Algebraic surfaces containing an ample divisor of arithmetic genus two

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Introduction

After some progress was made in the study of the adjunction process [19], these last few years many papers [12], [15], [16] have appeared on the classical subject of classifying projective algebraic surfaces whose general hyperplane section has a given genus.

A more general and intrinsic version of this problem can be stated as follows: classify all pairs $(S, \mathscr{L})$ where S is a smooth complex projective algebraic surface and $\mathscr{L}$ an ample line bundle on S whose arithmetic genus $g(\mathscr{L})=1+\frac{1}{2}(\mathscr{L}^2+\mathscr{L}K_S)$ is a given number g. Of course $g(\mathscr{L})\geqq 0$. In the cases $g=0$ and $g=1$ this classification is known [13] (see also [7], [8]) and, in a sense, gives nothing new with respect to the classical case where $\mathscr{L}$ is assumed a very ample line bundle.

In the case $g=2$ the situation is quite different since some meaningful new pairs appear with respect to the classical case, e.g. the pair $(J(C), \mathcal{O}_{J(C)}(C))$ defined by a smooth curve C of genus two embedded in its Jacobian $J(C)$ and the pair $(\Sigma, \pi^* \mathcal{O}_{\mathbf{P}^2}(1))$ where $\pi: \Sigma \to \mathbf{P}^2$ is a double cover branched along a smooth sextic.

In this paper we give a classification of the polarized pairs $(S, \mathscr{L})$ with $g(\mathscr{L})=2$. Just a few words about what we mean by classifying pairs $(S, \mathscr{L})$. First of all S is classified birationally, according to the Enriques—Kodaira classification. As far as the line bundle $\mathscr{L}$ is concerned, since the definition of $g(\mathscr{L})$ involves numerical characters only, it seems reasonable to classify $\mathscr{L}$ up to numerical equivalence. The results we find are too complicated to be outlined here, so after noticing that they are summarized in various tables section by section, we use this introduction to point out the defects of our classification. Indeed here we find necessary conditions for polarized pairs $(S, \mathscr{L})$ to exist, but we are not always able to decide whether all the pairs with the characters we find do exist. However pairs $(S, \mathscr{L})$ where S is a minimal model of Kodaira dimension $\varkappa(S)\neq 1$ really occur.

When $\varkappa(S)\geqq 1$, S is forced to be a minimal model. So as to surfaces of general type our description is rather complete. In the case of elliptic surfaces, the complexity of $\mathrm{Num}(S)$ makes us describe $\mathscr{L}$ only in terms of fibres and m-sections of the elliptic fibration. When $\varkappa(S)\leqq 0$, S is not necessarily a minimal model and so we classify the pairs $(S, \mathscr{L})$ by means of their minimal pairs. This means that we consider a birational morphism $\eta\colon S\to S_0$ onto a minimal model S_0, describe the sequence of the blowing-ups η factors through and classify the pair $(S_0, \eta_*\mathscr{L})$ even if $g(\eta_*\mathscr{L})\geqq g(\mathscr{L})$ in general. Since the ampleness of a line bundle does not necessarily lift to S, we can only give necessary conditions for $(S, \mathscr{L})$ to exist, e.g. the description of $\eta_*\mathscr{L}$ in $\mathrm{Num}(S_0)$, some restrictions on the factorization of η and further restrictions on the centers of the blowing-ups.

In the rational case the behaviour of the adjunction mapping is unknown in its full generality, so we can give no restrictions on the possible factorization of η (unlike when $\mathscr{L}$ is very ample).

In Sec. 1 we consider surfaces of general type and elliptic surfaces. In Sec. 2 we consider surfaces of Kodaira dimension zero, while Sections 3 and 4 are devoted to non-rational ruled surfaces and rational surfaces respectively.

Finally a reason why so many people as authors. Actually this paper started from some contributions independently given by the second and the other two authors; afterwards it was developed together, so as to avoid more publications on the same topic.

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0. Notation, definitions and preliminary results

By surface we shall always mean a smooth complex projective algebraic variety of dimension two. Let S be a surface and D, D' divisors on S. The symbol DD' will denote the intersection index of D and D'; D^2 will denote the self-intersection of D. The linear and the numerical equivalence of D and D' will be expressed by writing $D\equiv D'$ and $D\doteqdot D'$ respectively. We shall use the following other standard symbols: $\mathcal{O}_S(D)$=the invertible sheaf associated to D, $h^q(D)=\dim_{\mathbf{C}} H^q(S, \mathcal{O}_S(D))$; $|D|$=the complete linear system defined by D; $\varphi_{|D|}\colon S\to \mathbf{P}^{h^0(D)-1}$=the rational map associated to $|D|$; K_S=a canonical divisor on S; $p_g(S)=h^0(K_S)$=the geometric genus of S; $p_n(S)=h^0(nK_S)$=the n-genus ($n\geqq 1$) of S; $q(S)=h^1(\mathcal{O}_S)$=the irregularity of S; $\chi(\mathcal{O}_S)=1-q(S)+p_g(S)$.

For a divisor D on S, the integer $g(D)=1+\frac{1}{2}(D^2+DK_S)$ will be called the arithmetic genus of D. Of course, if $|D|$ contains a smooth curve C, then $g(D)=g(C)$

coincides with the geometric genus of C. Moreover, if D is an ample divisor, then (cf. [13]) $g(D)\geqq\max(0, q(S)-p_g(S))$.

Throughout all the paper we deal with *polarized pairs* (S, A) where S is a surface and A is an *ample divisor* on S. We shall denote by $\mathcal{S}_g$ the class of the pairs (S, A) as above such that $g(A)=g$. We shall denote by $\mathcal{B}$ the class of pairs (S, A) where S is a $\mathbf{P}^1$-bundle and $\mathcal{O}_f(A)=\mathcal{O}_{\mathbf{P}^1}(1)$ for any fibre f of S and by $\mathcal{C}$ the class of pairs (S, A) where S is a conic bundle (i.e. a $\mathbf{P}^1$-bundle blown up at a finite number $s\geqq 0$ of points lying on distinct fibres) and $\mathcal{O}_f(A)=\mathcal{O}_{\mathbf{P}^1}(2)$ for the general fibre f of S.

Let us point out some basic facts about ample divisors.

Proposition 0.1 ([13], Th. 1.1). *Assume* $(S, A)\notin\mathcal{B}$ *and* $g(A)\geqq 2$. *Then* $A^2\leqq 4(g(A)-1)+K_S^2$, *and equality holds if and only if* $(S, A)\in\mathcal{C}$. □

Proposition 0.2 ([13, Th. 2.2). *Assume* $p_g(S)=0$ *and* $g(A)\geqq 1$. *Then* $g(A)=q(S)$ *if and only if* $(S, A)\in\mathcal{B}$. □

Let $(S, A)\in\mathcal{S}_g$ and let $\eta\colon S\to S_0$ be a birational morphism onto a minimal model S_0 of S. Then $A_0=\eta_*A$ is an ample divisor on S_0 by the Nakai—Moishezon criterion and $g_0=g(A_0)\geqq g$. Such a pair $(S_0, A_0)\in\mathcal{S}_{g_0}$ will be said a *minimal pair* of (S, A).

Finally, for a $\mathbf{P}^1$-bundle S, e will denote the invariant and C_0 and f a fundamental section and a fibre respectively (cfr. [10], p. 372). We also denote by $\mathbf{F}_e$ the rational $\mathbf{P}^1$-bundle of invariant e.

1. The case $\varkappa(S)\geqq 1$

In this section we deal with a polarized pair (S, A) where the surface S has Kodaira dimension $\varkappa(S)\geqq 1$. First of all, let us state some preliminary results.

Lemma 1.1. *If* $(S, A)\in\mathcal{S}_2$ *and* $\varkappa(S)\geqq 1$, *then*

(1.1.1) $$A^2 = AK_S = 1$$

and S *is a minimal model.*

Proof. Since $\varkappa(S)\geqq 1$, there exists an integer $m\gg 0$ such that mK_S is effective. Then $K_SA>0$ and $A^2>0$ as A is ample; so $g(A)=2$ gives (1.1.1). Now assume that S is not a minimal model and consider a birational morphism $\eta\colon S\to S_0$ from S to its minimal model S_0. One has $K_S\equiv\eta^*K_{S_0}+E$ where E is an effective non-trivial divisor whose connected components are contracted by η. Since $\varkappa(S)=\varkappa(S_0)\geqq 1$ then mK_{S_0} is effective for $m\gg 0$ and so $A\eta^*K_{S_0}>0$, which gives $1=AK_S=A\eta^*K_{S_0}+AE$, contradiction. q.e.d.

Lemma 1.2. *If* $(S, A)\in\mathcal{S}_2$ *and* $\varkappa(S)=1$ *then*

(1.2.1) $$h^0(A) \geqq \chi(\mathcal{O}_S)$$

and either $\chi(\mathcal{O}_S)=0$ *or* $q(S)=p_g(S)=0$.

Proof. In view of Lemma 1.1, S is a minimal model and $(K_S-A)A=0$. Then the ampleness of A implies $h^2(A)=h^0(K_S-A)=0$, so (1.2.1) is a consequence of the Riemann–Roch theorem.

Clearly $q(S)\leqq p_g(S)+1$ because $\chi(\mathcal{O}_S)\geqq 0$. Assume that $\chi(\mathcal{O}_S)>0$. Therefore $h^0(A)>0$, so there exists a curve $A\in|A|$ which is irreducible and reduced since $A^2=1$. We want to prove that $q(S)=0$. By contradiction suppose that $q(S)\neq 0$. Then we can choose an element $\gamma\in \mathrm{Pic}^0(S)\setminus\{0\}$ and $h^2(K_S+\gamma)=h^0(-\gamma)=0$ as $\gamma A=0$. Hence the Riemann–Roch theorem yields

$$h^0(K_S+\gamma) \geqq \chi(\mathcal{O}_S) \geqq 1.$$

First suppose that $h^0(K_S+\gamma)\geqq 2$, so $|K_S+\gamma|$ contains a pencil Φ. Note that Φ does not contain A as a fixed component. Otherwise for every $\Gamma\in\Phi$ we should have $\Gamma\equiv A+C$ with C effective divisor, therefore $1=\Gamma A=A^2+AC=1+AC$ which implies $C\equiv 0$, i.e. $\Gamma\equiv A\equiv K_S+\gamma$. This gives $1=(A-\gamma)^2=K_S^2=0$, contradiction. It thus follows that for every point $p\in A$ there exists a curve $\Gamma\in\Phi$ passing through p and such that $\Gamma A=1$. This implies that A is not singular. Moreover Φ cuts a g_1^1 on A and then A is rational, contradiction.

Now assume that $h^0(K_S+\gamma)=1$ for every $\gamma\in \mathrm{Pic}^0(S)\setminus\{0\}$. Hence the restricted divisor $(K_S+\gamma)|_A$ is effective and its support consists of a single point. By extending to zero the map

$$\mathrm{Pic}^0(S) \to A$$

sending γ to the support of $(K_S+\gamma)|_A$ we get a morphism ϱ, which is injective in view of [23], p. 120. Therefore, as A is irreducible and $\mathrm{Pic}^0(S)\neq(0)$, ϱ is an isomorphism. It follows that $\dim \mathrm{Pic}^0(S)=q(S)=1$ and so A is an elliptic curve, which is again a contradiction. Hence $q(S)=0$.

Now suppose that $p_g(S)\geqq 1$. The Riemann–Roch theorem gives

$$h^0(K_S+A) = p_g(S)+2-q(S) \geqq 3.$$

Let $K\in|K_S|$ and let Φ' be a pencil contained in $|K_S+A|$. Then the relations $1=AK=\Gamma'K$ for every $\Gamma'\in\Phi'$ imply that K is an irreducible nonsingular curve containing a g_1^1. Once again this gives a contradiction because the genus formula implies that $g(K)=1$. q.e.d

Lemma 1.3. *Let* $(S, A)\in\mathcal{S}_2$ *with* $\varkappa(S)=1$, $\chi(\mathcal{O}_S)=0$. *Let* $\Psi\colon S\to\Delta$ *be an elliptic fibration and suppose that* $\Delta\cong\mathbf{P}^1$. *Then* $q(S)=1$, $p_g(S)=0$.

Proof (essentially due to Sommese [20]). Since $q(S)=p_g(S)+1$ we have only to show that $q(S)=1$. Suppose that $q(S)>1$ and $\dim \alpha(S)=1$ where $\alpha\colon S\to \mathrm{Alb}(S)$ is the Albanese map. Then $\alpha(S)$ is a nonsingular curve C of genus $g(C)>1$, so the fibres of Ψ are contracted by α. Hence we get a surjective morphism $\mathbf{P}^1\to C$, contradiction. Therefore we can assume that $\dim \alpha(S)=2$. Let E be the general fibre of Ψ. The image $\alpha(E)$ is neither a point nor a rational curve; hence it is an elliptic curve and we can consider the abelian variety $\mathscr{A}=\mathrm{Alb}\,(S)/\alpha(E)$. Clearly $\dim \mathscr{A}\geqq 1$ as $\dim \mathrm{Alb}\,(S)\geqq 2$. Let $\pi\colon S\to\mathscr{A}$ be the canonical projection composed with α. The image $\pi(E)$ is a point since Ψ has connected fibres. Then we get a surjective morphism $\mathbf{P}^1\to\mathscr{A}$, wihch is again a contradiction. q.e.d

Now we are going to consider the case $\varkappa(S)=2$. Any smooth rational curve $C\subset S$ such that $C^2=-2$ will be called (-2)*-curve*. Of course for such a curve one has $CK_S=0$. Recalling some well known facts about surfaces of general type (cfr. [2], [3], .11]) we can state the following

Theorem 1.4. *Assume that* $(S, A)\in\mathscr{S}_2$ *and* $\varkappa(S)=2$. *Then S is a minimal surface, the canonical bundle is ample (in particular S does not contain any (-2)-curves), the numerical invariants are* $K_S^2=1$, $q(S)=0$, $p_g(S)\leqq 2$ *and S is as follows:*

(i) $p_g(S)=0$, *S is a (numerical) Godeaux surface and* $A\equiv K_S+\varepsilon$, *where* $\varepsilon\in\mathrm{Tors}\,(\mathrm{Pic}\,(S))$. *Moreover if* $\varepsilon\neq 0$ *then* $h^0(A)=1$.

(ii) $p_g(S)=1, A\equiv K_S$. *In this case S is isomorphic to a (smooth) complete intersection of type* (6,6) *in the weighted projective space* $\mathbf{P}(1, 2, 2, 3, 3)$.

(iii) $p_g(S)=2$, $A\equiv K_S$ *and S is a Horikawa surface. In this case let* $\sigma\colon\tilde{S}\to S$ *be the blowing-up at the unique base point of* $|A|$. *Then $\tilde{S}$ is the minimal resolution of singularities of a double covering of the rational* $\mathbf{P}^1$*-bundle* $\mathbf{F}_2$ *branched along a curve* $B\in|6C_0+10f|$ *with no infinitely close triple points.*

Proof. The surface S is a minimal model by Lemma 1.1. Furthermore we have (cf. [10], 1.9, p. 368).

$$A^2(K_S+A)^2 \leqq \big(A(K_S+A)\big)^2 = 4. \tag{1.4.1}$$

On the other hand, $A^2(K_S+A)^2=K_S^2+3$ again by means of Lemma 1.1, so that $K_S^2\leqq 1$. Then, since S is a minimal surface of general type, one gets $K_S^2=1$. This implies $q(S)=0$ and therefore by Noether's inequality we find $p_g(S)\leqq 2$ (see [2], Thm. 9). Moreover, since A is an ample divisor and $A(K_S-A)=AK_S-A^2=0$, the Hodge index theorem implies either $(K_S-A)^2<0$ or $K_S\equiv A$, but as $(K_S-A)^2=K_S^2-2K_SA+A^2=0$, one concludes that $K_S\equiv A$ is ample. In particular it follows that S cannot contain any (-2)-curves. If $p_g(S)=0$, then S is a numerical Godeaux surface. The last part of (i) follows from a result due to M. Reid (cfr. [5], p. 158). Whenever $p_g(S)=1$ or 2 the Picard group $\mathrm{Pic}\,(S)$ is torsion free by [2], Thms. 14, 15. In both cases $A \equiv K_S$ and every divisor belonging to $|A|$ is an irreducible reduced curve since $A^2=1$; so, if $p_g(S)=2$, $|A|=|K_S|$ is a pencil of irreducible curves of genus two. If $p_g(S)=1$ the canonical model M of S is (isomorphic to) a complete intersection of type (6,6) in the weighted projective space $\mathbf{P}(1, 2, 2, 3, 3)$, having double rational points at most, as proved in [3], §1. Now since S contains no (-2)-curves it has to be $S \cong M$ and we get (ii). If $p_g(S)=2$ all that we specified in (iii) follows from [11], Thm. 2.1. q.e.d.

Example. The numerical Godeaux surfaces with $T = \mathrm{Tors}(\mathrm{Pic}(S)) \cong \mathbf{Z}_r$, $r=4$, or 5, are well known. Following [5], III § 2 we can explicitly describe the surfaces occuring in 1.4 i) in these cases.

Let $S' \to S$ be the unramified covering of degree r ($r=4$ or 5) corresponding to the torsion group T. Since S contains no (-2)-curves, S' is isomorphic to its canonical model.

If $T=\mathbf{Z}_5$, the canonical model of S' is a quintic surface $\Sigma \subset \mathbf{P}^3$ and then S is the quotient of Σ with respect to the action of a cyclic group of order 5 (the classical Godeaux surface). Since $\mathcal{O}_\Sigma(K_\Sigma) \cong \mathcal{O}_\Sigma(1)$, K_S is ample and $g(K_S)=2$.

If $T=\mathbf{Z}_4$, the canonical model of S' is a smooth complete intersection V of type (4,4) in the weighted projective space $\mathbf{P}(1, 1, 1, 2, 2)$. Thus S is isomorphic to the quotient of V with respect to the action of a cyclic group of order 4. The weighted adjunction formula gives $\mathcal{O}_V(K_V) \cong \mathcal{O}_V(4+4-1-1-1-2-2) = \mathcal{O}_V(1)$. Furthermore $\mathcal{O}_V(1)$ is ample since V can be chosen as a nonsingular weighted complete intersection in the sense of Mori [17]. Then K_S is ample and $g(K_S)=2$. For instance one can consider the group μ_4 of the 4-th roots of 1 and its action on $\mathbf{P}(1, 1, 1, 2, 2)$ defined by $(T_0, T_1, T_2, T_3, T_4) \to (\zeta T_0, \zeta^2 T_1, \zeta^3 T_2, \zeta T_3, \zeta^3 T_4)$, ($\zeta$ a primitive 4-th root of 1). The surface V whose equations are

$$T_0^4+T_1^4+T_2^4+T_3T_4 = T_0^2T_1^2+T_1^2T_2^2+T_3^2+T_4^2 = 0$$

is invariant under this action and contains no fixed points of it. The quotient of V with respect to this action of μ_4 is just an example as above. □

The last part of this section is devoted to the case of a surface S with $\varkappa(S)=1$. Let us consider an elliptic fibration $\Psi\colon S \to \Delta$. Let F be the general fibre of Ψ and f_i the reduced component of a fibre of multiplicity m_i. Then $F \equiv m_i f_i$ and the canonical divisor can be written as follows (cf. [9], p. 572).

$$K_S = \Psi^* \delta + \textstyle\sum_i (m_i-1) f_i, \tag{1.5.1}$$

where $\delta \in \mathrm{Div}(\Delta)$ and $\deg \delta = 2g(\Delta)-2+\chi(\mathcal{O}_S)$. As $K_S A=1$, by Lemma 1.1. one gets

$$(2g(\Delta)-2+\chi(\mathcal{O}_S))AF + \textstyle\sum_i (m_i-1)Af_i = 1. \tag{1.5.2}$$

We shall divide our analysis of formula (1.5.2) into three parts according to whether $2g(\Delta)-2+\chi(\mathcal{O}_S)\gtreqless 0$.

a) Case $2g(\Delta)-2+\chi(\mathcal{O}_S)>0$. Since $\sum_i(m_i-1)Af_i\geqq 0$, it is $AF=1$, $2g(\Delta)-2+\chi(\mathcal{O}_S)=1$ and Ψ has no multiple fibres. In view of Lemma 1.2 it can only be $g(\Delta)=\chi(\mathcal{O}_S)=1$ and so $|\Delta|$ contains an effective divisor D. Now the equalities $A^2=AF=1$ imply that D is an irreducible section of Ψ, but this gives $2=g(A)=g(\Delta)\leqq 1$, contradiction.

b) Case $2g(\Delta)-2+\chi(\mathcal{O}_S)=0$. In this case formula (1.5.2) becomes $\sum_i(m_i-1)Af_i=1$, hence, in view of the ampleness of A, the morphism Ψ has a unique multiple fibre f_1 of multiplicity $m_1=2$. By (1.5.1) we get $K_S=f_1$ and so $p_g(S)=1$. Then Lemma 1.2 gives $\chi(\mathcal{O}_S)=0$, which implies $g(\Delta)=1$ and then $p_2(s)=h^0(2K_s)=1$. This case however cannot occur due to a result of Kollar [24], Props. 4.5 and 5.1.

c) Case $2g(\Delta)-2+\chi(\mathcal{O}_S)<0$. We can assume $m_i\geqq 2$ for each $i=1,\dots,k$ in view of (1.5.2) and let $m_1\leqq\dots\leqq m_k$. Since $\chi(\mathcal{O}_S)\geqq 0$, we have $g(\Delta)=0$ and by Lemma 1.2 it can only be either $p_g(S)=q(S)=0$ or $\chi(\mathcal{O}_S)=0$; in the latter case it has to be $p_g(S)=0$, $q(S)=1$ in view of Lemma 1.3. Put $\chi(\mathcal{O}_S)=1-\varepsilon$, $\varepsilon=0,1$. Then as $F\equiv m_if_i$, formula (1.5.2) becomes

$$(1.5.3)\qquad \Big(k-1-\varepsilon-\textstyle\sum_{i=1}^k 1/m_i\Big)AF=1.$$

In particular it follows that $AF=\text{l.c.m.}\ \{m_i\}$. Let $m=\max\{m_i\}$. Then $AF\geqq m$ and since $m_i\geqq 2$ for each i, one gets the inequality

$$(1.5.4)\qquad 1\geqq m\big(k-1-\varepsilon-(k-1)/2-1/m\big)=m\big((k-1)/2-\varepsilon\big)-1,$$

where the equality holds if and only if both $AF=m$ and $m_1=\dots=m_k=2$. Hence

$$(1.5.5)\qquad m(k-1-2\varepsilon)\leqq 4$$

and again the equality holds if and only if both $AF=m$ and $m_1=\dots=m_k=2$. As $m\geqq 2$, one gets $k\leqq 3+2\varepsilon$ by (1.5.5) and if $k=3+2\varepsilon$, then $m_1=m_2=\dots=m_k=2$. On the other hand, it has to be $k>1+\varepsilon$ in view of (1.5.3). Therefore, when $\chi(\mathcal{O}_S)=1$ we have the following possibilities:

c_{11}) $k=3,\quad m_1=m_2=m_3=2$ and $AF=2$;

c_{12}) $k=2,\quad m_1=m_2=3$ and $AF=3$;

c_{13}) $k=2,\quad m_1=2,\quad m_2=4$ and $AF=4$;

c_{14}) $k=2,\quad m_1=2,\quad m_2=3$ and $AF=6$.

When $\chi(\mathcal{O}_S)=0$, if $k>3$, in the same way we get the following possibilities (recall that $AF=$l.c.m. $\{m_i\}$):

c_{21}) $k=5, \quad m_1=m_2=m_3=m_4=m_5=2$ and $AF=2$;

c_{22}) $k=4, \quad m_1=m_2=m_3=2, \quad m_4=4$ and $AF=4$;

c_{23}) $k=4, \quad m_1=m_2=m_3=2, \quad m_4=3$ and $AF=6$.

For $k=3$ more care is needed. Indeed in this case (1.5.3) reduces to

(1.5.6) $$(1-1/m_1-1/m_2-1/m_3)AF=1.$$

To make positive the left hand of (1.5.6) it has to be $m_i \geqq 4$ for some i. Assume, for instance, $m_3 \geqq 4$; then (1.5.6) gives $1 \geqq AF(3/4-1/m_1-1/m_2)$, and the equality holds if and only if $m_3=4$. For shortness, write $p=m_1$, $q=m_2$. Since $AF \geqq m_3 \geqq 4$, we get $pq \leqq 2(p+q)$. Then $p \leqq 4$ by the assumption $p \geqq q$ and furthermore $3 \leqq q \leqq 6$ when $p=3$, $q=4$ when $p=4$, whereas we get no restrictions on q when $p=2$. In case $p=3$, $3 \leqq q \leqq 6$ we obtain the 4-tuples (3, 3, 4; 12), (3, 3, 6; 6) and when $p=q=4$ we get (4, 4, 4; 4) only. Whenever $p=2$ by looking at the low values of m_3 we find the following 4-tuples (2, 4, 5; 20), (2, 4, 6; 12), (2, 4, 8; 8), (2, 5, 5; 10), (2, 6, 6,; 6), (2, 3, 7; 42), (2, 3, 8; 24), (2, 3, 9; 18), (2, 3, 12; 12), while for $m_3 \geqq 13$ one gets the inequality $1 \geqq AF(1-1/2-1/q-1/13)=AF(11/26-1/q)$. Since $AF \geqq m_3 \geqq 13$, it follows that $13 \leqq AF \leqq 26q/(11q-26)$, then $q \leqq 2$, i.e. $q=2$; but (1.5.6) cannot hold if $p=q=2$. In conclusion for $k=3$ we have the following cases:

c_{31}) $m_1=2, \quad m_2=4, \quad m_3=8$ and $AF=8$

c_{32}) $m_1=2, \quad m_2=m_3=6,$ and $AF=6$;

c_{33}) $m_1=2, \quad m_2=3, \quad m_3=12$ and $AF=12$:

c_{34}) $m_1=m_2=3, \quad m_3=6,$ and $AF=6$;

c_{35}) $m_1=m_2=m_3=4$ and $AF=4$;

c_{41}) $m_1=2, \quad m_2=4, \quad m_3=5$ and $AF=20$;

c_{42}) $m_1=2, \quad m_2=4, \quad m_3=6$ and $AF=12$;

c_{43}) $m_1=2, \quad m_2=m_3=5$ and $AF=10$;

c_{44}) $m_1=2, \quad m_2=3, \quad m_3=7$ and $AF=42$;

c_{45}) $m_1=2, \quad m_2=3, \quad m_3=8$ and $AF=24$;

c_{46}) $m_1=2, \quad m_2=3, \quad m_3=9$ and $AF=18$;

c_{47}) $m_1=m_2=3, \quad m_3=4$ and $AF=12$.

To obtain further information about the pair (S, A) we have to analyze the above case c) more closely. First of all, by putting $n=AF$ and recalling (1.5.1), (1.5.3), one gets

$$nK_S \equiv \left(n(k-1-\varepsilon)-\sum_{i=1}^{k} n/m_i\right)F \equiv F,$$

hence $p_n(S)=2$, as $g(\Delta)=0$. In each case c_{1l}), since $\chi(\mathcal{O}_S)>0$, we know that $h^0(A)>0$ by (1.2.1). Moreover, as $A^2=1$, any effective divisor $C\in|A|$ is an irreducible curve which is a n-section of the morphism $\Psi\colon S\to\Delta$. On the contrary, when $\chi(\mathcal{O}_S)=0$, we do not know whether A is an effective divisor or not. Put

$$f=\begin{cases} f_k & \text{in cases } c_{2h}),\quad h\neq 3 \quad\text{and}\quad c_{3s}),\\ f_1-f_k & \text{in cases } c_{23}) \quad\text{and}\quad c_{47}),\\ f_1-f_2-f_3 & \text{in cases } c_{4t}) \quad t\neq 7.\end{cases}$$

Now consider the divisor $A+f$. Of course $h^2(A+f)=h^0(K_S-A-f)=0$ since $(K_S-A-f)A=-fA<0$, so that, by the Riemann—Roch theorem we get $h^0(A+f)\geqq Af=1$. Let $D\in|A+f|$; as $DA=2$ and $D^2=3$, D cannot have more than two irreducible reduced components. So, if D is not a curve, then it can be written as $D=C_1+C_2$, where C_1, C_2 are irreducible curves such that $C_1A=C_2A=1$. Furthermore as $Df=(A+f)f=1$, one of the two curves C_i's for instance C_2, is contained in a union of fibres of Ψ. Since $Af=1$ any fibre of Ψ is numerically equivalent to a multiple of f; so we have $C_2\equiv rf$, r positive rational number. On the other hand, $2=DA=(C_1+rf)A=C_1A+r$, so $r=1$, in view of the ampleness of A. Therefore, if we are not in cases c_{23}) and c_{4t}) we conclude that C_2 itself is a multiple fibre of maximal multiplicity m_k. Hence we get $C_1\in|A|$ in cases c_{22}), c_{31}), c_{33}) and c_{34}), where $f=f_k$ is the reduced component of the unique fibre of maximal multiplicity; while in cases c_{21}), c_{32}) and c_{35}) we get only $A\equiv C_1+(f'-f'')$ where f' and f'' are the reduced components of any two fibres of maximal multiplicity. In each of these cases C_1 is a m_k-section of Ψ. In cases c_{23}) and c_{47}) in the same way we get $A\equiv C_1+f_i-f_j$ where $i,j=1,2,3$ and $i,j=1,2$ respectively. As to the remaining cases c_{4t}), $t\neq 7$, we get either $A\equiv C_1$ or $A\equiv C_1+2f_2-f_1$ in cases c_{41}) and c_{42}), $A\equiv C_1+f_3-f_2$ in case c_{43}), $A\equiv C_1+4f_3-f_1$ in case c_{45}), $A\equiv C_1+f_2-3f_3$ in case c_{46}). Note that, in case c_{44}) as the three multiplicities are relatively prime, it can only be $A\equiv C_1$. In all these cases C_1 is a n-section of Ψ.

What we have proven is summarized by the following

Theorem 1.5. *Assume that $(S, A)\in\mathcal{S}_2$ and $\varkappa(S)=1$. Then S is a minimal surface endowed with an elliptic fibration $\Psi\colon S\to\mathbf{P}^1$, $A^2=1$ and (S, A) belongs to one of the classes described below (there Γ_s denotes a s-section of Ψ). Moreover if m_i are the multiplicities of the multiple fibres of Ψ and $n=$l.c.m. $\{m_i\}$, then $p_n(S)=2$.*

Class	Description of S			$g(\Delta)$	Description of A in terms of n-sections
	$q(S)$	$p_g(S)$	$(m_1, m_2, \ldots, m_k)$		
c_{11}			$(2, 2, 2)$		
c_{12}			$(3, 3)$		
c_{13}	0	0	$(2, 4)$	0	Γ_n, effective
c_{14}			$(2, 3)$		
c_{21}			$(2, 2, 2, 2, 2)$		$\Gamma_2+f_i-f_j$ or Γ_2-f_i
c_{22}	1	0	$(2, 2, 2, 4)$	0	Γ_4 or Γ_4-f_4
c_{23}			$(2, 2, 2, 3)$		$\Gamma_6+f_i-f_j (i, j=1, 2, 3)$ or $\Gamma_6-f_i+f_4$
c_{31}			$(2, 4, 8)$		Γ_8 or Γ_8-f_3
c_{32}			$(2, 6, 6)$		$\Gamma_6+f_i-f_j (i, j=2, 3)$ or Γ_6-f_i
c_{33}			$(2, 3, 12)$		Γ_{12} or $\Gamma_{12}-f_3$
c_{34}	1	0	$(3, 3, 6)$	0	Γ_6 or Γ_6-f_3
c_{35}			$(4, 4, 4)$		Γ_4 or Γ_4-f_i
c_{41}			$(2, 4, 5)$		$\Gamma_{20}, \Gamma_{20}\pm(2f_2-f_1)$ or $\Gamma_{20}-f_1+f_2+f_3$
c_{42}			$(2, 4, 6)$		$\Gamma_{12}, \Gamma_{12}\pm(2f_2-f_1)$ or $\Gamma_{12}-f_1+f_2+f_3$
c_{43}			$(2, 5, 5)$		$\Gamma_{10}, \Gamma_{10}\pm(f_3-f_2)$ or $\Gamma_{10}-f_1+f_2+f_3$
c_{44}	1	0	$(2, 3, 7)$	0	Γ_{42} or $\Gamma_{42}-f_1+f_2+f_3$
c_{45}			$(2, 3, 8)$		$\Gamma_{24}, \Gamma_{24}\pm(4f_3-f_1)$ or $\Gamma_{24}-f_1+f_2+f_3$
c_{46}			$(2, 3, 9)$		$\Gamma_{18}, \Gamma_{18}\pm(3f_3-f_2)$ or $\Gamma_{18}-f_1+f_2+f_3$
c_{47}			$(3, 3, 4)$		$\Gamma_{12}+f_i-f_j$ $(i, j=1, 2)$ or $\Gamma_{12}-f_i+f_3$

2. The case $\varkappa(S)=0$

Throughout this section we deal with a polarized pair (S, A) where S is a surface of Kodaira dimension $\varkappa(S)=0$. To begin with we state the following

Proposition 2.1. *Let* $(S, A)\in\mathcal{S}_2$ *with* $\varkappa(S)=0$. *Then one of the following cases holds:*

(2.1.1) *S is a minimal model and* $A^2=2$, $AK_S=0$;

(2.1.2) *S is not a minimal model and* $A^2=AK_S=1$.

Furthermore, whenever a pair $(S, A)\in\mathcal{S}_2$ *verifying* (2.1.2) *exists, then there exists* $(S_0, A_0)\in\mathcal{S}_2$ *verifying* (2.1.1) *and* η: $S=B_p(S_0)\to S_0$ *is the blowing-up of* S_0 *at a point* p, $E=\eta^{-1}(p)$, $A\equiv\eta^*A_0-E$.

Proof. (2.1.1), (2.1.2) follow from the genus formula recalling that A is an ample divisor. Now assume that there exists (S, A) verifying (2.1.2) and consider the morphism $\eta\colon S\to S_0$ onto the minimal model S_0. Thus $K_S\equiv\eta^*K_{S_0}+E$, where E is an effective divisor contracted by η and K_{S_0} is numerically trivial. Then we get $AE=1$, hence E is an exceptional curve of the first kind and η is a blowing-up at a single point. Moreover $A_0=\eta_*A$ is an ample divisor and $\eta^*A_0\equiv A+E$ so we find $g(A_0)=g(A)$.

q.e.d.

To go on we need some preliminary results.

Lemma 2.2. *Let $(S, A)\in\mathcal{S}_2$ with S minimal model. Then one has*

$$h^0(A)=\begin{cases}1 & \textit{if } S \textit{ is either an abelian or a hyperelliptic surface;}\\ 2 & \textit{if } S \textit{ is an Enriques surface,}\\ 3 & \textit{if } S \textit{ is a } K3\textit{-surface.}\end{cases}$$

Proof. Indeed $A-K_S$ is an ample divisor as $K_S\equiv 0$, therefore $h^i(A)=0$, $i=1,2$. Hence the Riemann–Roch theorem gives $h^0(A)=\chi(\mathcal{O}_S)+A^2/2$ and the assertion follows. q.e.d.

Lemma 2.3. *Let $(S, A)\in\mathcal{S}_2$ with S minimal model and let D be an effective divisor on S, $D\equiv A$. If D is reducible, then it can be written as $D=C_1+C_2$ where C_1, C_2 are the irreducible, reduced components and either*

$$g(C_1)=g(C_2)=0,\quad C_1^2=C_2^2=-2,\quad C_1C_2=3$$

or

$$g(C_1)=g(C_2)=1,\quad C_1^2=C_2^2=0,\quad C_1C_2=1.$$

Proof. The ampleness of A and the condition $A^2=2$ show that D has only two irreducible, reduced components C_1, C_2. Moreover since $2=A^2=A(C_1+C_2)$ and $AC_i>0$, we see that $AC_i=1$ and from the equalities $1=AC_i=C_i^2+C_1C_2$, $i=1,2$, we get $C_1^2=C_2^2$. Furthermore the genus formula gives $C_i^2=2g(C_i)-2$, so that C_i^2 is even and $C_i^2\geqq -2$. Hence, as $C_1C_2\geqq 0$ and $C_1C_2=1-C_i^2$, it can only be either $C_1C_2=1$ and $C_i^2=0$, or $C_1C_2=3$ and $C_i^2=-2$, $i=1,2$. q.e.d.

Lemma 2.4. *Let $(S, A)\in\mathcal{S}_2$ with S minimal model. If $h^0(A)\geqq 2$ then $|A|$ has no fixed components.*

Proof. Whenever $h^0(A)\geqq 2$, then S is either a $K3$-surface or an Enriques surface by Lemma 2.2. Assume that $|A|$ has a fixed component C_1 and write $|A|=C_1+|C_2|$. Lemma 2.3 implies that $C_1^2=C_2^2=0$: otherwise $C_2^2=-2$, $g(C_2)=0$ and $|C_2|$ would contain a pencil of rational curves. Then S would be a ruled surface in view of the Noether–Enriques theorem, contradiction.

If S is a $K3$-surface, then $K_S\equiv 0$, so $h^2(C_1)=h^0(K_S-C_1)=0$ and by the Riemann—Roch theorem we find $h^0(C_1)-h^1(C_1)=\chi(\mathcal{O}_S)=2$. Since C_1 is a fixed component of $|A|$, one has $h^0(C_1)=1$ so we get a contradiction. Assume now that S is an

Enriques surface. Then $h^0(C_2)=h^0(A)=2$ and the pencil $|C_2|$ is base point free as $C_2^2=0$. Furthermore for every point $p\in C_1$ there is a curve $C\in|C_2|$ passing through p and such that $1=C_1C$. This implies that C_1 is smooth. On the other hand C_1 is an elliptic curve since $g(C_1)=g(C_2)=1$, while the map which sends every $C\in|C_2|$ to the point $p=C_1\cap C$ defines a surjective morphism $\mathbf{P}^1\to C$, contradiction q.e.d

Proposition 2.5. *Let $(S, A)\in\mathcal{S}_2$, where S is an Enriques surface. Then S is the quotient of a $K3$ surface X which is the double cover of a smooth quadric $Q\subset\mathbf{P}^3$ branched along a curve of bidegree $(4, 4)$, with respect to a fixed-point free involution τ. Moreover, up to numerical equivalence, A is the quotient of the inverse image of a hyperplane section of Q, by τ.*

Proof. In view of the ampleness, $|A|$ contains an irreducible smooth curve A ([4], Thm. 8.3.1 and Thm. 4.1). Let $\pi\colon X\to S$ be the $K3$ universal cover of S and put $B=\pi^*A$. B is ample, hence 1-connected and then $|B|$ has to contain irreducible (and hence smooth, by [18]) curves, which are hyperelliptic of genus 3, since $B^2=4$ and $K_B\equiv\pi^*K_A$. Moreover, $h^0(B)=4$, $|B|$ is base-point free by well known properties of $K3$ surfaces and the corresponding morphism $\varphi_{|B|}\colon X\to\mathbf{P}^3$ exhibits X as a double cover of a quadric surface Q. We claim that Q is smooth. Actually, were Q a quadric cone, then S would be of special type ([4], Lemma 4.4.3.4), i.e. S would contain an elliptic pencil $|2E|$, a nodal curve θ with $\theta E=1$ and $|A|=|2E+\theta+K_S|$. But this leads to a contradiction. Indeed, since nodal curve θ satisfies $\theta^2=-2$ ([4], Prop. 1.6.1), we would get $A\theta=(2E+\theta+K_S)\theta=0$, contradicting the ampleness of A. This proves the claim. Now, since $K_X\equiv 0$, the branch locus of $\varphi_{|B|}$ has bidegree $(4, 4)$. q.e.d.

Example. Let Q be a smooth quadric surface and let $\varphi\colon X\to Q$ be a double cover branched along a smooth curve of bidegree $(4, 4)$. Then X is a $K3$ surface. One can find an involution of Q fixing the branch locus, a smooth hyperplane section H, and inducing a fixed-point free involution τ on X (e.g. [1], p. 184); then τ acts on $B=\varphi^*H$. Now let $S=X/(\tau)$ and let $\pi\colon X\to S$ be the natural projection. Then S is an Enriques surface and $A=\pi(B)$ is an ample divisor with $g(A)=2$, its ampleness following from the one of B and from the finiteness of π.

Proposition 2.6. *Let $(S, A)\in\mathcal{S}_2$ where S is an abelian surface. Then either*

(2.6.1) *$S=J(C)$ is the Jacobian of a smooth curve C of genus 2 and A is numerically equivalent to C embedded in $J(C)$, or*

(2.6.2) *S is the product of two elliptic curves and A is numerically equivalent to the sum of the factors.*

Proof. By Lemma 2.2 there exists a unique $C\in|A|$. Firstly assume C to be irreducible; then C is smooth since it can be neither a rational nor a singular elliptic curve ([22], p. 117) and the inclusion $C \hookrightarrow S$ factors through the commutative diagram

$$\begin{array}{ccc} C & \xrightarrow{i} & S \\ \downarrow{\scriptstyle \alpha_C} & & \downarrow{\scriptstyle \alpha_S} \\ \mathrm{Alb}\,(C) & \xrightarrow{i_*} & \mathrm{Alb}\,(S) \end{array},$$

where α_C, α_S are Albanese morphisms and i_* is a surjection with connected fibres, due to the Lefschetz theorem ([6], p. 155). Since C has genus $2=q(S)$, this implies that i_* is an isomorphism; on the other hand α_S is an isomorphism too, S being an abelian surface and then (2.6.1) holds. Now assume that C is reducible. By Lemma 2.3 we get $C=C_1+C_2$, where C_1, C_2 are smooth elliptic curves such that $C_1^2=C_2^2=0$ and $C_1C_2=1$. The curves C_1, C_2 can be identified with subgroups of S, up to the translation sending the point $p=C_1\cap C_2$ to the zero of the group. Hence we get an isomorphism $S\cong C_1\times C_2$ and (2.6.2) holds. q.e.d

Now let us recall some general facts about hyperelliptic surfaces. As is known, a hyperelliptic surface S is the quotient of the product $F\times C$ of two elliptic curves with respect to a translation group G of F, which acts on C and is isomorphic to $\mathbf{Z}_m\oplus \mathbf{Z}_n$ where the pair (m, n) is one of the following: (2, 1), (2, 2), (4, 1), (4, 2), (3, 1), (3, 3), (6, 1) (for details see [9], p. 586—590). In any case the canonical divisor K_S is torsion of order m and the surface S contains two pencils of elliptic curves. The first one is elliptic and consists of the fibres of the Albanese map of S, all of which are isomorphic to C. The second one is rational and consists of the fibres of the morphism ψ which makes the diagram

$$\begin{array}{ccc} F\times C & \xrightarrow{\pi} & C \\ \downarrow & & \downarrow \\ S & \xrightarrow{\psi} & \mathbf{P}^1 \end{array}$$

commute, where π is the projection and the vertical arrows are defined by the action of G. Of course this fibration has multiple fibres corresponding to the fixed points of the action of G on C. Let $\bar{F}$ and $\bar{f}$ be the general fibre and the reduced component of the fibre of maximal multiplicity of ψ and let $\bar{C}$ be a fibre of the Albanese map. It can be shown that $\bar{F}\equiv m\bar{f}$ and $\bar{C}\bar{F}=mn$.

Coming back to the pair (S, A), let us introduce the following notation

$$\mathscr{X}(S, A) = \bigcup_D \{\mathrm{Supp}\,(D),\ D \text{ reducible divisor, } D \equiv A\}.$$

Theorem 2.7. *Consider a polarized pair* $(S, A)\in\mathscr{S}_2$, *with* $\varkappa(S)=0$.
(2.7.1) *If S is a minimal model, then (S, A) belongs to one of the classes listed in the table below.*

Class	S	Description of (S, A)	A^2	$h^0(A)$
a)	$K3$	$\pi: S \to \mathbf{P}^2$ is a double cover branched along a smooth sextic; $A \equiv \pi^*$ (line)	2	3
b)	Enriques	The universal cover $\pi: X \to S$ is a $K3$ surface s.t. $\varphi: X \to Q \subset \mathbf{P}^3$ is a double cover of a smooth quadric surface with branch locus of bidegree (4, 4); $A \equiv \pi(\varphi^*$ (hyperplane section))	2	2
c_1)	abelian	$S = J(C)$ the jacobian of a smooth curve C of genus 2; $A \equiv$ image of C in its jacobian	2	1
c_2)		$S = E_1 \times E_2$, the product of two elliptic curves; $A \equiv E_1 + E_2$		
d)	hyper-elliptic	$nA \equiv r\bar{C} + s\bar{f}$, $r, s \in \mathbf{Z}$, $rs = n$, $n = 1, 2, 3$	2	1

(2.7.2) *If S is not a minimal model, then $\eta: S = B_p(S_0) \to S_0$ is the blowing-up at a point p of the minimal model S_0, $A = \eta^* A_0 - \eta^{-1}(p)$ and (S_0, A_0) is as in cases* a), b), c_1) *and* d) *with $nA_0 \equiv \bar{C} + n\bar{f}$, $n = 2, 3$, listed in* (2.7.1). *Furthermore, p lies in $A_0 \setminus \mathscr{L}(S_0, A_0)$,*

Proof. Assume that S is a minimal model. If S is a $K3$ surface then $h^0(A) = 3$, by Lemma 2.2 and the map $\varphi_{|A|}: S \to \mathbf{P}^2$ is a morphism exhibiting S as a double cover of $\mathbf{P}^2$ branched along a smooth sextic, as was proved in [18], Sec. 5. Whenever S is either an Enriques or an abelian surface, Propositions 2.5 and 2.6 give the result. So it only remains to consider the case where S is a hyperelliptic surface. In this case, since the 2^{nd} Betti number $b_2(S) = 2$, the classes of $\bar{C}$ and $\bar{f}$ generate over $\mathbf{Q}$ the algebraic 1-cycles modulo numerical equivalence. It thus follows that $A \equiv a\bar{C} + b\bar{f}$, $a, b \in \mathbf{Q}$. Hence one obtains $2 = A^2 = 2abn$; moreover an, bn are integers, as we can see by computing $A\bar{f}$ $A\bar{C}$, respectively. Therefore we find that $nA \equiv r\bar{C} + s\bar{f}$, where $r, s \in \mathbf{Z}$, $rs = n$, $n = 1, 2, 3$ and (2.7.1) is proved.

If S is not a minimal model, then, by Proposition 2.1, $\eta: S = B_p(S_0) \to S_0$ is the blowing-up at a point p of the minimal model S_0, $A = \eta^* A_0 - \eta^{-1}(p)$ and (S_0, A_0) is one of the pairs listed in (2.7.1). Now let D be a reducible divisor, $D \equiv A_0$, and suppose that $p \in \mathrm{Supp}(D)$. Then $D = C_1 + C_2$, where C_i is an irreducible curve and $C_i D = 1$, by Lemma 2.3; so, if $p \in C_i$, one sees that $\eta^{-1}(C_i) A \leqq 0$, $i = 1, 2$, which contradicts the ampleness of A(*). Hence $p \notin \mathscr{L}(S_0, A_0)$. In particular, case c_2) cannot occur, since in that case $\mathscr{L}(S_0, A_0) = S_0$. Whenever (S_0, A_0) belongs to class

(*) η^{-1} (D) always denotes the proper transform of a divisor D.

d), consider the fibre $\bar{C}$ through p; since $\eta^{-1}(\bar{C})A>0$, one easily sees that $s\neq 1$, hence $n\neq 1$, $r=1$ and (2.7.2) is proved. q.e.d.

Example. Case (2.7.2) with (S_0, A_0) as in c$_1$) does really occur ([14], Remark 3.10). It can be obtained by taking $S=C^{(2)}$ the two-fold symmetric power of a smooth curve C of genus 2 and A as the image in $C^{(2)}$ of a factor of $C\times C$. Then A is a smooth ample curve of genus 2 and the natural map $C^{(2)}\to J(C)$ is merely the contraction of the unique exceptional curve of $C^{(2)}$ corresponding to the g_2^1 of C.

3. The case $\varkappa(S)<0$, $q(S)>0$

Let (S, A) be a polarized pair belonging to $\mathscr{S}_2$ and assume S to be a ruled surface. The Riemann–Roch and the Kodaira vanishing theorems give $h^0(K_S+A)=g(A)-q(S)$, so $q(S)\leqq g(A)=2$.

First of all, by Proposition 0.2 we have

Proposition 3.1. *Let* $(S, A)\in\mathscr{S}_2$ *and assume that* $\varkappa(S)<0$. *Then* $q(S)=2$ *if and only if* $(S, A)\in\mathscr{B}$. □

Therefore it remains to study the cases $q(S)=0, 1$. Throughout this section we shall suppose $q(S)=1$.

Let $\eta\colon S\to S_0$ be a birational morphism from S to a minimal model S_0. Then S_0 is a $\mathbf{P}^1$-bundle over an elliptic curve B and $A_0\equiv aC_0+bf$, $a, b\in\mathbf{Z}$, where C_0 and f stand for a fundamental section and a fibre respectively. The ampleness conditions for A_0 say that

$$\text{(3.2.1)}\qquad \begin{aligned} &a>0 \quad\text{and}\quad b>ae \quad\text{if}\quad e\geqq 0,\\ &a>0 \quad\text{and}\quad b>ae/2 \quad\text{if}\quad e=-1, \end{aligned}$$

where $e=-C_0^2$ is the invariant of S_0 (cf. [10], p. 377).

We need some more facts, which we shall be using in the sequel. Let $\eta\colon S=B_{p_1,\dots,p_s}(S_0)\to S_0$ be the blowing-up of S_0 at s distinct points $p_1, \dots, p_s$. One has $A\equiv\eta^*A_0-\sum_i r_iE_i$ where $E_i=\eta^{-1}(p_i)$, $r_i=AE_i$ and

$$\text{(3.2.2)}\qquad 0<r_i<a.$$

Indeed $0<AE_i<A\eta^*f=\eta^*A_0\eta^*f-r_iE_i\eta^*f=A_0f=a$. Furthermore

$$\text{(3.2.3)}\qquad A^2=A_0^2-\textstyle\sum_i r_i^2$$

and $AK_S=A_0K_{S_0}+\sum_i r_i$ recalling that $K_S\equiv\eta^*K_{S_0}+\sum_i E_i$. It thus follows that $A_0^2+A_0K_{S_0}=A^2+AK_S+\sum_i r_i(r_i-1)$, that is

$$\text{(3.2.4)}\qquad (a-1)(2b-ae)=2+\textstyle\sum_i r_i(r_i-1).$$

On the other hand $A_0^2=a(2b-ae)=A^2+\sum_i r_i^2$ by (3.2.3), so (3.2.4) yields

(3.2.5) $$2b-ae = A^2-2+\sum_i r_i.$$

Now come back to the original pair $(S, A)\in\mathcal{S}_2$. Of course $(S, A)\notin\mathcal{B}$ since $1=q(S)\neq g(A)=2$. Then by Proposition 0.1 we get

(3.2.6) $$A^2 \leqq 4+K_S^2$$

and the equality holds if and only if $(S, A)\in\mathcal{C}$. Since $K_S^2\leqq K_{S_0}^2-s$, (3.2.6) gives

(3.2.7) $$s \leqq 3, \text{ and if } s = 3 \text{ then } (S, A)\in\mathcal{C}.$$

First we describe class $\mathcal{C}$.

Proposition 3.2. *With the notation as above, let us consider a pair $(S, A)\in\mathcal{S}_2$. Then $(S, A)\in\mathcal{C}$ if and only if $S=B_{p_1,\dots,p_s}(S_0)$ is S_0 blown up at s $(0\leqq s\leqq 3)$ points p_i lying on distinct fibres, $A\equiv\eta^*A_0-\sum_i E_i$, the minimal pair (S_0, A_0) belongs to $\mathcal{S}_2\cap\mathcal{C}$ and is described as follows:*

i) S_0 has invariant $e=0$, $A_0\equiv 2C_0+f$, $S_0\not\cong B\times\mathbf{P}^1$ and none of the points p_i belongs to a fundamental section;

ii) S_0 has invariant $e=-1$, $A_0\equiv 2C_0$ and two or three points p_i do not lie on the same fundamental section.

Proof. The "if part" is clear, so it is enough to describe S_0 since (3.2.7) already gives the bound $s\leqq 3$. As $(S, A)\in\mathcal{C}$ then $r_i=1$ in (3.2.2), so $(S_0, A_0)\in\mathcal{S}_2$. On the other hand $K_S+A\equiv\eta^*(K_{S_0}+A_0)$, therefore $(S_0, A_0)\in\mathcal{C}$ in view of Proposition 0.1. Now put $A_0\equiv 2C_0+bf$, $b\in\mathbf{Z}$, and recall that $K_{S_0}\equiv -2C_0-ef$. Condition $g(A_0)=2$ yields $b=e+1$, which as to A_0 gives the listed cases by means of (3.2.1). Finally, assume that $t\leqq s$ points $p_1, \dots, p_t$ belong to a fundamental section C_0. Then $0<A\eta^{-1}(C_0)=1-e-t$, so $t<1-e$. In particular if $e=0$, S_0 cannot be $B\times\mathbf{P}^1$, since in that case the whole S_0 is covered by fundamental sections. q.e.d.

Now we shall assume that $(S, A)\notin\mathcal{C}$. We are going to characterize the minimal pairs (S_0, A_0) according to the possible values of s; of course it can only be $s\leqq 2$ in view of (3.2.7) and (3.2.6) becomes $A^2<4+K_S^2$.

a) Let $s=2$. First of all assume that $S=B_{p_1,p_2}(S_0)$ is obtained by blowing up S_0 at two distinct points p_1, p_2. Then $K_S^2=-2$ so $A^2\leqq 1$; hence $A^2=1$ and (3.2.4), (3.2.5) give

(3.3.1) $$2 = (a-1)+(r_1-1)(a-r_1-1)+(r_2-1)(a-r_2-1),$$

where every summand on the right hand is non-negative. As $A^2=1$, (3.3.1) gives $a=3$ and either $r_1=r_2=1$, $r_1=r_2=2$ or $r_1r_2=2$. Again from (3.2.5), (3.2.1) one deduces the following cases

a$_1$) $A^2=1, \quad r_1=r_2=1, \quad e=-1, \quad A_0 \equiv 3C_0-f,$

a$_2$) $A^2=1, \quad r_1=2, \quad r_2=1, \quad e=0, \quad A_0 \equiv 3C_0+f,$

a$_3$) $A^2=1, \quad r_1=r_2=2, \quad e=-1, \quad A_0 \equiv 3C_0,$

Note that if the centers p_1, p_2 are on the same fibre of S_0 then the stronger condition $r_1+r_2<a$ holds. Therefore in cases a$_2$), a$_3$) the points p_1, p_2 have to belong to distinct fibres.

It remains to consider the case when p_2 is infinitely near p_1. In this case the birational morphism $\eta: S \xrightarrow{\sigma_1} S_1 \xrightarrow{\sigma_2} S_0$ is the composition of the blowing-ups σ_1 of S_0 at p_1 and σ_2 of S_1 at $p_2 \in \sigma_1^{-1}(p_1)=E$. After putting $E_1=\sigma_2^{-1}(E)$ and $E_2=\sigma_2^{-1}(p_2)$ we get

(3.3.2) $$A \equiv \eta^* A_0 - r_1 E_1 - (r_1+r_2) E_2,$$

where $2 \leqq r_1 = A(E_1+E_2) < a$, $1 \leqq r_2 = AE_2 \leqq a-1$ and $a \geqq 3$ since $(S, A) \notin \mathscr{C}$. Formally the equations we obtain in this case are still (3.2.4), (3.2 5) and so the new cases we find are characterized by the same numbers as in a$_2$), a$_3$), though r_1, r_2 have a different meaning according to (3.3.2).

b) Let $s=1$. Then $K_S^2=-1$ and $A^2<4+K_S^2=3$. So one deduces the following possibilities from (3.3.1): $a=5$, $r=2, 3$ if $A^2=1$ and $a=3$, $r=1, 2$ if $A^2=2$. Therefore (3.2.5), (3.2.1) yield the cases

b$_1$) $A^2=1, \quad r=2, \quad e=-1, \quad A_0 \equiv 5C_0-2f;$

b$_2$) $A^2=1, \quad r=3, \quad e=0, \quad A_0 \equiv 5C_0+f;$

b$_3$) $A^2=2, \quad r=1, \quad e=-1, \quad A_0 \equiv 3C_0-f;$

b$_4$) $A^2=2, \quad r=2, \quad e=0, \quad A_3 \equiv 3C_0+f.$

c) If $S=S_0$, the genus formula gives $(a-1)(2b-ae)=2$. As we are assuming $(S, A) \notin \mathscr{C}$, we have $a \geqq 3$ and the previous equality implies $a=3$ and $b=(3e+1)/2$. Then $e \neq 0$, so ampleness conditions (3.2.1) give

c$_1$) $e=-1, \quad A=A_0 \equiv 3C_0-f.$

The analysis made in this section can be summarized by the following

Theorem 3.3. *Let $(S, A) \in \mathscr{S}_2$ be a polarized pair such that $\varkappa(S)<0$, $q(S)=1$. Let (S_0, A_0) be a minimal pair of (S, A) and let B be the base curve of the elliptic $\mathbf{P}^1$-bundle S_0. Then the pair (S, A) is one of those described in the table below by means of (S_0, A_0). Moreover $S_0 \not\cong \mathbf{P}^1 \times B$ whenever $e(S_0)=0$.*

S	K_S^2	$e(S_0)$	Points	A	A^2	A_0 numerically
$B_{p_1,\dots,p_s}(S_0)$ $0 \leqq s \leqq 3$	$-s$	-1	on distinct fibres, no $1-e(S_0)$ on the same fundamental section	$\eta^* A_0 - \sum_i E_i$	$4-s$	$2C_0$
		0				$2C_0+f$
S_0	0	-1		A_0	3	$3C_0-f$
$B_p(S_0)$	-1	-1		$\eta^* A_0 - E$	2	$3C_0-f$
		0	p not on a fundamental section	$\eta^* A_0 - 2E$		$3C_0+f$
		-1		$\eta^* A_0 - 2E$	1	$5C_0-f$
		0	p not on a fundamental section	$\eta^* A_0 - 3E$		$5C_0+f$
$B_{p_1,p_2}(S_0)$	-2	-1	distinct, no both on the same fundamental section	$\eta^* A_0 - E_1 - E_2$	1	$3C_0-f$
		0	on distinct fibres; no one on a fundamental section	$\eta^* A_0 - E_1 - 2E_2$		$3C_0+f$
		-1	on distinct fibres, no both on the same fundamental section	$\eta^* A_0 - 2E_1 - 2E_2$		$3C_0$
		-1	p_2 infinitely near p_1; p_2 not on a fundamental section	$\eta^* A_0 - 2E_1 - 4E_2$		$3C_0$
		0	p_2 infinitely near p_1; p_1 not on a fundamental section	$\eta^* A_0 - 2E_1 - 3E_2$		$3C_0+f$

Remark 3.4. As far as the surface S_0 is concerned, note that S_0, which is the projectivization $\mathbf{P}(\mathscr{E})$ of a rank-two vector bundle $\mathscr{E}$, is defined up to isomorphisms if $e=-1$, while if $e=0$, either S_0 is unique or else it has a moduli space isomorphic to $\mathbf{P}^1$ according to whether $\mathscr{E}$ is indecomposable or not (see [21], p. 295).

In case $e=0$, and $\mathscr{E}$ decomposable, S_0 is said of type (2, 2) if there exists an elliptic fibration $S_0 \to \mathbf{P}^1$ whose fibres are smooth elliptic 2-sections $\Phi \equiv 2C_0$. Note also that in case $e=0$ with $\mathscr{E}$ indecomposable S_0 has no smooth 2-sections while if $e=-1$, S_0 has exactly three smooth 2-sections $\Phi_i \equiv 2C_0 - f$, $i=1, 2, 3$.

Further restrictions on the position of the points p_i with respect to such 2-sections can easily be obtained. For instance whenever $S=B_p(S_0)$ one sees that p cannot belong to any 2-sections. In case $S=B_{p_1,p_2}(S_0)$ with $e=0$ then p_2 cannot belong to a 2-section $\Phi \equiv 2C_0$ if p_1, p_2 are on distinct fibres, while p_1 cannot belong to such a 2-section if p_2 is infinitely near p_1. In particular this implies that S_0 cannot be of type (2, 2).

4. The rational case

Now assume that $\varkappa(S)<0$ and $q(S)=0$, i.e. S is a rational surface. First of all note that if $(S, A)\in\mathcal{S}_2$, then $S\not\cong\mathbf{P}^2$. Otherwise if we had $A\equiv nL$ where L is a line in $\mathbf{P}^2$, we should obtain $A^2+AK_S=n(n-3)\neq 2$. Hence S dominates a geometrically ruled surface S_0 (possibly $S=S_0$) and just as in the previous section one can consider the corresponding pair (S_0, A_0) which is a minimal pair if S_0 has invariant $e\neq 1$, while S_0 is $\mathbf{P}^2$ blown up at a point p if $e=1$. In this section we shall deal with geometrically ruled pairs instead of minimal pairs.

As usual denote by C_0 and f respectively the fundamental section and the fibre of the geometrically ruled surface S_0. One has $A_0\equiv aC_0+bf$ for some $a, b\in\mathbf{Z}$ and the ampleness of A_0 implies (see again [10]).

(4.1.1) $$a>0 \quad\text{and}\quad b>ae.$$

We shall restrict our considerations to the case in which S is S_0 blown up at s distinct points i.e. the birational morphism $\eta\colon S\to S_0$ factors via s simultaneous blowing-ups.

Since $K_S\equiv\eta^*K_{S_0}+\sum_i E_i$ one has $K_S^2=K_{S_0}^2-s$, so by Proposition 0.1 we know that $A^2\leqq 12-s$ and the equality holds if and only if $(S, A)\in\mathcal{C}$. Since $A^2\geqq 1$ it thus follows that $s\leqq 11$. Let E_i, $i=1, \dots, s$, be an exceptional curve contracted by η. We have $A\equiv\eta^*A_0-\sum_i r_iE_i$ where $r_i=AE_i$ and $0<r_i<a$ by ampleness. Moreover we find

$$A^2 = A_0^2-\textstyle\sum_i r_i^2, \quad AK_S = A_0K_{S_0}+\sum_i r_i$$

and then

$$A_0^2+A_0K_{S_0} = A^2+AK_S+\textstyle\sum_i r_i(r_i-1).$$

Hence, as $A_0(K_{S_0}+A_0)=(a-1)(2b-e)-2a$, condition $g(A)=2$ yields

(4.1.2) $$(a-1)(2b-ae) = 2(a+1)+\textstyle\sum_i r_i(r_i-1).$$

On the other hand, as $A_0^2=a(2b-ae)=A^2+\sum_i r_i^2$, from (4.1.2) we get

(4.1.3) $$2b-ae = \textstyle\sum_i r_i+A^2-2(a+1).$$

Then formulas (4.1.2), (4.1.3) give

(4.1.4) $$(a-1)(A^2+s)+\textstyle\sum_i(r_i-1)(a-r_i-1) = 2a(a+1).$$

Firstly assume that $(S, A)\in\mathcal{C}$. Then it has to be $A^2+s=12$ and $r_i=1, i=1, \dots s$. In this case (4.1.4) gives either $a=2$ or $a=3$. If $a=2$, by (4.1.3), we get $2(b-e)=s+A^2-6=6$ i.e. $b=e+3$. So $e=0, 1, 2$ in view of the ampleness conditions (4.1.1). Therefore the geometrically ruled pairs (S_0, A_0) corresponding to the pairs $(S, A)\in\mathcal{C}$, when $a=2$, are characterized as follows:

$$e=0, 1, 2 \quad\text{and}\quad A_0\equiv 2C_0+(3+e)f.$$

If $a=3$ it can only be $e=0$ and we have nothing to add to the above list. Indeed in view of the symmetry between C_0 and f when $e=0$, we fall again in one of the cases listed above. Henceforth the pairs $(S, A)\in\mathscr{C}$ and the corresponding geometrically ruled pairs are the following ones:

S	K_S^2	A	A^2	$e=e(S_0)$	A_0
$B_{p_1,\dots,p_s}(S_0)$ $0\leqq s\leqq 11$	$8-s$ $0\leqq s\leqq 11$	$A\equiv\eta^*A_0-\sum_i E_i$	$12-s$	$e=0,1,2$	$A_0\equiv 2C_0+(3+e)f$

Remark 4.1. 1) We can prove that whenever A is (numerically) 2-connected (i.e. $D_1D_2\geqq 2$ for every splitting $A=D_1+D_2$, D_1 and D_2 effective) and $h^0(A)\geqq 4$, then $(S, A)\in\mathscr{C}$. Indeed $|A+K_S|$ is base point free and $\varphi_{|A+K_S|}\colon S\to\mathbf{P}^1$ gives the structure of a conic bundle.

2) Let $\Phi\in|aC_0+bf|$ be a curve of S_0 passing through $t\leqq s$ of the points $p_1,\dots,p_s$. Since $A\eta^{-1}(\Phi)>0$ one easily finds that $t<2b+3a-ae$. In particular $2-e$ points at most can lie on the same fundamental section.

3) Whenever $e=1$ one has $S=B_{p,p_1,\dots,p_s}(\mathbf{P}^2)$ and $A=(\eta\circ\sigma)^*4L-2E-\sum_i E_i$, where L is a line in $\mathbf{P}^2$, σ denotes the blowing-up of $\mathbf{P}^2$ at a point p and $E=(\eta\circ\sigma)^{-1}(p)$. Note that the pairs (S, A) consisting of a Castelnuovo surface and a hyperplane section can be found in this class when $0\leqq s\leqq 7$ (cfr. [12], Thm. 2.1). □

Now consider the number

$$\varrho=\sum_i(r_i-1)(a-r_i-1)$$

which appears at the left hand of (4.1.4).

Remark 4.2. The pair (S, A) belongs to $\mathscr{C}$ if and only if $\varrho=0$. Indeed if $(S, A)\in\mathscr{C}$, then $r_i=1$ for every i. On the other hand, if $\varrho=0$, recalling that $A^2+s\leqq 12$, it follows from (4.1.4) that $12(a-1)\geqq 2a(a+1)$, which implies $a=2, 3$ and then $A^2+s=12$ by using again (4.1.4). This means $(S, A)\in\mathscr{C}$ by Proposition 0.1. □

Finally assume $(S, A)\notin\mathscr{C}$; then $A^2+s\leqq 11$, so that

(4.3.1) $$s\leqq 10,$$

in view of the ampleness of A. As ϱ attains its maximum value when $r_i-1=[(a-2)/2]$ for all $i=1,\dots,s$ (where [ ] is the least integer function), from (4.1.4) one gets the inequality

$$11(a-1)+s[(a-2)^2/4]\geqq 2a(a+1).$$

This implies

$$(8-s)a^2-4(9-s)a+4(11-s)\leqq 0 \quad\text{if } a \text{ is even},$$

$$(8-s)a^2-4(9-s)a+44-3s\leqq 0 \quad\text{if a is odd}.$$

The latter inequality is never satisfied when $s \leqq 7$ while the former one fails to be satisfied for $s \leqq 6$ and if $s=7$ gives $a=4$. Unfortunately both inequalities say nothing when $8 \leqq s \leqq 10$. This makes evident that in the rational case, still under the assumption that η factors via s simultaneous blowing-ups, we are dealing with infinitely many pairs (S_0, A_0). The integer a can assume all the positive values $\geqq 4$. Note that if $a=3$ one has $\varrho=0$, hence by Remark 4.2 we fall again into the case $(S, A) \in \mathscr{C}$. Let us give an idea of what happens when $a=4$. By (4.1.4), recalling that $A^2+s \leqq 11$, we get $\varrho=40-3(A^2+s) \geqq 7$. Moreover since

$$(r_i-1)(3-r_i) = \begin{cases} 0 & \text{if} \quad r_i = 1 \quad \text{or} \quad 3 \\ 1 & \text{if} \quad r_i = 2, \end{cases}$$

one concludes that ϱ is the number of the r_i's equal to 2. So $\varrho \leqq s$ and we can assume $r_1 = \ldots = r_7 = 2$. On the other hand $A^2+s=(40-\varrho)/3$ by (4.1.4), so that it can only be $\varrho=7$ and then $A^2+s=11$. Indeed, recalling (4.3.1), $\varrho \leqq 10$ and if it were $\varrho=10$ then we should have $\varrho=s=10$, which leads to the contradiction $A^2+10=10$. To conclude, if $a=4$, one has

$$A^2 = 11-s, \quad 7 \leqq s \leqq 10, \quad A \equiv \eta^* A_0 - 2\sum_{i=1}^{7} E_i - \sum_{j>7} r_j E_j$$

and in view of (4.1.3) the corresponding geometrically ruled pairs (S_0, A_0) are classified in the following table according to the values of the last $s-7$ integers r_j's.

s	r_8	r_9	r_{10}	$e=e(S_0)$	A_0 (up to linear equivalence)
7				$e \leqq 1$	$4C_0+(4+2e)f$
8	1			$e \leqq 1$	$4C_0+(4+2e)f$
	3			$e \leqq 2$	$4C_0+(5+2e)f$
9	1	1		$e \leqq 1$	$4C_0+(4+2e)f$
	1	3		$e \leqq 2$	$4C_0+(5+2e)f$
	3	3			$4C_0+(6+2e)f$
10	1	1	1	$e \leqq 1$	$4C_0+(4+2e)f$
	1	1	3	$e \leqq 2$	$4C_0+(5+2e)f$
	1	3	3		$4C_0+(6+2e)f$
	3	3	3	$e \leqq 3$	$4C_0+(7+2e)f$

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