=(f_n^*)_1^\infty$ and $g^*=(g_n^*)_1^\infty$ be the non increasing rearrangement sequences of f and g, that is, for each $n\geqq 1$, $f_n^*=f^*(t)$, $t\in[n-1, n)$ and similarly for g_n^*. We shall show that (1) implies the existence of an operator $V\in\mathscr{L}(l^p)\cap\mathscr{L}(l^\infty)$ with $Vf^*=g^*$. Let us first establish the existence of operators S and T in $\mathscr{L}_1(l^p)\cap\mathscr{L}_1(l^\infty)$ such that $Sf=f^*$ and $Tg^*=g$. The desired operator will then be given by $Q=TVS$.

Let $\alpha=\lim_{n\to\infty} f_n^*$. Let $J=\{n|f_n>\alpha\}$ and $J^*=\{n|f_n^*>\alpha\}$. J and J^* are either both infinite or both finite with the same cardinality. On J^* f_n^* assumes any given value $\beta>\alpha$ at most finitely many times. The sets $\{n|f_n=\beta\}$ and $\{n|f_n^*=\beta\}$ have the same finite cardinality and so there exists a one to one map π of J^* onto J such that $f_{\pi(n)}=f_n^*$ for all $n\in J^*$. There exists an infinite subsequence $(m(n))_{n=1}^\infty$ of the positive integers such that $\lim_{n\to\infty} f_{m(n)}=\alpha$. Let ω be a Banach limit, that is $\omega\in(l^\infty)^*$ with norm 1 and $\omega((h_n))=\lim_{n\to\infty} h_n$ for all convergent sequences $(h_n)_1^\infty$.

We can now define the operator S which maps any given sequence $(h_k)_{k=1}^\infty$ to the sequence

$$\begin{aligned}(Sh)_k &= h_{\pi(k)} && \text{for}\quad k\in J^*\\ &= \omega((h_{m(n)})) && \text{for}\quad k\notin J^*.\end{aligned}$$

It is clear that S is in $\mathscr{L}(l^p)$ and $\mathscr{L}(l^\infty)$ with bound 1 and $Sf=f^*$. The operator T is constructed in an almost identical fashion. We may take $m(n)=n$ in this case. For each sequence $(h_k)_{k=1}^\infty$ we define

$$\begin{aligned}(Th)_k &= h_{\pi^{-1}(k)} && \text{for}\quad k\in J\\ &= \omega((h_n))\,g_k/\alpha && \text{for}\quad k\notin J,\end{aligned}$$

where here α, J, J^* and π are defined exactly as above but for the sequence $(g_n)_{n=1}^\infty$ instead of for $(f_n)_1^\infty$.

We now turn to the construction of the operator V which maps $f^*=(f_n^*)$ to $g^*=(g_n^*)_{n=1}^\infty$. At this point we may drop the asterisks and consider non increasing sequences $f=(f_n)_1^\infty$, $g=(g_n)_1^\infty$, with non increasing rearrangement functions $f^*(t)=f(t)$ and $g^*(t)=g(t)$ satisfying (1). Let Γ be the finite or infinite sequence of consecutive positive integers n for which $g_n>0$. The inequalities (1) imply the existence of a strictly increasing sequence $(a_n)_{n\in\Gamma}$ of positive numbers such that:

$$a_n\leqq n \quad\text{and}\quad \int_0^{a_n} f(s)^p\,ds=\int_0^n g(s)^p\,ds \tag{2}$$

for each $n\in\Gamma$. Let $a_0=0$ and for each $n\in\Gamma$ let $k(n)$ be the integer for which $2^{k(n)}\leqq a_n-a_{n-1}<2^{k(n)+1}$. Then

$$g_n^p=\int_{n-1}^{n} g(s)^p\,ds=\int_{a_{n-1}}^{a_n} f(s)^p\,ds\leqq\int_{a_{n-1}}^{a_{n-1}+2^{k(n)+1}} f(s)^p\,ds\leqq 2\int_{a_{n-1}}^{a_{n-1}+2^{k(n)}} f(s)^p\,ds.$$

Partition Γ into $X=\{n\in\Gamma|2^{k(n)}\leqq 1/2\}$, $\Gamma_0=\{n\in\Gamma|2^{k(n)}=1\}$ and $Y=\{n\in\Gamma|2^{k(n)}\geqq 2\}$. We shall need the following lemma.

Lemma A. *There exists a one to one map ξ of Y into X such that $\xi(n)<n$ for each $n\in Y$.*

Proof. Let $y(1), y(2), y(3), \dots$ be a list in increasing order of the elements of Y and let $x(1), x(2), x(3), \dots$ be a list in increasing order of the elements of X. The map ξ will be given by $\xi(y(n))=x(n)$ for $n=1, 2, \dots$ card Y. We need only show that $x(n)<y(n)$ for all such n. Fix n and let m be the greatest integer such that $x(m)<y(n)$. In other words, of the $y(n)$ integers $1, 2, \dots, y(n)$, n integers are in Y, m are in X and $y(n)-n-m$ are in Γ_0. Clearly $\sum_{\nu=1}^{\lambda} 2^{k(\nu)}\leqq a_\lambda\leqq\lambda$ for each $\lambda\in\Gamma$ and so

$$\begin{aligned} y(n) &\geqq \sum_{\nu=1}^{y(n)} 2^{k(\nu)}=\sum_{\nu\in Y,\,\nu\leqq y(n)} 2^{k(\nu)}+\sum_{\nu\in\Gamma_0,\,\nu\leqq y(n)} 2^{k(\nu)}+\sum_{\nu\in X,\,\nu\leqq y(n)} 2^{k(\nu)}\\ &\geqq 2n+y(n)-n-m. \end{aligned}$$

It follows that $n\leqq m$ and so $x(n)\leqq x(m)<y(n)$. This completes the proof of Lemma A, and we proceed with the proof of Theorem 3.

The next step is to construct an operator V_0 which maps f to $g\chi_{X\cup\Gamma_0}$. Let $z(1), z(2), z(3), \dots$ be a list in increasing order of the elements of $X\cup\Gamma_0$ and let $I_1, I_2, \dots$ be intervals defined by $I_1=[0, 2^{k(z(1))})$ and $I_n=\left[\sum_{\nu=1}^{n-1} 2^{k(z(\nu))}, \sum_{\nu=1}^{n} 2^{k(z(\nu))}\right)$ for $n>1$. Each interval I_n meets at most two of the intervals $J_m=[m, m+1)$. Let $j(n)$ be the smallest integer m such that $J_m\cap I_n$ is non empty. For any fixed m, $\bigcup_{j(n)=m} I_n$ is an interval of length less than 2.

Let V_1 be a linear operator mapping l^p+l^∞ into the space of step functions on $[0, \infty)$ which are constant on each interval I_n. For each $h=(h_n)_{n=1}^\infty$, $V_1 h$ will be the function which equals $h_{j(n)}$ on I_n. Let (M, μ) be the measure space generated by the sets I_n acting as atoms with $\mu(I_n)=|I_n|=2^{k(z(n))}$. Then V_1 maps l^∞ into $L^\infty(\mu)$ with norm 1. V_1 also maps l^p boundedly into $L^p(\mu)$ with bound $2^{1/p}$ since

$$\int_M |V_1 h|^p\,d\mu=\sum_{n=1}^\infty |h_n|^p\,\mu\Big(\bigcup_{j(m)=n} I_m\Big)\leqq 2\sum_{n=1}^\infty |h_n|^p.$$

We observed earlier that

$$g_n^p\leqq 2\int_{a_{n-1}}^{a_{n-1}+2^{k(n)}} f(s)^p\,ds \quad\text{for each}\quad n\in\Gamma. \tag{3}$$

Now if $n=z(m)\in X\cup\Gamma_0$ we also have

$$\int_{a_{n-1}}^{a_{n-1}+2^{k(n)}} f(s)^p\,ds \leqq \int_{I_m} f(s)^p\,ds \leqq \int_{I_m} (V_1 f)^p\,ds. \tag{4}$$

These inequalities follow immediately from the facts that $f(s)$ is non increasing, that $|I_m|=2^{k(n)}$, and that the left endpoint of I_m is

$$\sum_{\nu=1}^{m-1} 2^{k(z(\nu))} \leqq \sum_{\nu=1}^{z(m-1)} 2^{k(\nu)} \leqq a_{z(m-1)} \leqq a_{z(m)-1} = a_{n-1}.$$

Let V_2 be a linear operator from $L^p(\mu)+L^\infty(\mu)$ into l^p+l^∞ such that for any step function $h=\sum_{m=1}^{\infty} h_m\chi_{I_m}\in L^p(\mu)+L^\infty(\mu)$ the sequence $V_2h=\{(V_2h)_n\}_{n=1}^{\infty}$ is given by

$$(V_2h)_n = 0 \quad \text{whenever} \quad n\in Y \quad \text{or} \quad n\notin\Gamma.$$

$$(V_2h)_n = h_m|I_m|^{1/p} \quad \text{whenever} \quad n=z(m)\in X\cup\Gamma_0.$$

V_2 clearly has bound 1 as an element of both $\mathscr{L}(L^p(\mu), l^p)$ and $\mathscr{L}(L^\infty(\mu), l^\infty)$. The operator $V_2V_1\in\mathscr{L}_{2^{1/p}}(l^p)\cap\mathscr{L}_1(l^\infty)$ and, by (3) and (4), $((V_2V_1f)_n)^p=\int_{I_m}(V_1f)^p ds \geqq (1/2)g_n^p$ (where $n=z(m)$) for each $n\in X\cup\Gamma_0$. Let V_0 be given by V_2V_1 followed by the operator which multiplies the n^{th} element of the sequence by $g_n/(V_2V_1f)_n$, $n\in X\cup\Gamma_0$. Then $V_0\in\mathscr{L}_{4^{1/p}}(l^p)\cap\mathscr{L}_{2^{1/p}}(l^\infty)$, and $(V_0f)_n=g_n$ for all $n\in X\cup\Gamma_0$ and $(V_0f)_n=0$ for $n\notin X\cup\Gamma_0$. Finally we may construct V with the help of the mapping ξ.

For each $h\in l^p+l^\infty$ let $Vh=\{(Vh)_n\}_{n=1}^{\infty}$ be given by

$$\begin{aligned}(Vh)_n &= (V_0h)_n && \text{for} \quad n\in X\cup\Gamma_0\\ &= (g_n/g_{\xi(n)})(V_0h)_{\xi(n)} && \text{for} \quad n\in Y\\ &= 0 && \text{for} \quad n\notin\Gamma.\end{aligned}$$

Clearly $Vf=g$ and $V\in\mathscr{L}_{8^{1/p}}(l^p)\cap\mathscr{L}_{2^{1/p}}(l^\infty)$. This completes the proof of Theorem 3.

5. Further comments and questions

It seems likely that one can establish that quite a number of other interpolation couples are Calderón, perhaps using techniques related to those of the preceeding sections. In attempting to work towards the solution of Peetre's problem [12] of determining general conditions to characterize Calderón couples the following questions seem natural, if rather difficult.

(1) Is every pair of rearrangement invariant spaces a Calderón couple?

(2) Does there exist a mutually closed couple (see [3], p. 218) $\bar{A}=(A_0, A_1)$ which is not Calderón but nevertheless all of the complex interpolation spaces $\bar{A}_{[\theta]}$ are K-monotone? We note the existence of couples such as $(L^1(\mathbf{R}), C(\mathbf{R}))$ ([3] p. 217) which are not mutually closed but do have the latter two properties.

(3) In every case where it has been possible to show that a given couple $\bar{A}$ is Calderón the proof has been related to Sparr's result for pairs of weighted L^p spaces or a special case of that result. Is the Calderón property very much an "L^p" phenomenon or can one find examples of couples $\bar{A}$ which bear little or no relation to L^p spaces and yet are Calderón pairs? Initially for example one might consider couples of Orlicz spaces.

In the remainder of this section we discuss a sequel to the investigations begun in Section 3 of [3]. There we discovered that for various non-Calderón couples $\bar{A}$ all interpolations spaces A have a property weaker than K-monotonicity; if $f\in A$ and $g\in\Sigma(\bar{A})$ with $K(t,g;\bar{A})\leqq w(t)K(t,f;\bar{A})$ for all t where the function $w(t)$ has certain properties then $g\in A$. We wish to consider such results for the important couple $\bar{W}=(L^p, W^{1,p})$ where $W^{1,p}$ denotes the usual Sobolev space of functions which together with their first derivatives are in L^p, the underlying space being $\mathbf{R}^n$ or $\mathbf{T}^n$, (as in [1], Chapter 6) with $1<p<\infty$. This is not a Calderón pair when $p\neq 2$ ([3] p. 218) but one can show that it has the following weaker property (cf. [3] Section 3, Theorems 1 and 2).

Theorem 4. *Let $w(t)$ be a positive measurable function such that for some positive number ε $\int_0^1[\min(\varepsilon, w(t))]^{p_*}dt/t<\infty$, where $p_*=\min(p,2)$. Let A be an interpolation space for $\bar{W}$. Then if $f\in A$ and $g\in\Sigma(\bar{W})$ such that $K(t,g;\bar{W})\leqq w(t)K(t,f;\bar{W})$ for $0\leqq t\leqq 1$ then $g\in A$.*

Proof. As a first simplification we may deduce that in fact $K(t,g;\bar{W})\leqq w_1(t)K(t,f;\bar{W})$ where $\int_0^1 w_1(t)^{p_*}dt/t<\infty$ using an argument identical to that in the proof of Theorem 1 of [3] p. 221. Then, using "averages" $w_m=\int_{2^{m-1}}^{2^m} w_1(t)\,dt/t$ we deduce further that $K(2^m,g;\bar{W})\leqq \text{const.}\, w_m K(2^m,f;\bar{W})$ for every non-positive integer m, with $\sum_{m\leqq 0} w_m^{p_*}<\infty$. Using this condition we shall construct a bounded linear operator $T\colon \bar{W}\to\bar{W}$ such that $Tf=g$.

As a second simplification we observe that it suffices to carry out the analogous construction when $\bar{W}$ is replaced by the couple $\bar{L}=(L^p(l_0^2), L^p(l_1^2))$, where $L^p(l_\alpha^2)$ denotes the space of sequence valued functions $u(x)=\{u_k(x)\}_{k=0}^\infty$ on $\mathbf{R}^n$ with norm $\left(\int_{\mathbf{R}^n}\left(\sum_{k=0}^\infty |2^{k\alpha}u_k(x)|^2\right)^{p/2}dx\right)^{1/p}$. From this one may deduce the result for $\bar{W}$ using the operators $\mathscr{P}$ and $\mathscr{I}$ defined in [1], p. 150, and the fact that $\bar{W}$ is a retract of $\bar{L}$ ([1] Theorem 6.4.3 p. 151). For any $u=\{u_k(x)\}\in\Sigma(\bar{L})=L^p(l_0^2)$ it is a routine matter to show that for each $t>0$:

$$\frac{1}{2}K(t,u;\bar{L}) \leqq \left\|\left(\sum_{k=0}^\infty |\min(1,t2^k)u_k(x)|^2\right)^{1/2}\right\|_{L^p(\mathbf{R}^n)} \leqq K(t,u;\bar{L}).$$

Now suppose $f=\{f_k(x)\}$ and $g=\{g_k(x)\}$ are in $\Sigma(\bar{L})$ and $K(t,g;\bar{L})\leqq w(t)K(t,f;\bar{L})$ for $0\leqq t\leqq 1$ where $w(t)$ is as above and so for some sequence w_m,

$\sum_{m\leq 0} w_m^{p_*}<\infty$, we have for each $m\leqq 0$,

$$\|g_{-m}(x)\|_{L^p} \leqq \left\|\left(\sum_{k=0}^{\infty} |\min(1, 2^{m+k}) g_k(x)|^2\right)^{1/2}\right\|_{L^p}$$
$$\leqq w_m \left\|\left(\sum_{k=0}^{\infty} |\min(1, 2^{m+k}) f_k(x)|^2\right)^{1/2}\right\|_{L^p}.$$

We now proceed rather as in the proof of Lemma 1 of [3], p. 219. For each negative integer m $L^p(l_0^2)$ can be renormed by $\|u\|_m = \left\|\left(\sum_{k=0}^{\infty} |\min(1, 2^{m+k}) u_k(x)|^2\right)^{1/2}\right\|_{L^p}$. Thus there exists a continuous linear functional l_m on $L^p(l_0^2)$ with

$$l_m(f) = \|f\|_m \quad \text{and} \quad |l_m(u)| \leqq \|u\|_m$$

for all $u=\{u_k(x)\}$ in $L^p(l_0^2)$.

We now define the operator $T\colon \bar{L}\to\bar{L}$ by

$$Tu(x) = \{l_{-k}(u)\, g_k(x)/\|f\|_{-k}\}_{k=0}^{\infty}.$$

It is clear that $Tf=g$ and it remains only to verify that T is bounded on $L^p(l_0^2)$ and $L^p(l_1^2)$. In fact if $\alpha=0$ or 1

$$2^{-m\alpha}\|u\|_m \leqq \|u\|_{L^p(l_\alpha^2)} \quad \text{for any} \quad u = \{u_k(x)\}\in L^p(l_\alpha^2).$$

Therefore

$$\|Tu\|_{L^p(l_\alpha^2)} \leqq \left\|\left(\sum_{k=0}^{\infty} |2^{k\alpha}\|u\|_{-k}\, g_k(x)/\|f\|_{-k}|^2\right)^{1/2}\right\|_{L^p}$$
$$\leqq \left\|\left(\sum_{k=0}^{\infty} |g_k(x)/\|f\|_{-k}|^2\right)^{1/2}\right\|_{L^p} \|u\|_{L^p(l_\alpha^2)}$$
$$\leqq \left(\sum_{k=0}^{\infty} \left\|g_k(x)/\|f\|_{-k}\right\|_{L^p}^2\right)^{1/2} \|u\|_{L^p(l_\alpha^2)} \leqq \left(\sum_{m\leq 0} w_m^2\right)^{1/2} \|u\|_{L^p(l_\alpha^2)}$$

provided $p\geqq 2$.

For $p<2$ we have similarly

$$\|Tu\|_{L^p(l_\alpha^2)} \leqq \left\|\left(\sum_{k=0}^{\infty} |g_k(x)/\|f\|_{-k}|^2\right)^{1/2}\right\|_{L^p} \|u\|_{L^p(l_\alpha^2)}$$
$$\leqq \left\|\left(\sum_{k=0}^{\infty} |g_k(x)/\|f\|_{-k}|^p\right)^{1/p}\right\|_{L^p} \|u\|_{L^p(l_\alpha^2)}$$
$$\leqq \left(\sum_{k=0}^{\infty} (\|g_k\|_{L^p}/\|f\|_{-k})^p\right)^{1/p} \|u\|_{L^p(l_\alpha^2)}$$
$$\leqq \left(\sum_{m\leq 0} w_m^p\right)^{1/p} \|u\|_{L^p(l_\alpha^2)}.$$

This shows that T is bounded on $L^p(l_\alpha^2)$ with norm not exceeding $\left(\sum_{m\leq 0} w_m^{p_*}\right)^{1/p_*}$ and completes the proof.

It is well known that for each $\theta\in(0, 1)$ and $1<p<\infty$,

$$\overline{W}_{\theta, \min(2,p)} \subseteqq \overline{W}_{[\theta]} \subseteqq \overline{W}_{\theta, \max(2,p)}$$

(cf. [1], p. 152, Theorem 6.4.4), and, using Hölder's inequality and these two continuous inclusions, we can readily see that in the case where $A=\overline{W}_{[\theta]}$, Theorem 4 holds with the condition on $w(t)$ weakened to $\int_0^1 [\min(\varepsilon, w(t))]^r dt/t<\infty$, where $r=p_{**}=2p/|p-2|$. Indeed in this case p_{**} is the best possible exponent. Suppose that $2<p<\infty$ and $1/p<\alpha<1$; then for any $r>p_{**}$ there exist functions $f(x)$ and $k(x)$ on the torus $\mathbf{T}$ such that $f\in\overline{W}_{[\alpha]}(\mathbf{T})$ but $k\notin\overline{W}_{[\alpha]}(\mathbf{T})$ even though $I=\int_0^1 [K(t, k; \overline{W})/K(t, f; \overline{W})]^r dt/t$ is finite. In fact these are precisely the functions which we used to show that $\overline{W}$ is not Calderón (see [3] p. 218 and [18] pp. 472—474). As already noted in [3] p. 218 $\|k(x+h)-k(x)\|_{L^p}\leqq \text{const.}\ |h|^\alpha \log^{-1/2}(1/|h|)$ and $\|f(x+h)-f(x)\|_{L^p}\geqq \text{const.}\ |h|^\alpha \log^{-1/p-\varepsilon}(1/|h|)$, for all h, $x\in[0, 2\pi]\cong\mathbf{T}$. It follows that

$$K(t, k; \overline{W})/K(t, f; \overline{W}) \leqq \text{const.} \log^{1/p-1/2+\varepsilon}(1/t).$$

If $(1/p-{}^1/_2+\varepsilon)r<-1$ it is easy to deduce that the integral I is finite, (we have only to estimate the integrand for small values of t) and the last inequality can be fulfilled by choosing ε to satisfy $0<\varepsilon<1/p_{**}-1/r$. (The construction of f and k works for all such ε.)

On the basis of these remarks we are naturally led to ask whether in Theorem 4 one can weaken the hypothesis on $w(t)$ by replacing p_* by p_{**}, and thus obtain in some sense a best possible weak K-monotonicity result for the couple $\overline{W}$.

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