Estimates for maximal functions along hypersurfaces

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1. Introduction

Let $x_{n+1}=F(x_1, \ldots, x_n)$ be the equation of a surface in $\mathbf{R}^{n+1}$. We shall study the mean values

$$m_h f(x) = \frac{1}{\prod_1^n h_i} \int_{0<y_i<h_i} f(x'-y, x_{n+1}-F(y))\, dy.$$

Here $h_i>0$, $i=1, \ldots, n$, $x=(x', x_{n+1})\in\mathbf{R}^{n+1}$ and $y\in\mathbf{R}^n$. Assuming $F(0)=0$, we ask whether $m_h f\to f$ a.e. as $h_i\to 0$ when $f\in L^p$, $p>1$. This was proved for $F(x')=\prod_1^n x_i^{\alpha_i}$, $\alpha_i>0$, in Carlsson, Sjögren, and Strömberg [1]. Convergence of this type follows from the L^p boundedness of the corresponding maximal function operator

$$M_F f = \sup_{0<h_i<\delta} m_h|f|,$$

where $\delta>0$. Stein and Wainger asked in [2, Problem 8, p. 1289] for which F the operator M_F is bounded on L^p, as a natural extension of the known results for curves. We shall give some answers to this question.

Theorem 1. *Let $F\in C^{2+\varepsilon}$ in a neighborhood of $0\in\mathbf{R}^n$, for some $\varepsilon>0$. If $\partial^2F(0)/\partial x_i^2\neq 0$, $i=1, \ldots, n$, then there exists a δ making M_F bounded on $L^p(\mathbf{R}^{n+1})$, $p>1$.*

Under stronger assumptions on the Hessian of F at 0, the regularity hypothesis can be weakened.

Theorem 2. *Let $F\in C^2$ in a neighborhood of $0\in\mathbf{R}^n$, $n\geqq 2$. Assume that the matrix $(\partial^2F(0)/\partial x_i\,\partial x_j)_{i,j\in A}$ is nonsingular for any nonempty proper subset A of $\{1, \ldots, n\}$. Then M_F is bounded on $L^p(\mathbf{R}^{n+1})$, $p>1$, for some $\delta>0$.*

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Notice that the condition in Theorem 2 is satisfied if the Hessian of $F\in C^2$ is positive or negative definite. Also if $n=2$, the assumption about the Hessian in these two theorems are the same. In general, the assumption of Theorem 1, $\partial^2F(0)/\partial x_i^2\neq 0$, cannot be weakened. In fact with prescribed values of $\partial^2F(0)/\partial x_i\partial x_j$ such that some $\partial^2F(0)/\partial x_i^2=0$, we can find a smooth F for which M_F is unbounded on L^p. For this surface $m_h f$ will not converge a.e. even for $f\in L^\infty$.

On the other hand, if F is a second-degree polynomial, no hypothesis on the Hessian is needed.

Theorem 3. *Let F be a polynomial of degree at most 2. Then M_F is bounded on $L^p(\mathbf{R}^{n+1})$, $p>1$, even with $\delta=+\infty$.*

For $n=2$, this was proved in [1].

By and large, our proof of Theorem 1 follows that of Theorem 1 in [1]. In Section 2, the proof is reduced to three lemmas which are proved in Sections 3 and 4. The main part of our proof is contained in the third of these lemmas, whose analogue in [1] is trivial. Section 5 briefly describes the modifications needed for Theorem 2. The proof of Theorem 3 is also in Section 5, as well as the counterexample mentioned above.

In this paper, C denotes various positive constants, and $\alpha\sim\beta$ means $C^{-1}\leqq \alpha/\beta\leqq C$.

2. Structure of the proof of Theorem 1

We use induction in the dimension. The case $n=1$ is well-known [2]. This case also follows directly from our proof. From now on, we assume the theorem to be true for $n-1$, although this assumption will be used only in the proof of Lemma 3 below.

We need only treat the case $F(0)=0$. Considering the transformation $(x', x_{n+1})\to(x', x_{n+1}-x'\cdot\operatorname{grad} F(0))$, we see that it can also be assumed that $\operatorname{grad} F(0)=0$. We next show that we may assume $\partial^2F(0)/\partial x_i\partial x_j\neq 0$ for all i and j, by making a change of variables which depends on the relative sizes of the h_i. Let $\max h_i=h_q$. For any fixed η, the transformation

$$(2.1)\qquad \begin{aligned} x_q &= x_q'+\eta \textstyle\sum_{i\neq q} x_i \\ x_i &= x_i', \quad i=1,\dots,n, \quad i\neq q, \end{aligned}$$

is admissible, see the proof of Theorem 2 in [1]. Since $\partial^2F(0)/\partial x_q^2\neq 0$, it can be seen that small nonzero values of η will give $\partial^2F(0)/\partial x_i'\partial x_j'\neq 0$, as required. *Choosing* δ suitably, we shall always work in a small neighborhood of the origin where

$$(2.2)\qquad \frac{\partial^2 F}{\partial x_i\partial x_j}\sim\frac{\partial^2F(0)}{\partial x_i\partial x_j}\neq 0, \quad 1\leqq i,\ j\leqq n.$$

The mean value $m_h f$ can be replaced by that over the rectangle $\{\frac{1}{2}h_i<y_i<h_i, i=1, ..., n\}$, and we can take $h_i=2^{-j_i}$ for large integers j_i. In the sequel, we shall write $j=(j_1, ..., j_n)\in \mathbf{N}^n$ and $k=\min j_i$, and k will always be large. Let $0\leqq\psi\in C_0^\infty(\mathbf{R})$ be 1 in $[\frac{1}{2}, 1]$ and have support in $]0, \infty[$. (In this proof, we could actually use the rectangles $\{0<y_i<h_i\}$ and hence take $\psi\in C_0^\infty$ with $\psi=1$ in $[0, 1]$, but this is not convenient in the proof of Theorem 2.) Define a measure μ_j by

$$\int \varphi\, d\mu_j = \int \varphi(y, F(y)) \Pi_1^n \psi_{j_i}(y_i)\, dy, \tag{2.3}$$

where $g_m(t)=2^m g(2^m t)$ for any function g in $\mathbf{R}$. It is enough to estimate the maximal function operator

$$M_\mu f = \sup |\mu_j * f|,$$

the supremum taken over those j with all j_i sufficiently large.

As in [1], we shall compare the μ_j to measures ν_j whose maximal function is easier to control. Take $0\leqq\varphi\in C_0^\infty(\mathbf{R})$ with $\int \varphi\, dt=1$. Define

$$\nu_j = \mu_j - \mu_j * (\otimes_{i=1}^n (\delta_0 - \varphi_{j_i}) \otimes \delta_0), \tag{2.4}$$

δ_0 being the Dirac measure at 0 in $\mathbf{R}$.

We use anisotropic dilations of the Bessel kernel G^z to improve and worsen our operators. With $z\in\mathbf{C}$ and

$$\hat{G}^z(\xi) = (1+|\xi|^2)^{-(1/2)z}, \quad \xi\in\mathbf{R}^{n+1},$$

we let

$$G_j^z(x) = 2^{\Sigma j_i + 2k} G^z(2^{j_1}x_1, ..., 2^{j_n}x_n, 2^{2k}x_{n+1}).$$

The reason for the factor 2^{2k} in the last variable is that 2^{-2k} is in general the order of magnitude of $|F|$ in supp μ_j. Notice that the μ_j and ν_j are no longer dilations of fixed measures as in [1]. Now set $\mu_j^z=G_j^z*\mu_j$ and similarly for ν_j^z. We shall study the maximal function operator

$$M_{\mu-\nu}^z f = \sup_j |(\mu_j^z - \nu_j^z) * f|,$$

where f is assumed to be in the Schwartz class S, and its analogues M_μ^z and M_ν^z.

The following two lemmas give L^p estimates for $M_{\mu-\nu}^z$. They are similar to the corresponding lemmas in [1], and their proofs are given in the next section.

Lemma 1. *There exists a* $\sigma>0$ *such that for* $-\sigma<\operatorname{Re} z<0$

$$\|M_{\mu-\nu}^z f\|_2 \leqq C\|f\|_2, \quad f\in S.$$

Lemma 2. *For* $0<\operatorname{Re} z<1$ *and each* $p>1$

$$\|M_{\mu-\nu}^z f\|_p \leqq C(z)\|f\|_p, \quad f\in S,$$

where the constant $C(z)$ *increases at most polynomially in* $\operatorname{Im} z$ *for fixed* $\operatorname{Re} z$.

Interpolating as in, e.g., [1], we conclude that the operator $M^0_{\mu-\nu}$ is bounded on L^p for $p>1$. Defining M_ν like M_μ, we see that Theorem 1 follows from the next lemma.

Lemma 3. *The operator M_ν is bounded on L^p for $p>1$.*

In [1], the measures ν_j were found to be dilations of a C_0^∞ function, and so M_ν was easy to control. In our case however, the density of ν_j may be unbounded near the surface when some derivative of F vanishes at points in supp μ_j. This is the main difficulty in the proof of Lemma 3, given in Section 4.

3. Estimates for $M^z_{\mu-\nu}$

Proof of Lemma 1. As in the proof of Lemma 1 in [1], it is enough to show that

$$\sum_j |\hat{\mu}_j^z - \hat{\nu}_j^z|^2 \leqq C. \tag{3.1}$$

Clearly,

$$|\hat{\mu}_j(\xi) - \hat{\nu}_j(\xi)| \leqq C2^{-j_i}|\xi_i|, \quad i = 1, \ldots, n. \tag{3.2}$$

We shall use van der Corput's lemma, see [2, Lemma 2.3], to estimate $\hat{\mu}_j(\xi)$ for large ξ. One has

$$\hat{\mu}_j(\xi) = \int e^{-2\pi i(\Sigma y_i \xi_i + F(y)\xi_{n+1})} \prod_1^n \psi_{j_i}(y_i)\, dy. \tag{3.3}$$

Take $q \in \{1, \ldots, n\}$ such that $i_q = k$. In the region we are interested in, $\partial^2 F/\partial y_q^2$ is bounded away from zero. By van der Corput's lemma, the integral in y_q of the exponential in (3.3) over any interval near the origin is at most $C|\xi_{n+1}|^{-1/2}$. Integrating by parts in y_q, we conclude that

$$|\hat{\mu}_j(\xi)| \leqq C(2^{-2k}|\xi_{n+1}|)^{-1/2}.$$

The first derivative with respect to y_i of the parenthesis in (3.3) is $\xi_i + F_i'(y)\xi_{n+1}$. Notice that $|F_i'(y)| \leqq C2^{-k}$ here. Hence, if

$$|\xi_i| > C2^{-k}|\xi_{n+1}|, \tag{3.4}$$

van der Corput's lemma gives

$$|\hat{\mu}_j(\xi)| \leqq C(2^{-j_i}|\xi_i|)^{-1}.$$

Now $\hat{\mu}_j$ is bounded and these estimates imply

$$|\hat{\mu}_j(\xi)| \leqq C\big(1 + \sum_1^n 2^{-j_i}|\xi_i| + 2^{-2k}|\xi_{n+1}|\big)^{-1/2}$$

for all ξ, since the last term dominates $2^{-j_i}|\xi_i|$ when (3.4) is false. The same estimate then follows for $\hat{\nu}_j$.

Combining this with (3.2), we get

$$|\hat{\mu}_j(\xi)-\hat{\nu}_j(\xi)| \leqq C\min\left((1+\textstyle\sum_1^n 2^{-j_i}|\xi_i|+2^{-2k}|\xi_{n+1}|)^{-1/2}, 2^{-j_1}|\xi_1|, \dots, 2^{-j_n}|\xi_n|\right).$$

Arguing now as in [1], last part of the proof of Lemma 1, we obtain (3.1) and thus Lemma 1.

Proof of Lemma 2. We have

$$|G^z(x)| \leqq C(z)\textstyle\sum_{m\in\mathbf{Z}} 2^{-|m|\operatorname{Re} z+m(n+1)}\chi_{|x|<2}2^{-m},$$

see [1], proof of Lemma 2. Let

$$\lambda_j^m(x) = \begin{cases} 2^{\Sigma j_i+2k+m(n+1)} & \text{if } |x_i| \leqq 2^{-j_i-m},\ i=1,\dots,n, \text{ and } |x_{n+1}| \leqq 2^{-2k-m} \\ 0 & \text{otherwise.}\end{cases}$$

We estimate M_μ^z and M_ν^z separately. For $f\geqq 0$,

$$|\mu_j^z * f| \leqq C(z)\textstyle\sum_{m\in\mathbf{Z}} 2^{-|m|\operatorname{Re} z}\mu_j*\lambda_j^m*f \tag{3.5}$$

Let

$$\mathcal{M}^m f = \sup_j \mu_j*\lambda_j^m*f.$$

When $m\leqq 0$, the support of $\mu_j*\lambda_j^m$ is contained in the box $\{|x_i|\leqq C2^{-j_i-m}, i=1, \dots, n, |x_{n+1}|\leqq C2^{-2k-m}\}$, and the density of $\mu_j*\lambda_j^m$ is seen to be bounded by a constant divided by the volume of this box. Hence, $\mathcal{M}^m f$ is bounded by a constant times the strong maximal function $M_s f$, and thus $\mathcal{M}^m$ is bounded on L^p uniformly for $m\leqq 0$.

Now let $m>0$. We use (2.3) with the change of variables $y_i=2^{-j_i}s_i$, getting

$$\mu_j*\lambda_j^m*f(x) \leqq \int \Pi_1^n \psi(s_i)\,ds\, 2^{\Sigma j_i+2k+m(n+1)} \int_{\substack{|v_i|\leqq 2^{-j_i-m},\ i=1,\dots,n \\ |v_{n+1}|\leqq 2^{-2k-m}}} f(x'-2^{-j}s-v', x_{n+1}-F(2^{-j}s)-v_{n+1})\,dv. \tag{3.6}$$

Here $s=(s_1, \dots, s_n)$ and $2^{-j}s=(2^{-j_1}s_1, \dots, 2^{-j_n}s_n)$, and we write $v=(v_1, \dots, v_{n+1})=(v', v_{n+1})\in\mathbf{R}^{n+1}$. When taking the supremum in the j_i, we shall start by fixing the non-negative integers $l_i=j_i-k$, and vary k. Let $\Lambda=\{i\colon l_i<m\}$. Denote by π_Λ the projection $\mathbf{R}^n\to\mathbf{R}^n$ obtained by replacing the i: th coordinate by 0 for $i\notin\Lambda$. With $F_\Lambda=F\circ\pi_\Lambda$, we have

$$F(2^{-j}s) = F_\Lambda(2^{-j}s)+O(2^{-2k-m})$$

for $s_i\in\operatorname{supp}\psi$. Therefore, we can replace F by F_Λ in (3.6), provided we integrate in v_{n+2} over a longer interval $|v_{n+1}|\leqq C2^{-2k-m}$.

We now recall a one-dimensional lemma from [1]. If $\omega=(\omega_k)$ is a decreasing sequence of positive numbers and $\tau=(\tau_k)$ a sequence of real numbers, let

$$M^{\omega,\tau} g(t) = \sup_k \frac{1}{2\omega_k} \int_{-\omega_k}^{\omega_k} |g(t-\tau_k-s)|\, ds. \tag{3.7}$$

Lemma 4 in [1] says that if for each k the inequalities $|\tau_l|>\omega_k$, $l\geqq k$ hold for at most $m\geqq 1$ values of l, then $M^{\omega,\tau}$ is bounded on $L^p(\mathbf{R}), p>1$, with norm at most a constant times $m^{1/p}$. In particular, this is satisfied when $|\tau_{k+m}|\leqq\omega_k$ for all k.

We shall estimate the modified integral in (3.6) and start by integrating in v_{n+1}:

$$2^{2k+m} \int_{|v_{n+1}|\leqq C2^{-2k-m}} f(x'-2^{-j}s-v', x_{n+1}-F_A(2^{-j}s)-v_{n+1})\, dv_{n+1}$$
$$\leqq CM^{n+1} f(x'-2^j s-v', x_{n+1}).$$

Here M^{n+1} is $M^{\omega,\tau}$ applied to the $n+1$:st variable, with $\omega_k=C2^{-2k-m}$ and $\tau_k=F_A(2^{-k-l_1}s_1, \dots, 2^{-k-l_n}s_n)$. Now fix $p>1$. One finds $|\tau_{k+m+C}|\leqq C\,2^{-2k-2m-2C}\leqq\omega_k$ for some C. Thus M^{n+1} is bounded on $L^p(\mathbf{R}^{n+1})$ with norm at most $Cm^{1/p}$.

Integrating then in v_i, $i=1, \dots, n$, we can apply similar maximal operators M^i, defined as $M^{\omega,\tau}$ acting in the i:th variable, with $\omega_k=2^{-k-l_i-m}$ and $\tau_k=2^{-k-l_i}s_i$. The norm of M^i on $L^p(\mathbf{R}^{n+1})$ is bounded by $Cm^{1/p}$.

Summing up, we obtain

$$\mu_j*\lambda_j^m*f(x) \leqq C\int M^1 \dots M^{n+1} f(x) \textstyle\prod_1^n \psi(s_i)\, ds_i.$$

The right-hand side here defines an operator with norm at most $Cm^{(n+1)/p}$ on $L^p(\mathbf{R}^{n+1})$.

Having thus varied k, we shall also let the l_i vary, first in such a way that A is fixed. Observe that $M^1, \dots, M^n$ are independent of the l_i. Moreover, M^{n+1} depends only on those l_i for which $i\in A$. Such an l_i can take only m different values, and the number of possible A is finite. Replacing the supremum in these remaining variables by a sum, we see that the operator $\mathcal{M}^m$ for $m>0$ is bounded on $L^p(\mathbf{R}^{n+1})$, with norm at most $Cm^n m^{(n+1)/p}=Cm^C$.

From (3.5) and our estimates for $\mathcal{M}^m$, it follows that

$$\|\sup_j |\mu_j^z*f|\|_p \leqq C(z) \textstyle\sum_{m\in\mathbf{Z}} 2^{-|m|\,\mathrm{Re}\, z} \|\mathcal{M}^m f\|_p \leqq C(z)\, \|f\|_p.$$

To conclude the proof, we need a similar estimate for ν_j^z. Because of (2.4), ν_j^z is a sum of convolutions in certain variables of μ_j^z with normalized dilations φ_{j_i} of $\varphi\in C_0^\infty(\mathbf{R})$. These convolutions can be estimated by means of one-dimensional maximal operators. Hence $|\nu_j^z*f|\leqq CM(\mu_j^z*f)$, where M is a sum of products of maximal operators in the coordinate directions. Since M is bounded on L^p, so is $M_{\nu_j}^z$ and Lemma 2 is proved.

4. Proof of Lemma 3.

Expanding the tensor product in (2.4), we see that the measure ν_j is a sum of convolutions in one or more variables of μ_j with one-dimensional functions φ_{j_i}, $i=1, \ldots, n$. Let ν_j^r be the convolution in the r:th variable of μ_j with φ_{j_r}. The terms in the above sum are either of type ν_j^r or convolutions in certain variables of some ν_j^r with functions φ_{j_i}. These last convolutions can be estimated by means of one-dimensional operators acting on ν_j^r, cf. the last lines of Section 3. Therefore, it is enough to estimate the maximal function associated with the measures $(\nu_j^r)_j$ for each r. To simplify notations, we take $r=1$.

We have

$$\nu_j^1 * f(x) = 2^{j_1+\Sigma j_i} \times$$

$$\times \iint f(x_1-y_1-u, x_2-y_2, \ldots, x_n-y_n, x_{n+1}-F(y))\, \varphi(2^{j_1}u) \prod_1^n \psi(2^{j_i}y_i)\, du\, dy.$$

In this integral we want to make the change of variables $(u, y_1)\to(s, t)$ given by $s=u+y_1$, $t=F(y_1, y_2, \ldots, y_n)$. It is therefore necessary to study the zero set of the Jacobian $\partial(s, t)/\partial(u, y_1)=F_1'(y)$. Because of (2.2), F_{11}'' is of constant sign near 0, say $F_{11}''>0$. Hence, the implicit function theorem shows that the function $y_1\to F_1'(y_1, \ldots, y_n)$ has a unique zero $y_1=\xi=\xi(y_2, \ldots, y_n)$ for $(y_1, \ldots, y_n)$ in a neighborhood of 0. Further, $\xi\in C^1$ and

$$\xi_i' = -\frac{F_{1i}''}{F_{11}''} \sim -\frac{F_{1i}''(0)}{F_{11}''(0)} \sim \pm 1, \quad i = 2, \ldots, n. \tag{4.1}$$

Later we shall need the function $T=T(y_2, \ldots, y_n)=F(\xi, y_2, \ldots, y_n)$. Notice that

$$|T_i'|=|F_i'(\xi, y_2, \ldots, y_n)| \leqq C \max |y_i|, \quad i = 2, \ldots, n. \tag{4.2}$$

The indicated change of variables can be carried out in each of the domains $\{y_1<\xi\}$ and $\{y_1>\xi\}$. It follows that we can estimate $\nu_j^1 * f(x)$ by at most two integrals of type

$$\begin{aligned} &2^{j_1+\Sigma j_i} \int f(x_1-s, x_2-y_2, \ldots, x_n-y_n, x_{n+1}-t) \\ &\cdot \varphi(2^{j_1}u) \prod_1^n \psi(2^{j_i}y_i) \frac{1}{|F_1'(y)|}\, ds\, dt\, dy_2 \ldots dy_n. \end{aligned} \tag{4.3}$$

Here $|s|\leqq C\, 2^{-j_1}$, because the same is true for u and y_1. Since y_1 is independent of s and φ is bounded, we can estimate the integral in s in terms of the standard maximal function operator M_1 taken in the first variable. Thus the expression (4.3) is at most a constant times

$$\begin{aligned} &2^{\Sigma j_i} \int M_1 f(x_1, x_2-y_2, \ldots, x_n-y_n, x_{n+1}-t) \\ &\cdot \prod_1^n \psi(2^{j_i}y_i) \frac{1}{|F_1'(y)|}\, dt\, dy_2 \ldots dy_n = I(x). \end{aligned}$$

We consider first the case when $F_1'(y)$ stays away from 0. Let $I'(x)$ be that part of $I(x)$ obtained by restricting the integration in $(y_2, \dots, y_n)$ to those points for which $\xi \notin [-C\,2^{-j_1}, C\,2^{-j_1}]$. Here C is chosen so large that $\operatorname{supp} \psi \subset [-C/2, C/2]$. Because of (2.2), F_1' is not far from linear and, therefore, essentially constant as we integrate dt in $I'(x)$. By the mean value theorem, the variable t in $I'(x)$ stays within the interval

$$|t - F(0, y_2, \dots, y_n)| \leqq C 2^{-j_1} |F_1'|.$$

Now we can estimate the integral in t by means of a one-dimensional maximal function:

$$I'(x) \leqq C 2^{\Sigma_2^n j_i} \times$$

$$\times \int_{0 \leqq y_i \leqq C2^{-j_i}} M_{n+1} M_1 f(x_1, x_2 - y_2, \dots, x_n - y_n, x_{n+1} - F(0, y_2, \dots, y_n))\, dy_2 \dots dy_n.$$

The supremum in $j_2, \dots, j_n$ of this expression is dominated by a lower dimensional maximal function of the type of Theorem 1. This is controlled by our induction assumption, and thus

$$\|\sup_j I'(x)\|_p \leqq C \|f\|_p, \quad p > 1. \tag{4.4}$$

Consider next $I''(x) = I(x) - I'(x)$. The function $y_1 \to F(y_1, \dots, y_n)$ now has a minimum T at $\xi \in [-C2^{-j_1}, C2^{-j_1}]$. Hence, $t - T \sim (y_1 - \xi)^2$ so that $0 \leqq t - T \leqq C2^{-2j_1}$ in $I''(x)$. Moreover $|F_1'(y)| \sim \sqrt{t-T}$, and thus

$$I''(x) \leqq C 2^{\Sigma j_i} \int_{\substack{0 \leqq t - T \leqq C2^{-2j_1} \\ |\xi| \leqq C2^{-j_1}}} M_1 f(x_1, x_2 - y_2, \dots, x_n - y_n, x_{n+1} - t)$$

$$\cdot \Pi_2^n \psi(2^{j_i} y_i) \frac{1}{\sqrt{t-T}}\, dt\, dy_2 \dots dy_n$$

$$\leqq \sum_{m=1}^{\infty} C 2^{j_1 + m/2 + \Sigma j_i} \int_{\substack{C2^{-2j_1 - m} \leqq t - T \leqq C2^{-2j_1 - m + 1} \\ |\xi| \leqq C2^{-j_1}}} M_1 f(x_1, x_2 - y_2, \dots, x_n - y_n, x_{n+1} - t)$$

$$\cdot \Pi_n^2 \psi(2^{j_i} y_i)\, dt\, dy_2 \dots dy_n = \sum_{m=1}^{\infty} J_m(x).$$

Fixing m, we estimate J_m. Write $l_i = j_i - k$ as before. Consider those i for which $l_i > m + 2l_1$. From now on, we assume that this happens precisely when $2 \leqq i \leqq d$ for some d with $1 \leqq d \leqq n$. This is no restriction. In particular,

$$0 \leqq l_i \leqq m + 2l_1, \quad i = d+1, \dots, n. \tag{4.5}$$

Define

$$\xi^* = \xi^*(y_2, \dots, y_n) = \xi(0, \dots, 0, y_{d+1}, \dots, y_n)$$

and

$$T^* = T^*(y_2, \dots, y_n) = T(0, \dots, 0, y_{d+1}, \dots, y_n).$$

Then (4.1–2) imply

$$|\xi^*-\xi| \leqq C2^{-k-m-2l_1} \leqq C2^{-j_1}$$

and

$$|T^*-T| \leqq C2^{-2k-m-2l_1} = C2^{-2j_1-m},$$

when $2^{j_i}y_i \in \operatorname{supp}\psi$, $i=2, \dots, n$. Extending the domain of integration in the definition of J_m, we get for some C

$$J_m(x) \leqq C2^{j_1+m/2+\Sigma j_i} \int_{\substack{|t-T^*| \leqq C2^{-2j_1-m} \\ |\xi^*| \leqq C2^{-j_1}}} M_1 f(x_1, x_2-y_2, \dots, x_n-y_n, x_{n+1}-t)$$

$$\cdot \Pi_2^n \psi(2^{j_i}y_i)\, dt\, dy_2 \dots dy_n.$$

Now $y_2, \dots, y_d$ appear only in the argument of $M_1 f$, and one can apply the standard maximal function operators in these variables. Hence,

(4.6)
$$J_m(x) \leqq C2^{2j_1+m/2+\Sigma_{d+1}^n j_i}$$

$$\cdot \int_{\substack{|t-T^*| \leqq C2^{-2j_1-m} \\ |\xi^*| \leqq C2^{-j_1}}} M_d \dots M_2 M_1 f(x_1, \dots, x_d, x_{d+1}-y_{d+1}, \dots, x_n-y_n, x_{n+1}-t)$$

$$\cdot \Pi_{d+1}^n \psi(2^{j_i}y_i)\, dt\, dy_{d+1} \dots dy_n.$$

We shall estimate $\sup_j J_m(x)$ and its L^p norm, for a fixed $p>1$. Notice that the right-hand side of (4.6) is independent of $j_2, \dots, j_d$, so that the supremum need only be taken in $j_1, j_{d+1}, \dots, j_n$.

If $d=n$, we have $T^*=0$ and

$$J_m(x) \leqq C2^{-m/2} M_{n+1} M_n \dots M_1 f(x).$$

Hence,

(4.7)
$$\|\sup_{j:d=n} J_m\|_p \leqq C2^{-m/2}\|f\|_p.$$

The remaining case $d<n$ is divided into two parts. In the first part, we can replace the supremum by a sum. In the second part, T^* is almost linear in $y_2, \dots, y_d$, which will allow us to apply the operator $M^{\omega,\tau}$ defined in (3.7).

Part 1: $d<n$ and $j_1>(1+\varepsilon)k$, or equivalently $l_1>\varepsilon k$. The right-hand side of (4.6) is the convolution of $M_d \dots M_1 f$ with a positive measure σ_j.

We shall estimate $\|\sigma_j\|$ and consider first the size of supp σ_j. Because of (4.1), $|\partial\xi^*/\partial y_q| \sim 1$, where as before q is chosen so that $j_q=k$. Notice that now $d<q\leqq n$. For fixed y_i, $i\neq q$, the inequality $|\xi^*| \leqq C2^{-j_1}$ can thus hold only for y_q in an interval of length $C2^{-j_1}$. It follows that

$$\|\sigma_j\| \leqq C2^{-m/2+j_q-j_1} = C2^{-m/2-l_1}.$$

Clearly,

$$\|\sup \sigma_j * M_d \dots M_1 f\|_p \leqq \left(\sum \|\sigma_j\|\right) \|M_d \dots M_1 f\|_p,$$

where the supremum and the sum are taken over those $j_1, j_{d+1}, \ldots, j_n$ satisfying (4.5) and $l_1 > \varepsilon k$. If we sum $\|\sigma_j\|$ in $l_{d+1}, \ldots, l_n$ with l_1 and k fixed, we get at most $C(m+l_1)^{n-d} 2^{-m/2-l_1}$. Taking then the sum in l_1 and k, we see that

$$\sum \|\sigma_j\| \leqq Cm^C 2^{-m/2}$$

so that

$$\|\sup_{\text{Part 1}} J_m\|_p \leqq Cm^C 2^{-m/2} \|f\|_p. \tag{4.8}$$

Part 2: $d<n$ and $j_1 \leqq (1+\varepsilon)k$. We fix $l_1, l_{d+1}, \ldots, l_n$ and vary k. The main difficulty in estimating the right-hand side of (4.6) is now that ξ^* and T^* depend on $y_{d+1}, \ldots, y_n$. We shall therefore divide the range of these variables into small cubes in which ξ^* and T^* are essentially constant.

Using (4.1) and the fact that $F \in C^{2+\varepsilon}$, we get

$$\xi^*(y_2, \ldots, y_n) = \sum_{d+1}^n b_i y_i + O(2^{-k(1+\varepsilon)}), \tag{4.9}$$

if $2^{-j_i} y_i \in \operatorname{supp} \psi$. Here $b_i = -F''_{1i}(0)/F''_{11}(0) \neq 0$. The remainder in (4.9) is at most $C2^{-j_1}$ by the assumptions of Part 2, so that $|\xi^*| \leqq C2^{-j_1}$ implies

$$\left|\sum_{d+1}^n b_i y_i\right| \leqq C2^{-j_1}. \tag{4.10}$$

Consider the lattice of cubes in $\mathbf{R}^{n-d}$ having side $2^{-k-m-2l_1}$ and centers at those points whose coordinates are integer multiples of $2^{-k-m-2l_1}$. In (4.6) we make the integral larger by deleting the factor $\prod \psi(2^{-j_i})$ and extending the integration in $y_{d+1}, \ldots, y_n$ to the union of those lattice cubes which intersect the set

$$\left\{(y_{d+1}, \ldots, y_n) \colon |y_i| \leqq C2^{-j_i} \ \text{ and } \ \left|\sum_{d+1}^n b_i y_i\right| \leqq C2^{-j_1}\right\}.$$

Let these cubes be Q_k^r, $r=1, \ldots, N$. Their centers can be written as $2^{-k}\eta^r = (2^{-k}\eta^r_{d+1}, \ldots$ $\ldots, 2^{-k}\eta^r_n)$, and η^r and N do not depend on k. Since $q>d$ and $j_1 - j_q = l_1$, a comparison of volumes shows that

$$N \leqq C2^{-\sum_{d+1}^n j_i - l_1} 2^{(n-d)(k+m+2l_1)}. \tag{4.11}$$

From (4.2) we see that if $(y_{d+1}, \ldots, y_n) \in Q_k^r$, then $T^*(y)$ differs from $T_k^r = T^*(2^{-k}\eta^r)$ by at most $C2^{-2j_1-m}$. Now (4.6) implies

$$J_m(x) \leqq C2^{-m/2+\sum_{d+1}^n j_i - (n-d)(k+m+2l_1)} \sum_{r=1}^N 2^{2j_1+m} \int_{|t-T_k^r| \leqq C2^{-2j_1-m}} dt \cdot$$

$$\cdot |Q_k^r|^{-1} \int_{Q_k^r} M_d \ldots M_1 f(x_1, \ldots, x_d, x_{d+1}-y_{d+1}, \ldots, x_n-y_n, x_{n+1}-t)\, dy_{d+1} \ldots dy_n.$$

To estimate these integrals, we shall use operators of type (3.7). For $i=d+1, \ldots$ $\ldots, n$ we let M_i^r be $M^{\omega,\tau}$ acting in the i:th variable, with $\omega_k = 2^{-k-m-2l_1-1}$ and $\tau_k = 2^{-k}\eta_i^r$. Since $|2^{-k}\eta_i^r| \leqq C2^{-k}$, the norm of M_i^r in $L^p(\mathbf{R}^{n+1})$ is bounded by

$C(m+l_1)^{1/p}$. Let similarly M^r_{n+1} be $M^{\omega,\tau}$ acting in the $n+1$:*st* variable, with $\omega_k = C2^{-2k-2l_1-m}$ and $\tau_k = T^r_k$. The quantity T^r_k is the value of F at some point with coordinates at most $C2^{-k}$, so that $|T^r_k| \leqq C2^{-2k}$. Hence, the norm of M^r_{n+1} is less than $C(m+l_1)^{1/p}$.

We conclude that

$$J_m(x) \leqq C2^{-m/2+\sum_{d+1}^n l_i-(n-d)(k+m+2l_1)} \sum_{r=1}^N M^r_{n+1} \dots M^r_{d+1} M_d \dots M_1 f(x).$$

For the norms, we have in view of (4.11)

$$\|\sup_k J_m\|_p \leqq C2^{-m/2}2^{-l_1}(m+l_1)^C \|f\|_p.$$

The supremum in $l_1, l_{d+1}, \dots, l_n$ is now estimated by the corresponding sum. Because of (4.5),

$$\|\sup_{\text{Part 2}} J_m\|_p \leqq Cm^C 2^{-m/2} \|f\|_p. \tag{4.12}$$

From (4.7, 8, 12) we conclude

$$\|\sup_j J_m\|_p \leqq Cm^C 2^{-m/2} \|f\|_p.$$

Summing in m, we get

$$\|\sup_j I''(x)\|_p \leqq C\|f\|_p.$$

Together with (4.4), this estimate ends the proof of Lemma 3.

5. C^2 surfaces, quadratic surfaces, and a counterexample

Proof of Theorem 2. We start with some linear algebra. Let $\emptyset \neq \Lambda \subset \{1, \dots, n\}$, and take $q \in \Lambda$.

Lemma 4. *Let F satisfy the assumptions of Theorem 2, and take $\varepsilon > 0$. Then there exists a linear change of variables of type*

$$\begin{aligned} x_i' &= x_i + \textstyle\sum_{j \notin \Lambda} a_{ij} x_j, \quad && i \in \Lambda, \\ x_i' &= x_i, && i \notin \Lambda, \end{aligned} \tag{5.1}$$

such that $|\partial^2 F(0)/\partial x_i' \partial x_j' - \delta_{iq}| < \varepsilon$ for $i \in \Lambda$, $j \notin \Lambda$ and such that the assumptions of Theorem 2 remain valid if F is considered as a function of $(x_1', \dots, x_n')$.

To prove this lemma, one can assume $\Lambda = \{1, \dots, q\}$. Using block matrix computations and the fact that the matrix $(\partial^2 F(0)/\partial x_i \partial x_j)_{i,j=1}^q$ is nonsingular, one finds that there exists exactly one transformation of type (5.1) giving $\partial^2 F(0)/\partial x_i' \partial x_j' = \delta_{iq}$, $i \in \Lambda$, $j \notin \Lambda$. A slight perturbation produces the desired transformation.

In the proof of Theorem 2, one can assume $F(0)=0$, $\operatorname{grad} F(0)=0$, as in Theorem 1. Let j_i, $i=1, \dots, n$, be as before, with $k=\min j_i=j_q$. Further, N will be a large natural number determined later. We first change coordinates according to Lemma 4 with $\Lambda=\{i: j_i \leqq k+N\}$, and then make a change of variables of type (2.1), with a suitably small η. It is therefore no restriction to assume that

$$\left|\frac{\partial^2 F(0)}{\partial x_i \partial x_j}-\delta_{iq}\right| < \varepsilon, \quad i\in\Lambda, \quad j\notin\Lambda,$$

and

$$\frac{\partial^2 F(0)}{\partial x_i \partial x_j} \neq 0, \quad \text{all} \quad i,j,$$

in addition to the conditions of Theorem 2. We then follow the pattern of the proof of Theorem 1. The only part of that proof where $F\in C^2$ is not sufficient is the estimate for $I''(x)$ is Section 4.

Consider first the case when $1\notin\Lambda$. Assuming $y_2, \dots, y_n$ as in $I(x)$, i.e. $y_i \sim 2^{-j_i}$, we shall make sure that $|\xi(y_2, \dots, y_n)| > C2^{-j_1}$ in $I(x)$, so that $I''(x)=0$. The mean value theorem and (4.1) imply

$$F''_{11}(\eta)\xi = -\sum_{i=2}^n F''_{1i}(\eta)\, y_i$$

for some $\eta\in\mathbf{R}^n$ with $|\eta|\leqq C2^{-k}$. In this sum, term number q is $-F''_{1q}(\eta)y_q \sim -2^{-k}$. The terms with $i\in\Lambda, i\neq q$ are at most $C\varepsilon 2^{-k}$. Those terms with $i\notin\Lambda$ are bounded by $C2^{-k-N}$, because $|y_i|\leqq C2^{-k-N}$ for these i. Since $|F''_{11}(\eta)|\leqq C$, it is then clear that we can choose ε and N so that this implies $|\xi|\sim 2^{-k} > C2^{-k-N} \geqq C2^{-j_1}$, as desired. Notice that this choice depends only on Λ and q. Thus by finiteness there exists one choice of ε and N which will do for all Λ and q.

Next we indicate how to estimate $I''(x)$ when $1\in\Lambda$. When $d=n$, we proceed as in Section 4. For $d<n$, we always use the argument of Part 2. Instead of (4.9), we now get

$$\xi^* = \sum_{d+1}^n b_i y_i + o(2^{-k}).$$

Since $2^{-j_1} \geqq 2^{-k-N}$, the remainder here is bounded by $C2^{-j_1}$ if we stay near enough to the origin. This implies (4.10), and we can argue as in Section 4 to complete the proof.

Proof of Theorem 3. As in the proof of Theorem 1, we can assume that the terms of order 0 and 1 in F vanish. We may further assume $h_1 \leqq h_2 \leqq \dots \leqq h_n$. There exists an m such that F is independent of $x_{m+1}, \dots, x_n$ but not independent of x_m.

If $F''_{mm}\neq 0$, we make the change of variables

$$x_m = x'_m + \eta \sum_1^{m-1} x_i$$

$$x_i = x'_i, \quad i \neq m.$$

It is easy to see that for a.a. η this transforms F to a quadratic form with nonvanishing $(x_i')^2$ terms for $i=1, \ldots, m$. Now apply Theorem 1 with $n=m$ to $x_1', \ldots, x_m', x_{n+1}'$ and the strong maximal function in the remaining variables. The conclusion follows, since we can have $\delta=\infty$ in the proof of Theorem 1 when F is a quadratic form.

Assume next $F''_{mm}=0$. Then F can be written

$$F = x_m \sum_{i=1}^{l} a_i x_i + P_1(x_1, \ldots, x_{m-1}),$$

where $l<m$ and $a_l \neq 0$, and P_1 is a quadratic form. The change of variables

$$x_l' = \sum_1^l a_i x_i$$

$$x_i' = x_i, \quad i \neq l,$$

gives

$$F = \left(x_m' + \sum_{i<m} b_i x_i'\right) x_l' + P_2,$$

where P_2 is a quadratic form in x_i', $1 \leq i < m$, $i \neq l$. Now let

$$x_m'' = x_m' + \sum_{i<m} b_i x_i'$$

$$x_i'' = x_i', \quad i \neq m,$$

so that

$$F = x_m'' x_l'' + P_2.$$

As in the proof of Theorem 2 in [1], M_F will be a superposition of two maximal function operators for quadratic surfaces in $\mathbf{R}^3$ and $\mathbf{R}^{m-1}$. Since Theorem 3 holds for $n=1$ [2, Theorem D p. 1248] and $n=2$ [1, Theorem 2], an obvious induction argument ends the proof.

A counterexample. We shall construct an $F \in C^\infty$ with prescribed values for F''_{ij} not satisfying the hypothesis of Theorem 1. The corresponding M_F is unbounded on all L^p, $p<\infty$, and not even L^∞ can be differentiated along the surface $x_{n+1} = F(x')$. We make the construction for $n=2$, since the general case is analogous.

From [2, Sect. III, 3] we know that there exists a C^∞ curve $t=\varphi(s)$ in the plane which does not differentiate L^∞ functions in $\mathbf{R}^2$. Moreover, φ and all its derivatives vanish at 0. We take

$$F(x_1, x_2) = \varphi(x_1) + a x_1 x_2 + b x_2^2.$$

For $f \in C^\infty$ one has

$$\frac{1}{h_1 h_2} \int_0^{h_1} dy_1 \int_0^{h_2} dy_2 f(x_1 - y_1, x_2 - y_2, x_3 - F(y_1, y_2)) \tag{5.2}$$

$$\to \frac{1}{h_1} \int_0^{h_1} f(x_1 - y_1, x_2, x_3 - \varphi(y_1))\, dy_1, \quad h_2 \to 0.$$

This gives an estimate for the maximal operator M_φ for the curve $(s, 0, \varphi(s))$ in terms of M_F. Since M_φ is unbounded on L^p, so is M_F.

Take an L^∞ function g in the plane which cannot be differentiated along the curve $t=\varphi(s)$. If the function $f(x_1, x_2, x_3)=g(x_1, x_3)$ satisfies (5.2) for a.a. x, then f cannot be differentiated along the surface, and we have the desired counter-example. Let us thus verify (5.2) for a.a. x when $f\in L^\infty$. Because of bounded convergence, it suffices to show that

$$\text{(5.3)} \qquad \frac{1}{h_2}\int_0^{h_2} f(x_1-y_1, x_2-y_2, x_3-\varphi(y_1)-ay_1y_2-by_2^2)\,dy_2$$
$$\to f(x_1-y_1, x_2, x_3-\varphi(y_1)), \quad h_2\to 0$$

for a.a. $(x, y_1)\in\mathbf{R}^4$. For each y_1, one can differentiate f a.e. along the curve $y_2\to ay_1y_2+by_2^2$. Hence, (5.3) holds for all x outside a null set E_{y_1}. Since the set $\{(y, y_1)\colon x\in E_{y_1}\}$ is measurable, (5.3) follows by Fubini's theorem.

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